MATHEMATICAL PHYSICS

for

M.Sc. Entrance

IIT-JAM | JNU | TIFR | JEST | BHU | ISM | JNCASR | DU

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Chapter 1

VECTOR ALGEBRA

1.1 Basic Review of Vectors

Definition:

Physical quantities having both magnitude and a definite direction in space. It should follow the law of vector addition.

Example: Velocity, Acceleration, Momentum, Force, Electric Field, Torque, etc.

Note: Current is a physical quantity that has both magnitude and direction but it does not follow the law of vector addition. So, current is a scalar quantity.

Various type of vectors:

(1) Equal vectors: Vectors having same magnitude and same direction.

(2) Null Vectors: Vectors having coincident initial and terminal point i.e. its magnitude is zero and it has any arbitrary direction.

(3) Unit Vector: Vector having unit magnitude. Unit vector along $\vec{a}$ is $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$

(4) Reciprocal Vector: Vector having same direction as $\vec{a}$ but magnitude reciprocal to that of $\vec{a}$, is known as the reciprocal vector of $\vec{a}$. Reciprocal vector of $\vec{a}$ is $\vec{a}^{-1} = \dfrac{1}{|a|}\hat{a}$

(5) Negative Vector: Vectors having same magnitude as $\vec{a}$ but direction opposite to that of $\vec{a}$, is known as the negative vector of $\vec{a}$. Negative vector as $\vec{a}$ is $-\vec{a} = -|a|\hat{a}$

Orthogonal Resolution of Vectors:

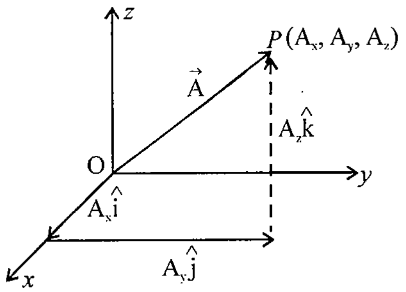
Any vector $\vec{A}$ in the 3-D right- handed rectangular cartesian coordinate system can be represented as

$$\overrightarrow{OP} = \vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k},$$

where, $\hat{i}, \hat{j}$ and $\hat{k}$ are the unit vectors in direction of x, y and z axis respectively and A_x, A_y, A_z are the rectangular components of vector $\vec{A}$ along x, y, z axis.

Magnitude of vector $\vec{A}$ is $|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$

Unit vector along $\vec{A}$ is $\hat{A} = \vec{A}/|\vec{A}| = \left(A_x\hat{i} + A_y\hat{j} + A_z\hat{k}\right)/\sqrt{A_x^2 + A_y^2 + A_z^2}$



Direction cosines of vector $\vec{A}$:

If $\vec{A}$ makes angles α, β, γ with x, y and z axes respectively, then direction cosines of $\vec{A}$ are defined as

$$l = \cos\alpha = \frac{A_x}{A};\ m = \cos\beta = \frac{A_y}{A};\ n = \cos\gamma = \frac{A_z}{A} \quad \text{and } l^2 + m^2 + n^2 = 1$$

Note: Unit vector along $\vec{A}$ can be written as $\hat{A} = l\hat{i} + m\hat{j} + n\hat{k}$

1.2 Products of Vectors

Scalar Product or Dot Product:

Dot product of two vectors $\vec{a}$ and $\vec{b}$ are defined as

$$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta \Rightarrow \cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$$

Note:

(i) Dot product is commutative in nature i.e. $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$

(ii) For two mutually perpendicular vectors $\vec{a}$ and $\vec{b}$, $\vec{a}\cdot\vec{b} = 0$

(iii) $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0,\ \hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

(iv) If $\vec{a} = a_z\hat{i} + a_y\hat{j} + a_z\hat{k}$ and $\vec{b} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$, then $\vec{a}.\vec{b} = a_x b_x + a_y b_y + a_z b_z$

(v) Projection of $\vec{A}$ on $\vec{B} = \vec{A}.\hat{B}$

Vector Product or Cross Product :

Cross product of two vectors $\vec{a}$ and $\vec{b}$ are defined as $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$

where $\hat{n}$ is unit vector normal to the plane containing $\vec{a}$ and $\vec{b}$.

Note:

(i) Cross product is not commutative i.e. $\vec{a}\times\vec{b} = -(\vec{b}\times\vec{a})$

(ii) For two collinear vectors (parallel or anti-parallel vectors) $\vec{a}\times\vec{b} = 0$.

(iii) $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0,\ \hat{i}\times\hat{j} = \hat{k}, \hat{j}\times\hat{k} = \hat{i}, \hat{k}\times\hat{i} = \hat{j}$

(iv) If $\vec{a} = a_z\hat{i} + a_y\hat{j} + a_z\hat{k}$ and $\vec{b} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$, then $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$

Example 1: The range of x for which the angle between the vectors

$$\vec{A} = 2x^2\hat{i} + 4x\hat{j} + \hat{k} \text{ and } \vec{B} = 7\hat{i} - 2\hat{j} + x\hat{k}$$

is obtuse, is equal to

(a) $x < 0$ (b) $x > 1/2$ (c) $0 < x < 1/2$ (d) None of these

Soln: Since, $90^0 < \theta < 180^0$, then $\cos\theta < 0 \Rightarrow \vec{A}.\vec{B} < 0 \Rightarrow 14x^2 - 7x < 0 \Rightarrow x(2x-1) < 0$

Case I: $x > 0$ and $(2x-1) < 0 \Rightarrow x < \frac{1}{2}$. Therefore, $0 < x < \frac{1}{2}$

Case II: $x < 0$ and $(2x-1) > 0 \Rightarrow x > \frac{1}{2}$. Both of them cannot be satisfied simultaneously.

Correct option is (c)

Example 2: Given: $\vec{a} = 3\hat{i} - 2\hat{j} + \hat{k}, \vec{b} = \hat{i} - 3\hat{j} + 5\hat{k}$ and $\vec{c} = 2\hat{i} + \hat{j} - 4\hat{k}$ are the sides of

(a) an isosceles triangle (b) a right-angled triangle

(c) an equilateral triangle (d) none of these

Soln: For the given set of vectors, $\vec{a}.\vec{c} = 0 \Rightarrow \vec{a}$ is perpendicular to $\vec{c}$ and

$$|\vec{a}|^2 + |\vec{c}|^2 = 14 + 21 = 35 = |\vec{b}|^2$$

Therefore, $\vec{a}$, $\vec{c}$ will correspond to base and height respectively and $\vec{b}$ will correspond to hypotenus i.e. the given set of vectors are sides of a right-angled triangle.

Correct option is (b)

Example 3: Given : $\vec{P} = 2\hat{i} - 3\hat{j}, \vec{Q} = \hat{i} + \hat{j} - \hat{k}, \vec{R} = 3\hat{i} - \hat{k}$, the vector $\vec{V}$, which is orthogonal to $\vec{P}$ and $\vec{Q}$ and having unit scalar product with $\vec{R}$, can be estimated to be

(a) $(3\hat{i} + 2\hat{j} + 5\hat{k})/4$ (b) $(3\hat{i} + 2\hat{j} - 5\hat{k})/4$ (c) $(3\hat{i} - 2\hat{j} + 5\hat{k})/4$ (d) $(3\hat{i} - 2\hat{j} - 5\hat{k})/4$

Soln: Unit vector along $\vec{V}$ is $\hat{V} = \frac{\vec{P}\times\vec{Q}}{|\vec{P}\times\vec{Q}|} = \frac{3\hat{i} + 2\hat{j} + 5\hat{k}}{\sqrt{38}}$

According to the question, $\vec{V}.\vec{R} = 1 \Rightarrow (V\hat{V}).\vec{R} = 1 \Rightarrow V = \frac{\sqrt{38}}{4}$

Correct option is (a)

Example 4: For any vector $\vec{a}$, the value of $|\vec{a}\times\hat{i}|^2 + |\vec{a}\times\hat{j}|^2 + |\vec{a}\times\hat{k}|^2$ is equal to

(a) $|\vec{a}|^2$ (b) $2|\vec{a}|^2$ (c) $\vec{a}^2$ (d) $2\vec{a}^2$

Soln: Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ (Here, a_1, a_2, a_3 are not constants)

$\therefore \quad \vec{a}\times i = a_2\hat{k} - a_3\hat{j}$

$$\left|\vec{a}\times\hat{i}\right| = \sqrt{a_2^2 + a_3^2}$$

Similarly, $\left|\vec{a}\times\hat{j}\right| = \sqrt{a_1^2 + a_3^2}$ and $\left|\vec{a}\times\hat{k}\right| = \sqrt{a_1^2 + a_2^2}$

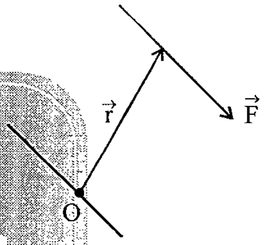
$\therefore \quad \left|\vec{a}\times\hat{i}\right|^2 + \left|\vec{a}\times\hat{j}\right|^2 + \left|\vec{a}\times\hat{k}\right|^2 = 2\left(a_1^2 + a_2^2 + a_3^2\right) = 2\left|\vec{a}\right|^2$

Correct option is (b)

Example 5: If $\vec{F} = \hat{i} + 3\hat{j} - \hat{k}$ acts at the point (1, 1, 1), then the torque of $\vec{F}$ about the line $\vec{r} = 3\hat{i} + 2\hat{k} + (2\hat{i} - 2\hat{j} + \hat{k})t$ is_________.

Soln: We first find the vector torque about a point on the line, say the point (3, 0, 2).

This is $\vec{r}\times\vec{F}$, where $\vec{r}$ is the vector from the point about which we want the torque, to the point at which $\vec{F}$ acts, that is, from (3, 0, 2) to (1, 1, 1).



$\therefore \quad \vec{r} = \left(\hat{i} + \hat{j} + \hat{k}\right) - \left(3\hat{i} + 2\hat{k}\right) = -2\hat{i} + \hat{j} - \hat{k}$

$$\therefore \quad \vec{r}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & -1 \\ 1 & 3 & -1 \end{vmatrix} = 2\hat{i} + 3\hat{j} - 7\hat{k}$$

The torque about the line is $\hat{n}\cdot\left(\vec{r}\times\vec{F}\right)$ where $\hat{n}$ is a unit vector along the line $\hat{n} = \frac{1}{3}\left(2\hat{i} - 2\hat{j} + \hat{k}\right)$

Therefore, torque about the line, $\hat{n}\cdot\left(\vec{r}\times\vec{F}\right) = 1$

Correct answer is (1)

Example 6: Which of the following vectors is orthogonal to the vector $\left(a\hat{i} + b\hat{j}\right)$, where a and b $\left(a \neq b\right)$ are constants, and $\hat{i}$ and $\hat{j}$ are unit orthogonal vectors?

(a) $-b\hat{i} + a\hat{j}$ (b) $-a\hat{i} + b\hat{j}$ (c) $-a\hat{i} - b\hat{j}$ (d) $-b\hat{i} - a\hat{j}$

Soln: $\left(a\hat{i} + b\hat{j}\right).\left(-b\hat{i} + a\hat{j}\right) = -ab + ab = 0$

Correct option is (a)

Scalar Triple Product:

Scalar triple product of three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are defined as

$$\vec{a}\cdot(\vec{b}\times\vec{c})=\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}=[abc]$$

Note:

(i) $\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b})$ i.e. $[abc]=[bca]=[cab]$

(ii) Volume of a parallelopiped having $\vec{a},\vec{b},\vec{c}$ as concurrent edges is :V=$\left|\vec{a}\cdot(\vec{b}\times\vec{c})\right|$

(iii) If $\vec{a},\vec{b},\vec{c}$ are coplanar vectors, then, $\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b}) = 0$

Vector Triple Product:

Vector triple product of three vectors $\vec{a},\vec{b}$ and $\vec{c}$ are defined as

$$\vec{a}\times(\vec{b}\times\vec{c})=\vec{b}(\vec{a}\cdot\vec{c})-\vec{c}(\vec{a}\cdot\vec{b})$$

It represents a vector coplanar with $\vec{b}$ and $\vec{c}$ and perpendicular to $\vec{a}$.

Note:

(i) $\vec{a}\times\left(\vec{b}\times\vec{c}\right)+\vec{b}\times\left(\vec{c}\times\vec{a}\right)+\vec{c}\times\left(\vec{a}\times\vec{b}\right)=0$

Example 7: The value of m for which the vectors $\vec{A}=\hat{i}-\hat{j}-2\hat{k}, \vec{B}=3\hat{i}+5\hat{j}+6\hat{k}, \vec{C}=-\hat{i}+4\hat{j}+m\hat{k}$ will be co-planar is

(a) 13/2 (b) 7/2 (c) 5/2 (d) 11/2

Soln: Since, $\vec{A},\vec{B},\vec{C}$ are co-planar, $\vec{A}.\left(\vec{B}\times\vec{C}\right)=0\Rightarrow\begin{vmatrix} 1 & -1 & -2 \\ 3 & 5 & 6 \\ -1 & 4 & m \end{vmatrix}=0\Rightarrow m=\frac{13}{2}$

Correct option is (a)

Example 8: If $\vec{b}=\hat{i}\times\left(\vec{a}\times\hat{i}\right)+\hat{j}\times\left(\vec{a}\times\hat{j}\right)+\hat{k}\times\left(\vec{a}\times\hat{k}\right)$, then $\vec{b}$ can be simplified to

(a) 0 (b) $\vec{a}$ (c) $2\vec{a}$ (d) None of these

Soln: $\vec{b}=\hat{i}\times\left(\vec{a}\times\hat{i}\right)+\hat{j}\times\left(\vec{a}\times\hat{j}\right)+\hat{k}\times\left(\vec{a}\times\hat{k}\right)$

$=\vec{a}\left(\hat{i}.\hat{i}\right)-\hat{i}\left(\hat{i}.\vec{a}\right)+\vec{a}\left(\hat{j}.\hat{j}\right)-\hat{j}\left(\hat{j}.\vec{a}\right)+\vec{a}\left(\hat{k}.\hat{k}\right)-\hat{k}\left(\hat{k}.\vec{a}\right)=3\vec{a}-\vec{a}=2\vec{a}$

$=\left(\vec{a}+\vec{a}+\vec{a}\right)-\hat{i}\left(a_x\right)-\hat{j}\left(a_y\right)-\hat{k}\left(a_z\right)$

$=3\vec{a}-\vec{a}-=2\vec{a}$

Correct option is (c)

Example 9: Three unit vectors $\vec{a},\vec{b},\vec{c}$ ($\vec{b}$ and $\vec{c}$ are not parallel) are such that $\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\frac{\sqrt{3}}{2}\vec{c}$. The angles which $\vec{a}$ makes with $\vec{b}$ and $\vec{c}$, respectively are

(a) $30^0, 90^0$ (b) $150^0, 90^0$ (c) $60^0, 90^0$ (d) $90^0, 30^0$

Soln: $\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\frac{\sqrt{3}}{2}\vec{c}\Rightarrow\vec{b}\left(\vec{a}.\vec{c}\right)-\vec{c}\left(\vec{a}.\vec{b}\right)=\frac{\sqrt{3}}{2}\vec{c}$

Comparing coefficients of $\vec{b}$ and $\vec{c}$ on both sides.

$\vec{a}\cdot\vec{c}=0 \;\Rightarrow\; \vec{a}\perp\vec{c}$

$$\vec{a}\cdot\vec{b}=\frac{\sqrt{3}}{2}\Rightarrow ab\cos\theta=-\frac{\sqrt{3}}{2}\;\Rightarrow\;\cos\theta=-\frac{\sqrt{3}}{2}\Rightarrow\theta=150^\circ$$

The angle which $\vec{a}$ makes with $\vec{b}$ and $\vec{c}$ are $150^0, 90^0$ respectively.

Correct option is (b)

Example 10: If $\vec{a},\vec{b}$ and $\vec{c}$ are non-coplanar unit vectors such that $\vec{a}\times(\vec{b}\times\vec{c})=\frac{\vec{b}+\vec{c}}{\sqrt{2}}$, then the angle between $\vec{a}$ and $\vec{b}$ is

(a) $\frac{\pi}{2}$ (b) π (c) $\frac{\pi}{3}$ (d) $\frac{3\pi}{4}$

Soln: Since, $\vec{a}\times(\vec{b}\times\vec{c})=\frac{\vec{b}+\vec{c}}{\sqrt{2}}$ or $\vec{b}(\vec{a}\cdot\vec{c})-\vec{c}(\vec{a}\cdot\vec{b})=\frac{\vec{b}}{\sqrt{2}}+\frac{\vec{c}}{\sqrt{2}}$

On comparing both side, we have $-(\vec{a}\cdot\vec{b})=\frac{1}{\sqrt{2}}\;\Rightarrow\;\vec{a}\cdot\vec{b}=-\frac{1}{\sqrt{2}}$

If θ is the angle between $\vec{a}$ and $\vec{b}$, then $\cos\theta=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}=\vec{a}\cdot\vec{b}=-\frac{1}{\sqrt{2}}$

Therefore, $\theta=\frac{3\pi}{4}$.

Correct option is (d)

1.3 Gradient, Divergence and Curl

Gradient of a Scalar Field:

Gradient of a continuously differentiable scalar function $\phi\,(x, y, z)$ is mathematically defined as:

$$grad\ \phi=\vec{\nabla}\phi=\left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)\phi=\frac{\partial\phi}{\partial x}\hat{i}+\frac{\partial\phi}{\partial y}\hat{j}+\frac{\partial\phi}{\partial z}\hat{k}$$

where, $\vec{\nabla}=\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\Rightarrow$ 'del' or 'grad' or 'nabla' operator

Physical interpretation: Gradient of scalar function ϕ at any point P(x, y, z) is a vector quantity, whose magnitude is equal to the rate of change of scalar function ϕ with distance along the normal to level surface and its direction is along the normal to the level surface at that point. The direction of $\vec{\nabla}\phi$ is that along which the rate of change of ϕ is maximum.

Level surface: At each point of the level surafce, the value of scalar function ϕ will be same.

Example: Equipotential surface on which value of electrostatic potential is same at all points

Note:

(i) Normal vector to the level surface: $\vec{\nabla}\phi$

(ii) Unit normal vector to the level surface: $\hat{n} = \dfrac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|}$

Directional Derivative: Directional derivative of ϕ in the direction of $\vec{A}$ is defined as rate of change of ϕ with distance along the direction of $\vec{A}$. It is mathematically defined as the component of $\vec{\nabla}\phi$ in the direction of vector $\vec{A}$ i.e. $\vec{\nabla}\phi\cdot\hat{A} = \vec{\nabla}\phi\cdot\dfrac{\vec{A}}{|\vec{A}|}$

Tangent Plane and Normal to the level surface:

Consider $\phi(x,y,z) = c$ be the equation of a level surface. and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector of any point P (x,y,z) on this surface.

Tangent plane at P: $\vec{\nabla}\phi$ is a vector normal to the surface i.e. it is perpendicular to the tangent plane at P. Let, $\vec{R} = X\hat{i} + Y\hat{j} + Z\hat{k}$ be the position vector of any point on the tangent plane at P to the surface. Therefore, $\vec{R} - \vec{r} = (X-x)\hat{i} + (Y-y)\hat{j} + (Z-z)\hat{k}$ lies in the tangent plane at P and it will be perpendicular to $\vec{\nabla}\phi$

i.e. $(\vec{R} - \vec{r}).\vec{\nabla}\phi = 0$

$\Rightarrow (X-x)\dfrac{\partial\phi}{\partial x} + (Y-y)\dfrac{\partial\phi}{\partial y} + (Z-z)\dfrac{\partial\phi}{\partial z} = 0$, which is the tangent plane at point P.

Example 11: Find the unit vector perpendicular to the surface $x^2 + y^2 + z^2 = 6$ at point P $(2,1,1)$.

Soln: Unit normal to the surface $\hat{n} = \left.\dfrac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|}\right|_{(2,1,1)} = \left.\dfrac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}}\right|_{(2,1,1)} = \dfrac{2\hat{i} + \hat{j} + \hat{k}}{\sqrt{6}}$

Example 12: Find the angle between the two surfaces $5x + y + z = 1$ and $3x + 3y + 3z = 5$.

Soln: Given, $\phi_1 : 5x + y + z = 1$ and $\phi_2 : 3x + 3y + 3z = 5$

Therefore, $\cos\theta = \dfrac{\vec{\nabla}\phi_1.\vec{\nabla}\phi_2}{|\vec{\nabla}\phi_1||\vec{\nabla}\phi_2|} = \dfrac{(5\hat{i} + \hat{j} + \hat{k}).(3\hat{i} + 3\hat{j} + 3\hat{k})}{\sqrt{27}.\sqrt{27}} = \dfrac{21}{27} \Rightarrow \theta = \cos^{-1}\left(\dfrac{7}{9}\right)$

Example 13: Find the directional derivative of $f\left(x,y\right)=x^2y^3-y^4$ at (2,1) in the direction given by the angle $\theta=\frac{\pi}{4}$.

Soln: Unit vector along which the directioanl derivative is to be calculated is

$$\hat{A}=\cos\frac{\pi}{4}\hat{i}+\sin\frac{\pi}{4}\hat{j}=\frac{1}{\sqrt{2}}\left(\hat{i}+\hat{j}\right)$$

Therefore, directional derivative will be

$$\vec{\nabla}f\Big|_{(2,1)}.\hat{A}=\left[\left(2xy^3\right)\hat{i}+\left(3x^2y^2-4y^3\right)\hat{j}\right]_{(2,1)}.\frac{1}{\sqrt{2}}\left(\hat{i}+\hat{j}\right)=6\sqrt{2}$$

Example 14: If grad $\phi=\frac{\hat{r}}{r^4}$ & $\phi\left(r\right)=0$ at $r=1$, then find ϕ.

Soln: $\vec{\nabla}\phi=\frac{\hat{r}}{r^4}\Rightarrow\frac{\partial\phi}{\partial r}\hat{r}=\frac{\hat{r}}{r^4}\Rightarrow\frac{\partial\phi}{\partial r}=\frac{1}{r^4}\Rightarrow\phi=-\frac{1}{3r^3}+C$

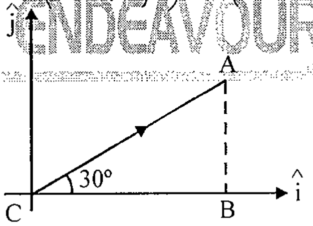
Since, at $r=1$, $\phi\left(r\right)=0\Rightarrow0=-\frac{1}{3}+C\Rightarrow\text{C}=\frac{1}{3}$

Putting this we get $\phi\left(r\right)=-\frac{1}{3r^3}+\frac{1}{3}$

Example 15: For the function $\phi\left(x,y\right)=\frac{x}{x^2+y^2}$, find the magnitude of the directional derivative along a line making an angle 30° with the positive x-axis at (0, 2)

Soln: $\vec{\nabla}\phi=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\frac{x}{x^2+y^2}=\hat{i}\left(\frac{1}{x^2+y^2}-\frac{x(2x)}{\left(x^2+y^2\right)^2}\right)-\hat{j}\frac{x(2y)}{\left(x^2+y^2\right)^2}$

$$=\hat{i}\frac{y^2-x^2}{\left(x^2+y^2\right)^2}-\hat{j}\frac{2xy}{\left(x^2+y^2\right)^2}$$



$\vec{\nabla}\phi$ at the point (0, 2)

$$=\hat{i}\frac{4-0}{\left(0+4\right)^2}-\hat{j}\frac{2(0)(2)}{\left(0+4\right)^2}=\frac{\hat{i}}{4}$$

Direction along which Directional derivative is to be calculated is

$$\overline{CA}=\cos30°\hat{i}+\sin30°\hat{j}=\frac{\sqrt{3}}{2}\hat{i}+\frac{1}{2}\hat{j}$$

Directional derivative at the point (0, 2) in the direction $\overline{CA}$

$$=\frac{\hat{i}}{4}.\left(\frac{\sqrt{3}}{2}\hat{i}+\frac{1}{2}\hat{j}\right)=\frac{\sqrt{3}}{8}$$

Example 16: Find the directional derivative of $\vec{V}^2$, where $\vec{V} = xy^2\hat{i} + zy^2\hat{j} + xz^2\hat{k}$, at the point (2, 0, 3) in the direction of the outward normal to the sphere $x^2 + y^2 + z^2 = 14$ at the point (3, 2, 1).

Soln: $V^2 = \vec{V}.\vec{V} = V^2 = \vec{V}.\vec{V} = \left(xy^2\hat{i} + zy^2\hat{j} + xz^2\hat{k}\right).\left(xy^2\hat{i} + zy^2\hat{j} + xz^2\hat{k}\right) = x^2y^4 + z^2y^4 + x^2z^4$

$$\nabla V^2 = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(x^2y^4 + z^2y^4 + x^2z^4\right)$$

$$= \left(2xy^4 + 2xz^4\right)\hat{i} + \left(4x^2y^3 + 4y^3z^2\right)\hat{j} + \left(2y^4z + 4x^2z^3\right)\hat{k}$$

∇V^2 at (2, 0, 3) $= \left(0 + 2\times2\times81\right)\hat{i} + \left(0+0\right)\hat{j} + \left(0 + 4\times4\times27\right)\hat{k}$

$$= 324\hat{i} + 432\hat{k} = 108\left(3\hat{i} + 4\hat{k}\right)$$

Normal to $x^2 + y^2 + z^2 - 14 = \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(x^2 + y^2 + z^2 - 14\right)$

$$= \left(2x\hat{i} + 2y\hat{j} + 2z\hat{k}\right)$$

Normal vector at (3, 2, 1) $= 6\hat{i} + 4\hat{j} + 2\hat{k}$

Unit normal vector $= \dfrac{6\hat{i} + 4\hat{j} + 2\hat{k}}{\sqrt{36+16+4}} = \dfrac{2\left(3\hat{i} + 2\hat{j} + \hat{k}\right)}{2\sqrt{14}} = \dfrac{3\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{14}}$

Direction derivative along the normal $= 108\left(3\hat{i} + 4\hat{k}\right).\dfrac{3\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{14}} = \dfrac{108\times(9+4)}{\sqrt{14}} = \dfrac{1404}{\sqrt{14}}$

Example 17: The plane tangent to $x^2 + y^2 + z^2 = 4$ at the point $\left(1, 1, \sqrt{2}\right)$ is

(a) $z = \frac{1}{\sqrt{2}}(x-1) - \frac{1}{\sqrt{2}}(y-1) + \sqrt{2}$ (b) $z = -\frac{1}{\sqrt{2}}(x-1) - \frac{1}{\sqrt{2}}(y-1) + \sqrt{2}$

(c) $z = -(x-1) - (y-1) + \sqrt{2}$ (d) $z = -\frac{1}{\sqrt{2}}(x-1) + (y-1) + \sqrt{2}$

Soln: Let $\phi : x^2 + y^2 + z^2 - 4 = 0$

$\therefore \quad \frac{\partial\phi}{\partial x} = 2x \Rightarrow \left.\frac{\partial\phi}{\partial x}\right|_{1,1,\sqrt{2}} = 2, \ \frac{\partial\phi}{\partial y} = 2y \Rightarrow \left.\frac{\partial\phi}{\partial y}\right|_{1,1,\sqrt{2}} = 2, \ \frac{\partial\phi}{\partial z} = 2z \Rightarrow \left.\frac{\partial\phi}{\partial y}\right|_{1,1,\sqrt{2}} = 2\sqrt{2}$

Therefore, equation of tangent plane at the point $\left(1, 1, \sqrt{2}\right)$

$$(x-1)\times2 + (y-1)\times2 + \left(z - \sqrt{2}\right)\times2\sqrt{2} = 0$$

$$\Rightarrow z - \sqrt{2} = -\frac{1}{\sqrt{2}}(x-1) - \frac{1}{\sqrt{2}}(y-1)$$

Correct option is (b)

Example 18: Suppose the temperature at (x, y, z) is given by $T = xyz$. In what direction can you go from the point $(1,1,1)$ to maintain the same temperature?

(a) $-\hat{i} + 2\hat{j} - \hat{k}$ (b) $\hat{i} + \hat{j} + \hat{k}$ (c) $\hat{i} - \hat{j} + \hat{k}$ (d) none

Soln: To maintain the same temperature, we must travel perpendicular to the direction of $grad\left(T\right)$.

Here, $\vec{\nabla} T = yz\hat{i} + xz\hat{j} + xy\hat{k}$

$\therefore \quad \vec{\nabla} T\Big|_{(1,1,1)} = \hat{i} + \hat{j} + \hat{k}$

and $\left(\hat{i} + \hat{j} + \hat{k}\right) \cdot \left(-\hat{i} + 2\hat{k} - \hat{k}\right) = 0$

Correct option is (a)

Example 19: If 'f' is a scalar function such that $\vec{\nabla} f = (3 + 2xy)\,\hat{i} + (x^2 - 3y^2)\,\hat{j}$, then the value of f at the point (1, 1) is_________.

Soln: Since, $\vec{\nabla} f = (3 + 2xy)\,\hat{i} + (x^2 - 3y^2)\,\hat{j}$

$$\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} = (3 + 2xy)\,\hat{i} + \left(x^2 - 3y^2\right)\hat{j}$$

$$\therefore \quad \frac{\partial f}{\partial x} = 3 + 2xy \Rightarrow f = 3x + x^2 y \qquad \text{... (i)}$$

$$\text{and} \quad \frac{\partial f}{\partial y} = x^2 - 3y^2 \Rightarrow f = x^2 y - y^3 \qquad \text{... (ii)}$$

Hence, $f(x, y) = x^2 y - y^3 + 3x$

Therefore, value of $f(x, y)$ at (1, 1) is 3.

Correct answer is (3)

Example 20: The unit vector normal to the surface $3x^2 + 4y = z$ at the ponit (1, 1, 7) is:

(a) $\left(-6\hat{i} + 4\hat{j} + \hat{k}\right)/\sqrt{53}$ (b) $\left(4\hat{i} + 6\hat{j} - \hat{k}\right)/\sqrt{53}$

(c) $\left(6\hat{i} + 4\hat{j} - \hat{k}\right)/\sqrt{53}$ (d) $\left(4\hat{i} + 6\hat{j} + \hat{k}\right)/\sqrt{53}$

Soln: Unit vector normal to the surface $3x^2 + 4y = z$ at the ponit (1, 1, 7) is

$$\hat{n} = \frac{\vec{\nabla}\phi}{\left|\vec{\nabla}\phi\right|} = \frac{\left(6x\,\hat{i} + 4\hat{j} - \hat{k}\right)}{\sqrt{36x^2 + 16 + 1}}\Bigg|_{(1,1,7)} = \frac{\left(6\,\hat{i} + 4\hat{j} - \hat{k}\right)}{\sqrt{53}}$$

Correct option is (c)

Example 21: For the function $\phi = x^2 y + xy$ the value of $\left|\vec{\nabla}\phi\right|$ at $x = y = 1$ is

(a) 5 (b) $\sqrt{5}$ (c) 13 (d) $\sqrt{13}$

Soln: $\phi = x^2 y + xy \Rightarrow \vec{\nabla}\phi = (2xy + y)\,\hat{i} + \left(x^2 + x\right)\hat{j}$

$\vec{\nabla}\phi\Big|_{(1,1,1)} = 3\hat{i} + 2\hat{j}$; $\left|\vec{\nabla}\phi\right|_{(1,1,1)} = \sqrt{13}$

Correct option is (d)

Example 22: The direction of $\vec{\nabla} f$ for a scalar field $f(x,y,z)=\frac{1}{2}x^2-xy+\frac{1}{2}z^2$ at the point $P(1, 1, 2)$ is

(a) $\frac{(-\hat{j}-2\hat{k})}{\sqrt{5}}$ (b) $\frac{(-\hat{j}+2\hat{k})}{\sqrt{5}}$ (c) $\frac{(\hat{j}-2\hat{k})}{\sqrt{5}}$ (d) $\frac{(\hat{j}+2\hat{k})}{\sqrt{5}}$

Soln: We have, $f(x,zy,z)=\frac{1}{2}x^2-xy+\frac{1}{2}z^2$

$$\therefore \quad \vec{\nabla} f=\left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)\left(\frac{1}{2}x^2-xy+\frac{1}{2}z^2\right)=(x-y)\hat{i}+(-x)\hat{j}+z\hat{k}$$

$$\therefore \quad \vec{\nabla} f\Big|_{(1,1,2)}=\left(-\hat{j}+2\hat{k}\right)$$

Therefore, direction of $\vec{\nabla} f$ is $\frac{-\hat{j}+2\hat{k}}{\sqrt{5}}$

Correct option is (b)

Example 23: If $\vec{A}$ and $\vec{B}$ are constant vectors, then $\vec{\nabla}\left[\vec{A}.\left(\vec{B}\times\vec{r}\right)\right]$ is

(a) $\vec{A}.\vec{B}$ (b) $\vec{A}\times\vec{B}$ (c) $\vec{r}$ (d) zero

Soln: $\vec{\nabla}\left[\vec{A}\cdot\left(\vec{B}\times\vec{r}\right)\right]=\vec{\nabla}\left[\vec{r}\cdot\left(\vec{A}\times\vec{B}\right)\right]=\vec{\nabla}\left[\left(\vec{A}\times\vec{B}\right)\cdot\vec{r}\right]$

If $\vec{A}$ and $\vec{B}$ are constant vectors, then $\left(\vec{A}\times\vec{B}\right)$ is also constant vector. For constant vector $\vec{a}$

$$\vec{\nabla}\left(\vec{a}\cdot\vec{r}\right)=\vec{a}$$

Therefore, $\vec{\nabla}\left[\left(\vec{A}\times\vec{B}\right)\cdot\vec{r}\right]=\left(\vec{A}\times\vec{B}\right)$

Correct option is (b)

Divergence of a vector field:

Divergence of a continuous differentiable vector point function $\vec{V}$ specified in a vector field is given by,

$$\vec{\nabla}.\vec{V}=\left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)\cdot\left(V_x\hat{i}+V_y\hat{j}+V_z\hat{k}\right)=\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z}=\text{Scalar quantity}$$

Physical Interpetation: Divergence of $\vec{V}$ at point $P(x, y, z)$ is defined as the outward flux of the vector field $\vec{V}$ per unit volume enclosed by an infinitesimal closed surface surrounding point P.

Note:

(i) If $\vec{\nabla}.\vec{V}=0$, then $\vec{V}$ is known as solenoidal vector field.

(ii) If $\vec{\nabla}.\vec{V}=$ negative, then $\vec{V}$ is known as sink field i.e. vector lines are going inward.

(iii) If $\vec{\nabla}.\vec{V}=$ positive, then $\vec{V}$ is known as source field i.e. vector lines are the going outward.

(iv) $\vec{\nabla}\cdot\left(\vec{U}+\vec{V}\right)=\vec{\nabla}\cdot\vec{U}+\vec{\nabla}\cdot\vec{V}$ (v) $\vec{\nabla}.\left(u\vec{V}\right)=(\vec{\nabla}u)\cdot\vec{V}+u(\vec{\nabla}.\vec{V})$

Example 24: A vector $\vec{A}=(5x+2y)\hat{i}+(3y-z)\hat{j}+(2x-az)\hat{k}$ is solenoidal if the constant a has a value.

(a) 4 (b) –4 (c) 8 (d) –8

Soln: $\vec{A}$ will solenoidal if $\vec{\nabla}\cdot\vec{A}=0\Rightarrow 5+3-a=0\ \Rightarrow a=8$

Correct option is (c)

Example 25: If $\vec{r}=x\hat{i}+y\hat{j}$, then

(a) $\vec{\nabla}\cdot\vec{r}=0$ and $\vec{\nabla}|\vec{r}|=\vec{r}$ (b) $\vec{\nabla}\cdot\vec{r}=2$ and $\vec{\nabla}|\vec{r}|=\vec{r}$

(c) $\vec{\nabla}\cdot\vec{r}=2$ and $\vec{\nabla}|\vec{r}|=\frac{\vec{r}}{r}$ (d) $\vec{\nabla}\cdot\vec{r}=3$ and $\vec{\nabla}|\vec{r}|=\frac{\vec{r}}{r}$

Soln: $\vec{r}=x\hat{i}+y\hat{j}\ \Rightarrow\vec{\nabla}\cdot\vec{r}=2$

$$\vec{\nabla}|\vec{r}|=\vec{\nabla}r=1\cdot\hat{r}=\frac{\vec{r}}{|\vec{r}|}$$

Correct option is (c)

Curl of a vector field:

Curl of a continuous differentiable vector point function $\vec{V}$ specified in a vector field is given by,

$$\vec{\nabla}\times\vec{V}=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\times\left(V_x\hat{i}+V_y\hat{j}+V_z\hat{k}\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ V_x & V_y & V_z\end{vmatrix}$$

It is the measurement of rotation of vector field and the direction of curl of vector is along axis of rotation.

Note:

(i) If $\vec{\nabla}\times\vec{V}=0$, then $\vec{V}$ is known as an irrotational vector and we can write $\vec{V}=\vec{\nabla}\phi$

(ii) If $\vec{\nabla}\times\vec{V}\neq 0$, then $\vec{V}$ is known as rotational vector.

(iii) $\vec{\nabla}\times(\vec{U}+\vec{V})=\vec{\nabla}\times\vec{U}+\vec{\nabla}\times\vec{V}$

(iv) $\vec{\nabla}\times(u\vec{V})=u\vec{\nabla}\times\vec{V}+(\vec{\nabla}u)\times\vec{V}$

Example 26: The curl of the vector $\vec{A}=z\hat{i}+x\hat{j}+y\hat{k}$ is given by

(a) $\hat{i}+\hat{j}+\hat{k}$ (b) $\hat{i}-\hat{j}+\hat{k}$ (c) $\hat{i}+\hat{j}-\hat{k}$ (d) $-\hat{i}-\hat{j}-\hat{k}$

Soln: $\vec{\nabla}\times\vec{A}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ z & x & y\end{vmatrix}=\left(\hat{i}+\hat{j}+\hat{k}\right)$

Correct option is (a)

Example 27: If a vector field $\vec{F} = x\hat{i} + 2y\hat{j} + 3z\hat{k}$, then $\vec{\nabla}\times\left(\vec{\nabla}\times\vec{F}\right)$ is

(a) 0 (b) $\hat{i}$ (c) $2\hat{j}$ (d) $3\hat{k}$

Soln: $\vec{F} = x\hat{i} + 2y\hat{j} + 3z\hat{k}$

$\vec{\nabla}\times\vec{F} = 0 \;\Rightarrow \vec{\nabla}\times\left(\vec{\nabla}\times\vec{F}\right) = 0$

Correct option is (a)

Example 28: If a force $\vec{F}$ is derivable from a potential function $V(r)$, where r is the distance from the origin of the coordinate system, it follows that

(a) $\vec{\nabla}\times\vec{F} = 0$ (b) $\vec{\nabla}\cdot\vec{F} = 0$ (c) $\vec{\nabla}V = 0$ (d) $\nabla^2 V = 0$

Soln: If force $\vec{F}$ is derivable from a potential function $V(r)$, then it will be conservative in nature i.e. $\vec{\nabla}\times\vec{F} = 0$

Correct option is (a)

Example 29: A fluid motion is given by

$$\vec{v} = (y\sin z - \sin x)\hat{i} + (x\sin z + 2yz)\hat{j} + \left(xy\cos z + y^2\right)\hat{k}$$

is the motion irrotational? If so, find the velocity potential

Soln: $\text{Curl}\,\vec{v} = \vec{\nabla}\times\vec{v}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y\sin z - \sin x & x\sin z + 2yz & xy\cos z + y^2 \end{vmatrix}$$

$$= (x\cos z + 2y - x\cos z - 2y)\hat{i} - [y\cos z - y\cos z]\hat{j} + (\sin z - \sin z)\hat{k} = 0$$

Hence, the motion is irrotational.

So, $\vec{v} = \vec{\nabla}\phi$ where ϕ is called velocity potential.

$$\Rightarrow \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right) = \vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$$

Comparing both sides we get,

$$\frac{\partial\phi}{\partial x} = (y\sin z - \sin x) \Rightarrow \phi = xy\sin z + \cos x + f_1(y,z)$$

$$\frac{\partial\phi}{\partial y} = (x\sin z + 2yz) \Rightarrow \phi = xy\sin z + y^2 z + f_1(x,z)$$

$$\frac{\partial\phi}{\partial z} = \left(xy\cos z + y^2\right) \Rightarrow \phi = xy\sin z + y^2 z + f_1(x,y)$$

Therefore, the velocity potential will be $\phi = xy\sin z + \cos x + y^2 z + c$

Important Vector Identities:

If ϕ, ψ are scalar point functions and $\vec{A}, \vec{B}$ are vector point functions in certain region then.

(1) $\vec{\nabla}(\phi+\psi)=\vec{\nabla}\phi+\vec{\nabla}\psi$

(2) $\vec{\nabla}(\phi\psi)=\phi\vec{\nabla}\psi+\psi\vec{\nabla}\phi$

(3) $\vec{\nabla}\cdot(\vec{A}+\vec{B})=\vec{\nabla}\cdot\vec{A}+\vec{\nabla}\cdot\vec{B}$

(4) $\vec{\nabla}\times(\vec{A}+\vec{B})=\vec{\nabla}\times\vec{A}+\vec{\nabla}\times\vec{B}$

(5) $\vec{\nabla}(\vec{A}\cdot\vec{B})=\vec{A}\times(\vec{\nabla}\times\vec{B})+(\vec{A}\cdot\vec{\nabla})\vec{B}+\vec{B}\times(\vec{\nabla}\times\vec{A})+(\vec{B}\cdot\vec{\nabla})\vec{A}$

(6) $\vec{\nabla}\cdot(\phi\vec{A})=\vec{\nabla}\phi\cdot\vec{A}+\phi\left(\vec{\nabla}\cdot\vec{A}\right)$

(7) $\vec{\nabla}\times(\phi\vec{A})=\phi(\vec{\nabla}\times\vec{A})+(\vec{\nabla}\phi)\times\vec{A}$

(8) $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A})=0$

(9) $\vec{\nabla}\times(\vec{\nabla}\phi)=0$

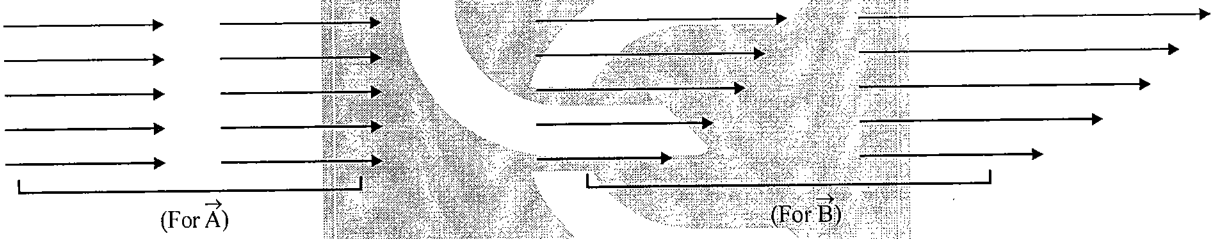
(10) $\vec{\nabla}\times(\vec{A}\times\vec{B})=\vec{B}\cdot(\vec{\nabla}\times\vec{A})-\vec{A}\cdot(\vec{\nabla}\times\vec{B})$

(11) $\vec{\nabla}\times(\vec{A}\times\vec{B})=(\vec{B}\cdot\vec{\nabla})\vec{A}-(\vec{A}\cdot\vec{\nabla})\vec{B}+\vec{A}(\vec{\nabla}\cdot\vec{B})-\vec{B}(\vec{\nabla}\cdot\vec{A})$

(12) $\vec{\nabla}\times(\vec{\nabla}\times\vec{A})=\vec{\nabla}(\vec{\nabla}\cdot\vec{A})-\nabla^2\vec{A}$

(13) For a constant vector $\vec{A}, \vec{\nabla}\left(\vec{A}.\vec{r}\right)=\vec{A}$ and $\vec{\nabla}\times\left(\vec{A}\times\vec{r}\right)=2\vec{A}$

Example 30: Consider the field lines drawn below for two different vector fields $\vec{A}$ and $\vec{B}$



Now choose correct statement(s)

(a) vector field $\vec{A}$ is solenoidal in nature
(b) vector field $\vec{A}$ is conservative in nature
(c) vector field $\vec{B}$ is not solenoid in nature
(d) vector field $\vec{B}$ is conservative in nature

Soln: The length of the vector $\vec{A}$ is not changing in a direction either parallel or perpendicular to the vector. So,

$$\vec{\nabla}\cdot\vec{A}=0,\ \vec{\nabla}\times\vec{A}=0,$$

Hence, $\vec{A}$ is solenoidal and conservative in nature.

and the length of the vector $\vec{B}$ is changing in a direction both parallel and perpendicular to the vector. So,

$$\vec{\nabla}\cdot\vec{B}\neq 0,\ \vec{\nabla}\times\vec{B}\neq 0,$$

Hence, $\vec{B}$ is neither solenoidal nor conservative in nature.

Correct options are (a), (b) and (c)

Example 31: If $\vec{A}=x\hat{e}_x+y\hat{e}_y+z\hat{e}_z$, then $\nabla^2\vec{A}$

(a) 1
(b) 3
(c) 0
(d) –3

Soln: $\vec{A}=x\hat{e}_x+y\hat{e}_y+z\hat{e}_z$

$$\nabla^2\vec{A}=\frac{\partial^2}{\partial x^2}\left(x\hat{e}_x+y\hat{e}_y+z\hat{e}_z\right)+\frac{\partial^2}{\partial y^2}\left(x\hat{e}_x+y\hat{e}_y+z\hat{e}_z\right)+\frac{\partial^2}{\partial z^2}\left(x\hat{e}_x+y\hat{e}_y+z\hat{e}_z\right)=0$$

Correct option is (c)

Example 32: For a scalar function ϕ satisfying the laplace equation, $\nabla\phi$ has

(a) zero curl and non-zero divergence (b) non-zero curl and zero divergence

(c) zero curl and zero divergence (d) mon-zero curl and non-zero divergence

Soln: $\vec{\nabla}\times\vec{\nabla}\phi = 0$. From Laplacian equation $\nabla^2\phi = 0 \Rightarrow \vec{\nabla}\cdot(\vec{\nabla}\phi) = 0$

Correct option is (c)

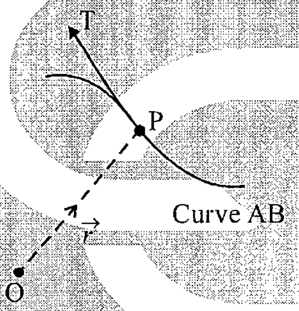
Example 33: Show that, $\nabla^2 r^n = n(n+1)r^{n-2}$.

Soln:
$$\nabla^2 r^n = \vec{\nabla}.\left(\vec{\nabla}r^n\right) = \vec{\nabla}.\left(nr^{n-1}\hat{r}\right) = \vec{\nabla}.\left(nr^{n-2}\vec{r}\right) = \vec{\nabla}\left(nr^{n-2}\right).\vec{r} + nr^{n-2}\left(\vec{\nabla}.\vec{r}\right)$$
$$= n(n-2)r^{n-3}\hat{r}.\vec{r} + 3nr^{n-2} = n(n-2)r^{n-2} + 3nr^{n-2} = n(n+1)r^{n-2}$$

1.4 Line Integration of Vectors

The integration of a vector along a curve is known as line integration of vectors.

Let $\vec{F}(x,y,z)$ be a vector function and a curve AB.



Line integral of a vector function $\vec{F}$ along the curve AB is defined as integral of the component of $\vec{F}$ along the tangent to the curve AB.

Component of $\vec{F}$ along a tangent PT at P = Dot product of $\vec{F}$ and unit vector along PT

$$= \vec{F}\cdot\frac{d\vec{r}}{ds}\left(\frac{d\vec{r}}{ds} \text{ is a unit vector along tangent PT}\right)$$

Line integral of $\vec{F}$ from A to B along the curve C will be $= \int_c \left(\vec{F}\cdot\frac{d\vec{r}}{ds}\right)ds = \int_c \vec{F}\cdot d\vec{r}$

Note:

(i) If $\vec{F}$ represents the variable force acting on a particle along arc AB, then the total work done

$$W_{AB} = \int_A^B \vec{F}\cdot d\vec{r}$$

(ii) If $\vec{V}$ represents the velocity of a liquid then $\oint_c \vec{V}\cdot d\vec{r}$ is called the circulation of $\vec{V}$ round closed curve C.

(iii) When the path of integration is a closed curve then notation of integration is $\oint$ in place of $\int$.

(iv) Work done by a conservative field $\vec{A}$ in moving a particle from point P to Q will be

$$\int_P^Q \vec{F}\cdot\overrightarrow{dr} = \int_P^Q \nabla\phi\cdot\overrightarrow{dr} = \int_P^Q d\phi = \phi_Q - \phi_P = \text{independent of path.}$$

(v) Work done by a conservative field $\vec{A}$ in moving a particle around a closed path C is $\oint_c \vec{F}\cdot\overrightarrow{dr} = 0$

Example 34: If $\vec{F} = \left(5xy - 6x^2\right)\hat{i} + \left(2y - 4x\right)\hat{j}$, then evaluate the line integral $\oint_C \vec{F}.\,d\vec{r}$ along the curve C in the x-y plane given by $y = x^3$ from the point $(1,1)$ to $(2,8)$.

Soln: $\vec{r} = x\,\hat{i} + x^3\hat{j} \Rightarrow d\vec{r} = dx\,\hat{i} + 3x^2dx\,\hat{j}$

$$So,\ \int_C \vec{F}.\,d\vec{r} = \int_1^2 \left[\left(5x^4 - 6x^2\right)\hat{i} + \left(2x^3 - 4x\right)\hat{j}\right].\left[dx\,\hat{i} + 3x^2dx\,\hat{j}\right]$$

$$= \int_1^2 \left(6x^5 + 5x^4 - 12x^3 - 6x^2\right)dx = 35$$

Example 35: If a force field $\vec{A} = y\hat{i} - x\hat{j}$ is applied to a particle, then the work done by the force field in traversing the particle around a circle C in x-y plane, with center at the origin and radius 1 units is equal to (C is traversed in the counter-clockwise direction)

(a) 2π (b) -2π (c) 8π (d) -8π

Soln: $\oint_C \vec{A}.d\vec{r} = \oint_C \left[ydx - xdy\right]$

Putting $x = \cos\theta$ and $y = \sin\theta$ we get,

$$= \int_0^{2\pi}\left[\sin\theta\left(-\sin\theta d\theta\right) - \cos\theta\left(\cos\theta d\theta\right)\right] = -\int_0^{2\pi}\left(\sin^2\theta + \cos^2\theta\right)d\theta = -2\pi$$

Correct option is (b)

Example 36: Evaluate $\int_C \vec{F}.\overrightarrow{dr}$, where $F = x^2\hat{i} + y^3\hat{j}$ and curve C is the arc of parabola $y = x^2$ in the x-y plane from (0, 0) to (1, 1).

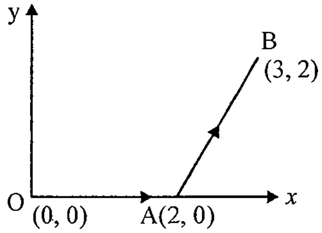
Soln: Along the curve C, $y = x^2 \Rightarrow dy = 2xdx$; So, $\vec{r} = x\hat{i} + x^2\hat{j} \Rightarrow d\vec{r} = dx\hat{i} + 2xdx\hat{j}$

Therefore, $\int_C \vec{F}.d\vec{r} = \int_{x=0}^1\left[x^2dx + x^6\left(2x\right)dx\right]$

$$= \int_0^1\left(x^2 + 2x^7\right)dx = \left[\frac{x^3}{3} + \frac{2x^8}{8}\right]_0^1 = \frac{7}{12}$$

Example 37: If $\vec{F}=(2x+y)\hat{i}+(3y+x)\hat{j}$, evaluate $\int_C \vec{F}.\overrightarrow{dr}$ where C is the curve in the xy-plane consisting of the straight lines from (0, 0) to (2, 0) and then to (3, 2).

Soln: The path of integration C has been shown in the figure below:



We have, $\int_C \vec{F}.d\vec{r} = \int_C\left[(2x+y)\hat{i}+(3y-x)\hat{j}\right].\left(dx\,\hat{i}+dy\,\hat{j}\right)$

$$=\int_C\left[(2x+y)\,dx+(3y-x)\,dy\right]$$

Path 1: Along the straight line OA; $y=0 \Rightarrow dy=0$ and x varies from 0 to 2.

Path 2: Equation of the straight line AB is $y-0=\frac{2-0}{3-2}(x-2)$ *i.e.* $y=2x-4$

Therefore, along AB, $y=2x-4 \Rightarrow dy=2dx$ and x varies from 2 to 3.

$$\therefore \quad \int_C \vec{F}.d\vec{r}=\int_0^2\left[(2x+0)\,dx+0\right]+\int_2^3\left[(2x+2x-4)\,dx+(6x-12-x)2dx\right]$$

$$=\left[x^2\right]_0^2+\int_0^2(14x-28)\,dx=4+14\int_2^3(x-2)\,dx=11$$

Example 38: Find the total work done in moving a particle in a force field given by $F=3xy\hat{i}-5z\hat{j}+10x\hat{k}$ along the curve $x=t^2+1,\ y=2t^2,\ z=t^3$ from $t=1$ to $t=2$.

Soln: The total work done

$$=\int_C \vec{F}.d\vec{r}=\int_C\left(3xy\hat{i}-5z\hat{j}+10x\hat{k}\right).\left(dx\hat{i}+dy\hat{j}+dz\hat{k}\right)=\int_1^2\left(3xydx-5zdy+10xdz\right)$$

$$=\int_1^2\left[3\left(t^2+1\right)\left(2t^2\right)(2t)-\left(5t^3\right)(4t)+10\left(t^2+1\right)\left(3t^2\right)\right]dt=\int_1^2\left(12t^5+12t^3-20t^4+30t^4+30t^2\right)dt=303$$

Example 39: Evaluate $\int_C\left\{\left(2xy^3-y^2\cos x\right)dx+\left(1-2y\sin x+3x^2y^2\right)dy\right\}$

Where C is the arc of the parabola $2x=\pi y^2$ from (0, 0) to $\left(\frac{1}{2}\pi,1\right)$.

Soln: The given integral in the question is the line integral of the following vector field:

$$\vec{F}=\left(2xy^3-y^2\cos x\right)\hat{i}+\left(1-2y\sin x+3x^2y^2\right)\hat{j}$$

This vector field is conservative in nature as $\vec{\nabla}\times\vec{F}=0$ and it can be written as $\vec{F}=\vec{\nabla}\phi$

$$\frac{\partial\phi}{\partial x}=2xy^3-y^2\cos x \Rightarrow \phi=x^2y^3-y^2\sin x+f_1(y)$$

$$\frac{\partial\phi}{\partial y}=\left(1-2y\sin x+3x^2y^2\right) \Rightarrow \phi=y-y^2\sin x+x^2y^3+f_2(x)$$

Therefore, $\phi = y - y^2 \sin x - x^2 y^3 + c$

Since, work done by a conservative field i.e value of the given integral is independent of path and will be equal to the difference of scalar potential ϕ at initial and final point.

So, $$I = \phi\left(\frac{\pi}{2}, 1\right) - \phi(0,0) = \left[y - y^2 \sin x + x^2 y^3\right]_{(0,0)}^{(\pi/2,1)} = \frac{\pi^2}{4}$$

Example 40: Suppose an object moves from $(-1,1)$ to $(2,4)$ along the path $\vec{r}(t) = \langle t, t^2 \rangle$, subject to the force $\vec{F} = \langle x \sin y, y \rangle$, then the work done by $\vec{F}$ is

(a) $\frac{15}{2} + \frac{\cos(1) - \cos(4)}{2}$

(b) $\frac{15}{2} - \frac{\cos(1) + \cos(4)}{2}$

(c) $\frac{15}{2} + \frac{\sin(1) - \sin(4)}{2}$

(d) $\frac{15}{2} - \frac{\sin(1) + \sin(4)}{2}$

Soln: $\vec{r}(t) = t\hat{i} + t^2\hat{j} \Rightarrow x = t,\ y = t^2$ and $\vec{F} = x \sin y \hat{i} + y\hat{j} \Rightarrow \vec{F} = t \sin t^2 \hat{i} + t^2 \hat{j}$

Work done $W = \int \vec{F} \cdot d\vec{r}$, Here, $d\vec{r} = \left(\hat{i} + 2t\hat{j}\right)dt$

Therefore, $W = \int_{t=-1}^{2} \left(t \sin t^2 + 2t^3\right) dt = \int_{-1}^{2} t \sin t^2\, dt + \int_{-1}^{2} 2t^3\, dt$

Assuming $t^2 = z,\ 2t\,dt = dz$ in the first integration, we get

$$\int_{-1}^{2} t \sin t^2\, dt = \int_{1}^{4} \sin z \frac{dz}{2} = \frac{1}{2}(-\cos z)_1^4 = \frac{1}{2}(\cos 1 - \cos 4)$$

Therefore, $W = \frac{1}{2}(\cos 1 - \cos 4) + 2\left(\frac{t^4}{4}\right)_{-1}^{2} = \frac{1}{2}(\cos 1 - \cos 4) + \frac{1}{2}(15)$

Correct option is (a)

Example 41: Value of the circulation of the field $\vec{F} = -x^2 y\hat{i} + xy^2\hat{j} + (y^3 - x^3)\hat{k}$ around the curve C, where C is the intersection of the sphere $x^2 + y^2 + z^2 = 25$ and the plane $z = 4$ and is oriented counterclockwise, is

(a) 81π

(b) -81π

(c) $\frac{81\pi}{4}$

(d) $-\frac{81\pi}{4}$

Soln: Here, $C: x^2 + y^2 = 9$ (curve of intersection of sphere and plane $z = 4$)

$$\therefore \quad \oint_C \vec{F} \cdot d\vec{r} = \oint_C -x^2 y\,dx + xy^2\,dy + \left(y^3 - x^3\right)dz$$

Let $x = 3\cos\theta \Rightarrow dx = -3\sin\theta\, d\theta$

$y = 3\sin\theta \Rightarrow dy = 3\cos\theta\, d\theta$

and $z = 4 \Rightarrow dz = 0$

$$\oint \vec{F} \cdot d\vec{r} = \int_{\theta=0}^{2\pi} \left(-9\cos^2\theta \times 3\sin\theta \times -3\sin\theta\, d\theta + 3\cos\theta \times 9\sin^2\theta \times 3\cos\theta\, d\theta\right)$$

$$=81\int_{\theta=0}^{2\pi}\cos^2\theta\sin^2\theta\,d\theta=\frac{81}{4}\int_{\theta=0}^{2\pi}(\sin 2\theta)^2\,d\theta$$

$$=\frac{81}{4}\int_{0}^{2\pi}\frac{1-\cos 4\theta}{2}\,d\theta=\frac{81}{8}\left(\theta-\frac{\sin 4\theta}{4}\right)_0^{2\pi}=\frac{81}{8}\times 2\pi=\frac{81\pi}{4}$$

Correct option is (c)

Example 42: If the value of line integral $I=\int_A^B \vec{a}\cdot d\vec{r}$, where $\vec{a}=(xy^2+z)\hat{i}+(x^2y+2)\hat{j}+x\hat{k}$, A is the point (c,c,h) and B is the point $\left(2c,\frac{c}{2},h\right)$ along the path c_1, given by $x=cu,\ y=\frac{c}{u},\ z=h$ is I_1, and along the path c_2 given by $2y=3c-x,\ z=h$ is I_2. Then, the ratio I_1/I_2 is equal to (u is a parameter)

(a) $\frac{2c(h-1)}{h}$ (b) $\frac{2h}{c(h-1)}$ (c) 2 (d) 1

Soln: $\vec{a}=\left(xy^2+z\right)\hat{i}+\left(x^2y+2\right)\hat{j}+x\hat{k}$

$$\vec{\nabla}\times\vec{a}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ xy^2+z & x^2y+2 & x\end{vmatrix}=\hat{i}(0-0)-\hat{j}(1-1)+\hat{k}(2xy-2xy)=0$$

$\because \quad \vec{\nabla}\times\vec{a}=0$

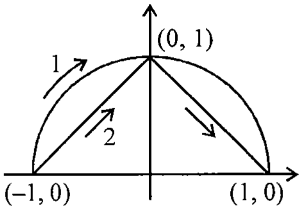
Hence, $\vec{a}$ is a conservative vector field. Therefore, it's line integral will be independent of path and will depend only on initial and final point.

Here, C_1 and C_2 are two different paths taken between the same points A and B. Hence, the line integrals I_1 and I_2 would have same value.

Therefore, $\frac{I_1}{I_2}=1$

Correct option is (d)

Example 43: Consider the integral $I=\int\frac{x\,dy-y\,dx}{x^2+y^2}$. Then which of the following statement(s) is/are correct about the value of I along each of the two paths indicated in figure :



(a) For the path-1, $I_1=-\pi$ (b) For the path-1, $I_1=\pi$

(c) For the path-2, $I_2=-\pi$ (d) For the path-2, $I_2=\pi$

Soln: $I=\int\frac{xdy-ydx}{x^2+y^2}=\int\left(\frac{x\hat{j}-y\hat{i}}{x^2+y^2}\right)\cdot d\vec{r}=\int\vec{a}.d\vec{r}$

$\because \quad \vec{\nabla}\times\vec{a}=0$

Hence, $\vec{a}$ is a conservative vector field. Therefore, it's line integral will be independent of path and will depend only on initial and final point.

Here, Path 1 & Path 2 are two different paths taken between the same points (-1, 0) and (1, 0). Hence, the line integrals I_1 and I_2 would have same value.

Along path 1: (Semi circle of radius 1 unit)

Put $x=\cos\theta\Rightarrow dx=-\sin\theta d\theta$ and $y=\sin\theta\Rightarrow dy=\cos\theta d\theta$

$$\therefore \quad I_1=\int_{\theta=\pi}^{0}\frac{(\cos\theta\times\cos\theta-\sin\theta\times-\sin\theta)d\theta}{\sin^2\theta+\cos^2\theta}$$

$$I_1=\int_{\pi}^{0}d\theta=-\pi=I_2$$

Correct options are (a) and (c)

Example 44: Let a vector $\vec{V}$ is given by $\vec{V}=\frac{\vec{r}}{r^3}\ (r\neq 0)$

Which one of the following is/are CORRECT?

(a) Value of the line integral of this vector around any closed curve is zero

(b) This vector can be written as the gradient of some scalar function

(c) This is a conservative vector field.

(d) This vector can represent the magnetic field of some current distribution.

Soln: Here, $\vec{\nabla}\times\left(\frac{\vec{r}}{r^3}\right)=0$

Therefore, $\vec{V}=\frac{\vec{r}}{r^3}$ is a conservative field.

So, the value of the line integral of this vector around any closed curve C is zero i.e. $\oint_C\vec{V}\cdot d\vec{r}=0$,

Vector can be written as the gradient of some scalar function $\vec{V}=\vec{\nabla}\phi$.

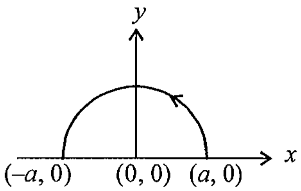
Since, magnetic field $\vec{B}$ is not conservative, therefore this vector cannot represent the magnetic field of some current distribution.

Correct option is (a,b,c)

Example 45: The value of the integral $\int_C\vec{F}\cdot d\vec{r}$, where $\vec{F}=\frac{y^2}{x^2+y^2}\hat{i}-\frac{x^2}{x^2+y^2}\hat{j}$ and $C: y=\sqrt{a^2-x^2}$ is_________$\times a^3$.

Soln: Given curve C is a half circle above x-axis

Hence, $\oint_C\vec{F}\cdot d\vec{r}=\int_C\frac{y^2dx}{x^2+y^2}-\frac{x^2}{x^2+y^2}dy$



Put $x = a\cos\theta \Rightarrow dx = -a\sin\theta d\theta$ and $y = a\sin\theta \Rightarrow dy = a\cos\theta d\theta$

$$\therefore \quad \int \vec{F}\cdot d\vec{r} = \int_{\theta=0}^{\pi}\left(a^2\sin^2\theta \times -a\sin\theta\right)d\theta - \left(a^2\cos^2\theta \times a\cos\theta\right)d\theta \; = -a^3\int_{\theta=0}^{\pi}\left(\sin^3\theta + \cos^3\theta\right)d\theta$$

$$= -a^3\left[\int_{\theta=0}^{\pi}\frac{1}{4}(3\sin\theta - \sin 3\theta)\,d\theta + \int_{0}^{\pi}\frac{1}{4}(\cos 3\theta - 3\cos\theta)\,d\theta\right]$$

$$= -\frac{a^3}{4}\left[3\times -\cos\theta + \frac{\cos 3\theta}{3} + \frac{\sin 3\theta}{3} - \frac{3\sin\theta}{3}\right]_0^{\pi}$$

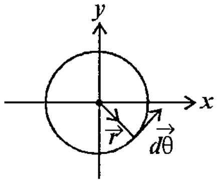
$$= -\frac{a^3}{4}\left[3 - \frac{1}{3} + 3 - \frac{1}{3}\right] = -\frac{a^3}{4}\left[\frac{16}{3}\right] = -\frac{4a^3}{3}$$

Correct answer is (1.33)

Example 46: The value of contour integral $\left|\int_C \vec{r}\times d\vec{\theta}\right|$, for a circle C of radius r with centre at the origin is

(a) $2\pi r$ (b) $\frac{r^2}{2}$ (c) πr^2 (d) r

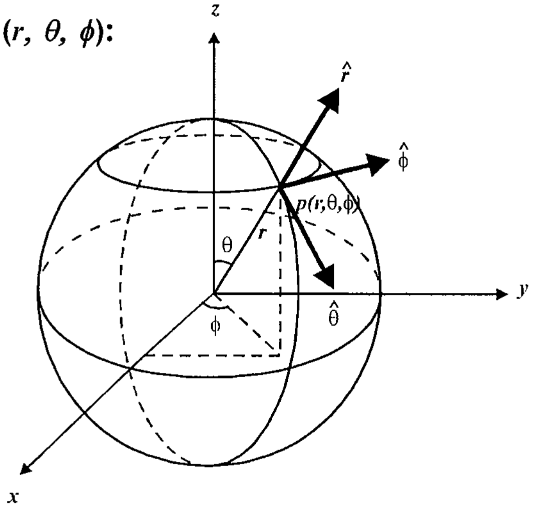
Soln: $\left|\int_C \vec{r}\times d\vec{\theta}\right| = \int_C r d\theta \sin 90^\circ = r\int_0^{2\pi} d\theta = 2\pi r$



Correct option is (a)

1.5 Orthogonal Currvilinear Co-ordinates

(A) Spherical Polar co-ordinates (r, θ, ϕ):



Let the point P (in the above figure) has cartesian coordinates (x, y, z) and spherical polar coordinates (r, θ, ϕ).

Relation between cartesian and spherical polar co-ordiantes:

$$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\ \phi,\ z = r\cos\theta$$

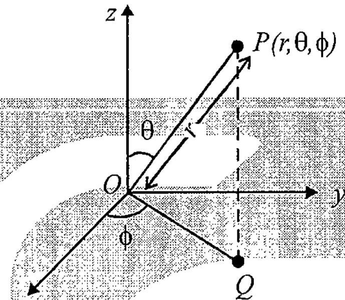
In spherical polar co-ordinates,

$$r = \sqrt{x^2+y^2+z^2} = \text{radial distance of the point } P \text{ from the origin}$$

$$\theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z} = \text{angle which } OP \text{ makes with } z\text{-axis}$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right) = \text{angle which } OQ \text{ (projection of } OP \text{ in } x\text{-}y \text{ plane) makes with positive } x\text{-axis}$$

Corresponding limits: $0 \le r < \infty$, $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$



Unit vectors in Spherical Polar coordinates:

The position vector of P is given by

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

Putting (x, y, z) in the above relations,

$$\vec{r} = (r\sin\theta\cos\phi)\,\hat{i} + (r\sin\theta\sin\phi)\,\hat{j} + r\cos\theta\hat{k}$$

The unit vector along r is given by

$$\hat{r} = \frac{\vec{r}}{|r|} = \frac{r\sin\theta\cos\phi\hat{i} + r\sin\theta\sin\phi\hat{j} + r\cos\theta\hat{k}}{r}$$

$$\Rightarrow \quad \boxed{\hat{r} = \sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \cos\theta\hat{k}}$$

The unit vector along θ is given by

$$\hat{\theta} = \frac{\partial\vec{r}/\partial\theta}{|\partial r/\partial\theta|} = \frac{r\cos\theta\cos\phi\hat{i} + r\cos\theta\sin\phi\hat{j} + r\sin\theta\hat{k}}{\sqrt{r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta}}$$

$$\Rightarrow \quad \boxed{\hat{\theta} = \cos\theta\cos\phi\hat{i} + \cos\theta\sin\phi\hat{j} - \sin\theta\hat{k}}$$

The unit vector along ϕ is given by

$$\hat{\phi} = \frac{\partial\vec{r}/\partial\phi}{|\partial r/\partial\phi|} = \frac{-r\sin\theta\sin\phi\hat{i} + r\sin\theta\cos\phi\hat{j}}{\sqrt{r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi}}$$

$$\Rightarrow \quad \boxed{\hat{\phi} = -\sin\phi\hat{i} + \cos\phi\hat{j}}$$

Note: The unit vector $\hat{r}, \hat{\theta}, \hat{\phi}$ are orthogonal vectors and it thus follows,

$$\boxed{\hat{r}\times\hat{\theta} = \hat{\phi},\ \hat{\theta}\times\hat{\phi} = \hat{r}, \hat{\phi}\times\hat{r} = \hat{\theta}}$$

Line elements:

Line element between r to $r + dr$: $\vec{dl}_r = dr\,\hat{r}$ (along r curve)

Line element between θ to $\theta + d\theta$: $\vec{dl}_\theta = rd\theta\,\hat{\theta}$ (along θ curve)

Line element between ϕ to $\phi + d\phi$: $\vec{dl}_\phi = r\sin\theta d\phi\,\hat{\phi}$ (along ϕ curve)

Surface elements:

Constant r surface : $\vec{ds}_r = \vec{dl}_\theta \times \vec{dl}_\phi = r^2\sin\theta d\theta d\phi\,\hat{r}$ (surface of a sphere)

Constant θ surafce : $\vec{ds}_\theta = \vec{dl}_\phi \times \vec{dl}_r = r\sin\theta dr d\phi\,\hat{\theta}$

Constant ϕ surface : $\vec{ds}_\phi = \vec{dl}_r \times \vec{dl}_\theta = rdrd\theta\,\hat{\phi}$

Volume element:

$$dV = \vec{dl}_r.\left(\vec{dl}_\theta \times \vec{dl}_\phi\right) = r^2\sin\theta dr d\theta d\phi$$

Gradient: $\vec{\nabla} = \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\hat{\phi}$

Divergence: $\vec{\nabla}.\vec{a} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 a_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta a_\theta) + \frac{1}{r^2\sin\theta}\frac{\partial a_\phi}{\partial\phi}$

Curl: $\vec{\nabla}\times\vec{a} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ a_r & ra_\theta & r\sin\theta a_\phi \end{vmatrix}$

Laplacian: $\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin\theta}\frac{\partial^2}{\partial\phi^2}$

Conversion of a vector from Cartesian to Spherical polar co-ordiantes:

$$\hat{r} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\hat{k}$$

$$\hat{\theta} = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\hat{k}$$

$$\hat{\phi} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$

$$\begin{bmatrix}\hat{r}\\ \hat{\theta}\\ \hat{\phi}\end{bmatrix}=\begin{bmatrix}\sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta\\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta\\ -\sin\phi & \cos\phi & 0\end{bmatrix}\begin{bmatrix}\hat{i}\\ \hat{j}\\ \hat{k}\end{bmatrix}$$

Suppose the vector is given in cartesian coordinates i.e. $\vec{r}=a_x\hat{i}+a_y\hat{j}+a_z\hat{k}$ and we have to convert it in sphereical polar coordinates i.e $\vec{r}=a_r\hat{r}+a_\theta\hat{\theta}+a_\phi\hat{\phi}$, the components are related by the following equation.

i.e. $$\begin{bmatrix}a_r\\ a_\theta\\ a_\phi\end{bmatrix}=\begin{bmatrix}\sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta\\ \cos\theta\cos\phi & \cos\theta\sin & -\sin\theta\\ -\sin\phi & \cos\phi & 0\end{bmatrix}\begin{bmatrix}a_x\\ a_y\\ a_z\end{bmatrix}$$

Limits for various types of integrations:

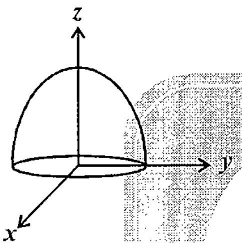


Fig. (a)
Upper hemisphere

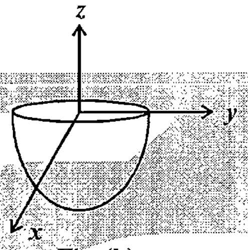


Fig. (b)
Lower hemisphere

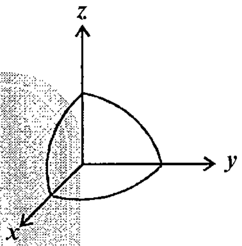


Fig. (c)
First Octant of sphere

Case-I: Full sphere $x^2+y^2+z^2=a^2$

Limit of volume integration : $0\le r\le a;\ \ 0\le\theta\le\pi;\ \ 0\le\phi\le 2\pi$

Limit of surface integration : $r=\text{const}=a;\ \ 0\le\theta\le\pi;\ \ 0\le\phi\le 2\pi$

Case-II: Upper Hemisphere $x^2+y^2+z^2=a^2, z>0$

Limit of volume integration : $0\le r\le a;\ \ 0<\theta\le\frac{\pi}{2};\ \ 0\le\phi\le 2\pi$

Limit of surface integration : $r=\text{const}=a;\ \ 0\le\theta\le\frac{\pi}{2};\ \ 0\le\phi\le 2\pi$

Case-III: Lower hemisphere $x^2+y^2+z^2=a^2, z<0$

Limit of volume integration : $0\le r\le a;\ \ \frac{\pi}{2}<\theta\le\pi;\ \ 0\le\phi\le 2\pi$

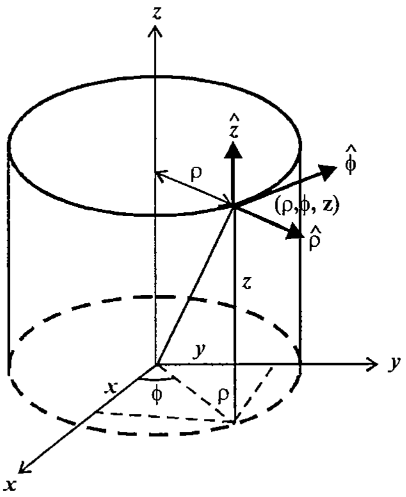
Limit of surface integration : $r=\text{const}=a;\ \ \frac{\pi}{2}\le\theta\le\pi;\ \ 0\le\phi\le 2\pi$

Case-IV: First octant of sphere $x^2+y^2+z^2=a^2,\ x,y,z>0$

Limit of volume integration : $0\le r\le a;\ \ 0<\theta\le\frac{\pi}{2};\ \ 0\le\phi\le\frac{\pi}{2}$

Limit of surface integration : $r=\text{const}=a;\ \ 0\le\theta\le\frac{\pi}{2};\ \ 0\le\phi\le\frac{\pi}{2}$

(B) Cylindrical co-ordinates (ρ, ϕ, z):



Let the point P (in above figure) has cartesian coordinates (x, y, z) and cylinderical coordinates (ρ, ϕ, z). So, the relation between cartesian and cyllindrical co-ordiantes:

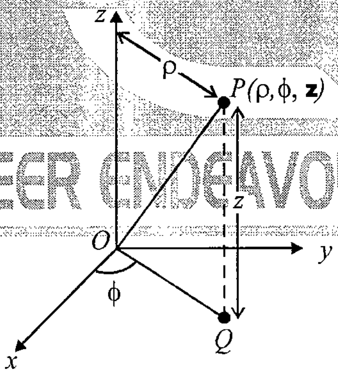
$$x = \rho \cos\phi,\ y = \rho \sin\phi,\ z = z$$

In cylindrical co-ordinates,

$\rho = \sqrt{x^2 + y^2}$ = perpendicular distance point 'P' from the z axis

$\phi = \tan^{-1}\left(\dfrac{y}{x}\right)$ = angle which OQ (projection of OP in x-y plane) makes with positive x-axis

z = height of the point 'P' above the x-y plane



Unit Vector in cylindrical coordinates:

The position vector of the point P is

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

Putting the value of (x, y, z) in the above relation

$$\vec{r} = \rho\cos\phi\hat{i} + \rho\sin\phi\hat{j} + z\hat{k}$$

The unit vector along ρ is given by

$$\hat{\rho} = \frac{\partial\vec{r}/\partial\rho}{|\partial\vec{r}/\partial\rho|} = \cos\phi\hat{i} + \sin\phi\hat{j} \quad \Rightarrow \quad \boxed{\hat{\rho} = \cos\phi\hat{i} + \sin\phi\hat{j}}$$

The unit vector along ϕ is given by

$$\hat{\phi} = \frac{\partial\vec{r}/\partial\phi}{|\partial\vec{r}/\partial\phi|} = -\sin\phi\hat{i} + \cos\phi\hat{j} \quad \Rightarrow \quad \boxed{\hat{\phi} = -\sin\phi\hat{i} + \cos\phi\hat{j}}$$

The unit vector along z is given by

$$\hat{z} = \frac{\partial\vec{r}/\partial z}{|\partial\vec{r}/\partial z|} = \hat{k} \quad \Rightarrow \quad \boxed{\hat{z} = \hat{k}}$$

Note: $\hat{\rho}, \hat{\phi}$ and $\hat{z}$ are orthogonal to each other and they follow

$$\boxed{\hat{\rho}\times\hat{\phi} = \hat{z}, \ \hat{\phi}\times\hat{z} = \hat{\rho}, \quad \hat{z}\times\hat{\rho} = \hat{\phi}}$$

Line elements:

Line element between ρ to $\rho + d\rho$: $d\vec{l}_\rho = d\rho\ \hat{\rho}$ (along ρ curve)

Line element between ϕ to $\phi + d\phi$: $d\vec{l}_\phi = \rho d\phi\ \hat{\phi}$ (along ϕ curve)

Line element between z to $z + dz$: $d\vec{l}_z = dz\ \hat{z}$ (along z curve)

Surface elements:

Constant ρ surface : $d\vec{s}_\rho = d\vec{l}_\phi \times d\vec{l}_z = \rho d\phi dz\ \hat{\rho}$ (curved surface of a circular cyllinder)

Constant ϕ surface : $d\vec{s}_\phi = d\vec{l}_z \times d\vec{l}_r = d\rho dz\ \hat{\phi}$

Constant z surafce : $d\vec{s}_z = d\vec{l}_\rho \times d\vec{l}_\phi = \rho d\rho d\phi\ \hat{z}$ (plane surface of a circular cyllinder parallel to x - y plane)

Volume element :

$$dV = d\vec{l}_\rho\cdot\left(d\vec{l}_\phi \times d\vec{l}_z\right) = \rho d\rho d\phi dz$$

Gradient: $\vec{\nabla} = \frac{\partial}{\partial\rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial}{\partial\phi}\hat{\phi} + \frac{\partial}{\partial z}\hat{z}$

Divergence: $\vec{\nabla}.\vec{a} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho a_\rho) + \frac{1}{\rho}\frac{\partial a_\phi}{\partial\phi} + \frac{\partial a_z}{\partial z}$

Curl: $\vec{\nabla}\times\vec{a} = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ a_\rho & \rho a_\phi & a_z \end{vmatrix}$

Laplacian: $\nabla^2 = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}$

Conversion of a vector from Cartesian to Cylindrical co-ordiantes:

$$\hat\rho=\cos\phi\,\hat i+\sin\phi\,\hat j$$

$$\hat\phi=-\sin\phi\,\hat i+\cos\phi\,\hat j$$

$$\hat z=\hat k$$

Suppose the vector is given in cartesian coordinates i.e. $\vec r=a_x\hat i+a_y\hat j+a_z\hat k$ and we have to convert it in sphereical polar coordinates i.e $\vec r=a_\rho\hat\rho+a_\phi\hat\phi+a_z\hat z$, the components are related by the following equation.

$$\begin{bmatrix}a_\rho\\a_\phi\\a_z\end{bmatrix}=\begin{bmatrix}\cos\phi&\sin\phi&0\\-\sin\phi&\cos\phi&0\\0&0&1\end{bmatrix}\begin{bmatrix}a_x\\a_y\\a_z\end{bmatrix}$$

Example 47: Four forces are given below in Cartesian and spherical polar coordinates.

(i) $\vec F_1=K\exp\left(\frac{-r^2}{R^2}\right)\hat r$ (ii) $\vec F_2=K\left(x^3\hat y-y^3\hat z\right)$ (iii) $\vec F_3=K\left(x^3\hat x+y^3\hat y\right)$ (iv) $\vec F_4=K\left(\frac{\hat\phi}{r}\right)$

where K is a constant. Identify the correct option.

(a) (iii) and (iv) are conservative but (i) and (ii) are not

(b) (i) and (ii) are conservative but (iii) and (iv) are not

(c) (ii) and (iii) are conservative but (i) and (iv) are not

(d) (i) and (iii) are conservative but (ii) and (iv) are not

Soln: $$\vec\nabla\times\vec F_1=\frac{1}{r^2\sin\theta}\begin{vmatrix}\hat r&r\hat\theta&r\sin\theta\hat\phi\\ \frac{\partial}{\partial r}&\frac{\partial}{\partial\theta}&\frac{\partial}{\partial\phi}\\ k.\exp\left(-r^2/R^2\right)&0&0\end{vmatrix}=0$$

$$\vec\nabla\times\vec F_2=\begin{vmatrix}\hat i&\hat j&\hat k\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ 0&kx^3&-kz^3\end{vmatrix}=3kx^2\hat k\qquad\Rightarrow\vec\nabla\times\vec F_3=\begin{vmatrix}\hat i&\hat j&\hat k\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ kx^3&ky^3&0\end{vmatrix}=0$$

$$\vec\nabla\times\vec F_4=\begin{vmatrix}\hat r&r\hat\theta&r\sin\theta\hat\phi\\ \frac{\partial}{\partial r}&\frac{\partial}{\partial\theta}&\frac{\partial}{\partial\phi}\\ 0&0&\frac{k}{r}\end{vmatrix}=\frac{1}{r^2\sin\theta}r\hat\theta\left[-\frac{k}{r^2}\right]=-\frac{k}{r^3\sin\theta}\hat\theta$$

Correct option is (d)

Example 48: Consider two vector functions $\vec{A} = r^n\vec{r}$ and $\vec{B} = \frac{\vec{r}}{r}$ (where r is a spherical co-ordinate)

Which of the following statements is/are CORRECT?

(a) Vector function $\vec{A}$ can be written as $\vec{A} = grad(\phi)$ (where ϕ is any scalar function)

(b) $\nabla^2(A) = (n+1)(n+2)r^{n-1}$, where ∇^2 is a laplacian operator

(c) curl of $\vec{B}$ is equal to zero

(d) vector function $\vec{B}$ is conservative in nature

Soln: $\vec{A} = r^n\vec{r} = r^{n+1}\hat{r}$

$$\vec{\nabla}\times\vec{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ r^{n+1} & 0 & 0 \end{vmatrix}$$

$\boxed{\vec{\nabla}\times\vec{A} = 0}$ i.e. $\vec{A}$ is conservative and $\vec{A}$ can be written as grad (ϕ) i.e. $\vec{A} = grad(\phi)$

$$\nabla^2(A) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left(r^{n+1}\right)\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\times(n+1)r^n\right)$$

$$= \frac{(n+1)}{r^2}\frac{\partial}{\partial r}\left(r^{n+2}\right) = \frac{(n+1)}{r^2}(n+2)r^{n+1} = (n+1)(n+2)r^{n-1}$$

$curl\ \vec{B} = \vec{\nabla}\times\vec{B} = 0$

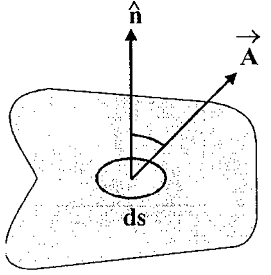
Hence, $\vec{B}$ is conservative in nature

Correct options are (a), (b), (c) and (d)

1.6 Surface Integration of Vectors

The integration of vector on an open or closed surface is known as surface integration of vectors.

Let $\vec{A}$ be a vector function and S be the given surface. Surface integral of a vector function $\vec{A}$ over the surface S is defined as the integral of the components of $\vec{A}$ along the normal to the surface.



Components of $\vec{A}$ along the normal

$= \vec{A}.\hat{n}$, where $\hat{n}$ is the unit normal vector to an element ds

Surface integral of $\vec{A}$ over S

$$= \Sigma\vec{A}.\hat{n} = \iint_S \left(\vec{A}.\hat{n}\right) ds$$

Physical Interpetation:

Example 49: Evaluate $\iint_s \vec{F}.\overrightarrow{\hat{n}ds}$, where $\vec{F} = yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k}$ and S is that part of the surface of the sphere $x^2 + y^2 + z^2 = 1$ which lies in the first octant.

Soln: For the surface of a sphere $(r = \text{radius} = 1 \text{ unit})$,

$$d\vec{S} = \hat{r}\left(r^2 \sin\theta\, d\theta d\varphi\right) = \hat{r}\left(\sin\theta\, d\theta d\varphi\right) = \left(\sin\theta\cos\varphi\,\hat{i} + \sin\theta\sin\varphi\hat{j} + \cos\theta\hat{k}\right)\left(\sin\theta\, d\theta d\varphi\right)$$

Hence, $\iint_S \vec{F}.\hat{n}dS = \iint_S \vec{F}.d\vec{S} = 3\int\limits_{\theta=0}^{\pi/2}\int\limits_{\varphi=0}^{\pi/2} \sin^3\theta\cos\theta\sin\varphi\cos\varphi\, d\theta\, d\varphi = 3\left(\frac{1}{4}\right)\left(\frac{1}{2}\right) = \frac{3}{8}$

Example 50: Evaluate $\iint_s \vec{F}.\overrightarrow{\hat{n}ds}$, where $\vec{F} = z\hat{i} + x\hat{j} - 3y^2z\hat{k}$ and S is the surface of the cylinder $x^2 + y^2 = 16$ included in the first octant between z = 0 and z = 5

Soln: For the curved surface of the cylinder $(\rho = \text{base radius} = 4 \text{ units})$,

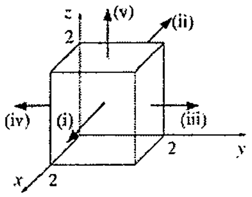
$$d\vec{S} = \hat{\rho}\left(\rho\, d\varphi\, dz\right) = \hat{\rho}\left(4\, d\varphi\, dz\right) = \left(\cos\varphi\,\hat{i} + \sin\varphi\hat{j}\right)\left(4\, d\varphi\, dz\right)$$

Hence, $\iint\limits_S \vec{F}.d\vec{S} = 4\int\limits_{z=0}^{5}\int\limits_{\varphi=0}^{2\pi}\left(z\cos\varphi + 4\cos\varphi\sin\varphi\right)dz\, d\varphi = 4\left(\frac{25}{2}\right)(1) + 16(5)(1) = 90$

Example 51: Calculate the surface integral of $\vec{v} = 2xz\,\hat{x} + (x+2)\hat{y} + y\left(z^2 - 3\right)\hat{z}$ over five sides (excluding the bottom) of the cubical box shown in figure.

Soln: (i) For the surface $x = 2$, $d\vec{s} = dy\, dz\, \hat{x}$, $\quad \vec{v}.d\vec{s} = 2xzdy\, dz = 4z\, dy\, dz$, So, $\int \vec{v}.d\vec{s} = 0$

(ii) For the surface $x = 0$, $d\vec{s} = -dy\, dz\, \hat{x}$, $\quad \vec{v}.d\vec{s} = -2xzdy\, dz = 0$, $so \int \vec{v}.d\vec{s} = 0$



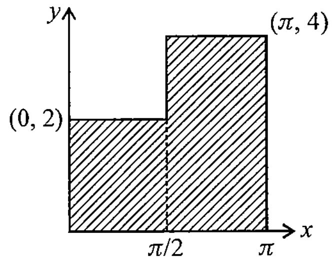
(iii) For the surface $y = 2$, $d\vec{s} = dx\, dz\, \hat{y}$, $\quad \vec{v}.d\vec{s} = (x+2)\, dx\, dz$, $so \int \vec{v}.d\vec{s} = \int_0^2 (x+2)\, dx \int_0^2 dz = 12$

(iv) For the surface $y = 0$, $d\vec{s} = -dx\, dz\, \hat{y}$, $\quad \vec{v}.d\vec{s} = -(x+2)\, dx\, dz$, $so \int \vec{v}.d\vec{s} = -\int_0^2 (x+2)\, dx \int_0^2 dz = -12$

(v) For the surface z = 2, $d\vec{s} = dxdy\, \hat{z}$, $\quad \vec{v}.d\vec{s} = ydxdy$, so $\int \vec{v}.d\vec{s} = \int_0^2 dx \int_0^2 ydy = 4$

Total flux $= \int\limits_{surface} \vec{v}.d\vec{s} = 16 + 0 + 12 - 12 + 4 = 20$

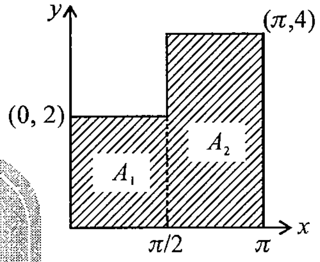
Example 52: The value of integral $\iint\limits_A y\,dx\,dy$, where A is the area as shown in figure :



(a) π (b) 2π (c) 4π (d) 5π

Soln: Let us divide the given area into two parts

$$\iint\limits_A ydxdy = \iint\limits_{A_1} ydxdy + \iint\limits_{A_2} ydxdy$$



$$= \int_{y=0}^{2}\int_{x=0}^{\pi/2} ydxdy + \int_{y=0}^{4}\int_{x=\pi/2}^{\pi} ydxdy$$

$$= \int_{y=0}^{2} (xy)_0^{\pi/2}\,dy + \int_{y=0}^{4} (xy)_{\pi/2}^{\pi}\,dy$$

$$= \frac{\pi}{2}\left(\frac{y^2}{2}\right)_0^2 + \frac{\pi}{2}\left(\frac{y^2}{2}\right)_0^4 = \frac{\pi}{2}\times 2 + \frac{\pi}{2}\times 8 = 10\frac{\pi}{2} = 5\pi$$

Correct option is (d)

Example 53: Consider the vector function $\vec{V} = \rho(2+\sin^2\phi)\,\hat{\rho} + \rho\sin\phi\cos\phi\,\hat{\phi} + 3z\,\hat{z}$ (ρ, ϕ, z are cylindrical co-ordinates), which of the following statement(s) is/are CORRECT ?

(a) $\vec{\nabla}\times\vec{V}\Big|_{\rho=1,\,\phi=\frac{\pi}{2},\,z=2}$ is equal to zero

(b) Flux of $\vec{V}$ through the curved surface of cylinder $x^2+y^2=4,\ 0\le z\le 5$ in first quadrant is 25π

(c) $\vec{\nabla}\times\vec{V}\Big|_{\rho=1,\,\phi=\frac{\pi}{2},\,z=2}$ is $3\hat{\rho}+2\hat{\phi}+\hat{z}$

(d) Flux of $\vec{V}$ through the curved surface of cylinder $x^2+y^2=4,\ 0\le z\le 5$ in first quadrant is zero

Soln: $\vec{V} = \rho\left(2+\sin^2\phi\right)\hat{\rho} + \rho\sin\phi\cos\phi\hat{\phi} + 3z\hat{z}$

$$\vec{\nabla}\times\vec{V} = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ \rho\left(2+\sin^2\phi\right) & \rho\cdot\rho\sin\phi\cos\phi & 3z \end{vmatrix}$$

$$= \frac{1}{\rho}\left[\hat{\rho}(0-0) - \rho\hat{\phi}(0-0) + \hat{z}\left(2\rho\sin\phi\cos\phi - \rho\times 2\sin\phi\cos\phi\right)\right] = 0$$

Flux of $\vec{V}$ through the curved surface of cylinder $x^2+y^2=4,\ 0\le z\le 5$ (i.e. $d\vec{S}_\rho = \rho d\phi dz\hat{\rho}$) in first quadrant, is

$$\therefore \quad \iint_S \vec{V}\cdot d\vec{S} = \int_{z=0}^{5}\int_{\phi=0}^{\pi/2} \rho\left(2+\sin^2\phi\right)\times \rho d\phi dz \qquad (\text{Here, } \rho=2)$$

$$=4\int_{z=0}^{5}\int_{\phi=0}^{\pi/2}\left(2d\phi+\sin^2\phi d\phi\right)dz = 4\int_{z=0}^{5}\left(2\times\frac{\pi}{2}+\frac{\pi}{4}\right)dz$$

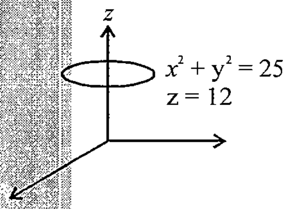
$$=4\times 5\frac{\pi}{4}\times 5 = 25\pi$$

Correct options are (a), (b)

Example 54: If $\vec{r}$ is the position vector field, $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$, then the value of surface integral $\iint_S \vec{r}\cdot d\vec{S}$, where S is the surface $z=12,\ x^2+y^2\le 25$ is ________ $\times\pi$.

Soln: Here, $d\vec{S}=dxdy\hat{k}$

$$\therefore \quad \iint_S \vec{r}\cdot d\vec{S} = \iint zdxdy = 12\iint dxdy = 12\times\pi(5)^2 = 300\,\pi$$



Correct answer is (300)

Example 55: The area of the part of the cone $z^2=3(x^2+y^2)$ (above xy-plane), which is inside the sphere $x^2+y^2+z^2=16$ is ________ $\times\pi$.

Soln: In spherical polar co-ordinates, cone is the surface for which θ is constant.

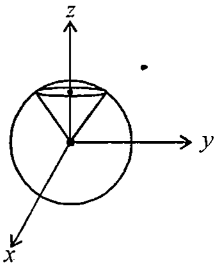
Therefore, area of cone,

$$=\iint_{S_\theta} dS_\theta = \iint r\sin\theta dr d\phi$$

For the cone, $z^2=3\left(x^2+y^2\right)$, $\quad \theta=\tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=30°=\frac{\pi}{6}$

Therefore, area $=\left(\sin\frac{\pi}{6}\right)\int_{r=0}^{4} rdr\int_{\phi=0}^{2\pi} d\phi$

$$=\frac{1}{2}\times\left(\frac{r^2}{2}\right)_0^4(2\pi)=\frac{1}{2}\times\frac{16}{2}\times 2\pi = 8\pi$$

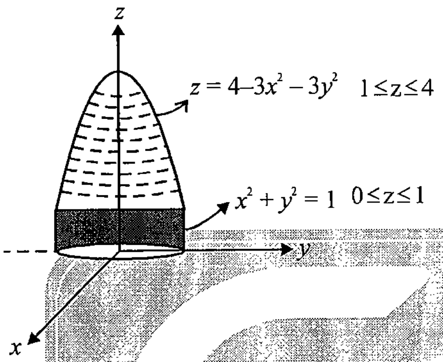


Correct answer is (8)

Example 56: Value of the integral $\iint_S \vec{F}\cdot d\vec{S}$, where $\vec{F} = xy\,\hat{i} - \frac{1}{2}y^2\,\hat{j} + z\,\hat{k}$ and the surface consists of the three surfaces $z = 4 - 3x^2 - 3y^2,\ 1 \le z \le 4$ on the top, $x^2 + y^2 = 1,\ 0 \le z \le 1$ on the sides and $z = 0$ on the bottom, is __________ $\times\pi$.

Soln: $\iint_S \vec{F}\cdot d\vec{S} = \iiint_V \left(\vec{\nabla}\cdot\vec{F}\right)dV$ (since, surface enclosed the volume, shown in the figure below)

$$\vec{\nabla}\cdot\vec{F} = \left(y - \frac{1}{2}\times 2y + 1\right) = 1$$


z
$z = 4-3x^2-3y^2 \quad 1\le z\le 4$
$x^2+y^2=1 \quad 0\le z\le 1$
y
x

$$\therefore \qquad \iint_S \vec{F}\cdot d\vec{S} = \iint \int_{z=0}^{4-3x^2-3y^2} 1\,dV = \iint \int_{z=0}^{4-3x^2-3y^2} dz\,dx\,dy = \iint\left(4 - 3x^2 - 3y^2\right)dx\,dy$$

In *x-y* plane, $x^2 + y^2 = 1$, Put $x = r\cos\theta$ and $y = r\sin\theta$, $dxdy = rdrd\theta$

$$\iint_S \vec{F}\cdot d\vec{S} = \int_{\theta=0}^{2\pi}\int_{r=0}^{1}\left(4 - 3r^2\right)rdrd\theta = 2\pi\left(\frac{4r^2}{2} - \frac{3r^4}{4}\right)_0^1 = \frac{5}{2}\pi$$

Correct answer is (2.5)

Example 57: The flux of vector field $\vec{V} = \left(r\sin\theta\,\hat{r} + \hat{\theta} + \hat{\phi}\right)$ through hemispherical bowl of radius one, base on x-y plane and the origin at the centre of the base is ______________ $\times\pi^2$

Soln: Flux $= \iint_{S_r} \vec{V}\cdot d\vec{S}_r$

Here, S_r is the curved surface of hemipsherical bowl and $d\vec{S}_r = r^2\sin\theta\,d\theta\,d\phi\,\hat{r}$

$$\therefore \qquad \text{Flux} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}\left(r\sin\theta\right)r^2\sin\theta\,d\theta\,d\phi$$

$$= r^3\int_0^{\pi/2}\sin^2\theta\,d\theta\int_0^{2\pi}d\phi = (1)^3\int_0^{\pi/2}\frac{1-\cos 2\theta}{2}d\theta\,(2\pi)$$

$$= 2\pi\left(\frac{1}{2}\theta - \frac{\sin 2\theta}{4}\right)_0^{\pi/2} = 2\pi\times\frac{\pi}{4} = \frac{\pi^2}{2}$$

Correct answer is (0.5)

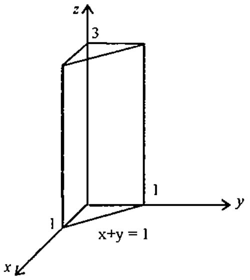
1.7 Volume Integration of Vectors

Volume integral of a vector field $\vec{A}$ within the volume V can be written as,

$$\iiint_V \vec{A}dV, \text{ where, } dV \text{ is the infinitesimal volume element}$$

Physical Interpetation:

Example 58: Calculate the volume integral of $T = xyz^2$ over the prism in figure.



Soln: $\int T dv = \int_0^3 z^2 \left\{ \int_0^1 y \left[\int_0^{1-y} x dx \right] dy \right\} dz = \frac{1}{2}\int_0^3 z^2 dz \int_0^1 (1-y)^2 y\, dy = \frac{1}{2}(9)\left(\frac{1}{12}\right) = \frac{3}{8}$

Example 59: The volume of the portion of the cyllinder $x^2 + y^2 = 4$ in the first octant between the planes z = 0 and $3x - z = 0$ is

(a) 2 (b) 4 (c) 8 (d) 16

Soln: Volume of the portion of the cyllinder in the first octant $= V = \int\limits_0^2 \int\limits_0^{\sqrt{4-x^2}} \int\limits_{z=0}^{3x} dxdydz = 8$

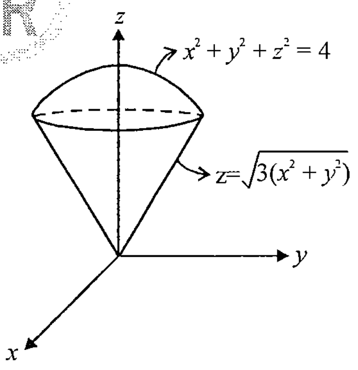
Example 60: Let 'u' be the 'ice cream cone' bounded below by $z = \sqrt{3(x^2 + y^2)}$ and above by $x^2 + y^2 + z^2 = 4$. Then the volume of 'u' is ______ $\times \pi$

Soln. Volume of 'ice-cream' cone $= \iiint_u dV$

Using spherical co-ordinates

$$\iiint_u dV = \int\limits_{\phi=0}^{2\pi} \int\limits_{\theta=0}^{\pi/6} \int\limits_{r=0}^{2} r^2 \sin\theta\, dr d\theta d\phi$$

$$= \left(\frac{r^3}{3}\right)^2 (-\cos\theta)_0^{\pi/6} (2\pi) = \frac{8}{3} \times \left(-\frac{\sqrt{3}}{2} + 1\right) 2\pi = 0.715\,\pi$$



Correct answer is (0.715)

Example 61: The volume integral of the function

$A(r,\theta,\phi)=r^2\cos\theta\cos\phi$ over the region $0\le r\le 1,\ 0\le\theta\le\frac{\pi}{2},\ 0\le\phi\le\pi$ is equal to _________.

Soln: Volume integral=

$$\iiint A dV=\int_{\phi=0}^{\pi}\int_{\theta=0}^{\pi/2}\int_{r=0}^{1} r^2\cos\theta\cos\phi r^2\sin\theta drd\theta d\phi$$

$$=\int_0^1 r^4 dr\int_0^{\pi/2}\sin\theta\cos\theta d\theta\int_0^{\pi}\cos\phi d\phi$$

$$=\left(\frac{r^5}{5}\right)_0^1\left(\frac{1}{2}\times\frac{-\cos 2\theta}{2}\right)_0^{\pi/2}(\sin\theta)_0^{\pi}=0$$

Correct answer is (0)

Example 62: Let D denote the solid bounded below by the plane $z+y=2$, above by the cylinder $z^2+y^2=4$ and on the sides $x=0$ and $x=2$. Then value of the integral $\iiint\limits_D x\,dx\,dy\,dz$ is

(a) $\pi-16$ (b) $\pi+16$ (c) π (d) $-\pi$

Soln: $I=\iint\int_{x=0}^{2} xdxdydz=2\int_{y=-2}^{2}\int_{z=2-y}^{\sqrt{4-y^2}} dzdy$

$$=2\int_{y=-2}^{2}\left(\sqrt{4-y^2}-2+y\right)dy=2\int_{-2}^{2}\sqrt{4-y^2}dy-4\int_{-2}^{2}dy+2\int_{-2}^{2}ydy$$

$$=2\times 2\int_0^2\sqrt{4-y^2}dy-4\times 4+2\times 0=4\times\frac{\pi}{4}-16 \qquad (\text{hint : put } y=2\sin\theta)$$

$$=\pi-16$$

Correct option is (a)

1.8 Vector Related Theorems

Divergence Theorem:

This theorem is applicable only for closed surfaces and this theorem converts surface integral into volume integral and vice versa.

If V is the volume bounded by a closed surface S and $\vec{A}$ is a vector function of position with continuous derivatives, then.

$$\oiint_S \vec{A}\cdot\vec{ds}=\oiint_S \vec{A}\cdot\hat{n}ds=\iiint_V(\vec{\nabla}\cdot\vec{A})dV$$

where $\hat{n}$ is the outward normal to 'S' indicating the positive direction of S.

Stokes Theorem:

This theorem is applicable only for open surfaces and this theorem converts surface integral into line integral and vice versa.

If *S* is an open, two-sided surface bounded by a closed, non-intersecting curve *C*, and $\vec{A}$ is a vector function of position with continuous derivatives; then

$$\oint_C \vec{A}\cdot\overrightarrow{dr} = \iint_S (\vec{\nabla}\times\vec{A}).\hat{n}ds = \iint_S (\vec{\nabla}\times\vec{A})\cdot\overrightarrow{ds}$$

where *C* is traversed in the positive (counterclockwise) direction.

Example 63: Calculate the flux of the vector field $\vec{V} = r^2\vec{r}$ through the surface S of a sphere of radius 'a', centered at the origin.

Soln: Flux of the vector field $\vec{V} = \oiint_S \vec{V}.d\vec{s} = \iiint_V \left(\vec{\nabla}.\vec{V}\right)dv$ (Using divergence theorem)

$$= \iiint_V \left(\vec{\nabla}.r^2\vec{r}\right)dv = \iiint_V 5r^2dv = 5\iiint_V r^2\left(r^2\sin\theta drd\theta d\phi\right) = 5\int_0^a r^4dr\int_0^\pi \sin\theta d\theta\int_0^{2\pi} d\phi = 4\pi a^5$$

Example 64: Calculate $\oint_C \vec{A}.d\vec{r}$ for the vector field $\vec{A} = (2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$ along the boundary of upper half of the sphere $x^2+y^2+z^2=1$.

Soln: The boundary is the circle of unit radius in the x-y plane i.e. $x^2+y^2=1$. Therefore, substituting $x=\cos\theta$, $y=\sin\theta$ and $z=0$ we get,

$$\oint_C \vec{A}.d\vec{r} = \oint_C\left[(2x-y)dx - yz^2dy - y^2zdz\right] = \oint_C (2x-y)dx = \int_0^{2\pi}(2\cos\theta-\sin\theta)(-\sin\theta d\theta) = \pi$$

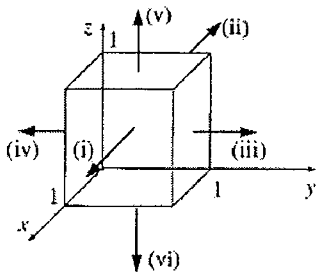
Example 65: Check the divergence theorem using the function

$$\vec{\mathrm{A}} = \mathrm{y}^2\hat{\mathrm{x}} + \left(2\mathrm{xy}+\mathrm{z}^2\right)\hat{\mathrm{y}} + (2\mathrm{yz})\hat{\mathrm{z}}$$

over the unit cube situated at the origin shown in figure.

Soln: $\iiint_V \left(\vec{\nabla}.\vec{A}\right)dv = \iiint_V 2(x+y)dv = 2\int_0^1\int_0^1\int_0^1 (x+y)\,dx\,dy\,dz$

$$= 2\left[\int_0^1\int_0^1\int_0^1 x\,dx\,dy\,dz + \int_0^1\int_0^1\int_0^1 y\,dx\,dy\,dz\right] = 2\left[\frac{1}{2}\times1\times1 + \frac{1}{2}\times1\times1\right] = 2$$



To evaluate the surface integral we must consider separately the six sides of the cube:

(i) $\int \vec{A}.d\vec{s} = \int_0^1\int_0^1 y^2 dy\, dz = \frac{1}{3}$ (ii) $\int \vec{A}.d\vec{s} = -\int_0^1\int_0^1 y^2 dy\, dz = -\frac{1}{3}$

(iii) $\int \vec{A}.d\vec{s} = -\int_0^1\int_0^1 \left(2x+z^2\right) dx\, dz = \frac{4}{3}$ (iv) $\int \vec{A}.d\vec{s} = -\int_0^1\int_0^1 z^2\, dx\, dz = -\frac{1}{3}$

(v) $\int \vec{A}.d\vec{s} = -\int_0^1\int_0^1 2y\, dx\, dy = 1$ (vi) $\int \vec{A}.d\vec{s} = -\int_0^1\int_0^1 0\, dx\, dy = 0$

So the total flux is:

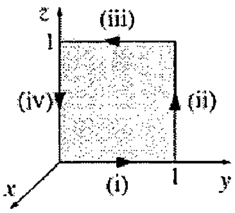
$\oint_S \vec{A}.d\vec{s} = \frac{1}{3}-\frac{1}{3}+\frac{4}{3}-\frac{1}{3}+1+0 = 2$. Hence, divergence theorem is verified.

Example 66: Suppose $\vec{A} = \left(2xz+3y^2\right)\hat{j} + \left(4yz^2\right)\hat{k}$. Check Stokes' theorem for the square surface shown in figure.

Soln: Here, $\nabla\times\vec{A} = \left(4z^2-2x\right)\hat{i} + 2z\,k$ and $d\vec{s} = dy\, dz\,\hat{i}$

Since x = 0 for this surface, $\int\left(\nabla\times\vec{A}\right).\vec{ds} = \int_0^1\int_0^1 4z^2 dy\, dz = \frac{4}{3}$

Now, for line integral, We must break this up into four segments:



(i) $x=0,\ z=0,\ \vec{A}.\vec{dl} = 3y^2 dy \Rightarrow \int \vec{A}.\vec{dl} = \int_0^1 3y^2 dy = 1$

(ii) $x=0,\ y=1,\ \vec{A}.\vec{dl} = 4z^2 dz \Rightarrow \int \vec{A}.\vec{dl} = \int_0^1 4z^2 dz = \frac{4}{3}$

(iii) $x=0,\ z=1,\ \vec{A}.\vec{dl} = 3y^2 dy \Rightarrow \int \vec{A}.\vec{dl} = \int_1^0 3y^2 dy = -1$

(iv) $x=0, y=0, \vec{A}.\vec{dl} = 0 \Rightarrow \int \vec{A}.\vec{dl} = \int_1^0 0 dz = 0$

So, $\oint \vec{A}.\vec{dl} = 1+\frac{4}{3}-1+0 = \frac{4}{3}$

Example 67: Evaluate $\oint_C \left(xy+y^2\right)dx + x^2 dy$ where C is the closed curve of the region bounded by $y = x$ and $y = x^2$.

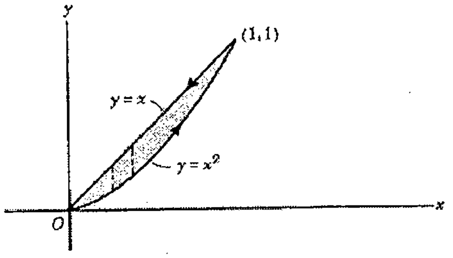
Soln: $\oint_C \left(xy+y^2\right)dx + x^2 dy = \oint_C\left[\left(xy+y^2\right)\hat{i} + x^2\hat{j}\right].d\vec{r} = \iint_S \vec{\nabla}\times\left[\left(xy+y^2\right)\hat{i}+x^2\hat{j}\right].d\vec{S}$ (Stokes theorem)

For any region in x-y plane, $d\vec{S} = dxdy\,\hat{k}$

$$= \iint_S (x-2y)\,dxdy = \int_{x=0}^{1}\int_{y=x^2}^{x}(x-2y)\,dydx$$

$$= \int_0^1\left[\int_{x^2}^{x}(x-2y)\,dy\right]dx = \int_0^1\left(xy-y^2\right)\Big|_{x^2}^{x} dx$$

$$= \int_0^1\left(x^4-x^3\right)dx = -\frac{1}{20}$$



Example 68: Calculate the flux of the vector field $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ through the surface bounded by the region $x^2 + y^2 = 4, z = 0, z = 3$.

Soln: Flux of the vector field $= \oiint\limits_S \vec{F}.d\vec{s} = \iiint\limits_V \left(\vec{\nabla}.\vec{F}\right) dv$ (according to divergence theorem)

$$= \iiint\limits_V (4 - 4y + 2z)\, dxdydz = \int\limits_{z=0}^{3} \int\limits_{\phi=}^{2\pi} \int\limits_{\rho=0}^{2} (4 - 4\rho\sin\phi + 2z)\, \rho d\rho d\phi dz = 84\pi$$

Example 69: Evalaute

$$\oint_C \left(x^2 - \cosh y\right) dx + \left(y + \sin x\right) dy$$

where C is the rectangle with vertices $(0,0), (\pi, 0), (\pi, 1), (0, 1)$

Soln: $\oint_C \left(x^2 - \cosh y\right) dx + \left(y + \sin x\right) dy = \oint\limits_C \left[\left(x^2 - \cosh y\right)\hat{i} + \left(y + \sin x\right)\hat{j}\right].d\vec{r}$

$$= \iint\limits_S \vec{\nabla} \times \left[\left(x^2 - \cosh y\right)\hat{i} + \left(y + \sin x\right)\hat{j}\right].d\vec{S} \text{ (using Stokes theorem)}$$

For any region in x-y plane, $d\vec{S} = dxdy\,\hat{k}$

$$= \iint_S (\cos x + \sinh y)\, dxdy = \int_{x=0}^{\pi} \int_{y=0}^{1} (\cos x + \sinh y)\, dxdy$$

$$= \int_{x=0}^{\pi} \left[y\cos x + \cosh y\right]_{y=0}^{1} dx = \int_{x=0}^{\pi} \left[\cos x + \cosh - 1\right] dx = \left[\sin x + x\cosh 1 - x\right]_0^x = \pi(\cosh 1 - 1)$$

Example 70: Evaluate, $I = \iint_S \left(x^3 dydz + x^2 ydzdx + x^2 z\, dx\, dy\right)$

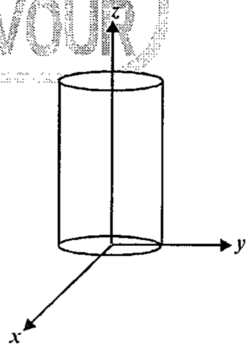
where S is the closed surface bounded by the planes $z = 0, z = b$ and the cylinder $x^2 + y^2 = a^2$.

Soln: By divergence theorem, the required surface integral I is equal to the volume integral.

$$\iiint_V \left[\frac{\partial}{\partial x}\left(x^3\right) + \frac{\partial}{\partial y}\left(x^2 y\right) + \frac{\partial}{\partial z}\left(x^2 z\right)\right] dV$$

$$= \iiint 5x^2 dxdydz = 5\iiint \rho^2 \cos^2\phi\, (\rho d\rho d\phi dz)$$

$$= 5\int\limits_{\rho=0}^{a} \rho^3 d\rho \int\limits_{\phi=0}^{2\pi} \cos^2\phi d\phi \int\limits_{z=0}^{b} dz = 5.\frac{a^4}{4}.\pi b = \frac{5}{4}\pi b a^4$$



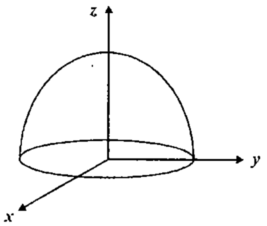
Example 71: Evaluate, $\iint_S \left(y^2 z^2\hat{i} + z^2 x^2\hat{j} + z^2 y^2\hat{k}\right).\hat{n}\, ds$

Where S is the part of the sphere $x^2 + y^2 + z^2 = 1$ above the xy-plane are bounded by this plane.

Soln: By divergence theorem, we have

$$\iint_S \left(y^2 z^2\hat{i} + z^2 x^2\hat{j} + z^2 y^2\hat{k}\right).\hat{n}ds = \iiint_V div\left(y^2 z^2\hat{i} + z^2 x^2\hat{j} + z^2 y^2\hat{k}\right) dV$$

$$= \iiint_V 2zy^2\, dV = 2\iiint_V zy^2 dV$$



Using spherical polar coordinates (r, θ, ϕ), the integral will be

$$=2\int_{r=0}^{1}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi}(r\cos\theta)\left(r^2\sin^2\theta\sin^2\phi\right)r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$=2\int_{r=0}^{1}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi}r^5\sin^3\theta\cos\theta\sin^2\phi\,dr\,d\theta\,d\phi=\frac{\pi}{2}$$

Example 72: If $F=\left(x^2+y-4\right)\hat{i}+3xy\hat{j}+\left(2xz+z^2\right)\hat{k}$, evaluate $\iint_S\left(\nabla\times\vec{F}\right).\hat{n}dS$ where S is the surface of the sphere $x^2+y^2+z^2=16$ above the xy-plane.

Soln: S is the hemispherical surafce $x^2+y^2+z^2=16$ bounded by the circle C given by $x^2+y^2=16, z=0$.

Using Stokes theorem,

$$\iint\limits_S\left(\vec{\nabla}\times\vec{F}\right).\hat{n}dS=\oint\limits_C\vec{F}.d\vec{r}=\oint\limits_C\left(x^2+y-4\right)dx+3xydy$$

Putting $x=4\cos\theta$ and $y=4\sin\theta$, we get

$$=\int\limits_{\theta=0}^{2\pi}\left(16\cos^2\theta+4\sin\theta-4\right)\left(-4\sin\theta d\theta\right)+3\left(4\cos\theta\right)\left(4\sin\theta\right)\left(4\cos d\theta\right)=-16\pi$$

Example 73: Evaluate by Stokes theorem

$$\oint_C\left(e^x dx+2ydy-dz\right)$$

where C is the curve $x^2+y^2=4, z=2$

Soln: $\oint\limits_C\left(e^x dx+2y\,dy-dz\right)=\oint\limits_C\left(e^x\hat{i}+2y\hat{j}-\hat{k}\right).\left(dx\hat{i}+dy\hat{j}+dz\hat{k}\right)=\oint\limits_C\vec{F}.d\vec{r}$, where $F=e^x\hat{i}+2y\hat{j}-\hat{k}$

Also, $curl\ F=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ e^x&2y&-1\end{vmatrix}=0$

Hence, $\oint\limits_C\vec{F}.d\vec{r}=\iint_S\left(curl\ \vec{F}\right).\hat{n}\,dS=0$, since curl F=0

Example 74: If $\vec{A}=\dfrac{-y\hat{x}+x\hat{y}}{y^2+\dfrac{x^2}{4}}$, then the value of $\iint\limits_S\left(\vec{\nabla}\times\vec{A}\right)\cdot d\vec{S}$, where S is the surface of paraboloid $z=4-(x^2+4y^2)$ centered at origin, is

(a) 0 (b) 4π (c) -2π (d) π

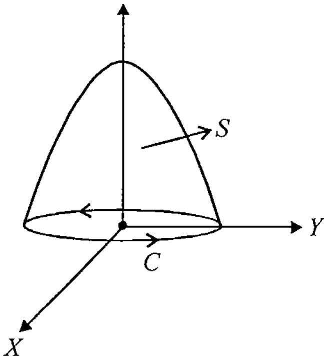
Soln: If you find $\vec{\nabla}\times\vec{A}$, it is zero. Hence, $\iint\limits_S\left(\vec{\nabla}\times\vec{A}\right)\cdot d\vec{S}=0$

But it is wrong because given vector field $\vec{A}$ is not defined at the point (0, 0, 4) on the surface. Hence curl of vector field $\vec{A}$ will not be equal to zero at (0, 0, 4).

Given surface S is bounded by the curve C on x-y plane and

$C : x^2 + 4y^4 = 4$ (taken anti-clockwise)

According to Stoke's theorem,



$$\iint_S \left(\vec{\nabla}\times\vec{A}\right)\cdot d\vec{S} = \oint_C \vec{A}\cdot d\vec{r} = \oint_C \left(\frac{-ydx}{\frac{x^2}{4}+y^2} + \frac{xdy}{\frac{x^2}{4}+y^2}\right)$$

Put $x = 2\cos\theta \Rightarrow dx = -2\sin\theta d\theta$ and $y = \sin\theta \Rightarrow dy = \cos\theta d\theta$

$$\therefore \quad \oint_C \vec{A}\cdot d\vec{r} = \int_{\theta=0}^{2\pi} \frac{-\sin\theta\times -2\sin\theta d\theta + 2\cos\theta\times\cos\theta d\theta}{\frac{4\cos^2\theta}{4}+\sin^2\theta}$$

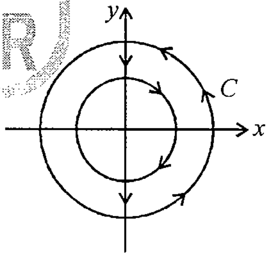
$$= \int_{\theta=0}^{2\pi} \left(2\sin^2\theta + 2\cos^2\theta\right)d\theta = 2\times 2\pi = 4\pi$$

Correct option is (b)

Example 75: The value of $\oint_C \vec{F}\cdot d\vec{r}$, where C is the curve bounded by $x^2+y^2 \geq 4$; $x^2+y^2 \leq 16$; $x \geq 0$ and $\vec{F} = -y\hat{i} + x\hat{j} + z\hat{k}$ is ________ $\times \pi$.

Soln: Using Stoke's theorem,

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \left(\vec{\nabla}\times\vec{F}\right)\cdot d\vec{S}$$



$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & z \end{vmatrix} = \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(1+1) = 2\hat{k}$$

and $d\vec{S} = dxdy\hat{k}$

$$\therefore \quad \iint_S \left(\vec{\nabla}\times\vec{F}\right)\cdot d\vec{S} = 2\iint dxdy$$

Put, $x = r\cos\theta$, $y = r\sin\theta$ and $dxdy = rdrd\theta$

$$= 2\int_{\theta=-\pi}^{\pi}\int_{r=2}^{4} rdrd\theta = 2\left(\frac{r^2}{2}\right)_2^4 (2\pi) = 2\pi\times 12 = 24\,\pi$$

Correct answer is (24)

Example 76: If S is the closed surface enclosing a volume V and $\hat{n}$ is the unit normal vector to the surface and $\vec{r}$ is the positive vector, then the value of the folllowing integral $\iint_S \vec{r}\cdot\hat{n}dS$ is

(a) V (b) 2V (c) 0 (d) 3V

Soln: $\iint_S \vec{r}\cdot\hat{n}dS = \iiint_V \left(\vec{\nabla}\cdot\vec{r}\right)dV$ (using Divergence Theorem)

$$= 3\iiint_V dV = 3V$$

Correct option is (d)

Example 77: The value of $\oint_S \frac{\vec{r}.d\vec{s}}{r^3}$, where $\vec{r}$ is the position vector ans S is the closed surface enclosing the origin, is

(a) zero (b) π (c) 4π (d) 8π

Soln: $\oint_s \frac{\vec{r}\cdot d\vec{s}}{r^3} = \int_V \vec{\nabla}\cdot\left(\frac{\vec{r}}{r^3}\right)dV$ (using divergence theorem)

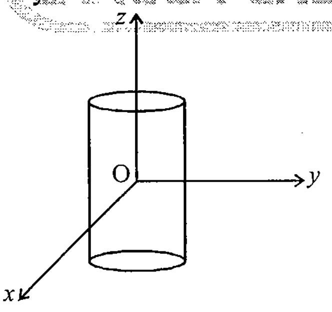
Since the closed surface encloses the origin i.e. $\vec{r} = 0$

Therefore, $\vec{\nabla}\cdot\left(\frac{\vec{r}}{r^3}\right) = 4\pi\,\delta^3(\vec{r})$

So, $\int_V \vec{\nabla}\cdot\left(\frac{\vec{r}}{r^3}\right)dV = 4\pi\int_V \delta^3(\vec{r})dV = 4\pi$

Correct option is (c)

Example 78: Consider a cylinder of height h and radius a, closed at both ends, centered at the origin. Let $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$ be the position vector and $\hat{n}$ a unit vector normal to the surface. The surface integral $\int_S \vec{r}\cdot\hat{n}\,dS$ over the closed surface of the cylinder is :



(a) $2\pi a^2(a+h)$ (b) $3\pi a^2 h$ (c) $2\pi a^2 h$ (d) zero

Soln: $\int_S \vec{r}\cdot\hat{n}\,dS = \int_V (\vec{\nabla}\cdot\vec{r})dV$ (Using divergence theorem)

$$= 3\int_V dV = 3V = 3\pi a^2 h$$

1.9 Linear Dependency of Vectors

Three vectors $\vec{u}, \vec{v}, \vec{w}$ will be linearly dependent if $\alpha_1\vec{u} + \alpha_2\vec{v} + \alpha_3\vec{w} = 0$, where α_1, α_2, α_3 are not all zero. These three independent vectors do not form the basis of $\mathbb{R}^3$ space.

If $\alpha_1\vec{u} + \alpha_2\vec{v} + \alpha_3\vec{w} = 0$ such that $\alpha_1 = \alpha_2 = \alpha_3 = 0$, then $\vec{u}$, $\vec{v}$, $\vec{w}$ are said to be linearly independent and they do form the basis of $\mathbb{R}^3$ space i.e. any vector in $\mathbb{R}^3$ space can be written as the linear combination of vectors $\vec{u}, \vec{v}, \mathrm{w}$ respectively.

How to check:

$$\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} \begin{cases} = 0 \text{ for linearly dependent vectors} \\ \neq 0 \text{ for linearly independent vectors} \end{cases}$$

Example 79: Consider the set of vectors $\frac{1}{\sqrt{2}}(1,1,0), \frac{1}{\sqrt{2}}(0,1,1)$ and $\frac{1}{\sqrt{2}}(1,0,1)$

(a) the three vectors are orthonormal

(b) the three vectors are linearly independent

(c) the three vectors cannot form a basis in a three-dimensional real vector space

(d) $\frac{1}{\sqrt{2}}(1,1,0)$ can be written as the linear combination of $\frac{1}{\sqrt{2}}(0,1,1)$ and $\frac{1}{\sqrt{2}}(1,0,1)$

Soln: $A = \frac{1}{\sqrt{2}}(1,1,0)$; $B = \frac{1}{\sqrt{2}}(0,1,1)$; $C = \frac{1}{\sqrt{2}}(1,0,1)$

$$\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \begin{vmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{vmatrix} = \frac{1}{\sqrt{2}}\left(\frac{1}{2}-0\right) - 0\left(\frac{1}{2}-0\right) + \frac{1}{\sqrt{2}}\left(\frac{1}{2}-0\right) = \frac{1}{\sqrt{2}} \neq 0$$

So, $\bar{A}, \bar{B}, \bar{C}$ are linearly independent.

Correct option is (b)

Example 80: Given the four vectors, $u_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, u_2 = \begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix}, u_3 = \begin{pmatrix} 2 \\ 4 \\ -8 \end{pmatrix}, u_4 = \begin{pmatrix} 3 \\ 6 \\ -12 \end{pmatrix}$, the linearly dependent pair is

(a) u_1, u_2 (b) u_1, u_3 (c) u_1, u_4 (d) u_3, u_4

Soln: $u_4 = \frac{3}{2}u_3$ i.e. u_3, u_4 are linearly dependent

Correct option is (d)

Example 81: Consider the set of vectors in three-dimensional real vector space R^3,

$$S=\{(1,1,1),(1,-1,1),(1,1,-1)\}$$

Which of the following statement is true?
(a) S is not a linearly independent set
(b) S is a basis for R^3
(c) The vectors in S are orthogonal
(d) An orthogonal set of vectors cannot be generated from S

Soln: $\begin{vmatrix}1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & -1\end{vmatrix}=1(1-1)-1(-1-1)+1(1+1)=4\neq 0$

Correct option is (b)

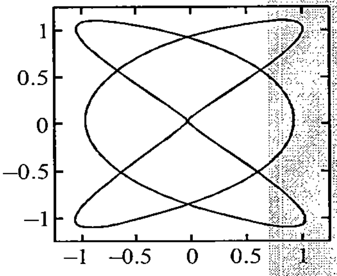
TIFR PREVIOUS YEAR QUESTIONS & SOLUTIONS

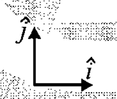
Question 1. A two-dimensional vector $\vec{A}(t)$ is given by $\vec{A}(t)=\hat{i}\sin 2t+\hat{j}\cos 3t$

Which of the following graphs best describes the locus of the tip of the vector, as t is varied from 0 to 2π?

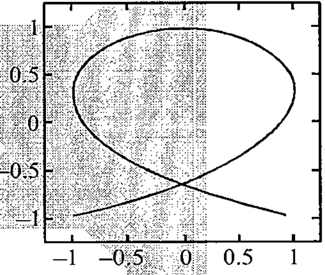
[TIFR 2013]

(a)

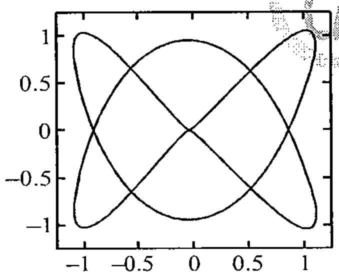




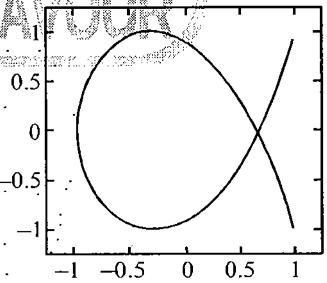
(b)



(c)



(d)



Soln: $\vec{A}(t)=\hat{i}\sin 2t+\hat{j}\cos 3t$

Here, $X(t)=\sin 2t$ and $Y(t)=\cos 3t$

Let us find the number of zeros along x-axis and y-axis.

Along y-axis	Along x-axis
$X(t)=0\Rightarrow\sin 2t=0$	$Y(t)=0\Rightarrow\cos 3t=0$
$2t=n\pi\ ;\ n=0,\pm1,\pm2,....$	$3t=(2n+1)\frac{\pi}{2}\ ;\ n=0,\pm1,\pm2,....$
$n=0\quad t=0$	$n=0\quad t=\frac{\pi}{6}$

$n=1 \quad t=\frac{\pi}{2}$

$n=2 \quad t=\pi$

$n=3 \quad t=\frac{3\pi}{2}$

$n=4 \quad t=2\pi$

Hence, $X(t)=0$ for four times in one period (2π).

$n=1 \quad t=\frac{\pi}{2}$

$n=2 \quad t=\frac{5\pi}{6}$

$n=3 \quad t=\frac{7\pi}{6}$

$n=4 \quad t=\frac{9\pi}{6}$

$n=5 \quad t=\frac{11\pi}{6}$

$n=6 \quad t=\frac{13\pi}{6}$

Hence, $Y(t)=0$ is zero for six times in one period (2π)

The curve given in option (c) passes thorugh $x=0$ four times and $y=0$ six times.

Correct option is (c)

Question 2. Consider the surface corresponding to the equation $4x^2+y^2+z=0$. A possible unit tangent to this surface at the point (1, 2, −8) is **[TIFR 2013]**

(a) $\frac{1}{\sqrt{5}}\hat{i}-\frac{2}{\sqrt{5}}\hat{j}$

(b) $\frac{1}{5}\hat{j}-\frac{4}{5}\hat{k}$

(c) $\frac{4}{9}\hat{i}-\frac{8}{9}\hat{j}+\frac{1}{9}\hat{k}$

(d) $-\frac{1}{\sqrt{5}}\hat{i}+\frac{3}{\sqrt{5}}\hat{j}-\frac{4}{\sqrt{5}}\hat{k}$

Soln: Given surface is $\phi(x,y,z)=4x^2+y^2+z=0$.

Hence normal to this surface is $\vec{\nabla}\phi=8x\hat{i}+2y\hat{j}+\hat{k}$. Therefore, $\vec{\nabla}\phi\big|_{(1,2,-8)}=8\hat{i}+4\hat{j}+\hat{k}$.

Now, the dot product of $\vec{\nabla}\phi$ with the tangent vector should be zero. Here, option (a), (b) and (d) satisfies the condition. But only the vector given in option (a) has unit magnitude.

Correct option is (a)

Question 3. Which of the following vectors is parallel to the surface $x^2y+2xz=4$ at the point (2,-2,3)? **[TIFR 2015]**

(a) $-6\hat{i}-2\hat{j}+5\hat{k}$ (b) $6\hat{i}+2\hat{j}+5\hat{k}$ (c) $6\hat{i}-2\hat{j}+5\hat{k}$ (d) $6\hat{i}-2\hat{j}-5\hat{k}$

Soln: Given surface is $\phi(x,y,z)=x^2y+2xz-4=0$.

Hence, normal to this surface is $\vec{\nabla}\phi=(2xy+2z)\hat{i}+x^2\hat{j}+2x\hat{k}$. Therefore, $\vec{\nabla}\phi\big|_{(2,-2,3)}=-2\hat{i}+4\hat{j}+4\hat{k}$

Now, the dot product of $\vec{\nabla}\phi$ with vector parallel to the surface, should be zero. Here, only option (c) satisfies the condition.

Correct option is (c)

JEST PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. The vector field $xz\hat{i}+y\hat{j}$ in cylinderical polar coordinates is **[JEST 2013]**

(a) $\rho\left(z\cos^2\phi+\sin^2\phi\right)\hat{e}_\rho+\rho\sin\phi\cos\phi\left(1-z\right)\hat{e}_\phi$

(b) $\rho\left(z\cos^2\phi+\sin^2\phi\right)\hat{e}_\rho+\rho\sin\phi\cos\phi\left(1+z\right)\hat{e}_\phi$

(c) $\rho\left(z\sin^2\phi+\cos^2\phi\right)\hat{e}_\rho+\rho\sin\phi\cos\phi\left(1+z\right)\hat{e}_\phi$

(d) $\rho\left(z\sin^2\phi+\cos^2\phi\right)\hat{e}_\rho+\rho\sin\phi\cos\phi\left(1-z\right)\hat{e}_\phi$

Soln: We know that in cylindrical coordinates, $x=\rho\cos\phi,\ \ y=\rho\sin\phi,\ \ z=z$

Also, $\hat{\rho}=\cos\phi\,\hat{i}+\sin\phi\,\hat{j}$ ...(1)

and $\hat{\phi}=-\sin\phi\,\hat{i}+\cos\phi\,\hat{j}$ ...(2)

and $\hat{z}=\hat{k}$

Solving equation (1) & (2), we can get

$$\hat{i}=\cos\phi\,\hat{\rho}-\sin\phi\,\hat{\phi}\ ;\ \hat{j}=\sin\phi\,\hat{\rho}+\cos\phi\,\hat{\phi}$$

Therefore, $\left(xz\,\hat{i}+y\,\hat{j}\right)=z\rho\cos\phi\left(\cos\phi\,\hat{\rho}-\sin\phi\,\hat{\phi}\right)+\rho\sin\phi\left(\sin\phi\,\hat{\rho}+\cos\phi\,\hat{\phi}\right)$

$$=\rho\left(z\cos^2\phi+\sin^2\phi\right)\hat{e}_\rho+\rho\sin\phi\cos\phi\left(1-z\right)\hat{e}_\phi$$

Correct option is (a)

Question 2. What is the equation of the plane which is tangent to the surface $xyz=4$ at the point (1, 2, 2) ? **[JEST 2017]**

(a) $x+2y+4z=12$ (b) $4x+2y+z=12$ (c) $x+4y+z=0$ (d) $2x+y+z=6$

Soln: Let $\phi: xyz-4=0$

$$\therefore\quad \frac{\partial\phi}{\partial x}=yz\Rightarrow\left.\frac{\partial\phi}{\partial x}\right|_{1,2,2}=4,\ \frac{\partial\phi}{\partial y}=xz\ \Rightarrow\left.\frac{\partial\phi}{\partial y}\right|_{1,2,2}=2,\ \frac{\partial\phi}{\partial z}=xy\ \Rightarrow\left.\frac{\partial\phi}{\partial y}\right|_{1,2,2}=2$$

Therefore, equation of tangent plane at the point (1, 2, 2)

$$\left(x-1\right)\times4+\left(y-2\right)\times2+\left(z-2\right)\times2=0\ \Rightarrow4x+2y+2z=12\ \Rightarrow2x+y+z=6$$

Correct option is (d)

JNU PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. The position vector of a particle moving in two dimensions, as a function of time t, is given as $\vec{r}\left(t\right)=\left(\hat{i}t+\hat{j}\sqrt{t}\right)e^{-t}$ for $t\geq0$. Starting at $t=0$, the distance of the particle from the origin reaches its maximum value at $t=t_m$, and at this instant, the velocity vector $\vec{v}$ makes an angle α with $\vec{r}$. The values of t_m and α are **[JNU 2014]**

(a) $t_m=\frac{1}{2}$ and $\alpha=\frac{\pi}{2}$

(b) $t_m=\frac{1}{\sqrt{2}}$ and $\alpha=\pi$

(c) $t_m=\frac{1}{2}$ and $\alpha=0$

(d) $t_m=\frac{1}{\sqrt{2}}$ and $\alpha=\frac{\pi}{2}$

(e) $t_m=1$ and $\alpha=\pi$

Soln: Given $\vec r(t)=\left(\hat i\, t+\hat j\sqrt{t}\right)e^{-t}$; $t\geq 0$

Distance of particle from origin $=|\vec r(t)|\Rightarrow|\vec r(t)|=\sqrt{(t^2+t)e^{-2t}}$.

For maxima and minima, $\frac{d|\vec r(t)|}{dt}=0\quad\Rightarrow\quad\frac{(2t+1)e^{-2t}+(t^2+t)\times-2e^{-2t}}{\sqrt{(t^2+t)e^{-2t}}}=0$

$$\Rightarrow(2t+1)-2(t^2+t)=0\qquad\Rightarrow-2t^2+1=0\Rightarrow t=\frac{1}{\sqrt{2}}$$

Therefore, $t_m=\frac{1}{\sqrt{2}}$.

Velocity of particle, $\vec V(t)=\frac{d\vec r(t)}{dt}=\hat i\left(e^{-t}-t\,e^{-t}\right)+\hat j\left(\frac{1}{2\sqrt{t}}e^{-t}-\sqrt{t}\,e^{-t}\right)=\left\{\hat i(1-t)+\hat j\left(\frac{1}{2\sqrt{t}}-\sqrt{t}\right)\right\}e^{-t}$

Here, $\vec V\cdot\vec r=\left[\left\{\hat i(1-t)+\hat j\left(\frac{1}{2\sqrt{t}}-\sqrt{t}\right)\right\}e^{-t}\right]\cdot\left[\left(\hat i\, t+\hat j\sqrt{t}\right)e^{-t}\right]=e^{-2t}\left[t(1-t)+\left(\frac{1}{2}-t\right)\right]$

If at $t=t_m$, the velocity vector $\vec v$ makes an angle α with $\vec r$, then

$$\cos\alpha=\left.\frac{\vec V.\vec r}{|\vec V||\vec r|}\right|_{t=t_m=\frac{1}{\sqrt{2}}}=0\Rightarrow\alpha=\frac{\pi}{2}$$

Correct option is (d)

Question 2. Consider a vector field $\vec F=xy^4\hat i+yz^2\hat j$. Its outward flux $\oint\vec F\cdot dS$ over the surface of a cube bounded by the planes $|x|=2$, $|y|=2$ and $|z|=2$ is nearest to **[JNU 2014]**

(a) 410 (b) −273 (c) 290 (d) −300 (e) 0

Soln: According to Divergence Theorem,

$$\oint_S\vec F.ds=\int_V\left(\vec\nabla.\vec F\right)dV=\int_{z=-2}^{2}\int_{y=-2}^{2}\int_{x=-2}^{2}\left(y^4+z^2\right)dxdydz$$

$$=x\Big|_{-2}^{2}\left.\frac{y^5}{5}\right|_{-2}^{2}z\Big|_{-2}^{2}+x\Big|_{-2}^{2}y\Big|_{-2}^{2}\left.\frac{z^3}{3}\right|_{-2}^{2}=\left(4\times\frac{64}{5}\times4\right)+\left(4\times4\times\frac{16}{3}\right)=291.13$$

Correct option is (c)

Question 3. Which of the following vector field can represents an electrostatic field? **[JNU 2015]**

(a) $\left(2xz-y^2\right)\hat{i}+\left(2xy-z^2\right)\hat{j}+\left(2yz-x^2\right)\hat{k}$ (b) $xy\hat{i}+2yz\hat{j}+3xz\hat{k}$

(c) $yz\hat{i}-xz\hat{j}+xy\hat{k}$ (d) $y^2\hat{i}+\left(2xy+z^2\right)\hat{j}+\left(2yz+x^2\right)\hat{k}$ (e) $y^2\hat{i}+\left(2xy+z^2\right)\hat{j}+2yz\hat{k}$

Soln: Since, curl of electrostatic field is zero, therefore check curl of all options.

$$\text{For option (c) ; } \vec{\nabla}\times\vec{E}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ yz & xz & xy\end{vmatrix}=\hat{i}(x-x)-\hat{j}(y-y)+\hat{k}(z-z)=0$$

Correct option is (c)

Question 4. Define I to be the value of the surface integral ($\int \mathbf{E}\cdot d\mathbf{S}$, where $d\mathbf{S}$ points outwards from the domain on integration) of a vector field $\mathbf{E}\,(\mathbf{E}=(x+y^2)\,\mathbf{i}+(z^3+y^3)\,\mathbf{j}+(x+z^4)\,\mathbf{k})$ over the entire surface of a cube which bounds the region $\{0<x<2,\ -1<y<1,\ 0<z<2\}$. The value of I is **[JNU 2016]**

(a) 0 (b) 16 (c) 72 (d) 80 (e) 32

Soln: According to Divergence Theorem,

$$\int_S \vec{E}.d\vec{s}=\int_V\left(\vec{\nabla}.\vec{E}\right)dV=\int_{z=0}^{2}\int_{y=-1}^{1}\int_{x=0}^{2}\left(1+3y^2+4z^3\right)dxdydz$$

$$=x\Big|_0^2\,y\Big|_{-1}^1\,z\Big|_0^2+x\Big|_0^2\,y^3\Big|_{-1}^1\,z\Big|_0^2+x\Big|_0^2\,y\Big|_{-1}^1\,z^4\Big|_0^2$$

$$=\left(2\times2\times2\right)+\left(2\times2\times2\right)+\left(2\times2\times16\right)=80$$

Correct option is (d)

Question 5. Which of the following force fields is/are conservative ? **[JNU 2016]**

I. $(4x^3y+3x^2z^2)\,\mathbf{i}+(x^4+3y^2z^2)\,\mathbf{j}+2z\,(x^3+y^3)\,\mathbf{k}$

II. $(yz^4+x^5)\,\mathbf{i}+(x^4z+x^2y^3)\,\mathbf{j}+(x^4y+z^5)\,\mathbf{k}$

III. $x(3xy^2+4z^2)\,\mathbf{i}+(z^4+2x^3y)\,\mathbf{j}+2z^2(2yz+3x^2)\,\mathbf{k}$

(a) II and III only (b) I, II and III (c) I and III only (d) I and II only (e) I only

Soln:

Question 6. The line integral $I=\int_\Gamma \vec{F}\cdot d\vec{r}$ of the vector field $\vec{F}=yz\,i+zx\,j+xy\,k$ is evaluated along the curve Γ parametrized by $(x=a\sin t,\ y=b\cos t,\ z=c)$, where a, b and c and c are constant. The value of I for $\frac{\pi}{4}\le t\le\frac{3\pi}{4}$ is **[JNU 2017]**

(a) abc (b) ab (c) bc (d) ac (e) 0

Soln: $I=\int_\Gamma \vec{F}\cdot d\vec{r}=\int_\Gamma (yz\,dx+zx\,dy+xy\,dz)$

Put $x=a\sin t \Rightarrow dx=a\cos t\,dt,\ y=b\cos t \Rightarrow dy=-b\sin t\,dt,\ z=c \Rightarrow dz=0$

$$\therefore \quad I = \int_{t=\pi/4}^{3\pi/4} b\cos t \times c\cdot a\cos t\,dt + c\cdot a\sin t \times -b\sin t\,dt + 0$$

$$= \int_{\pi/4}^{3\pi/4} abc(\cos^2 t - \sin^2 t)\,dt$$

$$= abc\int_{\pi/4}^{3\pi/4} \cos 2t\,dt = abc\left[\frac{\sin 2t}{2}\right]_{\pi/4}^{3\pi/4} = \frac{abc}{2}[-1-1] = -abc$$

Correct option is (a)

DU PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. A vector field is given by $\vec{A} = y\hat{i} + z\hat{j} + x\hat{k}$. The curl of the vector is: **[DU 2014]**

(a) $\hat{i}+\hat{j}-\hat{k}$ (b) $-\hat{i}-\hat{j}-\hat{k}$ (c) $-\hat{i}-\hat{j}+\hat{k}$ (d) $\hat{i}+\hat{j}+\hat{k}$

Soln: $\vec{A} = y\hat{i} + z\hat{j} + x\hat{k}$

$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & z & x \end{vmatrix} = \hat{i}(0-1) - \hat{j}(1-0) + \hat{k}(0-1) = -\hat{i}-\hat{j}-\hat{k}.$$

Correct option is (b)

Question 2. The magnitude of the cross product of two vectors $\overline{AB}$ and $\overline{AC}$ is equal to: **[DU 2014]**

(a) The area of the triangle
(b) The area of the parallelogram with adjacent sides AB and AC
(c) Twice the area of the rectangle with adjacent sides AB and AC
(d) The area of the circle passing through A, B and C

Soln: Correct option is (b)

Question 3. Consider a set of surfaces $f(x, y, z) = c$. Different values of c will generate different surfaces. The gradient of f at a point P has a direction: **[DU 2014]**

(a) Tangential to the surface at point P
(b) At 45 degrees to the surface at point P
(c) Normal to the surface at the point P in the direction of increasing value of f
(d) Normal to the surface at the point P in the direction of decreasing value of f

Soln: Correct option is (c)

Question 4. Let $\vec{r} = \hat{x}r\cos\omega t + \hat{y}r\sin\omega t$. For constant r and ω, calculate $\vec{r}\times\frac{d\vec{r}}{dt}$. **[DU 2015]**

(a) $r^2\omega\hat{x}$ (b) $r^2\omega\hat{y}$ (c) $r^2\omega\hat{z}$ (d) $r^2\omega\cos(2\omega t)\hat{z}$

Soln: $\vec{r} = \hat{x}\,r\cos\omega t + \hat{y}\,r\sin\omega t$

$$\frac{d\vec{r}}{dt} = \hat{x}\,r \times -\omega\sin\omega t + \hat{y}\,r\,\omega\cos\omega t$$

Therefore, $\vec{r}\times\dfrac{d\vec{r}}{dt}=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ r\cos\omega t & r\sin\omega t & 0\\ -r\,\omega\sin\omega t & r\,\omega\cos\omega t & 0\end{vmatrix}\hat{z}\left(r^2\omega\cos^2\omega t+r^2\omega\sin^2\omega t\right)$

$$=\hat{z}\,r^2\omega$$

Correct option is (c)

Question 5. The magnitude of $\oint\vec{r}\times\overline{dr}$ over the perimeter of an ellipse described by

$\vec{r}=\hat{x}a\cos\theta+\hat{y}b\sin\theta$, is **[DU 2015]**

(a) $\pi ab/2$ (b) πab (c) $2\pi ab$ (d) ab

Soln: $\vec{r}=\hat{x}\,a\cos\theta+\hat{y}\,b\sin\theta$ and $d\vec{r}=\hat{x}\,a\times-\sin\theta\,d\theta+\hat{y}\,b\cos\theta\,d\theta$

$$\vec{r}\times d\vec{r}=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ a\cos\theta & b\sin\theta & 0\\ -a\sin\theta\,d\theta & b\cos\theta\,d\theta & 0\end{vmatrix}=\hat{k}\left(a\,b\cos^2\theta+a\,b\sin^2\theta\right)=\hat{k}\,a\,b\,d\theta$$

Therefore, $\oint\vec{r}\times d\vec{r}=\hat{k}\,a\,b\oint d\theta=2\pi\,a\,b\,\hat{k}.$

Correct option is (c)

Question 6. Which of the following sets of vectors constitute a basis for the plane? **[DU 2016]**

(i) (0, 1), (1, 1) (ii) (1, 1), (1, −1) (iii) (1, 0), (0, −1)

(a) all three sets of vector (b) only (iii)

(c) only (i) and (iii) (d) only (ii) and (iii)

Soln: Set (i): Vectors (0,1) and (1,1) cannot be written as a scalar multiple of each other.

Set (ii): Vectors (1,1) and (1,-1) cannot be written as a scalar multiple of each other.

Set (iii): Vectors (1,0) and (0,-1) cannot be written as a scalar multiple of each other.

Therefore, all sets of vectors are sets of linearly independent vectors and they will constitute a basis for the plane.

Correct option is (a)

Question 7. Given the position vector $\vec{r}=x\hat{x}+y\hat{y}+z\hat{z}$ and $r^2=\vec{r}\cdot\vec{r}$, the value of $\vec{\nabla}\cdot(r^2\vec{r})$ is given by :

(a) $6r^2$ (b) $5r^2$ (c) $4r^2$ (d) $3r^2$ **[DU 2017]**

Soln: $\vec{\nabla}.\left(r^2\vec{r}\right)=\vec{\nabla}.\left(r^3\hat{r}\right)=\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2A_r\right)=\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^5\right)=5r^2$

Correct option is (b)

HCU PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. If $\vec{r}$ is a position vector, then $\vec{\nabla}\times\vec{r}$ is given by

(a) $r^{3/2}$ (b) $r^{1/2}$ (c) 0 (d) 3 **[H.C.U. 2010]**

Soln: $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$

$$\vec{\nabla}\times\vec{r}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ x & y & z\end{vmatrix}=0$$

Correct option is (c)

Question 2. For a given vector $\vec{A}=x^2\hat{i}+y^2\hat{j}+z^2\hat{k}$, the value of the integral $\iint_S \vec{A}\cdot d\vec{S}$, over the cube defined by $0\le x,\,y,\,z\le 1$ is given by

(a) 0 (b) 2 (c) 3 (d) 4 **[H.C.U. 2010]**

Soln: According to Divergence theorem,

$$\iint_S \vec{A}\cdot d\vec{S}=\iiint_V\left(\vec{\nabla}\cdot\vec{A}\right)dV=\int_{z=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1}(2x+2y+2z)\,dx\,dy\,dz$$

$$=x^2\Big|_0^1\,y\Big|_0^1\,z\Big|_0^1+x\Big|_0^1\,y^2\Big|_0^1\,z\Big|_0^1+x\Big|_0^1\,y\Big|_0^1\,z^2\Big|_0^1=1+1+1=3$$

Correct option is (c)

Question 3. Which of the following can represent a conservative force ? ($\hat{i},\,\hat{j},\,\hat{k}$ are unit vectors)

[H.C.U. 2011]

(a) $C(\hat{i}y+\hat{j}z+\hat{k}z)$ (b) $C(\hat{i}x+\hat{j}y+\hat{k}y)$ (c) $C\cdot xyz(\hat{i}+\hat{j}+\hat{k})$ (d) $C(\hat{i}x+\hat{j}y+\hat{k}z)$

Soln: For field $\vec{A}$ to be conservative, $\vec{\nabla}\times\vec{A}=0$. Only option (d) satifies the condition.

Correct option is (d)

Question 4. What should be the value of a such that the straight line $x+y=a$ will be a tangent to the circle $x^2+y^2=4$?

(a) $\frac{1}{\sqrt{2}}$ (b) $\sqrt{2}$ (c) $2\sqrt{2}$ (d) 4 **[H.C.U. 2011]**

Soln: For $x+y=a$ to be tangent to the circle, $x^2+y^2=4$,

Perpendicular distance to the line $x+y=a$ from the center of the circle (0,0) will be equal to radius of the circle i.e. $\left|\frac{x+y-a}{\sqrt{1+1}}\right|_{(0,0)}=2 \quad\Rightarrow\left|\frac{a}{\sqrt{2}}\right|=2\ \Rightarrow a=2\sqrt{2}$

Correct option is (c)

Question 5. $\nabla^2(\ln r)$ is equal to **[H.C.U.-2011]**

(a) $\frac{1}{r}$ (b) $\frac{1}{r^2}$ (c) $\frac{1}{r^3}$ (d) $\frac{1}{r}+\frac{1}{r^2}$

Soln: $\nabla^2(\ln r)=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\ln r\right)$

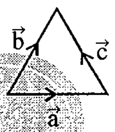
$$=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\times\frac{1}{r}\right)=\frac{1}{r^2}.$$

Correct option is (b)

Question 6. If two sides of triangle are represented by $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{b}=2\hat{j}+3\hat{k}$, then the third side is represented by **[H.C.U. 2012]**

(a) $\hat{i}-3\hat{j}+2\hat{k}$ (b) $\hat{i}+3\hat{j}+2\hat{k}$ (c) $-\hat{i}+\hat{j}-2\hat{k}$ (d) $-\hat{i}+\hat{j}+2\hat{k}$

Soln: In a triangle we can write, $\vec{a}+\vec{c}=\vec{b}\Rightarrow\vec{c}=\vec{b}-\vec{a}=-\hat{i}+\hat{j}+2\hat{k}$



Correct option is (d)

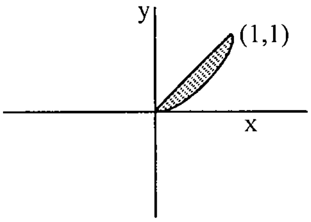
Question 7. A unit vector normal to the surface $z=x^2+y^2$ at the point (1, 2, 5) is given by **[H.C.U. 2012]**

(a) $\pm\frac{\hat{i}+2\hat{j}-\hat{k}}{\sqrt{6}}$ (b) $\pm\frac{2\hat{i}+2\hat{j}-2\hat{k}}{\sqrt{12}}$ (c) $\pm\frac{2\hat{i}+4\hat{j}-\hat{k}}{\sqrt{21}}$ (d) $\pm\frac{2\hat{i}+4\hat{j}-2\hat{k}}{\sqrt{24}}$

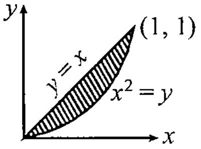
Soln: Unit normal to the surface $=\left.\frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|}\right|_{(1,2,5)}=\left.\frac{2x\,\hat{i}+2y\,\hat{j}-\hat{k}}{\sqrt{4x^2+4y^2+1}}\right|_{(1,2,5)}=\pm\frac{2\hat{i}+4\hat{j}-\hat{k}}{\sqrt{21}}.$

Correct option is (c)

Question 8. The area of the shaded region between the straight line and parabola as shown in the figure is given by **[H.C.U. 2012]**



(a) $\int_0^1 dy\int_{x^2}^{x} dx$ (b) $\int_0^1 dx\int_{x^2}^{x} dy$ (c) $\int_0^1 dx\int_0^{x} dy$ (d) $\int_0^1 dy\int_0^{x^2} dx$



Soln: Area $=\iint dx\,dy=\int_0^1 dy\int_{x=y}^{x=\sqrt{y}} dx=\int_0^1 dx\int_{x^2}^{x} dy$

Correct option is (b)

Question 9. The number of vectors orthogonal to $\vec{i}+2\vec{j}-\vec{k}$ is **[H.C.U. 2013]**

(a) One (b) Two (c) Three (d) Infinite

Soln: There can be many vectors orthogonal to a given vector.

Correct option is (d)

Question 10. $\vec{A}\cdot\left(\vec{B}\times\vec{C}\right)=0$ implies that **[H.C.U. 2013]**

(a) All the three vectors are parallel.
(b) All the three vectors are mutually perpendicular.
(c) Two of the three vectors are perpendicular
(d) The vectors lie in a plane.

Soln: $\vec{A}\cdot\left(\vec{B}\times\vec{C}\right)=0$

We know that $\vec{A}\cdot\left(\vec{B}\times\vec{C}\right)$ gives the volume of a parallelopiped whose adjacent sides are $\vec{A}$, $\vec{B}$ and $\vec{C}$.

Therefore, $\vec{A}\cdot\left(\vec{B}\times\vec{C}\right)=0$ means all vectors lie in a plane.

Correct option is (d)

Question 11. For a unit sphere, with center at the origin the equation of the tangent plane at the point $\left(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$ is **[H.C.U. 2013]**

(a) $x+y+z=\frac{1}{\sqrt{3}}$ (b) $x-y+z=\frac{1}{\sqrt{3}}$ (c) $x-y-z=\frac{1}{\sqrt{3}}$ (d) $x+y-z=\frac{1}{\sqrt{3}}$

Soln: Equation of unit sphere $\phi : x^2+y^2+z^2-1=0$

Equation of tangent plane at the point (x,y,z), will be

$$(x-x_1)\left.\frac{\partial\phi}{\partial x}\right|_{(x,y,z)}+(y-y_1)\left.\frac{\partial\phi}{\partial y}\right|_{(x,y,z)}+(z-z_1)\left.\frac{\partial\phi}{\partial z}\right|_{(x,y,z)}=0$$

$\Rightarrow$ Here, $\frac{\partial\phi}{\partial x}=2x \Rightarrow \left.\frac{\partial\phi}{\partial x}\right|_{\left(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)}=\frac{2}{\sqrt{3}}$, $\frac{\partial\phi}{\partial y}=2y \Rightarrow \left.\frac{\partial\phi}{\partial y}\right|_{\left(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)}=-\frac{2}{\sqrt{3}}$

and $\frac{\partial\phi}{\partial z}=2z \Rightarrow \left.\frac{\partial\phi}{\partial z}\right|_{\left(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)}=\frac{2}{\sqrt{3}}$

Therefore, equation of tangent plane to the given surface will be

$$\left(x-\frac{1}{\sqrt{3}}\right)\frac{2}{\sqrt{3}}+\left(y+\frac{1}{\sqrt{3}}\right)\left(-\frac{2}{\sqrt{3}}\right)+\left(z-\frac{1}{\sqrt{3}}\right)\frac{2}{\sqrt{3}}=0 \Rightarrow x-y+z=\frac{1}{\sqrt{3}}.$$

Correct option is (d)

Question 12. The divergence of $\vec{c}\times(\vec{r}\times\vec{c})$ where $\vec{c}$ is a numerical vector is given by

(a) zero (b) $|\vec{c}|^2$ (c) $2|\vec{c}|^2$ (d) $-2|\vec{c}|^2$ **[H.C.U. 2013]**

Soln:

$$\vec{c}\times(\vec{r}\times\vec{c})=\vec{r}(\vec{c}\cdot\vec{c})-\vec{c}(\vec{c}\cdot\vec{r})$$

$$\therefore\ \vec{\nabla}\cdot[\vec{c}\times(\vec{r}\times\vec{c})]=c^2\vec{\nabla}\cdot\vec{r}-\vec{\nabla}.[\vec{c}(\vec{c}\cdot\vec{r})]$$

$$=3c^2-\left[\vec{\nabla}(\vec{c}\cdot\vec{r}).\vec{c}+(\vec{c}\cdot\vec{r})(\vec{\nabla}.\vec{c})\right]$$

$$=3c^2-[\vec{c}.\vec{c}+0]=2c^2$$

Correct option is (c)

Question 13. The value of the line integral $\int(y^2+xy)dx+(x^2-xy)dy$ taken along the straight line joining (0, 0) to (2, 1) is **[H.C.U. 2013]**

(a) 64/3 (b) 32/3 (c) 16/3 (d) 8/3

Soln: Equation of straight line joining $A(0,0)$ and $B(2,1)$ is,

$$y-0=\frac{1-0}{2-0}(x-0)\Rightarrow\boxed{2y=x}\Rightarrow dx=2dy$$

$$\therefore\ \int(y^2+xy)dx+(x^2-xy)dy=\int_0^1\left[(y^2+2y.y)2dy+(4y^2-2y.y)dy\right]$$

$$=\left(\frac{6y^3}{3}+\frac{2y^3}{3}\right)_0^1=2+\frac{2}{3}=\frac{8}{3}.$$

Correct option is (d)

Question 14. The value of the integral, $\int_0^1 dx\int_0^x dyxy$ is **[H.C.U. 2014]**

(a) 1 (b) $\frac{1}{4}$ (c) $\frac{1}{8}$ (d) $\frac{1}{2}$

Soln: $\int_0^1 dx\int_0^x dy\,xy=\int_0^1 dx\int_0^x xy\,dy=\int_0^1 dx\left(\frac{xy^2}{2}\right)_0^x=\int_0^1\frac{x^3}{2}dx=\left(\frac{x^4}{2\times4}\right)_0^1=\frac{1}{8}.$

Correct option is (c)

Question 15. $\vec{F}=\frac{1}{x}\hat{i}+2y\hat{j}$, then $\vec{F}$ is a gradient of the following function **[H.C.U. 2014]**

(a) $f=-\frac{1}{x^2}+y^2+c$ (b) $f=y^2+c$ (c) $f=\log x+y^2+c$ (d) $f=2xy+c$

Soln: From option (c), $f=\log x+y^2+c$

$$\therefore\ \vec{\nabla}f=\frac{1}{x}\hat{i}+2y\hat{j}$$

Correct option is (c)

Question 16. The unit vector parallel to the resultant of the vectors $\vec{r}_1 = 2\hat{i}+4\hat{j}-5\hat{k}$ and $\vec{r}_2 = \hat{i}+2\hat{j}+3\hat{k}$ is given by **[H.C.U. 2015]**

(a) $\frac{1}{3}\hat{i}+\frac{1}{6}\hat{j}-\frac{1}{2}\hat{k}$ (b) $\frac{3}{7}\hat{i}+\frac{6}{7}\hat{j}-\frac{2}{7}\hat{k}$ (c) $\frac{2}{7}\hat{i}-\frac{3}{7}\hat{j}+\frac{6}{7}\hat{k}$ (d) $3\hat{i}+6\hat{j}-2\hat{k}$

Soln: Resultant of $\vec{r}_1$ and $\vec{r}_2$ is $\vec{r} = \vec{r}_1+\vec{r}_2 = 3\hat{i}+6\hat{j}-2\hat{k}$

Therefore, unit vector $\frac{\vec{r}}{|r|} = \frac{3\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{3^2+6^2+2^2}} = \frac{3}{7}\hat{i}+\frac{6}{7}\hat{j}-\frac{2}{7}\hat{k}$

Correct option is (b)

Question 17. If $T = \sin x \sin y \sin z$, $\nabla^2 T$ is given by

(a) 0 (b) $-2\sin x \sin y \sin z$ **[H.C.U. 2015]**

(c) $3\sin x \sin y \sin z$ (d) $-3\sin x \sin y \sin z$

Soln: $T = \sin x \sin y \sin z$

$$\nabla^2 T = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)(\sin x \sin y \sin z)$$

$$= -\sin x \sin y \sin z - \sin x \sin y \sin z - \sin x \sin y \sin z$$

$$= -3 \sin x \sin y \sin z$$

Correct option is (d)

Question 18. A force $\vec{F} = -A\left(x^2\hat{i}+y\hat{j}\right)$ is rotational. The potential energy function associated with it is given by **[H.C.U. 2015]**

(a) $U = A\left(\frac{x^3}{3}+\frac{y^2}{2}\right)$ (b) $U = A\left(\frac{x}{3}+\frac{y^2}{2}\right)$ (c) $U = A\left(\frac{x^3}{3}+\frac{y}{2}\right)$ (d) $U = A\left(\frac{x}{3}+\frac{y^3}{2}\right)$

Soln: Since, $\vec{F} = -A(x^2\hat{i}+y\hat{j})$ is irrotational i.e. $\vec{\nabla}\times\vec{F} = 0$, therefore it can be always written as gradient of a scalar function. i.e. $\vec{F} = \vec{\nabla}u \Rightarrow -A\left(x^2\hat{i}+y\hat{j}\right) = -\frac{\partial u}{\partial x}\hat{i}-\frac{\partial u}{\partial y}\hat{j}$

$$\frac{\partial u}{\partial x} = +Ax^2 \Rightarrow u = +\frac{Ax^3}{3} \text{ and } \frac{\partial u}{\partial y} = +Ay \Rightarrow u = +\frac{Ay^2}{2}$$

Therefore, $u = A\left(\frac{x^3}{3}+\frac{y^2}{2}\right)$

Correct option is (a)

Question 19. Two vectors in Cartesian coordinates are defined as

$$\vec{a}=\sin(t)\hat{i}+\hat{j}+\cos(t)\hat{k} \text{ and } \vec{b}=\sin(t)\hat{i}-\hat{j}+\cos(t)\hat{k}$$

for some paramater t. Then these two vectors are said to be **[H.C.U. 2016]**

(a) parallel to each other (b) perpendicular to each other

(c) make an angle t between them (d) not related to each other

Soln: Here, $\vec{a}\cdot\vec{b}=\left(\sin t\,\hat{i}+\hat{j}+\cos t\,\hat{k}\right).\left(\sin t\,\hat{i}-\hat{j}+\cos t\,\hat{k}\right)=\sin^2 t-1+\cos^2 t=0$

$\therefore$ $\vec{a}$ and $\vec{b}$ are perpendicular to each.

Correct option is (b)

Question 20. The value of the integral $\int\limits_0^1 dx\int\limits_0^x ydy$ is **[H.C.U. 2016]**

(a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) $\frac{1}{6}$ (d) $\frac{1}{8}$

Soln: $\int\limits_0^1 dx\int\limits_0^x dy=\int\limits_0^1 dx\left(\frac{y^2}{2}\right)_0^x=\int\limits_0^1\frac{x^2}{2}dx=\left(\frac{x^3}{2\times 3}\right)_0^1=\frac{1}{6}.$

Correct option is (c)

Question 21. Two vectors on a plane are given as $\vec{a}=3\hat{i}+5\hat{j}$ and $\vec{b}=4\hat{i}+10\hat{j}$. The plane is then defined by the vector **[H.C.U. 2016]**

(a) $50\,\hat{k}$ (b) $10\,\hat{k}$ (c) $0\,\hat{k}$ (d) $15\,\hat{k}$

Soln: Plane is $\vec{a}\times\vec{b}=\left(3\hat{i}+5\hat{j}\right)\times\left(4\hat{i}+10\hat{j}\right)=10\hat{k}$

Correct option is (b)

Question 21. If $\vec{V}=\vec{\nabla}\times\vec{A}$ and given that $\vec{V}=-\hat{k}$, then the vector $\vec{A}$ will be given by **[H.C.U. 2017]**

(a) $-\hat{i}-y\hat{j}+\hat{k}$ (b) $\hat{i}+\hat{j}-\hat{k}$ (c) $5\hat{i}+\hat{j}-2\hat{k}$ (d) $-y\hat{i}+\hat{j}+5\hat{k}$

Soln: From option (d), $\vec{A}=y\hat{i}+\hat{j}+5\hat{k}\Rightarrow\vec{\nabla}\times\vec{A}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ y & 1 & 5\end{vmatrix}=\hat{i}(0)-\hat{j}(0)+\hat{k}(0-1)=-\hat{k}$

Correct option is (d)

Question 22. The value of the surface integral $\int_S (\vec{\nabla}\times\vec{A})\cdot d\vec{S}$, where $\vec{A}=\hat{i}y+\hat{j}z+\hat{k}$ and S is the surface defined by the paraboloid $z=1-x^2-y^2$, with $z\geq 0$ is given by **[H.C.U. 2017]**

(a) $-\pi$ (b) π (c) 1 (d) -1

Soln:

Question 23. In plane polar coordinates, which of the following relation are true ? **[H.C.U. 2017]**

(a) $\frac{d\hat{r}}{d\theta}=\hat{\theta},\ \frac{d\hat{\theta}}{d\theta}=-\hat{r}$ (b) $\frac{d\hat{r}}{d\theta}=-\hat{\theta},\ \frac{d\hat{\theta}}{d\theta}=\hat{r}$ (c) $\frac{d\hat{r}}{d\theta}=\hat{r},\ \frac{d\hat{\theta}}{d\theta}=-\hat{\theta}$ (d) $\frac{d\hat{r}}{d\theta}=-\hat{r},\ \frac{d\hat{\theta}}{d\theta}=\hat{\theta}$

Soln: In plane polar coodinate system, we have

$\hat{r}=\cos\theta\,\hat{i}+\sin\theta\,\hat{j}\Rightarrow\frac{\partial\hat{r}}{\partial\theta}=-\sin\theta\,\hat{i}+\cos\theta\,\hat{j}=\hat{\theta}$ and

$\hat{\theta}=-\sin\theta\,\hat{i}+\cos\theta\,\hat{j}\Rightarrow\frac{\partial\hat{\theta}}{\partial\theta}=-\cos\theta\,\hat{i}-\sin\theta\,\hat{j}=-\hat{r}$

Correct option is (a)

BHU PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. In order to determine uniquely a vector field, it is necessary to specify **[B.H.U. 2010]**

(a) its divergence only (b) its curl only (c) curl of its curl (d) its divergence and curl both

Soln: **Correct option is (d)**

Question 23. A vector field $\vec{A}$ is said to be non-conservative if **[B.H.U. 2010]**

(a) $\oint\vec{A}\cdot\vec{dt}=0$ (b) $\vec{\nabla}\times\vec{A}=0$ (c) $\vec{A}\times\vec{\nabla}f$ (d) $\vec{\nabla}\cdot\vec{A}=0$

Soln: For non-conservative force $\vec{\nabla}\times\vec{A}\neq 0$ and $\oint\vec{A}\cdot\vec{dl}\neq 0$.

Therefore, options (a), (b) and (c) are incorrect.

Question 2. If two non-null vectors $\vec{A}$ and $\vec{B}$ are such that $|\vec{A}+\vec{B}|=|\vec{A}-\vec{B}|$, then the angle between the two vectors is

(a) 0° (b) $\frac{\pi}{2}$ (c) π (d) $\frac{\pi}{3}$ **[B.H.U. 2010]**

Soln: $|\vec{A}+\vec{B}|=|\vec{A}-\vec{B}|$

$|\vec{A}+\vec{B}|^2=|\vec{A}-\vec{B}|^2$

$A^2+B^2+2AB\cos\theta=A^2+B^2-2AB\cos\theta$

$\Rightarrow\ \cos\theta=0\ \Rightarrow\ \theta=\frac{\pi}{2}$

Correct option is (b)

Question 3. A vector field $\vec{A}$ is said to be non-conservative if ·**[B.H.U. 2011]**

(a) $\oint \vec{A}\cdot d\vec{\ell}=0$ (b) $\vec{\nabla}\times\vec{A}=0$ (c) $\vec{A}=\vec{\nabla}f$ (d) $\vec{\nabla}\cdot\vec{A}=0$

Question 4. For an incompressible fluid $\vec{\nabla}(\rho\vec{v})$, where ρ is the density and $\vec{v}$ is the velocity, is

(a) always positive (b) always negative (c) always zero **[B.H.U. 2011]**

(d) sometimes positive and sometimes negative

Soln: For an in compressible fluid $\nabla(\rho\vec{V})$ is always zero.

Correct option is (c)

Question 5. The vector $\vec{r}$ is directed from the point $p'(x',y',z')$ to the point $P(x,y,z)$. The point p' is fixed and P is allowed to move. If $\hat{r}$ is a unit vector in the direction of $\vec{r}$, then

[B.H.U-2011]

(a) $\vec{\nabla}\left(\frac{1}{r}\right)=-\frac{\hat{r}}{r^2}$ (b) $\vec{\nabla}\left(\frac{1}{r}\right)=\frac{\hat{r}}{r^2}$ (c) $\vec{\nabla}'\left(\frac{1}{r}\right)=-\frac{\hat{r}}{r^2}$ (d) $\vec{\nabla}'\left(\frac{1}{r}\right)=\frac{\hat{r}}{r^2}$

Soln: $\vec{r}=(x-x')\hat{i}+(y-y')\hat{j}+(z-z')\hat{k}$

$\vec{\nabla}\left(\frac{1}{r}\right)=\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\hat{r}$ (Using spherical polar co-ordinate)

$=-\frac{\hat{r}}{r^2}.$

Correct option is (a)

Question 6. If for a scalar function ϕ, $\vec{\nabla}\phi=\frac{\vec{r}}{r^2}$, then ϕ is equal to **[B.H.U. 2012]**

(a) $\frac{1}{r}$ (b) $\frac{1}{r^2}$ (c) $\ln(|r|)$ (d) $\ln(r)$

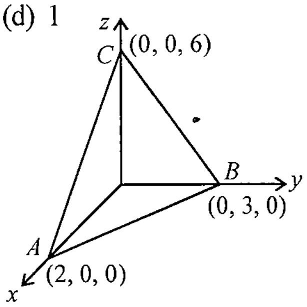
Soln: Since, $\vec{\nabla}\phi=\frac{\partial\phi}{\partial r}\hat{r}=\frac{\vec{r}}{r^2}=\frac{\hat{r}}{r}$. Therefore, $\frac{\partial\phi}{\partial r}=\frac{1}{r}$ integrating both side.

$\boxed{\phi=\ln(r)}$

Correct option is (d)

Question 7. Line integral of the vector $\vec{A}=(x+y)\hat{i}+(2x-z)\hat{j}+(y+z)\hat{k}$ along the sides of the triangle cut from the plane $3x+2y+z=6$ by the coordinate axes is **[B.H.U-2013]**

(a) 21 (b) 36 (c) −16 (d) 1



Soln: Given plane is $3x+2y+z=6\Rightarrow\frac{x}{2}+\frac{y}{3}+\frac{z}{6}=1$

Given vector is $\vec{A}=(x+y)\hat{i}+(2x-z)\hat{j}+(y+z)\hat{k}$

Therefore, $\int\vec{A}\cdot d\vec{l}=\int_{AB}\vec{A}\cdot d\vec{l}+\int_{BC}\vec{A}\cdot d\vec{l}+\int_{CA}\vec{A}\cdot d\vec{l}$

Along AB, equation of AB, $y-0=\frac{3-0}{0-2}(x-2)\Rightarrow-2y=3x-6\Rightarrow dy=-\frac{3}{2}dx$ and $z=0,\ dz=0$.

$$\int_{AB} \vec{A}\cdot d\vec{l} = \int_{AB} (x+y)\,dx + (2x-z)\,dy$$

$$= \int_{x=2}^{0} \left(x - \frac{3}{2}x + 3\right)dx + (2x-0)\times -\frac{3}{2}dx$$

$$= \left(-\frac{1}{2}\frac{x^2}{2} + 3x - 3\frac{x^2}{2}\right)_2^0 = 1.$$

Along BC, equation of BC, $x = dx = 0$

$$z - 0 = \frac{6-0}{0-3}(y-3) \Rightarrow z = -2y+6 \Rightarrow dz = -2dy.$$

Therefore, $\int_{BC} \vec{A}\cdot d\vec{l} = \int_{BC} (2x-z)\,dy + (y+z)\,dz$

$$= \int_{y=3}^{0} -(-2y+6)\,dy + (y-2y+6)\times -2dy$$

$$= \left(\frac{2y^2}{2} - 6y + \frac{2y^2}{2} - 2y\right)_3^0 = 36.$$

Along CA, equation of CA, $y = dy = 0$

$$x - 0 = \frac{2-0}{0-6}(z-6) \Rightarrow x = -\frac{1}{3}z + 2 \Rightarrow dx = -\frac{1}{3}dz$$

$$\int_{CA} \vec{A}\cdot d\vec{l} = \int_{CA} (x+y)\,dx + (y+z)\,dz$$

$$= \int (x\,dx + z\,dz) = \int_{x=0}^{2} x\,dx + \int_{6}^{0} z\,dz$$

$$= \left(\frac{x^2}{2}\right)_0^2 + \left(\frac{z^2}{2}\right)_6^0 = 2 - 18 = -16.$$

Therefore, $\int \vec{A}\cdot d\vec{l} = 1 + 36 - 16 = 21.$

Correct option is (a)

Question 8. The following vector is orthogonal to the vector $a\hat{i} + b\hat{j}$ (with $a \neq b$)

(a) $a\hat{i} - b\hat{j}$ (b) $-a\hat{i} - b\hat{j}$ (c) $-b\hat{i} + a\hat{j}$ (d) $-b\hat{i} - a\hat{j}$ **[B.H.U-2016]**

Soln: Check option (c) ; $\left(a\hat{i} + b\hat{j}\right)\cdot\left(-b\hat{i} + a\hat{j}\right) = -ab + ab = 0$

Correct option is (c)

Question 9. If $\vec{r}$ is a position vector, the value of $\vec{\nabla}^2(\vec{r}\cdot\vec{r}) = \vec{\nabla}^2 r^2$ is equal to

(a) 6 (b) 0 (c) 3 (d) –3 **[B.H.U-2016]**

Soln: $\nabla^2 r^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}r^2\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\times 2r\right) = \frac{2}{r^2}\times 3r^2 = 6.$

Correct option is (a)

Question 10. $r^n\vec{r}$ is solenoidal for **[B.H.U 2017]**

(a) $n=3$ (b) $n=-3$ (c) $n=2$ (d) $n=-2$

Soln: If $r^n\,\vec{r}$ or $r^{n+1}\,\hat{r}$ is solenoidal, then $\vec{\nabla}\cdot\left(r^{n+1}\,\hat{r}\right)=0$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\times r^{n+1}\right)=0$$

$$\frac{1}{r^2}(n+3)r^{n+2}=0$$

Therefore, $n+3=0 \Rightarrow n=-3.$

Correct option is (b)

Chapter 2

MATRICES

2.1 Basic Review of Matrices

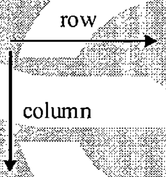
Definition:

A square or rectangular array of numbers or functions is known as matrices. Individual numbers or functions are known as elements of the matrix.

Example: $\begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}, \begin{bmatrix} 1-i & 2+i \\ 4+2i & x+iy \\ 2+4i & 3-2i \end{bmatrix}, \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}$

Order of a matrix:

A matrix of order $m \times n$ has 'm' number of rows and 'n' number of columns.



Every element of matrix is characterised by a row index and a column index:

a_{ij} : 'i' → row index , 'j' → column index, so, a_{ij} is the element of i^{th} row and j^{th} column

If a matrix A contains m rows and n columns, then it is written as $A = \left[a_{ij}\right]_{m\times n}$

Basic Review of matrix operation:

(i) Two matrices A and B are said to be equal if they are of the same order and each element of one is equal to corresponding element of the other matrix i.e. $a_{ij} = b_{ij}$ for all values of 'i' and 'j'.

(ii) Two matrices A and B are said to be conformable for addition/substraction if they have same order.

(iii) If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\lambda A = \begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix}$

(iv) Two matrices A and B are said to be comformable for multiplication if number of columns in A is equal to number of rows in B. Say, A matrix is of order '$m \times n$' and B matrix is of order '$p\times q$', then AB is possible only if $n = p$.

Various Type of Matrices:

(1) Square matrix: Matrix having number of rows equal to the number of columns i.e. $m = n$

Example: $\begin{bmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{bmatrix}_{2\times 2}$

(2) Diagonal matrix: Matrix having elements other than the principal diagonal elements are zero

i.e $a_{ij} = 0$ for $i \neq j$.

Example: $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$

(3) Scalar matrix: Diagonal matrix having all principal diagonal elements to be same

i.e. $a_{ij} = 0$ for $i \neq j$ and a_{ii} is same for all i.

Example: $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$

(4) Identity/unit matrix: Diagonal matrix having all principal diagonal elements equal to one.

i.e. $a_{ij} = 0$ for $i \neq j$ and a_{ii} is equal to 1 for all i.

Example: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(5) Row Matrix: Matrix having only one row.

Example: $\begin{bmatrix} a & b & c \end{bmatrix}_{1\times3}$

(6) Column matrix: Matrix having only one column.

Example: $\begin{bmatrix} a \\ b \\ c \end{bmatrix}_{3\times1}$

(7) Null matrix: Matrix having all elements equal to zero i.e. $a_{ij} = 0$ for i, j

Example: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(8) Upper triangular matrix: Matrix having elements below the principal diagonal equal to zero

i.e. $a_{ij} = 0$ for all $i > j$

Example: $\begin{bmatrix} a & d & e \\ 0 & b & f \\ 0 & 0 & c \end{bmatrix}$

(9) Lower triangular matrix: Matrix having elements above the principal diagonal equal to zero

i.e. $a_{ij} = 0$ for all $i < j$

Example: $\begin{bmatrix} a & 0 & 0 \\ d & b & 0 \\ e & f & c \end{bmatrix}$

(10) Periodic matrix: A square matrix 'A' for which $A^{k+1} = A$, is called periodic matrix of period k.

Example: $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Rightarrow A^5 = A$ i.e A is periodic matrix of period 4.

(11) Idempotent matrix: A square matrix 'A' for which $A^2 = A$, is called idempotent matrix. It is a periodic matrix of period 1. Any idempotent matrix will be either singular matrix or non-singular unit matrix.

Example: $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

(12) Nilpotent matrix: A square matrix for which $A^p = 0$, is called nilpotent matrix of index 'p'. The trace and determinant of the nilpotent matrix is always zero.

Example: $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ is a nilpotent matrix of index '2'.

(13) Involutory Matrix: A square matrix for which $A^2 = I$, is called involutory matrix. It is a self-inverse matrix.

Example: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Trace of a matrix

It is defined as the sum of the principal diagonal elements of the matrix i.e.

$$Tr(A) = a_{11} + a_{22} + a_{33} + \ldots\ldots\ldots\ldots + a_{nn} = \sum_{i=1}^{n} a_{ii}$$

Properties:

(i) $Tr(A+B) = Tr(A) + Tr(B)$ (ii) $Tr(cA) = cTr(A)$ [where c is constant]

(iii) $Tr(AB) = Tr(BA)$ (iv) $Tr(ABCD) = Tr(BCDA) = Tr(CDAB) = Tr(DABC)$

Transpose of a matrix:

By interchanging the rows and columns of a matrix, we will get the transpose of the original matrix.

Example: $A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$

Properties: (i) $(A^T)^T = A$, (ii) $(AB)^T = B^T A^T$

Transpose Conjugate of a Matrix:

It is obtained by taking the transpose of the matrix and then taking the complex conjugate of each element or vice versa i.e

$$A^{\dagger} = \left(A^T\right)^* = \left(A^*\right)^T$$

Example: $A=\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \Rightarrow A^{\dagger}=\begin{bmatrix} a^* & d^* & g^* \\ b^* & e^* & h^* \\ c^* & f^* & i^* \end{bmatrix}$

Properties: (i) $\left(A^{\dagger}\right)^{\dagger}=A$, (ii) $(AB)^{\dagger}=B^{\dagger}A^{\dagger}$

Example 1: If each element of '$k \times k$' square matrix A can be chosen as a Boolean Variable, then one can construct 'M' number of different combinations of matrix 'A'. Therefore, 'M' is equal to

(a) $2k$ (b) 2^{k^2} (c) k^2 (d) 2^k

Soln: Total number of elements $= k^2$

Each of the elements can be chosen to be 0 or 1 i.e. in 2 ways

Therefore, entire k^2 number of elements can be chosen in $2 \times 2 \times \ldots\ldots\ldots k^2$ times $= 2^{k^2}$

Correct option is (b)

Example 2: Which of the following statements is NOT CORRECT?

(a) Trace of a unit matrix is always a fixed natural number
(b) Trace of the sum of two matrices is equal to sum of their traces
(c) Trace of the scalar matrix is some multiple of its order
(d) $Tr(AB)=Tr(BA)$

Soln: Trace of a unit matrix of order 'n' is equal to 'n' i.e. it is natural number but not fixed.

Correct option is (a)

Example 3: Consider the matrix $B=\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, then B^{1023} equals

(a) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

Soln: $B=\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Rightarrow B^2=\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}=-I$ and $B^4=\left(B^2\right)^2=(-I)^2=I$

Therefore, $B^{1023}=B^{1020}.B^3=\left(B^4\right)^{255}.B^3=I.B^3=B^3=B^2.B=(-I).B=-B=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Correct option is (b)

Symmetric matrix: Matrix for which $A^T=A$ i.e. $a_{ij}=a_{ji}$.

Example: $\begin{bmatrix} a & b & c \\ b & c & d \\ c & d & e \end{bmatrix}$

Note: The number of independent components of a $n \times n$ symmetric matrix is $\frac{1}{2}n(n+1)$.

Skew-symmetric matrix: Matrix for which $A^T = -A$ i.e. $a_{ij} = -a_{ji}$.

Example: $\begin{bmatrix} 0 & b & c \\ -b & 0 & d \\ -c & -d & 0 \end{bmatrix}$

Note: (i) The number of independent components of a $n \times n$ antisymmetric matrix is $\frac{1}{2}n(n-1)$.

(ii) The principal diagonal elements of $n \times n$ skew-symmetric matrix are always zero.

- Every square matrix can be written as the sum of symmetric and antisymmetric matrix.

$$\mathrm{A} = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

- If A and B are symmetric matrices, then AB will be symmetric if and only if A and B commute.

Hermitan matrix: Matrix for which $A^\dagger = A$ i.e. $a_{ij} = a_{ji}^*$.

Example: $A = \begin{bmatrix} a & d & e \\ d^* & b & f \\ e^* & f^* & c \end{bmatrix}$ where $a, b, c \in R$

Note: (i) Principal diagonal elements of Hermitian matrix are always real.

(ii) Minimum number of real elements for the most general $n \times n$ hermitian matrix is n

Skew-Hermitan matrix: Matrix for which $A^\dagger = -A$ i.e. $a_{ij} = -a_{ji}^*$.

Example: $A = \begin{bmatrix} a & d & e \\ -d^* & b & f \\ -e^* & -f^* & c \end{bmatrix}$ where a, b, c can be either zero or purely imaginary.

Note: (i) Principal diagonal elements of a skew-hermitian matrix are either zero or purely imaginary.

(ii) Minimum number of real elements in a most general $n \times n$ skew hermitian matrix is zero.

(iii) If A is Hermitian matrix, iA and $-iA$ will be anti-Hermitian.

(iv) If A is Hermitian matrix, kA (k is real scalar) will be also Hermitian.

Example 4: The product MN of two Hermitian matrices M and N is anti-Hermitian. It follows that

(a) $\{M, N\} = 0$ (b) $[M, N] = 0$ (c) $M^\dagger = N$ (d) $M^\dagger = N^{-1}$

Soln: If MN is anti-Hermitian, then

$$(MN)^\dagger = -(MN) \Rightarrow N^\dagger M^\dagger = -MN \Rightarrow NM = -MN \Rightarrow MN + NM = 0 \Rightarrow \{M, N\} = 0$$

Correct option is (a)

Example 5: To convert a Hermitian matrix into Skew-Hermitian one, the Hermitian matrix must be multiplied by

(a) -1 (b) i (c) $-i$ (d) 1

Soln. To convert a Hermitian matrix into Skew-Hermitian we need to multiply it by an imaginary number.

Correct options are (b) and (c)

Example 6: If A is a $n \times n$ square matrix. Then, the correct statements is/are :

(a) If A is Skew-symmetric, then $Tr(A) = 0$

(b) If A is Hermitian, then $Tr(A)$ is a real number

(c) If A is Skew-symmetric, then $Tr(A)$ is zero

(d) If A is symmetric with distinct eigenvalues, then it is diagonalizable

Soln. All options are correct.

Correct options are (a), (b), (c) and (d)

Example 7: If $A = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}$ is an orthogonal matrix, then

(a) $a^2+b^2+c^2=1$ (b) $a+b+c=1$ (c) $a^2+b^2+c^2=0$ (d) $a+b+c=0$

Soln. Since, A is orthogonal, hence each row or column is a unit vector and every rows and columns are orthogonal to each.

Therefore, $\vec{R}_1 = a\hat{i}+b\hat{j}+c\hat{k}$ and $\vec{R}_2 = \hat{i}+a\hat{j}+c\hat{k}$.

Therefore, $\left|\vec{R}_1\right| = \sqrt{a^2+b^2+c^2} = 1 \Rightarrow a^2+b^2+c^2=1$ and $\vec{R}_1 \cdot \vec{R}_2 = 0 \Rightarrow ab+bc+ca=0$.

Now, $(a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca = 1+0 \Rightarrow a+b+c=1$.

Correct options are (a) and (b)

Example 8: For any operator $A, i\left(A^{+}-A\right)$ is :

(a) hermitian (b) anti-hermitian (c) unitary (d) orthogonal

Soln. $\left[i\left(A^{\dagger}-A\right)\right]^{\dagger} = (i)^{\dagger}\left[\left(A^{\dagger}\right)^{\dagger}-(A)^{\dagger}\right] = i\left(A^{\dagger}-A\right)$

Correct option is (a)

Example 9: For arbitrary matrices E, F, G and H, if $EF-FE=0$, then Trace ($EFGH$) is equal

(a) Trace (HFEG) (b) Trace (E), Tace (F), Trace (G), Trace (H)

(c) Trace (GFEH) (d) Trace (EGHF)

Soln. $Tr(EFGH) = Tr(HEFG) = Tr(HFEG)$

Correct option is (a)

Example 10: An unitary matrix $\begin{bmatrix} ae^{i\alpha} & b \\ ce^{i\beta} & d \end{bmatrix}$ is given, where a, b, c, d, α and β are real. The inverse of the matrix is

(a) $\begin{bmatrix} ae^{i\alpha} & -ce^{i\beta} \\ b & d \end{bmatrix}$ (b) $\begin{bmatrix} ae^{i\alpha} & ce^{i\beta} \\ b & d \end{bmatrix}$ (c) $\begin{bmatrix} ae^{i\alpha} & b \\ ce^{i\beta} & d \end{bmatrix}$ (d) $\begin{bmatrix} ae^{-i\alpha} & ce^{-i\beta} \\ b & d \end{bmatrix}$

Soln. For unitary matrix, $U^{-1} = U^{+} = \begin{bmatrix} a\,e^{-i\alpha} & c\,e^{-i\beta} \\ b & d \end{bmatrix}$

Correct option is (d)

Singular matrix: Matrix having zero determinant.

Example: $\begin{bmatrix} 2 & 1 & 5 \\ 3 & 5 & 7 \\ 2 & 1 & 5 \end{bmatrix}$

Non-singular matrix: Matrix having non-zero determinant.

Example: $\begin{bmatrix} 3 & 1 & 1 \\ 6 & -2 & 4 \\ 0 & 3 & -7 \end{bmatrix}$

Adjoint of a Matrix: Adjoint of a matrix is defined as the transpose of co-factor matrix.

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \text{then } Adj\ A = \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix}$$

Note: (i) $(Adj\ A).\ A = A.(Adj\ A) = |A|\ I_n$

(ii) If A is a square matrix of order 'n' having determinant 'm', then $|\ Adj\ A\ | = m^{n-1}$.

Inverse of a Matrix:

Let, A be a square matrix of order 'n', and B be a square matrix of order 'n' such that $AB = BA = I_{n,}$

then $B = A^{-1}$ = inverse of matrix $A = \dfrac{adj\ A}{|A|}$; Therefore, A is called invertible matrix.

Note: (i) Any invertible matrix possesses an unique inverse.

(ii) The necessary and sufficient condition for a square matrix A to posses an inverse is that it is non-singular.

(iii) $(ABC....)^{-1} = ...\ C^{-1}B^{-1}A^{-1}$

Example 11: If 'I' is an identity matrix of order '$n \times n$' and adj $(2I) = 2^k\,\mathrm{I}$, then 'k' is equal to

(a) 1 (b) 2 (c) n-1 (d) n

Soln. $A.(adj\ A) = |A|I \Rightarrow (2I).(adj\ 2I) = |2I|I \Rightarrow 2(adj\ 2I) = 2^n I \Rightarrow (adj\ 2I) = 2^{n-1}I \Rightarrow k = n-1$

Correct option is (c)

Example 12: If 'A' is a 3×3 matrix with determinant 3, then the determinant of $adj.[adj.[adj.(A^{-1})]]$ is equal to

(a) $\dfrac{1}{81}$ (b) $\dfrac{1}{729}$ (c) $\dfrac{1}{6561}$ (d) $\dfrac{1}{243}$

Soln: $|A|=3 \Rightarrow |A^{-1}|=\frac{1}{3} \Rightarrow \left|adj\left(A^{-1}\right)\right|=\left(\frac{1}{3}\right)^{3-1}=\frac{1}{9} \Rightarrow \left|adj\left\{adj\left(A^{-1}\right)\right\}\right|=\left(\frac{1}{9}\right)^{3-1}=\frac{1}{81}$

$$\Rightarrow \left|adj\left[adj\left\{adj\left(A^{-1}\right)\right\}\right]\right|=\left(\frac{1}{81}\right)^{3-1}=\frac{1}{6561}$$

Correct option is (c)

Example 13: Suppose A is $n \times n$ square matrix having determinant p. Which of the following statements is/are true for $A(\text{adj}A)$

(a) It has determinant p^{n-1}
(b) It is an unit matrix of order n
(c) It is a scalar matrix of order n
(d) It has determinant p^n

Soln. Taking determinant of $A(\text{adj}A)=|A(\text{adj}\,A)|$

$$= |A|\,|\text{adj}\,A|$$

$$= p\,p^{n-1} \quad \left(\because |\text{adj}\,A| = |A|^{n}\right)$$

$$= p^n$$

Also, $A(\text{adj}\,A)=(\text{adj}\,A)A=|A|I=pI$

Hence, it is a scalar matrix of order n.

Correct options are (c) and (d)

Example 14: If A and B are 3×3 square matrix such that $|A|=2$ and $|B|=3$, then the value of $|\text{adj}(AB)|$ is ________.

Soln. We have $|\text{adj}(AB)|=|\text{adj}\,A|\,|\text{adj}\,B|=|A|^{3-1}\,|B|^{3-1}=2^2\times 3^2=36.$

Correct answer is (36)

Example 15: Which one of the following is the inverse of the matrix $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$?

(a) $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$
(b) $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$
(c) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$
(d) $\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$

Soln. $A^{-1}=\frac{adjA}{|A|}=\frac{1}{1}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Correct option is (c)

Orthogonal Matrix: A square matrix 'A' is said to be orthogonal if $AA^T=A^TA=I$

Note: (i) Determinant of an orthogonal matrix will be either +1 or −1.

(ii) $A^T=A^{-1}$

(iii) Each row and column of an orthogonal matrix is a normalized vector.

(iv) Any two rows or any two columns of an orthogonal matrix are orthogonal to each other.

(v) If A and B are orthogonal matrices, then AB and BA will be always orthogonal.

Example: $A=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$

Unitary matrix: A square matrix 'A' is said to be orthogonal if $AA^{\dagger}=A^{\dagger}A=I$

Note: (i) Determinant of an orthogonal matrix will be of unit modulus..

(ii) $A^{\dagger}=A^{-1}$

(iii) Each row and column of an unitary matrix is a normalized vector.

(iv) Any two rows or any two columns of an unitary matrix are orthogonal to each other.

(v) If A and B are unitary matrices, then AB and BA will be always unitary.

Example: $A=\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0\\ 0 & 0 & 1\end{bmatrix}$

Example 16: The matrix A, defined by $A=\begin{pmatrix}1 & 0 & 0\\ 0 & a & b\\ 0 & -b & a\end{pmatrix}$ is orthogonal if

(a) $a=1, b=-1$ (b) $a=1/\sqrt{2}, b=-1/\sqrt{2}$ (c) $a=1/\sqrt{2}, b=-i/\sqrt{2}$ (d) $a=1,\ b=1$

Soln: Applying the condition for the orthogonal matrices i.e. $AA^{T}=I$ we get, $a^2+b^2=1$
Only option (b) satisfies this.

Correct option is (b)

Example 17: Given a 2 x 2 unitary matrix satisfying $U^{'}U=UU^{'}=I$ with $\det U=e^{i\varphi}$, one can construct a unitary matrix V $\left(V^{'}V=VV^{'}=I\right)$ with det V = 1 from it by

(a) Multiplying U by $e^{-i\varphi/2}$ (b) Multiplying a single element of U by $e^{-i\varphi}$

(c) Multiplying any row or column by $e^{-i\varphi/2}$ (d) Multiplying U by $e^{-i\varphi}$

Soln. Let $U=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ and $\det U=ad-bc=e^{i\phi}$ (Given)

Let, $V=e^{-i\phi/2}U \quad\Rightarrow\quad V=\begin{pmatrix}e^{-i\phi/2}a & e^{-i\phi/2}b\\ e^{-i\phi/2}c & e^{-i\phi/2}d\end{pmatrix}$

$$\det V=e^{-i\phi}ad-e^{-i\phi}bc=e^{-i\phi}\left(ad-bc\right)=e^{-i\phi}e^{i\phi}=1$$

We have to multiply U with $e^{-i\phi/2}$ to get V with determinant = 1

Correct answer is (a)

Example 18: The matrix $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ is

(a) orthogonal (b) symmetric (c) anti-symmetric (d) unitary

Soln. $AA^{+} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

So, A is an unitary matrix

Correct option is (d)

2.2 Eigenvalues and Eigenvectors:

Consider the following linear transformation, $AX = \lambda X \Rightarrow (A - \lambda I)\, X = 0$

The values of λ for which the above equation has non zero solution of X $(X \neq 0)$ are called eigen values or characteristic roots of matrix A and corresponding non-zero solution X is called eigenvector or characteristic vector of matrix A corresponding to that value of λ.

Here, $(A - \lambda I)$ is called Characteristic matrix of matrix A, $|A - \lambda I|$ is called Characteristic polynomial of matrix A, and $|A - \lambda I| = 0$ is called Characteristic equation or eigen equation of matrix A.

Example 19: Find the characteristic roots or eigenvalues and eigenvectors of matrix.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix}$$

Soln: Eigenvalue equation:

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 2 & 3 \\ 0 & 2-\lambda & 3 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)\left[(2-\lambda)^2 - 0\right] = 0 \Rightarrow \lambda = 1, 2, 2$$

Calculation of eigenvectors:

(i) $\lambda_1 = 1$: $AX_1 = \lambda_1 X_1$

$$\Rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 1\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 + 2x_2 + 3x_3 \\ 2x_2 + 3x_3 \\ 2x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow \begin{aligned} x_1 + 2x_2 + 3x_3 &= x_1 \\ 2x_2 + 3x_3 &= x_2 \\ 2x_3 &= x_3 \end{aligned}$$

Solving these equations, we get

$x_3 = 0,\ x_2 = 0$ & let $x_1 = k$. Therefore, $X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix}$

If this eigen vector is normalized to unity then,

$$X_1^{\dagger} X_1 = 1 \Rightarrow \begin{bmatrix} k^* & 0 & 0 \end{bmatrix}\begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix} = 1 \Rightarrow |k|^2 = 1 \Rightarrow k = 1 \Rightarrow X_1 = \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

(ii) $\lambda_2=\lambda_3=2$:

Let the eigen vector is $X_2=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$ $\quad\therefore\quad AX_2=\lambda_2 X_2$

$$\Rightarrow\quad \begin{pmatrix}1&2&3\\0&2&3\\0&0&2\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=2\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}\quad\Rightarrow\quad\begin{pmatrix}x_1+2x_2+3x_3\\2x_2+3x_3\\2x_3\end{pmatrix}=\begin{pmatrix}2x_1\\2x_2\\2x_3\end{pmatrix}\quad\Rightarrow\quad\begin{aligned}x_1+2x_2+3x_3&=2x_1\\2x_2+3x_3&=2x_2\\2x_3&=2x_3\end{aligned}$$

it gives $x_1=2x_2$ & x_3 can be arbitary, we choose x_3 to be zero.

Let $x_2=k\ \Rightarrow x_1=2k\ \therefore\ X_2=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}2k\\k\\0\end{pmatrix}$

If the eigen vector is normalized to unity, then

$$X_2^+X_2=1\Rightarrow\begin{pmatrix}2k^* & k^* & 0\end{pmatrix}\begin{pmatrix}2k\\k\\0\end{pmatrix}=1\Rightarrow 5|k|^2=1\Rightarrow k=\frac{1}{\sqrt5},\text{ therefore, }X_2=\begin{pmatrix}2/\sqrt5\\1/\sqrt5\\0\end{pmatrix}$$

Example 20: If $\begin{bmatrix}-4.5\\-4\\1\end{bmatrix}$ is an eigenvector of $\begin{bmatrix}8&-4&2\\4&0&2\\0&-2&-4\end{bmatrix}$, the eigenvalue corresponding to the eigenvector is

(a) 1 (b) 4 (c) –4.5 (d) 6

Soln. If λ is an eigenvalue, then $AX=\lambda X$

$$\Rightarrow\begin{bmatrix}8&-4&2\\4&2&2\\0&-2&-4\end{bmatrix}\begin{bmatrix}-4.5\\-4\\1\end{bmatrix}=\lambda\begin{bmatrix}-4.5\\-4\\1\end{bmatrix}$$

$$\begin{bmatrix}-18\\-16\\4\end{bmatrix}=\lambda\begin{bmatrix}-4.5\\-4\\1\end{bmatrix}$$

$\Rightarrow\ \lambda=4.$

Correct option is (b)

Example 21: Consider the matrix $P=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$, then the correct statements is/are ?

(a) eigenvalues of P are i and $-i$

(b) eigenvalues of P are $1-i$ and $-1+i$

(c) eigenvectors of P are $\begin{pmatrix}1\\i\end{pmatrix}$ and $\begin{pmatrix}-1\\i\end{pmatrix}$

(d) eigenvectors of P are $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}-1\\1\end{pmatrix}$

Soln. Characteristic equation, $|P-\lambda I|=0$

$$\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$$

Eigenvector for $\lambda = i$,

$$(A - \lambda I)X = 0 \Rightarrow \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$-iX + y = 0 \Rightarrow X = -iy$

If, $y = -1,\ x = i$

Therefore, $X_{\lambda = i} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$

Eigenvector for $\lambda = -i$, $\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$

$ix + y = 0 \Rightarrow x = -y$

Hence if $y = 1,\ x = i$. Therefore, $X_{\lambda = -i} = \begin{pmatrix} 1 \\ i \end{pmatrix}$.

Correct options are (a) and (c)

Example 22: The eiegenvalues of the anti-symmetric matrix $A = \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix}$ where n_1, n_2, n_3 are components of an unit vector, are

(a) 0, -i, i (b) 0, 1, -1 (c) 0, 1+i, -1-i (d) 0, 0, 0

Soln. Characteristic equation of the given matrix : $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -\lambda & -n_3 & n_2 \\ n_3 & -\lambda & -n_1 \\ -n_2 & n_1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^3 + \lambda\left(n_1^2 + n_2^2 + n_3^2\right) = 0$$

Since, n_1, n_2, n_3 are the components of an unit vector, so $n_1^2 + n_2^2 + n_3^2 = 0$

$\Rightarrow \lambda^3 + \lambda = 0 \quad \Rightarrow \lambda = 0, i, -i$

Correct answer is (a)

Example 23: The eigenvalues of the matrix $\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$ are

(a) +1 and +1 (b) zero and +1 (c) zero and +2 (d) −1 and +1

Soln. Eigenvalue equation : $\begin{vmatrix} 1-\lambda & i \\ -i & 1-\lambda \end{vmatrix} = 0 \Rightarrow (\lambda - 1)^2 + i^2 = 0 \Rightarrow \lambda^2 - 2\lambda = 0 \Rightarrow \lambda = 0, 2$

Correct option is (c)

Example 24: The eigenvalues of the matrix $\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$ are

(a) +1 and +1 (b) zero and +1 (c) zero and +2 (d) −1 and +1

Soln. Eigenvalue equation : $\begin{vmatrix} 1-\lambda & i \\ -i & 1-\lambda \end{vmatrix} = 0 \Rightarrow (\lambda-1)^2 + i^2 = 0$

$$\Rightarrow \lambda^2 - 2\lambda = 0 \Rightarrow \lambda = 0, 2$$

Correct option is (c)

Example 25 : The eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ are

(a) 6, 1 and $\begin{bmatrix} 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (b) 2, 5 and $\begin{bmatrix} 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(c) 6, 1 and $\begin{bmatrix} 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (d) 2, 5 and $\begin{bmatrix} 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Soln. $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$; eigenvalue equation : $|A - \lambda I| = 0$

$$\Rightarrow \quad \begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (\lambda-2)(\lambda-5) - 4 = 0 \Rightarrow \lambda^2 - 7\lambda + 6 = 0 \Rightarrow \lambda = 6, 1$$

Now, $(A - \lambda I)X = 0$

$$\Rightarrow \quad \begin{pmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

Putting $\lambda = 6 \Rightarrow \begin{pmatrix} -1 & 4 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow -x_1 + 4x_2 = 0 \Rightarrow x_1 = 4x_2$

So, $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

Putting $\lambda = 1 \Rightarrow \begin{pmatrix} 4 & 4 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow 4x_1 + 4x_2 = 0 \Rightarrow x_1 = -x_2$

So, $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Correct option is (a)

Example 26: The eigenvalues of the matrix $A=\begin{bmatrix}0 & i\\ i & 0\end{bmatrix}$ are

(a) real and distinct
(b) complex and distinct
(c) complex and coinciding
(d) real and coinciding

Soln. Eigenvalue equation : $\begin{vmatrix}-\lambda & i\\ i & -\lambda\end{vmatrix}=0 \Rightarrow \lambda^2-i^2=0 \Rightarrow \lambda=\pm i$

Correct option is (b)

Note:(i) Sum of the eigenvalues of a matrix is equal to trace of the Matrix i.e.

$\lambda_1+\lambda_2+\lambda_3+...............+\lambda_n=Tr(A)$

(ii) Product of the eigenvalues of a matrix is equal to determinant of the matrix i.e. $\lambda_1.\lambda_2.\lambda_3...............\lambda_n=Det(A)$

(iii) Any square matrix A and its transpose have same eigen values.

(iv) If λ_1, λ_2, λ_3, ... are the eigenvalues of A, then

(a) eigenvalues of the matrix kA will be $k\lambda_1, k\lambda_2, k\lambda_3$......

(b) eigenvalues of the matrix A^{-1} will be $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$......

(c) eigenvalues of the matrix A^m will be $\lambda_1^m, \lambda_2^m\ \lambda_3^m$ ....

(v) Eigenvalues of a real symmetric matrix or a Hermitian matrix are always real.

(vi) Eigen values of a real skew-symmetric matrix or a skew-Hermitian matrix are either zero or purely imaginary.

(vii) Eigen values of an real orthogonal matrix or Unitary matrix are of unit modulus.

(viii) Eigen values of a diagonal matrix or a upper triangular matrix or a lower triangular matrix are the principal diagonal elements.

(ix) Any two eigen vectors corresponding to two distinct eigen values of Hermitian matrix and a Unitary matrix are orthogonal to each other.

(x) Eigenvalues of a matrix is invariant under similarity transformation.

(xi) Eigenvalues of a nilpotent matrix are alwayszero.

(xii) Eigenvalues of idempotent matrix are either zero or unity.

(xiii) Consider a $n\times n$ matrix having all elements equal to one. One of the eigenvalue of the matrix will be equal to order of the matrix and all other eigenvalues are zero i.e $n, 0, 0, 0,$.....

(xiv) Consider a $n\times n$ matrix having rows and columns which are scalar multiple of a particular row and column respectively. One of the eigenvalue of the matrix will be equal to trace of the matrix and all other eigenvalues are zero i.e trace,$0, 0, 0,$............

Example 27: If $A=\begin{bmatrix}1 & 2 & -3\\ 0 & 3 & 2\\ 0 & 0 & -2\end{bmatrix}$, find the eigenvalues of $3A^3+5A^2-6A+2I$.

Soln: A is an upper triangular matrix, therefore eigenvalues of matrix A will be $\lambda=1, 3, -2$

Eigen values of $3A^3 = 3, 81, -24$

Eigen values of $5A^2 = 5, 45, 20$

Eigen values of $6A = 6, 18, -12$

Eigen values of $2I = 2, 2, 2$

Eigenvalue of given expression will be

$$(3+5-6+2)=4,(81+45-18+2)=110,(-24+20+12+2)=10$$

Example 28: The eigenvalues of $\begin{pmatrix} -5 & 2 \\ 2 & -2 \end{pmatrix}$ are

(a) 1, 3 (b) –1, –6 (c) 1, –6 (d) –2, –5

Soln: Sum of the eigenvalues is equal to the trace of matrix i.e. $\lambda_1 + \lambda_2 = -7$

Product of the eigenvalues is equal to the determinant of matrix i.e. $\lambda_1\lambda_2 = 6$

Only option (b) satisfies this.

Therefore, eigenvalues are = –1, –6

Correct option is (b)

Example 29: If trace of matrix M of order 3, is 13 and its determinant is 36, then the largest eigenvalue of M is equal to

(a) 4 (b) 6 (c) 9 (d) 12

Soln: Sum of the eigenvalues = trace of the matrix i.e. $\lambda_1 + \lambda_2 + \lambda_3 = 13$

Product of the eigenvalues = determinant of the matrix i.e. $\lambda_1\lambda_2\lambda_3 = 36$

Only the combiantion $\lambda_1 = 9, \lambda_2 = 2, \lambda_3 = 2$ satisfies the both conditions simultaneously.

Correct option is (c)

Example 30: What are the number of zero eigenvalues of matrix $\begin{pmatrix} a_1^2 & a_1a_2 & a_1a_3 \\ a_1a_2 & a_2^2 & a_2a_3 \\ a_1a_3 & a_2a_3 & a_5^2 \end{pmatrix}$?

Soln: For the given matrix, Second row $= \frac{a_2}{a_1} \times$ first row & third row $= \frac{a_3}{a_1} \times$ first row

Therefore, the eigenvalues of the matrix = trace of the matrix, 0, 0 = $\left(a_1^2 + a_2^2 + a_3^2\right)$, 0, 0

Example 31: The eigenvalues of the matrix $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ are given by λ_1, λ_2 *and* λ_3. Which one of th following statement is **NOT TRUE**?

(a) $\lambda_1 + \lambda_2 + \lambda_3 = 0$ (b) $\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = 0$

(c) All eigenvalues are real (d) Product of the eigenvalues is 1

Soln: We have a matrix, $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

Eigenvalue equation is,

$$\therefore \qquad |A - I\lambda| = 0 \Rightarrow \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \qquad -\lambda\left(\lambda^2 - 0\right) + 1(1) \Rightarrow \lambda^3 = 1$$

$$\Rightarrow \qquad \lambda_1 = 1, \lambda_2 = \omega, \lambda_3 = \omega^2$$

Therefore, all eigen values are not real.

Correct option is (c)

Example 32 : The matrix given by:

$$\begin{pmatrix} 3 & 2i & 1 \\ -2i & 4 & 1+2i \\ 1 & 1-2i & 5 \end{pmatrix}$$

(a) Has one real and one complex eigenvalues (b) Has two real and one complex eigenvalues
(c) Has three real eigenvalues (d) Has three complex eigenvalues

Soln: Since all $a_{ij} = a^*_{ji}$ of the given matrix. So, the matrix is Hamilton matrix.

So, all the eigen values are real.

Correct option is (c)

Example 33 : The eigenvalues of $\begin{bmatrix} 5 & 6 & 17 \\ 0 & -19 & 23 \\ 2 & 2 & 37 \end{bmatrix}$ are

(a) −19, 5, 37 (b) 19, −5, −37 (c) 2, −3, 7 (d) 3, −5, 37

Soln. Since, given matrix is an upper triangular matrix whose eigenvalues are principle diagonal element.
Hence, eigenvalues are 5, −19, 37.
Correct option is (a)

Example 34 : If one of the eigenvalues of $A_{n\times n}$ is zero, if implies

(a) The solution to $[A][X] = [C]$ system of equation is zero.

(b) The determinant of $[A]$ is zero.

(c) The solution to $[A][X] = [0]$ system of equation is trivial.

(d) The determinant of $[A]$ is non-zero.

Soln. As we know, determinant of a matrix = product of eigenvalues.
Hence, if any of one eigenvalue is zero, determinant of matrix = 0.
Correct option is (b)

Example 35 : Given that matrix $A = \begin{bmatrix} 8 & -4 & 2 \\ 4 & 0 & 2 \\ 0 & -2 & -3 \end{bmatrix}$ has an eigenvalue 4 with the corresponding eigenvectors of $X = \begin{bmatrix} -4.5 \\ -4 \\ 1 \end{bmatrix}$, then $A^5[X]$ is

(a) $\begin{bmatrix} -18 \\ -16 \\ 4 \end{bmatrix}$ (b) $\begin{bmatrix} -4.5 \\ -4 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} -4608 \\ -4096 \\ 1024 \end{bmatrix}$ (d) $\begin{bmatrix} -0.0043 \\ -0.003 \\ 0.0097 \end{bmatrix}$

Soln. As we know, if λ is an eigenvalue of matrix A and X is corresponding eigenvector, then $AX = \lambda X$ or $A^5 X = \lambda^5 X$.

Hence, $A^5 X = \lambda^5 X = 4^5 \begin{bmatrix} -4.5 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} -4608 \\ -4096 \\ 1024 \end{bmatrix}$.

Correct option is (c)

Example 36 : If matrix T has the eigenvalue λ with an associated eigenvector $\vec{V}$, then $\vec{V}$ is an eigenvector of $cT + dI$ associated with eigenvalue (here c and d are constants).

(a) $d\lambda - c$ (b) $c\lambda + d$ (c) $c\lambda - d$ (d) $d\lambda + c$

Soln. Since, $\vec{V}$ is an eigenvector of $(cT + dI)$.

Hence, $(cT + dI)\vec{V} = cT\vec{V} + dI\vec{V}$

$= c\lambda\vec{V} + d\vec{V} \qquad (\because T\vec{V} = \lambda\vec{V})$

$= (c\lambda + d)\vec{V}$

Hence, eigenvalue of $(cT + dI)$ is $(c\lambda + d)$.

Correct option is (b)

Example 37 : Let A be a 3×3 real orthogonal matrix with $\det(A) = 1$, if one of the eigenvalues is $\frac{-1+\sqrt{3}i}{2}$, then choose the correct statement(s) :

(a) Matrix A will have all eigenvalues real.
(b) Matrix A will have a complex conjugate eigenvalue.
(c) One of the eigenvalue of A is 1
(d) Product of the eigenvalues will be 1

Soln. Since, A is a real matrix and $\frac{-1+\sqrt{3}i}{2}$ is a complex eigenvalue, it's conjugate $\frac{-1-\sqrt{3}i}{2}$ is also an eigenvalues of A.

Also, product of eigenvalues = det(A)

$$\frac{-1+\sqrt{3}i}{2} \times \frac{-1-\sqrt{3}i}{2} \times \lambda = i \Rightarrow \lambda = 1$$

Correct options are (b), (c) and (d)

Example 38 : Consider a 3 × 3 matrix of the following form

$$\begin{pmatrix} a_1^2 & a_1a_2 & a_1a_3 \\ a_1a_2 & a_2^2 & a_2a_3 \\ a_1a_3 & a_2a_3 & a_3^2 \end{pmatrix}$$

Which of the following statements is/are TRUE ?

(a) has one degenerate eigenvalue

(b) has no degenerate eigenvalue

(a) is singular in matrix

(d) is trace less

Soln. $A = \begin{pmatrix} a_1^2 & a_1a_2 & a_1a_3 \\ a_1a_2 & a_2^2 & a_2a_3 \\ a_1a_3 & a_2a_3 & a_3^2 \end{pmatrix}$; Here, $R_2 \to \frac{a_2}{a_1}R_1$ and $R_3 \to \frac{a_3}{a_1}$

Hence, eigenvalue of A are $\left(a_1^2 + a_2^2 + a_3^2\right)$, 0, 0

Therefore, '0' is degenerate eigenvalue.

Also, $\det(A) = 0$

Correct options are (a) and (c)

Example 39 : Consider a $n \times n$ square matrix A whose elements a_{ij} are given by

$$a_{ij} = \begin{cases} i^2 \text{ if } i = j \\ 0 \text{ otherwise} \end{cases}$$

Now choose correct statement(s) about 'A'

(a) Trace of A is equal to 30 for $n = 4$

(b) Determinant of A is $(n!)^2$

(c) Eigenvectors of A are linearly independent

(d) Eigenvectors of A are linearly dependent.

Soln. $A = \begin{bmatrix} 1^2 & 0 & & \cdots & 0 \\ 0 & 2^2 & & \cdots & 0 \\ 0 & 0 & 3^2 & \cdots & 0 \\ & & & & n^2 \end{bmatrix}$; It is a diagonal matrix

Therefore, trace of $A = 1^2 + 2^2 + 3^2 + \cdots n^2 = \frac{n(n+1)(2n+1)}{6}$.

Put $n = 4$, trace of $A = 30$

Determinant of $A = 1^2 \cdot 2^2 \cdot 3^2 \cdot \ldots\ldots n^2 = (n!)^2$

Since, alll eigenvalues are distinct. Hence eigenvectors are linearly independent.

Correct options are (a), (b) and (c)

Example 40 : If $A = \begin{bmatrix} 1/7 & 3/7 & 3/7 \\ 3/7 & 1/7 & 3/7 \\ 3/7 & 3/7 & 1/7 \end{bmatrix}$, then the value of $\lim_{n\to\infty} (Tr(A^n))$ is equal to ________.

Soln. Matrix A is symmetric, hence diagonalizable.

Eigenvalues of matrix A are $1, -\frac{2}{7}, -\frac{2}{7}$

Since, the eigenvalue of A^n are $(1)^n, \left(-\frac{2}{7}\right)^n, \left(-\frac{2}{7}\right)^n$.

Therefore, $Tr(A^n) = 1^n + \left(-\frac{2}{7}\right)^n + \left(-\frac{2}{7}\right)^n$

$\Rightarrow \lim_{n \to \infty} Tr(A^n) = 1$

Correct answer is (1)

Example 41 : Consider the matrix $M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

A. The eigenvalues of M are

(a) 0,1,2 (b) 0,0,3 (c) 1,1,1 (d) -1,1,3

Soln. Eigenvalues of a n x n matrix having all elements equal to one, are n,0,0,0,.......... respectively.

Therefore, eigen values of M are : 0, 0, 3

Correct answer is (b) .

Example 42 : A 3 x 3 matrix M has $Tr[M] = 6, Tr[M^2] = 26, Tr[M^3] = 90$. Which of the following can be possible set of eigenvalues of M?

(a) {1,1,4} (b) {-1,0,7} (c) {-1,3,4} (d) {2,2,2}

Soln. $\lambda_1 + \lambda_2 + \lambda_3 = 6;$ $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 26;$ $\lambda_1^3 + \lambda_2^3 + \lambda_3^3 = 90$

It is trivial to check that only $\{-1, 3, 4\}$ satisfies there three equation.

Correct answer is (c).

Example 43 : The eigenvalues of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$

(a) (1,4,9) (b) (0,7,7) (c) (0,1,13) (d) (0,0,14)

Soln. Eigenvalues of a n x n matrix having rows and columns as a scalar multiple of a particular row and column respectively, are trace,0,0,0,............

Here, the second row and third row are twice and thrice of the first row and same is true for the columns.

So, the eigenvalues of the given matrix are 14, 0, 0

Correct answer is (d)

Example 44 : A 2×2 matrix 'A' has eigenvalues $e^{i\pi/5}$ and $e^{i\pi/6}$. The smallest value of 'n' such that $A^n = I$

(a) 20 (b) 30 (c) 60 (d) 120

Soln. A has eigenvalues $e^{i\frac{\pi}{5}}$ and $e^{i\frac{\pi}{6}}$

A^n has eigenvalues $e^{i\frac{n\pi}{5}}$ and $e^{i\frac{n\pi}{6}}$

So, $\left|A^n\right| = \text{product of eigenvalues} = e^{i\frac{11n\pi}{30}}$

Since, $A^n = I \quad \Rightarrow \left|A^n\right| = 1$

$\therefore \quad e^{i(11n\pi/30)} = 1 = e^{2m\pi i} \quad \Rightarrow \frac{11n\pi}{30} = 2m\pi \quad \Rightarrow \frac{n}{m} = \frac{60}{11}$

So, the smallest value of 'n' is 60.

Correct answer is (c)

Example 45 : A 3×3 matrix has eigenvalues 0, 2 + i and 2–i. Which one of the following statements is correct?

(a) the matrix is hermitian
(b) the matrix is unitary
(c) the inverse of the matrix exists
(d) the determinant of the matrix is zero

Soln. Determinant of matrix = product of the eigenvalues = 0

Correct option is (d)

Example 46 : A real traceless 4×4 unitary matrix has two eigen values –1 and 1. The other eigenvalues are
(a) zero and +2
(b) –1 and +1
(c) zero and +1
(d) +1 and +1

Soln. Sum of the eigenvalues = Trace of the matrix

$\Rightarrow \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0 \quad \Rightarrow 1 + (-1) + \lambda_3 + \lambda_4 = 0$

$\Rightarrow \lambda_3 + \lambda_4 = 0$

So, $\lambda_3 = +1,\ \lambda_4 = -1$

Correct option is (b)

Example 47 : A real traceless 4×4 unitary matrix has two eigen values –1 and 1. The other eigenvalues are
(a) zero and +2
(b) –1 and +1
(c) zero and +1
(d) +1 and +1

Soln. Sum of the eigenvalues = Trace of the matrix

$\Rightarrow \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0 \quad \Rightarrow 1 + (-1) + \lambda_3 + \lambda_4 = 0$

$\Rightarrow \lambda_3 + \lambda_4 = 0$

So, $\lambda_3 = +1,\ \lambda_4 = -1$

Correct option is (b)

Example 48 : Eigen value of the matrix $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2i \\ 0 & 0 & 2i & 0 \end{pmatrix}$ are

(a) –2, –1, 1, 2
(b) –1, 1, 0, 2
(c) 1, 0, 2, 3
(d) –1, 1, 0, 3

Soln. Sum of the eigenvalues = Trace of the matrix = 0
Only option (a) satisfies this.
Correct option is (a)

Example 49 : One of the eigen values of the matrix $\begin{pmatrix} 2 & 3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is 5. The other two eigen values are

(a) 0 and 0 (b) 1 and 1 (c) 1 and –1 (d) –1 and –1

Soln. Sum of the eigenvalues = Trace of the matrix = 5

$$\Rightarrow \quad \lambda_1 + \lambda_2 + \lambda_3 = 5$$

$$\Rightarrow \quad 5 + \lambda_2 + \lambda_3 = 5 \Rightarrow \lambda_2 + \lambda_3 = 0$$

So, $\lambda_2 = 1$ and $\lambda_2 = -1$

Correct option is (c)

Example 50 : The normalized eigen vector corresponding to the eigen value 5 is:

(a) $\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ (b) $\frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ (c) $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ (d) $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Soln. Eigenvalue equation : $(A - \lambda I)X = 0$

$$\Rightarrow \quad \begin{bmatrix} 2-\lambda & 3 & 0 \\ 3 & 2-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Putting $\lambda = 5$, we get

$$\Rightarrow \quad \begin{bmatrix} -3 & 3 & 0 \\ 3 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \quad \Rightarrow \quad -3x_1 + 3x_2 = 0 \text{ and } -4x_3 = 0$$

Therefore, $X = \begin{bmatrix} k \\ k \\ 0 \end{bmatrix}$, putting normalization condition $X^\dagger X = y \Rightarrow K = \frac{1}{\sqrt{2}}$

So, $X = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Correct option is (d)

Example 51 : The eigenvalues of a matrix are $i, -2i$ and $3i$. The matrix is

(a) unitary (b) anti-unitary (c) hermitian (d) anti-hermitian

Soln. Eigenvalues of an anti-hermitian matrix are either zero or purely imaginary

Correct option is (d)

Example 52 : The eigen values of the matrix $\begin{pmatrix} 2 & 3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ are

(a) 5, 2, –2 (b) –5, –1, 1 (c) 5, 1, –1 (d) –5, 1, 1

Soln. Sum of the eigenvalues = Trace of the matrix

i.e. $\lambda_1 + \lambda_2 + \lambda_3 = 5$

Product of the eigenvalues = Determinant of the matrix

i.e. $\lambda_1 \lambda_2 \lambda_3 = -5$

Only option (c) satisfy both relations.

Correct option is (c)

Example 53 : A 3 × 3 matrix has elements such that its trace is 11 and its determinant is 36. The eigenvalues of the matrix are all known to be positive integers. The largest eigenvalue of the matrix is :

(a) 18 (b) 12 (c) 9 (d) 6

Soln. Trace of the matrix = $\lambda_1 + \lambda_2 + \lambda_3 = 11$

Determinant of the matrix = $\lambda_1 \lambda_2 \lambda_3 = 36$

Combination of λ_1, λ_2 and λ_3 that will satisfy both relation will be $\lambda_1 = 6, \lambda_2 = 3, \lambda_3 = 2$.

Correct option is (d)

Example 54 : The eigenvalues of the matrix $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ are

(a) 0, 1, 1 (b) $0, -\sqrt{2}, \sqrt{2}$ (c) $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0$ (d) $\sqrt{2}, \sqrt{2}, 0$

Soln. Trace of the matrix = $\lambda_1 + \lambda_2 + \lambda_3 = 0$

Only option (b) satisfy this.

Correct option is (b)

Example 55 : The degenerate eigenvalues of the matrix $M = \begin{pmatrix} 4 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{pmatrix}$ is:.........

(Your answer should be an integer)

Soln. Trace of the matrix = $\lambda_1 + \lambda_2 + \lambda_3 = 12$

Determinant of the matrix = $\lambda_1 \lambda_2 \lambda_3 = 50$

The combination $\lambda_1 = 5, \lambda_2 = 5, \lambda_3 = 2$ satisfies the above two relations.

Correct answer is (5)

2.3 Cayley-Hamilton Theorem:

Every square matrix satisfies its own characteristic equation or eigenvalue equation.

Example 56: $A=\begin{bmatrix}1 & 2\\ 4 & -6\end{bmatrix} \Rightarrow$ It's characteristic equation is $|A-\lambda I|=0$

$$\Rightarrow\begin{vmatrix}1-\lambda & 2\\ 4 & -6-\lambda\end{vmatrix}=0\Rightarrow(\lambda+6)(\lambda-1)-8=0\Rightarrow\lambda^2+5\lambda-14=0$$

Therefore, Matrix A satisfies the above equation i.e. $A^2+5A-14I=0$

Calculation of the inverse of the given matrix:

Since, $A^2+5A=14I \Rightarrow A^{-1}A^2+5A^{-1}A=14A^{-1}I \quad \Rightarrow A^{-1}=\frac{1}{14}(A+5I)$

Example 57: Let P be a n×n diagonalizable matrix which satisfies the equations

$$P^2=P, Tr(P)=n-1$$

Det (P) is :

(a) n (b) 0 (c) 1 (d) $n-1$

Soln: Using Cayley-hamilton theorem, $P^2-P=0 \quad \Rightarrow\lambda^2-\lambda=0 \Rightarrow\lambda(\lambda-1)=0 \qquad \Rightarrow\lambda=0,1$

$\det(M)=\lambda_1.\lambda_2 \ldots\ldots\ldots\ldots\lambda_n=0$ (Since, one of the eigenvalue is zero)

Example 58 : Let M be a 3×3 Hermitian matrix which satisfies the matrix equation

$$M^2-5M+6I=0$$

Where I refers to the identity matrix. Which of the following are possible eigenvalues of M

(a) {1, 2, 3} (b) {2, 2, 3} (c) {2, 3, 5} (d) {5, 5, 6}

Soln: According to Cayley-Hamilton theorem, we can write $\lambda^2-5\lambda+6=0\Rightarrow\lambda=2,3$

Correct option is (b)

Example 59: The eigenvalues of matrix $\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}$ are :

(a) $+1,+1,+1$ and $+1$ (b) $+1,+1,-1$ and -1

(c) $0,0,\sqrt{2}$ and $-\sqrt{2}$ (d) $+\sqrt{2},-\sqrt{2},+1/\sqrt{2}$ and $-1/\sqrt{2}$

Soln: The given matrix A satisfies the relation $A^2=I$

According to Cayley-Hamilton theorem, we can write $\lambda^2=1\Rightarrow\lambda=\pm1$

Correct option is (b)

Example 60: The eigenvalues of the following matrix $A=\begin{bmatrix}3 & 2 & 9\\ 7 & 5 & 13\\ 6 & 17 & 19\end{bmatrix}$ are given by solving the cubic equation :

(a) $\lambda^3-27\lambda^2+167\lambda-285=0$ (b) $\lambda^3-27\lambda^2-122\lambda-313=0$

(c) $\lambda^3+27\lambda^2+167\lambda+285=0$ (d) $\lambda^3+23.23\lambda^2-158.3\lambda+313=0$

Soln. From Caley-Hamilton theorem, we know every square matrix satisfies it's own characteristic equation, i.e., $|A-\lambda I|=0$

Hence, $\det(A-\lambda I)=0$

$$\Rightarrow \det\left\{\begin{bmatrix}3 & 2 & 9\\ 7 & 5 & 13\\ 6 & 17 & 19\end{bmatrix}-\begin{bmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{bmatrix}\right\}=0$$

$$\begin{vmatrix}3-\lambda & 2 & 9\\ 7 & 5-\lambda & 13\\ 6 & 17 & 19-\lambda\end{vmatrix}=0$$

On solving this, we get $\lambda^3-27\lambda^2-122\lambda-313=0$

Correct option is (b)

Example 61: The trace of a real 4 x 4 matrix U = exp(A), where $A=\begin{pmatrix}0 & 0 & 0 & \frac{\pi}{4}\\ 0 & 0 & -\frac{\pi}{4} & 0\\ 0 & \frac{\pi}{4} & 0 & 0\\ -\frac{\pi}{4} & 0 & 0 & 0\end{pmatrix}$ is

(a) $2\sqrt{2}$ (b) $\frac{\pi}{4}$ (c) 0 (d) 2 **[TIFR 2011]**

Example 62: Let $A=\begin{bmatrix}1 & -1\\ 2 & 3\end{bmatrix}$, then the eigenvalues of the matrix $B=A^4-3A^3+3A^2-2A+8I$, are

(a) $2+i, 2-i$ (b) $i, -i$ (c) $1+i, 1-i$ (d) None of these

Soln. Characteristic equation of matrix A, $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix}1-\lambda & -1\\ 2 & 3-\lambda\end{vmatrix}=0 \Rightarrow \lambda^2-4\lambda+5$$

From Caley-Hamilton theorem matrix A will satisfy it's characteristic equation.

Hence, we have $A^2-4A+5=0$

Now, $B = A^4 - 3A^3 + 3A^2 - 2A + 8I$

$$= (A^2 - 4A + 5I)(A^2 + A + 2I) + (A - 2I)$$

$$B = A - 2I$$

$$B = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$$

Hence, eigenvalues of B

$$\Rightarrow \begin{vmatrix} -1-\alpha & -1 \\ 2 & 1-\alpha \end{vmatrix} = 0 \Rightarrow -(1+\alpha)(1-\alpha) + 2 = 0$$

$$\Rightarrow -1 + \alpha^2 + 2 = 0$$

Hence, $\alpha^2 = 1 \Rightarrow \alpha = \pm i.$

Correct option is (b)

Example 63: For the matrix $A = \begin{pmatrix} 0 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, A^{-1} is given by

(a) $A^2 = 2A$ (b) $A^2 + 2A + 3I$ (c) $A^2 - 2A - I$ (d) $A - 3I$

Soln. Characteristic equation of A, $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -\lambda & 0 & 1 \\ 2 & I-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0 \quad \Rightarrow -\lambda\left(1 + \lambda^2 - 2\lambda - 1\right) + 1(2 - 1 + \lambda) = 0$$

$$\Rightarrow -\lambda^3 + 2\lambda^2 + 1 + \lambda = 0 \quad \Rightarrow \lambda^3 - 2\lambda^2 - \lambda - 1 = 0$$

From Caley-Hamilton theorem, $A^3 - 2A^2 - A - I = 0$ or $A^{-1}A^3 - 2A^{-1}A^2 - A^{-1}A - A^{-1}I = 0$

$$\Rightarrow A^{-1} = A^2 - 2A - I.$$

Correct option is (c)

Example 64 : If a square matrix $A(3 \times 3)$ satisfies the equation $A^3 - 8A^2 + 17A - 10I = 0$ and it's eigenvalues are given by $\lambda_1, \lambda_2, \lambda_3$. Then, the value of $\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1$ is equal to ________.

Soln. $A^3 - 8A^2 + 17A - 10I = 0$

If $\lambda_1, \lambda_2, \lambda_3$ are eigenvalues of A, then they are the roots of this equation.

Hence, $\lambda_1 + \lambda_2 + \lambda_3 = 8$

$\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = 17.$

Correct answer is (17)

2.4 Rank of a matrix

A natural number 'r' is said to be the rank of the matrix A if it has the following properties:

(i) There exists at least one squre sub-matrix of the matrix A of order 'r' having non-zero determinant.

(ii) Every square sub-matrix of matrix A of order '$r + 1$' have zero determinant.

i.e. Rank of matrix ≡ order of the largest square sub-matrix of the given matrix having non-vanishing determinant.

Note: (i) The rank of the null matrix is zero.

(ii) The rank of any non-zero matrix is ≥ 1.

(iii) The rank of any non-singular matrix of order 'n' is 'n'.

(iv) The rank of a product of two matrices cannot exceed the rank of either matrix.

Example 65 : Find the rank of the matrix, (i) $A = \begin{bmatrix} 3 & 2 & -1 \\ 0 & 3 & 4 \\ 5 & 1 & 2 \end{bmatrix}$, (ii) $B = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & -2 \\ 2 & 4 & -3 \end{bmatrix}$

Soln: (i) $A = \begin{bmatrix} 3 & 2 & -1 \\ 0 & 3 & 4 \\ 5 & 1 & 2 \end{bmatrix} \Rightarrow |A| = 3(6-4) - 2(0-20) + (-1)(0-15) = 6 + 40 + 15 = 61 \neq 0$; So, $\rho(A) = 3$

(ii) $B = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & -2 \\ 2 & 4 & -3 \end{bmatrix} \Rightarrow |B| = 2(-9+8) - 1(0+4) + (-1)(0-6) = -2 - 4 + 6 = 0$; So, $\rho(B) \neq 3$

Now, a 2 × 2 sub-matrix of B: $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$, it's determinant = 6 , So, $\rho(B) = 2$

Example 66: The rank of matrix $\begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{bmatrix}$ (a, b, c, being all real), is 3 if

(a) $a = b = c$
(b) a, b, c are all different but $a + b + c = 0$
(c) Two of the numbers a, b, c are equal but are different from the third
(d) a, b, c are all different and $a + b + c \neq 0$

Soln: $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$ (applying $C'_3 = C_3 - C_1, C'_2 = C_2 - C_1$) $= \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^3 & b^3 - a^3 & c^3 - a^3 \end{vmatrix}$

$$= (b-a)(c-a)\begin{vmatrix} 1 & 0 & 0 \\ a & 1 & 1 \\ a^3 & b^2 + ab + a^2 & c^2 + ca + a^2 \end{vmatrix} = (b-a)(c-a)(b-c)(a+b+c)$$

The rank of the matrix is 3 if all a, b, c are different and a + b + c ≠ 0

Correct option is (d)

Example 67: The rank of the matrix $\begin{pmatrix} a & -1 & 0 \\ 0 & a & -1 \\ -1 & 0 & a \end{pmatrix}$ is 2 for the value of 'a' equal to ________.

Soln. Since, given matrix has rank 2. Hence, determinant of 3 × 3 submatrix will be zero.

i.e., $\begin{vmatrix} a & -1 & 0 \\ 0 & a & -1 \\ -1 & 0 & a \end{vmatrix} = 0 \Rightarrow a = 1.$

Correct answer is (1)

2.5 Diagonalization of Matrix

Let, $\lambda_1, \lambda_2, \lambda_3 ....\lambda_n$ be 'n' distinct eigen values of a diagonalisable matrix A and $X_1, X_2...X_n$ be the corresponding linearly independent eigenvectors of the matrix A.

Diagonalized matrix D corresponding to A: $D = P^{-1} AP$

where matrix P is given by,

$$P = \begin{bmatrix} x_{11} & x_{12} & \cdots \\ x_{21} & x_{22} & \cdots \\ x_{31} & x_{32} & \cdots \\ \vdots & \vdots & \cdots \end{bmatrix} \text{ where } X_1 = \begin{bmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{bmatrix}$$

Example 68: Diagonalize the matrix $A = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

Soln: Eigenvalue equation is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -\lambda & -i \\ i & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1 \text{ i.e. } \lambda_1 = 1, \lambda_2 = -1$$

Now the eigenvector corresponding to λ_1 is let X_1.

Therefore, $AX_1 = \lambda_1 X_1$

$$\Rightarrow \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 1 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \begin{pmatrix} -ix_2 \\ ix_1 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow x_1 = -ix_2 \ \& \ x_2 = ix_1$$

Let $x_1 = k,\ x_2 = ik \quad \therefore \quad X_1 = \begin{pmatrix} k \\ ik \end{pmatrix}$

Using the normalization condition i.e. $X_1^{\dagger} X_1 = 1$

$$\Rightarrow \begin{pmatrix} k^* & -ik^* \end{pmatrix} \begin{pmatrix} k \\ ik \end{pmatrix} = 1 \quad \Rightarrow \quad |k|^2 + |k|^2 = 1 \quad \Rightarrow \quad k = \frac{1}{\sqrt{2}}$$

This gives $X_1 = \begin{pmatrix} \dfrac{1}{\sqrt{2}} \\ +\dfrac{i}{\sqrt{2}} \end{pmatrix}$

Similarly, eigenvector corresponding to the eigenvalue $\lambda_2 = -1$ is $X_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix}$

Now, Diagonalizing matrix $P = (X_1 \;\; X_2) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix}$ and $P^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}$

This matrix P is the matrix which can diagonalize A and the diagonalized matrix will be

$$D = P^{-1}AP = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Altrenative Method:

(i) First find the eigen value of A i.e. $\lambda_1, \lambda_2 .... \lambda_n$

(ii) Now, if all eigenvalues distinct (no two eigenvalues are equal), then putting eigenvalues in place of principal diagonal elements and writting zero for all off diagonal elements, we will get D as

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots \\ 0 & \lambda_2 & \cdots & \cdots \\ 0 & 0 & \ddots & \cdots \\ \vdots & \vdots & \vdots & \lambda_n \end{bmatrix}$$

Calculation of Power of a Matrix:

According to similarity tranformation, $D = P^{-1}AP \Rightarrow A = PDP^{-1}$

$$\Rightarrow A^K = PD^KP^{-1} \text{ (}K\text{ is a positive integer)}$$

Calculation of Exponential of a Matrix:

According to similarity tranformation, $D = P^{-1}AP \Rightarrow A = PDP^{-1}$

$$\Rightarrow (\exp A) = P\,(\exp D)\,P^{-1}$$

Calculation of Logarithm of a Matrix:

According to similarity tranformation, $D = P^{-1}AP \Rightarrow A = PDP^{-1}$

$$\Rightarrow (\ln A) = P\,(\ln D)\,P^{-1}$$

Example 69: If $A = \begin{bmatrix} -2 & -1 & 0 \\ -2 & 2 & 0 \\ 2 & 2 & 3 \end{bmatrix}$ is diagonalizable. 'S' is a matrix such that $S^{-1}AS = D_A$.

(D_A is diagonal matrix of A), then D_A is

(a) $\begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

Soln. $A=\begin{bmatrix}-2 & -1 & 0\\ -2 & 2 & 0\\ 2 & 2 & 3\end{bmatrix}$

Hence, it's characteristic equation $|A-\lambda I|=0$

$$\begin{vmatrix}-2-\lambda & -1 & 0\\ -2 & 2-\lambda & 0\\ 2 & 2 & 3-\lambda\end{vmatrix}=0 \Rightarrow (1-\lambda)(3-\lambda)^2=0 \Rightarrow \lambda=1,3,3$$

Hence, $D_A=\begin{pmatrix}1 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 3\end{pmatrix}$.

Correct option is (d)

Example 70: If $A=\begin{pmatrix}-1 & 0 & 0\\ 1 & 2 & 0\\ 3 & 1 & 4\end{pmatrix}$ is a diagonalizable matrix and if exponential of A is expended as $e^A=\sum_{n=0}^{\infty}\frac{A^n}{n!}$, then the value of $\det(3e^A)$ is equal to

(a) e^5 (b) $27e^5$ (c) e^{-8} (d) $27e^{-8}$

Soln. We have $\det(e^A)=e^{Tr(A)}$

Therefore, $\det(3e^A)=3^3 e^{Tr(A)}=27e^5$.

Correct option is (b)

Example 71: Given a matrix $M=\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}$, which of the following represents $\cos(\pi M/6)$? **[JEST 2016]**

(a) $\frac{1}{2}\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}$ (b) $\frac{\sqrt{3}}{4}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}$ (c) $\frac{\sqrt{3}}{4}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$ (d) $\frac{1}{2}\begin{pmatrix}1 & \sqrt{3}\\ \sqrt{3} & 1\end{pmatrix}$

Soln: Given: $M=\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$ and $N=\frac{\pi M}{6}=\begin{bmatrix}\pi/3 & \pi/6\\ \pi/6 & \pi/3\end{bmatrix}$

Eigenvalues ofN are $\lambda_1=\frac{\pi}{6}, \lambda_2=\frac{\pi}{2}$

Eigenvectors of N are $X_1=\begin{bmatrix}1/\sqrt{2}\\ -1/\sqrt{2}\end{bmatrix}$ and $X_2=\begin{bmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{bmatrix}$

Now, we can use similarly transformation to diagonalize N i.e.

$$D=P^{-1}NP$$

$$\Rightarrow \quad N=PDP^{-1} \Rightarrow \cos N=P(\cos D)P^{-1}$$

Here, $D=\begin{bmatrix}\pi/6 & 0\\ 0 & \pi/2\end{bmatrix}$, $P=\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}$, $P^{-1}=\begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}$

$$\therefore \quad \cos N=\cos\left(\frac{\pi M}{6}\right)=\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}\cos\frac{\pi}{6} & 0\\ 0 & \cos\frac{\pi}{2}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}=\frac{\sqrt{3}}{4}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

Correct option is (b)

Example 72: Consider two matrices $A=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ and $B=\begin{pmatrix}1 & 1\\ 0 & 2\end{pmatrix}$. Now choose correct statement(s) :

(a) A and B both are diagonalizable.

(b) B is diagonalizable but not A.

(c) $Tr(\log B)=\log 2$

(d) $\det(e^{A})=e^{2}$

Soln. A and B both are upper triangular matrices.
Hence, eigenvalues of A are 1, 1 and eigenvalues of B are 1, 2.
Since, eigenvalues of B are distinct, it is diagonalizable.
Also, from similarly transformation $P^{-1}BP=D_B \Rightarrow B=P\,D_BP^{-1}$

Here, $D_B=\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}$. Hence, $\log B=P\log D_BP^{-1}$ or $Tr(\log B)=Tr(\log D_B)$.

Here, $\log D_B=\begin{pmatrix}\log 1 & 0\\ 0 & \log 2\end{pmatrix}$. Therefore, $Tr(\log D_B)=\log 1+\log 2=\log 2$.

Also, $\det(e^{A})=e^{Tr(A)}=e^{2}$.

Correct options are (b), (c) and (d)

Example 73: If $A_{n\times n}$ is symmetric matrix with real entries and all eigenvalues distinct, then which of the following is correct?

(a) The eigenvalues are real.

(b) Eigenvectors corresponding to distinct eigenvalues are orthogonal.

(c) A satisfies $A=P\,D\,P^{T}$ (Here D is a square matrix whose columns are the eigenvectors of the eigenvalues of A and D is a diagonal matrix).

(d) The eigenvalues are complex.

Soln. Symmetric matrix with real entries will always have real eigenvalues.
Also, eigenvectors corresponding to different eigenvalues are always orthogonal.
Hence, P is also an orthogonal matrix.

i.e., $P^{T}P=I$ or $P^{T}=P^{-1}$

Now, $A=PDP^{T}$

$A=PDP^{-1}$ or $P^{-1}AP=D$

(Similarly transformation for diagonalization of A).

Correct options are (a), (b) and (c)

Example 74: If $A=\begin{pmatrix}\pi/2 & 4\\ 0 & 3\pi\end{pmatrix}$ and $B=\sin\left(\frac{A}{3}\right)$. If $\sin A=\sum_{n=0}^{\infty}(-1)^{n+2}\frac{A^{2n+1}}{(2n+1)!}$, then trace of B is ________.

Soln. Here, $\frac{A}{3}=\begin{pmatrix}\pi/6 & 4/3\\ 0 & \pi\end{pmatrix}=C$ (let)

Eigenvalues of $\left(\frac{A}{3}\right)$ are $\frac{\pi}{6}, \pi$, hence it is diagonalizable.

(Since, eigenvalues are distinct, so corresponding eigenvectors will be linearly independent).

Therefore, $C=P\,D_C\,P^{-1}$ or $\sin C=P\sin D_C\,P^{-1}$

Here, $D_C=\begin{pmatrix}\pi/6 & 0\\ 0 & \pi\end{pmatrix}\Rightarrow \sin D_C=\begin{pmatrix}\sin\frac{\pi}{6} & 0\\ 0 & \sin\pi\end{pmatrix}$

Therefore, $B=\sin C=\sin\frac{A}{3}=P\sin D_C P^{-1}$

Therefore, trace of $B=$ trace of $\sin D_C=\sin\frac{\pi}{6}+\sin\pi=\frac{1}{2}$.

Correct answer is (0.5)

Example 75: Consider the matrix $M=\begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix}$

The exponential of M simplifies (I is the 3 x 3 identity matrix)

(a) $e^M=I+\left(\frac{e^3-1}{3}\right)M$ (b) $e^M=I+M+\frac{M^2}{2!}$ (c) $e^M=I+3^3M$ (d) $e^M=(e-1)M$

Soln. $e^M=I+M+\frac{M^2}{2!}+\frac{M^3}{3!}+..........$

Now, $M^2=\begin{pmatrix}3 & 3 & 3\\ 3 & 3 & 3\\ 3 & 3 & 3\end{pmatrix}=3M$, $M^3=\begin{pmatrix}9 & 9 & 9\\ 9 & 9 & 9\\ 9 & 9 & 9\end{pmatrix}=3^2M$ and so on

Therefore, $e^M=I+M+\frac{3M}{2!}+\frac{3^2M}{3!}+..........=I+\frac{M}{3}\left(3+\frac{3^2}{2!}+\frac{3^3}{3!}+.......\right)=I+\frac{M}{3}\left(e^3-1\right)$

Correct answer: (a)

Example 76: If two matrices A and B can be diagonalized simultaneously, which of the following is true?

(a) $A^2B=B^2A$ (b) $A^2B^2=B^2A$ (c) $AB=BA$ (d) $AB^2AB=BABA^2$

Soln. The diagonal matrices D_A and D_B corresponding respectively, will commute with each other.

$D_AD_B=D_BD_A$

$\Rightarrow P^{-1}APP^{-1}BP = P^{-1}BPP^{-1}AP$ [same matrix is used for A and B]

$\Rightarrow P^{-1}ABP = P^{-1}BAP \Rightarrow AB = BA$

Correct option is (c)

Example 77: Two matrices A and B are said to be similar if $B = P^{-1}AP$ for some invertible matrix P. Which of the following statements is **NOT TRUE** ?

(a) Det A = Det B (b) Trace of A = Trace of B

(c) A and B have the same eigenvectors (d) A and B have the same eigenvalues

Soln. Two similar matrices A and B have same trace, determinant eigenvalues and rank.

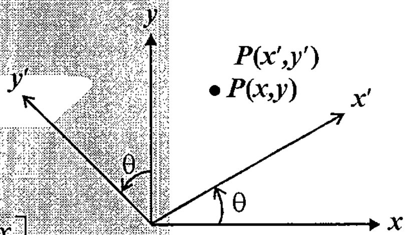
Correct option is (c)

2.6 Applications

Rotation Matrix:

Let (x, y, z) be a cartesian coordinate system in a three dimension space. Let $\vec{r} = x\hat{i} + y\hat{j}$ be a vector in x-y plane. If we consider a rotation of coordinate system about z-axis through an angle θ in anticlockwise sense and indicate the new axis by (x', y'), then the components of the same vector $\vec{r} = x'\hat{i}' + y'\hat{j}'$ relative to the new system may be expressed as

$$x' = x\cos\theta + y\sin\theta$$
$$y' = -x\sin\theta + y\cos\theta$$



Then equations in matrix form can be expressed as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = R_z(\theta) \begin{bmatrix} x \\ y \end{bmatrix}$$

where $R_z(\theta)$ is the transformation matrix corresponding to the rotation in x-y plane about z-axis.

Note: (i) $R_z(\theta)$ is orthogonal matrix.

(ii) Determinant of $R_z(\theta)$ is 1.

(iii) Eigenvalues of $R_z(\theta)$ are $e^{i\theta}$ and $e^{-i\theta}$.

Let us consider a vector in space i.e. $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. Here, three types of rotation will be possible:

CASE I: Rotation about z-axis:

Rotation matrix. $R_z(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

CASE II: Rotation about x-axis:

Rotation matrix. $R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$

CASE III: Rotation about y-axis:

Rotation matrix. $R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$

Example 78: Consider a vector $\vec{P} = \hat{i} + 2\hat{j} + \hat{k}$ in the co-ordinate system $(\hat{i}, \hat{j}, \hat{k})$. The axes are rotated in the clockwise direction about the z axis by an angle 45^0. The vector $\vec{P}$ in the rotated co-ordinate system will be

(a) $-\frac{1}{\sqrt{2}}\hat{i}' + \frac{3}{\sqrt{2}}\hat{j}' + \hat{k}'$ (b) $\frac{1}{\sqrt{2}}\hat{i}' - \frac{3}{\sqrt{2}}\hat{j}' + \hat{k}'$ (c) $\frac{3}{\sqrt{2}}\hat{i}' + \frac{1}{\sqrt{2}}\hat{j}' + \hat{k}'$ (d) $\frac{3}{\sqrt{2}}\hat{i}' - \frac{1}{\sqrt{2}}\hat{j}' + \hat{k}'$

Soln: $\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos 45^0 & -\sin 45^0 & 0 \\ \sin 45^0 & \cos 45^0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos 45^0 & -\sin 45^0 & 0 \\ \sin 45^0 & \cos 45^0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} \\ 1 \end{bmatrix}$

Correct Option is (a)

Example 79: Under a rotation by an angle θ about the x-axis, the vector that remains invariant is

(a) $2\hat{x} - \hat{y} + \hat{z}$ (b) $3\hat{z}$ (c) $5\sqrt{2}\hat{x}$ (d) $\sqrt{3}\hat{y}$

Soln: Since, rotation take place about x-axis.

Rotation matrix. $R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$

So, x-coordinate will remain invariant.

Correct Option is (c)

Example 80: Consider a vector $\vec{p} = 2\hat{i} + 3\hat{j} + 2\hat{k}$ in the coordinate system $(\hat{i}, \hat{j}, \hat{k})$. The axes are rotated anti-clockwise about the Y axis by an angle of 60°. The vector $\vec{p}$ in the rotated coordinate sytem $(\hat{i}', \hat{j}', \hat{k}')$ is

(a) $(1-\sqrt{3})\hat{i}' + 3\hat{j}' + (1+\sqrt{3})\hat{k}'$ (b) $(1+\sqrt{3})\hat{i}' + 3\hat{j}' + (1-\sqrt{3})\hat{k}'$

(c) $(1-\sqrt{3})\hat{i}' + (3+\sqrt{3})\hat{j}' + 2\hat{k}'$ (d) $(1-\sqrt{3})\hat{i}' + (3-\sqrt{3})\hat{j}' + 2\hat{k}'$

Soln. The rotation matrix corresponding to the anti-clockwise rotation about y-axis by an angle θ will be

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos 60^\circ & 0 & -\sin 60^\circ \\ 0 & 1 & 0 \\ \sin 60^\circ & 0 & \cos 60^\circ \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} \left(1-\sqrt{3}\right) \\ 3 \\ \left(1+\sqrt{3}\right) \end{bmatrix}$$

Correct option is (a)

Existance and uniqueness of the solution of system of Simultaneous Linear Non-Homogeneous Equations:

Consistent Equations: They have one or more solutions.

Example: (a) Unique Solution: $x+2y=4; 3x+2y=2$

(b) Infinite Solution: $x+2y=4; 3x+6y=12$

Inconsistent Equations: They have no solution.

Example: $x+2y=4; 3x+6y=5$

Suppose a system of 'n' simultaneous linear equations containing 'n' unknown variables as following:

$$\begin{aligned} &a_{11}x_1 + a_{12}x_2 + \dots\dots\dots\dots + a_{1n}x_n = b_1 \\ &a_{21}x_1 + a_{22}x_2 + \dots\dots\dots\dots + a_{2n}x_n = b_2 \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &a_{n1}x_1 + a_{n2}x_2 + \dots\dots\dots\dots + a_{nn}x_n = b_n \end{aligned}$$

where co-efficient matrix of the system $= A = \begin{bmatrix} a_{11} & a_{12} & \dots\dots & \dots\dots & a_{n1} \\ a_{21} & a_{22} & \dots\dots & \dots\dots & a_{n2} \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ a_{n1} & a_{n2} & \dots\dots & \dots\dots & a_{nn} \end{bmatrix}$

and Augamented matrix of the system $= C = \begin{bmatrix} a_{11} & a_{12} & \dots\dots & \dots\dots & a_{n1} & b_1 \\ a_{21} & a_{22} & \dots\dots & \dots\dots & a_{n2} & b_2 \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ a_{n1} & a_{n2} & \dots\dots & \dots\dots & a_{nn} & b_n \end{bmatrix}$

(1) If rank of A = rank of C, then the equations are consistent.

(a) If rank of A = rank of C = number of unknown variables, then there will be unique solution.

(b) If rank of A = rank of C $<$ number of unknown variables, then there will be infinite solution.

(2) If rank of A $\neq$ rank of C, then the equations are inconsistent and have no solutions.

TIFR PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. The matrix $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ **[TIFR 2010]**

can be relateed by a similarity transformation to the matrix.

(a) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 1 & 0 \\ 1 & -1 & -1 \\ 0 & -1 & 2 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

Soln: Since, the given matrix $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, and the matrices given in options are related by similarity transformation. Hence, their trace and determinant would be same.

Correct option is (d)

Question 2. Consider the matrix, $M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$

A 3-dimensional basis formed by eigenvectors of M is **[TIFR 2011]**

(a) $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

(c) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ (d) $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

Soln: Eigenvalue equation is $(M - \lambda I) = 0$

$$\begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & -1 \\ 0 & -1 & -\lambda \end{pmatrix} = 0 \text{ ; } (1-\lambda)(\lambda^2 - 1) = 0 \text{ ; } \lambda = 1, 1, -1$$

Now, eigenvectors of M.

For $\lambda = 1$; $(M - \lambda I)X = 0$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \text{ ; } -y - z = 0 \Rightarrow y = -z$$

x can have any value, if $z = k_1$, $y = -k_1$

Therefore, $X_{\lambda=1} = \begin{pmatrix} x \\ -k_1 \\ k_1 \end{pmatrix}$

For $\lambda = -1$

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \ ; \ 2x = 0 \ \boxed{x=0} \ y - z = 0 \ \boxed{y=z}$$

Hence, $X_{\lambda=-1} = \begin{pmatrix} 0 \\ k_2 \\ k_2 \end{pmatrix}$, which is satisfied by option (b) only.

Correct option is (b)

Question 3. The trace of the real 4 × 4 matrix $U = \exp(A)$, where $A = \begin{pmatrix} 0 & 0 & 0 & \pi/4 \\ 0 & 0 & -\pi/4 & 0 \\ 0 & \pi/4 & 0 & 0 \\ -\pi/4 & 0 & 0 & 0 \end{pmatrix}$

is equal to **[TIFR 2011]**

(a) $2\sqrt{2}$ (b) $\pi/4$ (c) $\exp(i\varphi)$ for $\varphi = 0, \pi$ (d) zero

Soln: $A^2 = -\frac{\pi^2}{16}I;\ A^3 = A^2.A = -\frac{\pi^2}{16}A;\ A^4 = \left(A^2\right)^2 = \frac{\pi^4}{256}I;$ and so on ....

$$\exp(A) = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \ldots\ldots\ldots = I + A - \frac{1}{2!}\frac{\pi^2}{16}I - \frac{1}{3!}\frac{\pi^2}{16}A + \frac{1}{4!}\frac{\pi^4}{256}I + \ldots\ldots.$$

$$Tr\left[\exp(A)\right] = Tr(I) + Tr(A) - Tr\left(\frac{1}{2!}\frac{\pi^2}{16}I\right) - Tr\left(\frac{1}{3!}\frac{\pi^2}{16}A\right) + Tr\left(\frac{1}{4!}\frac{\pi^4}{256}I\right) + \ldots\ldots..$$

$$= 4 + 0 - 4.\frac{1}{2!}\frac{\pi^2}{16} - 0 + 4.\frac{1}{4!}\frac{\pi^4}{256} + \ldots\ldots\ldots.. = 4\left[1 - \frac{1}{2!}\frac{\pi^2}{16} + \frac{1}{4!}\frac{\pi^4}{256} - \ldots\ldots.\right] = 4\cos\left(\frac{\pi}{4}\right) = 2\sqrt{2}$$

Question 4. Two different 2 × 2 matrices A and B are found to have the same eigenvalues. It is then correct to state that $A = SBS^{-1}$ where S can be a **[TIFR 2012]**

(a) traceless 2 × 2 matrix (b) Hermitian 2 × 2 matrix

(c) unitary 2 × 2 matrix (d) arbitrary 2 × 2 matrix

Soln: $A = S\,B\,S^{-1}$; since, matrix 'S' is formed by the eigenvectors of A which are orthogonal and normalized. Hence, 'S' would be an unitary matrix.

Correct option is (c)

Question 5. The product MN of two Hermitian matrices M and N is anti-Hermitian. It follows that **[TIFR 2014]**

(a) $\{M,N\}=0$ (b) $[M,N]=0$ (c) $M^{\dagger}=N$ (d) $M^{\dagger}=N^{-1}$

Soln: Since, M and N are Hermitian matrix. Therefore, $M^{+}=M$ and $N^{+}=N$.

Also, given $(MN)^{+}=-MN \Rightarrow N^{+}M^{+}=-MN$; $NM+MN=0 \Rightarrow \{M,N\}=0.$

Correct option is (a)

Question 6. If the eigenvalues of a symmetric 3×3 matrix A are 0, 1, 3 and the corresponding eigenvectors can be written as **[TIFR 2016]**

$$\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}1\\0\\-1\end{bmatrix},\begin{bmatrix}1\\-2\\1\end{bmatrix}$$

respectively, then the matrix A^4 is

(a) $\begin{bmatrix}41 & -81 & 40\\-81 & 0 & -81\\40 & -81 & 41\end{bmatrix}$ (b) $\begin{bmatrix}-82 & -81 & 79\\-81 & 81 & -81\\79 & -81 & 83\end{bmatrix}$ (c) $\begin{bmatrix}14 & -27 & 13\\-27 & 54 & -27\\13 & -27 & 14\end{bmatrix}$ (d) $\begin{bmatrix}14 & -13 & 27\\-13 & 54 & -13\\27 & -13 & 14\end{bmatrix}$

Soln: Since, eigenvalues of A are 0, 1, 3. Hence, eigenvalues of A^4 are 0, 1, 81

Therefore, Trace $(A^4)=82 \Rightarrow \det(A^4)=0$.

Check Trace and determinant of all options.

Correct option is (c)

Question 7. Denote the commutator of two matrices A and B by $[A,B]=AB-BA$ and the anti-commutator by $\{A,B\}=AB+BA$. If $\{A,B\}=0$, we can write $[A,BC]=$ **[TIFR 2017]**

(a) $-B[A,C]$ (b) $B\{A,C\}$ (c) $-B\{A,C\}$ (d) $[A,C]B$

Soln: Since, $\{AB\}=0 \Rightarrow AB+BA=0$ ... (1)

Now, $[A,BC]=0$

$\Rightarrow B[A,C]+[A,B]C=0 \Rightarrow B(AC-CA)+(AB-BA)C=0 \Rightarrow BAC-BCA+ABC-BAC=0$

From Eq. (1) ; $AB=-BA$

Therefore, $-BCA-BAC=0 \Rightarrow -B(AC+CA)=0$

Therefore, $-B\{A,C\}=0$

Correct option is (c)

Question 8. The matrix $\begin{pmatrix}100\sqrt{2} & x & 0\\-x & 0 & -x\\0 & x & 100\sqrt{2}\end{pmatrix}$, where $x>0$, is known to have two equal eigenvalues.

Find the value of x. **[TIFR 2017]**

Soln: $A=\begin{pmatrix}100\sqrt{2} & x & 0\\ -x & 0 & -x\\ 0 & x & 100\sqrt{2}\end{pmatrix}$

Eigenvalue equation is $(A-\lambda I)=0$, $A=\begin{pmatrix}100\sqrt{2}-\lambda & x & 0\\ -x & -\lambda & -x\\ 0 & x & 100\sqrt{2}-\lambda\end{pmatrix}=0$

$$\left(100\sqrt{2}-\lambda\right)\left(-\lambda\left(100\sqrt{2}-\lambda\right)+x^2\right)-x\left(-x\left(100\sqrt{2}-\lambda\right)-0\right)=0$$

$$\left(100\sqrt{2}-\lambda\right)\left\{2x^2+\lambda^2-100\sqrt{2}\lambda\right\}=0;\ \left(100\sqrt{2}-\lambda\right)\left\{\lambda^2-100\sqrt{2}\lambda+2x^2\right\}=0$$

Let us suppose quadratic terms in 2nd bracket will give equal roots or eigenvalues.

Hence, $\left(100\sqrt{2}\right)^2-4\times1\times2x^2=0\Rightarrow x=50$.

Correct answer is (50)

Question 9. A unitary matrix U is expanded in terms of a Hermitian matrix H, such that $U=e^{i\pi H/2}$. If we know that

$$H=\begin{pmatrix}\frac{1}{2} & 0 & \frac{\sqrt{3}}{2}\\ 0 & 1 & 0\\ \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2}\end{pmatrix}$$

then U must be **[TIFR 2017]**

(a) $\begin{pmatrix}i & \frac{1}{2} & \frac{\sqrt{3}}{2}\\ \frac{1}{2} & i & \frac{1}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2} & i\end{pmatrix}$ (b) $\begin{pmatrix}\frac{i}{2} & 0 & \frac{i\sqrt{3}}{2}\\ 0 & i & 0\\ \frac{i\sqrt{3}}{2} & 0 & -\frac{i}{2}\end{pmatrix}$ (c) $\begin{pmatrix}1 & 0 & \sqrt{3}\\ 0 & 2 & 0\\ \sqrt{3} & 0 & -1\end{pmatrix}$ (d) $\begin{pmatrix}2i & 1 & \frac{\sqrt{3}}{2}\\ 1 & 2i & 0\\ \frac{\sqrt{3}}{2} & 0 & 2i\end{pmatrix}$

Soln: Since, U is an unitary matrix, hence just check the normalization condition for each rows and orthogonalization condition for two rows.

Correct option is (b)

Question 10. If a 2×2 matrix M is given by, $M=\begin{pmatrix}1 & (1-i)/\sqrt{2}\\ (1+i)/\sqrt{2} & 0\end{pmatrix}$

Then Det(expM) = **[TIFR 2018]**

(a) e (b) e^2 (c) $2i\sin\sqrt{2}$ (d) $\exp\left(-2\sqrt{2}\right)$

Soln: We know that,

$$\det\left(e^M\right)=e^{Tr(M)}=e^{1+0}=e$$

Correct option is (a)

JEST PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. For an $N\times N$ matrix consisting of all ones, **[JEST 2012]**

(a) all eigenvalues = 1 (b) all eigenvalues = 0

(c) the eigenvalues are 1, 2, ..., N (d) one eigenvalue = N, the others = 0

Soln: $A=\begin{pmatrix}1&1&1&\cdots&1\\1&1&1&\cdots&1\\\vdots&\vdots&\vdots&\cdots&\vdots\\1&1&1&\cdots&1\end{pmatrix}$; Eigenvalues of A will be trace $(A), 0, 0, \ldots (n-1)$ times. i.e., $n, 0, 0, 0, \ldots (n-1)$ times.

Correct option is (d)

Question 2. The coordinate transformation

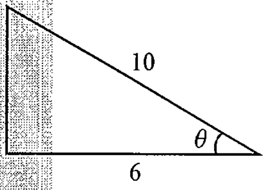
$$x' = 0.8x + 0.6y, \qquad y' = 0.6x - 0.8y$$

represents **[JEST 2013]**

(a) a translation (b) a proper rotation (c) a reflection (d) none of the above.

Soln: $x' = 0.8x + 0.6y,\ y' = 0.6x - 0.8y$

$$\begin{pmatrix}x'\\y'\end{pmatrix}=\begin{pmatrix}0.8&0.6\\0.6&-0.8\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$



Compare this with rotation matrix $R_z(\theta)$ about z-axis.

$$R_z(\theta)=\begin{pmatrix}\sin\theta&\cos\theta\\\cos\theta&-\sin\theta\end{pmatrix}, \text{ if } \sin\theta = 0.8 = \frac{8}{10}.$$

Therefore, $\cos\theta = \frac{6}{10} = 0.6$

Correct option is (b)

Question 3. Given a matrix $M=\begin{pmatrix}2&1\\1&2\end{pmatrix}$, which of the following represents $\cos(\pi M/6)$? **[JEST 2016]**

(a) $\frac{1}{2}\begin{pmatrix}1&2\\2&1\end{pmatrix}$ (b) $\frac{\sqrt{3}}{4}\begin{pmatrix}1&-1\\-1&1\end{pmatrix}$ (c) $\frac{\sqrt{3}}{4}\begin{pmatrix}1&1\\1&1\end{pmatrix}$ (d) $\frac{1}{2}\begin{pmatrix}1&\sqrt{3}\\\sqrt{3}&1\end{pmatrix}$

Soln: Given: $M=\begin{bmatrix}2&1\\1&2\end{bmatrix}$ and $N=\frac{\pi M}{6}=\begin{bmatrix}\pi/3&\pi/6\\\pi/6&\pi/3\end{bmatrix}$

Eigenvalues of N are $\lambda_1=\frac{\pi}{6}, \lambda_2=\frac{\pi}{2}$

Eigenvectors of N are $X_1=\begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix}$ and $X_2=\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$

Now, we can use similarly transformation to diagonalize N i.e.

$D = P^{-1}NP \Rightarrow \quad N = PDP^{-1} \Rightarrow \cos N = P(\cos D)P^{-1}$

Here, $D = \begin{bmatrix} \pi/6 & 0 \\ 0 & \pi/2 \end{bmatrix}$, $P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$, $P^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

$$\therefore \quad \cos N = \cos\left(\frac{\pi M}{6}\right) = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \cos\frac{\pi}{6} & 0 \\ 0 & \cos\frac{\pi}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{\sqrt{3}}{4}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Correct option is (b)

Question 4. Let $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 11 \end{bmatrix}$ and $M = \begin{bmatrix} 10 & 3i \\ -3i & 2 \end{bmatrix}$. Similarity transformation from M to Λ can be performed by

[JEST 2017]

(a) $\frac{1}{\sqrt{10}}\begin{pmatrix} 1 & 3i \\ 3i & 1 \end{pmatrix}$ (b) $\frac{1}{\sqrt{9}}\begin{pmatrix} 1 & -3i \\ 3i & 11 \end{pmatrix}$ (c) $\frac{1}{\sqrt{10}}\begin{pmatrix} 1 & 3i \\ -3i & 11 \end{pmatrix}$ (d) $\frac{1}{\sqrt{9}}\begin{pmatrix} 1 & 3i \\ -3i & 1 \end{pmatrix}$

Soln: $A = \begin{bmatrix} 1 & 0 \\ 0 & 11 \end{bmatrix}$ and $M = \begin{bmatrix} 10 & 3i \\ -3i & 2 \end{bmatrix}$

Since, 'M' and 'A' are related by similarly transformation. Hence, eigenvalues of M are 1 and 11.
Now, eigenvector corresponding to $\lambda = 1$

$$\begin{pmatrix} 10-\lambda & 3i \\ -3i & 2-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} 9 & 3i \\ -3i & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$9x + 3iy = 0 \qquad \ldots(1)$

$-3ix + y = 0 \qquad \ldots(2)$

Eq. (1) and Eq. (2) both are same.
$y = 3ix$, if $x = k$; $y = 3ik$

Therefore, $X_{\lambda=1} = \begin{pmatrix} k \\ 3ik \end{pmatrix} = \begin{pmatrix} 1 \\ 3i \end{pmatrix}$ $\{\text{if } k = 1\}$ and eigenvector corresponding to $\lambda = 11$

$$\begin{pmatrix} -1 & 3i \\ -3i & -9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \ ; \ -x + 3iy = 0 \ ; \ x = 3iy$$

Therefore, $X_{\lambda=11} = \begin{pmatrix} 3i \\ 1 \end{pmatrix}$

Therefore, similarly transformation from M to A will be performed by the matrix.

$\Rightarrow \frac{1}{\sqrt{10}}\begin{pmatrix} 1 & 3i \\ 3i & 1 \end{pmatrix}$ {normalized matrix}.

Correct option is (a)

JNU PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. The eigenvalues of the matrix $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ are **[JNU 2009]**

(a) –1, –1 and 2 (b) –1, 1 and 2 (c) –1, –2 and 2 (d) –1, 2 and 2

Soln: Let $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, then $\text{Tr}(A) = 0$ and $\det(A) = 2$.

Correct option is (a)

Question 2. The eigenvalues of matrix $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ are **[JNU 2009]**

(a) $+1, +1, +1$ and $+1$ (b) $+1, +1, -1$ and -1

(c) $0, 0, \sqrt{2}$ and $-\sqrt{2}$ (d) $+\sqrt{2}, -\sqrt{2}, +1/\sqrt{2}$ and $-1/\sqrt{2}$

Soln: Let $A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

Therefore, $A^2 = 1$ or eigenvalue equation is $\lambda^2 = 1 \Rightarrow \lambda = \pm 1$, other eigenvalues will be repeated.

Correct option is (b)

Question 3. You are given the following transformation of the coordinates (x,y) of a point in the two-dimensional plane :

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Which of the following statements describes the transformation ? **[JNU 2011]**

(a) Rotation by an angle of 45° about the z-axis followed by translation by one unit along both x-and y-axes

(b) Rotation by an angle of 45° about the z-axis followed by an expansion of both the x-and y-coordinates by a factor of $\sqrt{2}$

(c) Translation by one unit along both x- and y-axes followed by an expansion of both the x-and y-coordinates by a factor of $\sqrt{2}$

(d) Only a rotation by an angle 45° about the z-axis.

Soln: $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$; if axes is rotated about z-axis clockwise, then $R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, if $\theta = 45°$, then $R_z(\theta)=\begin{pmatrix}1/\sqrt{2} & -1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}$.

If both the axes are expanded by $\sqrt{2}$ times, then $R_z(\theta)=\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}$.

Correct option is (b)

Question 4. For which of the following matrices both the eigenvalues are positive? **[JNU 2011]**

(a) $\begin{pmatrix}-7 & 2\\ 2 & -2\end{pmatrix}$ (b) $\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}$ (c) $\begin{pmatrix}4 & -3\\ -3 & 5\end{pmatrix}$ (d) $\begin{pmatrix}-1 & 4\\ 4 & -1\end{pmatrix}$

Soln: Matrix whose trace and determinant both are positive will have both eigenvalues positive.

Correct option is (c)

Question 5. A homogeneous linear transformation takes the point (1, 1) in the xy-plane to the point (3, 3) and keeps the point (1, –1) fixed (i.e., it remains (1, –1) after the transformation). The matrix corresponding to this transformtion is **[JNU 2012]**

(a) $\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}$ (b) $\begin{pmatrix}3 & 0\\ 0 & 2\end{pmatrix}$ (c) $\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}$ (d) $\begin{pmatrix}2 & -1\\ -1 & 2\end{pmatrix}$

Soln: Let the transformation matrix is $T=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$

Hence, according to question,

$$\begin{pmatrix}3\\ 3\end{pmatrix}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix}$$

$$a+b=3 \qquad \ldots(1)$$

$$c+d=3 \qquad \ldots(2)$$

And $$\begin{pmatrix}1\\ -1\end{pmatrix}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1\\ -1\end{pmatrix}$$

$$a-b=1 \qquad \ldots(3)$$

$$-c+d=1 \qquad \ldots(4)$$

From equation (1) and (3), a = 2, b = 1 and from (2) and (4) d = 2, c = 1

$$\boxed{T=\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}}$$

Correct option is (c)

Question 6. Let $\lambda_i\,(i=1,2,3)$ be the eigenvalues of the matrix. The sum $\sum_{i=1}^{3}\lambda_i^2$ is equal to

$$\begin{bmatrix}2 & -1 & -3\\ -1 & 1 & 2\\ -3 & 2 & 3\end{bmatrix}$$

[JNU 2013]

(a) 14 (b) 42 (c) 6 (d) 0

Soln: Let $A=\begin{bmatrix}2 & -1 & -3\\ -1 & 1 & 2\\ -3 & 2 & 3\end{bmatrix}$. If $\lambda_1, \lambda_2, \lambda_3$ are eigenvalues of A, then $\lambda_1^2, \lambda_2^2, \lambda_3^2$ will be eigenvalues of A^2.

Here, $A^2=\begin{bmatrix}2 & -1 & -3\\ -1 & 1 & 2\\ -3 & 2 & 3\end{bmatrix}\begin{bmatrix}2 & -1 & -3\\ -1 & 1 & 2\\ -3 & 2 & 3\end{bmatrix}=\begin{bmatrix}14 & -9 & -19\\ -9 & 6 & 11\\ -17 & 11 & 22\end{bmatrix}$.

Now, $\sum\limits_{i=1,2,3} \lambda_1^2+\lambda_2^2+\lambda_3^2=\text{trace of } A^2=14+6+22=42$.

Correct option is (b)

Question 7. The exponential of a 2×2 matrix A is defined by the power series expansion $\exp(A)=\sum\limits_{n=0}^{\infty}\frac{1}{n!}A^n$, A^0 being the 2×2 identify matrix. If $S=\frac{1}{\sqrt{5}}\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}$ and $SAS^{-1}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, then the determinant of the matrix e^A is **[JNU 2014]**

(a) 1 (b) e^2 (c) $\frac{1}{e}$ (d) $e+\frac{1}{e}$ (e) −1

Soln: $S=\frac{1}{\sqrt{5}}\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}$ and $SAS^{-1}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=D_A$ (let)

Hence, eigenvalues of A are 1, 1

Therefore, $S^{-1}D_AS$ or $e^A=S^{-1}e^{D_A}S$. Here, $e^{D_A}=\begin{pmatrix}e & 0\\ 0 & e\end{pmatrix}$

Therefore, $\det(e^A)=\det(e^{D_A})=e\times e=e^2$.

Correct option is (b)

Question 8. The trace and the determinant of a 3×3 matrix A satisfy Tr A = 2 det A = 2. Further, the sum of two of the eigenvalues of A is equal to the third eigenvalue. Then the trace and the determination of the matrix A^2 are, respectively **[JNU 2015]**

(a) 2 and 1 (b) 0 and 1 (c) 4 and 1 (d) 4 and 2 (d) 4 and 4

Soln: Let $\lambda_1, \lambda_2, \lambda_3$ are three eigenvalues of A, then $\text{trace}(A)=\lambda_1+\lambda_2+\lambda_3$

$$\lambda_1+\lambda_2+\lambda_3=2 \qquad \dots(1)$$

and $\det(A)=\lambda_1\lambda_2\lambda_3$

$$\lambda_1\lambda_2\lambda_3=2 \qquad \dots(2)$$

Also, $\lambda_1+\lambda_2=\lambda_3 \qquad \dots(3)$

Solving Eqs (1), (2) and (3)

$\lambda_3=1;\ \lambda_1+\lambda_2=1;\ \lambda_1\lambda_2=2$

No options are matched.

DU PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. If the trace and the determinant of a 2 × 2 matrix are both zero; then, **[DU 2014]**

(a) Only one of eigenvalues needs to be zero (b) Both the eigenvalues need to be zero

(c) Both the eigenvalues are non-zero but finite (d) At least one of the eigenvalues should be infinite

Soln: Since, both trace and determinant of 2 × 2 matrix is zero, hence both eigenvalues must be zero.

Correct option is (b)

Question 2. The matrix given by: **[DU 2014]**

$$\begin{pmatrix} 3 & 2i & 1 \\ -2i & 4 & 1+2i \\ 1 & 1-2i & 5 \end{pmatrix}$$

(a) Has one real and one complex eigenvalues (b) Has two real and one complex eigenvalues

(c) Has three real eigenvalues (d) Has three complex eigenvalues

Soln: Given matrix is Hermitian in nature. Hence, it's all eigenvalues are real.

Correct option is (c)

Question 3. The trace of a 4 × 4 matrix is 8 and its determinant is –24. If two of its eigenvalues are 3 and 4, the other two eigenvalues are: **[DU 2014]**

(a) –2 and 1 (b) 2 and –1 (c) 2 and 1 (d) –2 and –1

Soln: Let eigenvalues of 4 × 4 matrix is $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, then given

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 8 \qquad \text{... (1)}$$

and $$\lambda_1 \lambda_2 \lambda_3 \lambda_4 = -24 \qquad \text{... (2)}$$

Also, $$\lambda_1 = 3,\ \lambda_2 = 4 \qquad \text{... (3)}$$

Hence, another eigenvalues must be 2 and –1.

Correct option is (b)

Question 4. Which of the following statements is true for a square matrix *A*?

(a) If $A^2 = 0$, it necessarily implies that A = 0 **[DU 2015]**

(b) If A is real and orthogonal, its eigenvalues will always be real

(c) If A is hermitian, its diagonal entries are always real

(d) If A is anti-hermitian (skew-hermitian), its diagonal entries are always zero

Soln: **Correct option is (c)**

Question 4. Under a rotation by an angle θ about the *x*-axis, the vector that remains invariant is

(a) $2\hat{x} - \hat{y} + \hat{z}$ (b) $3\hat{z}$ (c) $5\sqrt{2}\hat{x}$ (d) $\sqrt{3}\hat{y}$ **[DU 2015]**

Soln: Since, rotation is about *x*-axis, hence vector $5\sqrt{2}\,\hat{x}$ will remain invariant.

Correct option is (c)

Question 6. Consider the matrix

$$A = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}$$

If $B = 2e^A$, the determinant of B is **[DU 2016]**

(a) $4e^7$ (b) $4\left(e^7-e^4\right)$ (c) e^7 (d) e^7-e^4

Soln: $A=\begin{pmatrix}2 & 2\\ 2 & 5\end{pmatrix}$; $B=2e^A$

Therefore, $\det B=\det\left(2e^A\right)=2^2\det\left(e^A\right)$

$$=4e^{\text{tra}\,A} \qquad \left(\because \det e^A=e^{\text{tra}\,A}\right)$$

$$=4e^7.$$

Correct option is (a)

Question 7. If $A=\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$ and I is a 2 × 2 identity matrix, then e^A is : **[DU 2017]**

(a) $I+\left(\frac{e^2-1}{2}\right)A$ (b) $I+\left(\frac{e-1}{2}\right)A$ (c) $I+\left(\frac{e^2+1}{2}\right)A$ (d) $I+\left(\frac{e+1}{2}\right)A$

Soln: $A=\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$; $A^2=2A,\ A^3=A^2\cdot A=2A\cdot A=4A,\ A^4=A^2\cdot A^2=8A$ and so on ....

$$e^A=I+A+\frac{A^2}{2!}+\frac{A^3}{3!}+\frac{A^4}{4!}+\cdots$$

$$=I+A+\frac{2A}{2!}+\frac{4A}{3!}+\frac{8A}{4!}+\cdots$$

$$=I+A\left(I+\frac{2}{2!}+\frac{4}{3!}+\frac{8}{4!}+\cdots\right)$$

$$=I+\frac{A}{2}\left(2I+\frac{2^2}{2!}+\frac{2^3}{3!}+\frac{2^4}{4!}+\cdots\right)$$

$$e^A=I+\frac{A}{2}\left(e^2-1\right).$$

Correct option is (a)

HCU PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. The rank of the following matrix is

$$\begin{pmatrix}2 & -2 & 0 & 6\\ 4 & 2 & 0 & 2\\ 1 & -1 & 0 & 3\\ 1 & -2 & 1 & 2\end{pmatrix}$$

(a) 2 (b) 3 (c) 1 (d) 4 **[H.C.U.-2010]**

Soln: Given matrix is $A = \begin{pmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{pmatrix}$

$$|A|_{4\times4} = 2\begin{vmatrix} 2 & 0 & 2 \\ -1 & 0 & 3 \\ -2 & 1 & 2 \end{vmatrix} - (-2)\begin{vmatrix} 4 & 0 & 2 \\ 1 & 0 & 3 \\ 1 & 1 & 2 \end{vmatrix} + 0\begin{vmatrix} 4 & 2 & 2 \\ 1 & -1 & 3 \\ 1 & -2 & 2 \end{vmatrix} - 6\begin{vmatrix} 4 & 2 & 0 \\ 1 & -1 & 0 \\ 1 & -2 & 1 \end{vmatrix}$$

$$= 2(-1(6+2)) + 2(-1(12-2)) - 6(1(-4-2))$$

$$= -16 - 20 + 36 = 0$$

Let us take a 3 × 3 submatrix

$$A' = \begin{pmatrix} 4 & 2 & 0 \\ 1 & -1 & 0 \\ 1 & -2 & 1 \end{pmatrix} \; ; \; |A'|_{3\times3} = 1(-4-2) = -6 \neq 0$$

Hence rank of A is 3.

Correct option is (b)

Question 2. It is given that the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & -\alpha \\ 0 & \alpha & \alpha \end{pmatrix}$$

is orthogonal. The value of α must be equal to

(a) 1 (b) 0 (c) $\sqrt{2}$ (d) $\frac{1}{\sqrt{2}}$ **[H.C.U.-2010]**

Soln: If a matrix is orthogonal, then it's each row or column is a vector having unit magnitude.
Let us consider vector associated with second column.

$$\vec{c}_2 = 0\hat{i} + \alpha\hat{j} + \alpha\hat{k}$$

$$\therefore\ |\vec{c}_2| = \sqrt{d^2 + \alpha^2} = 1 \Rightarrow \alpha = \frac{1}{\sqrt{2}}$$

Correct option is (d)

Question 3. If A and B are $n \times n$ matrices, which of the following does not equal $(A+B)^2$?

(a) $A^2 + 2AB + B^2$ (b) $A^2 + AB + BA + B^2$

(c) $(A+B)A + (A+B)B$ (d) $(B+A)^2$ **[H.C.U.-2011]**

Soln: $(A+B)^2 = (A+B)(A+B)$

$$= A^2 + AB + BA + B^2$$

$$= (A+B)\cdot A + (A+B)B = (B+A)^2$$

Hence, option (a) is not equal to $(A+B)$

Correct option is (a)

Question 4. What should be the value of α so that the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & -\alpha \\ 0 & \alpha & \alpha \end{pmatrix}$ can become orthogonal ?

(a) 1
(b) $\frac{1}{2}$
(c) $\frac{1}{\sqrt{2}}$
(d) $\sqrt{2}$
[H.C.U.-2011]

Soln: If a matrix is orthogonal, then it's each row or column is a vector having unit magnitude.
Let us consider vector associated with second column.

$$\vec{c}_2 = 0\hat{i} + \alpha\hat{j} + \alpha\hat{k}$$

$$\therefore\ |\vec{c}_2| = \sqrt{d^2+\alpha^2} = 1 \Rightarrow \alpha = \frac{1}{\sqrt{2}}$$

Correct option is (d)

Question 5. The eigenvalues of the matrix $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ are given by

(a) 1, 0, 1
(b) 0, 1, 2
(c) 1, 2, 1
(d) 1, 1, 1
[H.C.U.-2011]

Soln: $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

Sum of eigenvalues = sum of principle diagonal element.
= 3.

Product of eigenvalues = det (A)

$$= 1\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - 0\begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

$$= 1 - 1 = 0$$

Therefore, option (b) satisfies these both condition.

Correct option is (b)

Question 6. Which one of the following statements is true for the matrix given below ?

$$A = \begin{pmatrix} 0 & 1 \\ -1 & i \end{pmatrix}$$

[H.C.U.-2012]

(a) A is symmetric
(b) A is hermitian
(c) A is anti-symmetric
(d) A is skew-Hermitian

Soln: $A = \begin{pmatrix} 0 & 1 \\ -1 & i \end{pmatrix}$

$\because\ (A^T)^C = -A$, Hence A is Skew Hermitian.

Correct option is (d)

Question 7. Which of the following cannot possibly be the eigenvalues of a real 3 × 3 matrix?

(a) 1, 2, –1 (b) 1, 1+i, 1–i (c) 1, 0, –1 (d) –1, 1+i, –1+i **[H.C.U.-2012]**

Soln: For a real matrix, sum of eigenvalues must be real. Hence option (d) cannot be eigenvalues.

Correct option is (d)

Question 8. The system of linear equations

$$\begin{pmatrix} 4 & 9 & 3 \\ 2 & 3 & 1 \\ 2 & 6 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 7 \end{pmatrix}$$

(a) has infinite number of solutions (b) has only trivial solution

(c) has a unique solution (d) is inconsistent and has no solution **[H.C.U.-2013]**

Soln: $D = \begin{pmatrix} 4 & 9 & 3 \\ 2 & 3 & 1 \\ 2 & 6 & 1 \end{pmatrix} \Rightarrow |D| = \begin{vmatrix} 4 & 9 & 3 \\ 2 & 3 & 1 \\ 2 & 6 & 1 \end{vmatrix}$

$$= 4(3-6) - 9(2-2) + 3(12-6)$$

$$= -12 - 0 + 18 = 6 \neq 0$$

Therefore, given sytem of linear equation will have unique solution.

Correct option is (c)

Question 9. Which of the following are eigenvalues of the matrix $\begin{pmatrix} 2 & i \\ -i & 2 \end{pmatrix}$

(a) 1, 0 (b) $i, -i$ (c) $1+i, 1-i$ (d) 1, 3 **[H.C.U.-2014]**

Soln: Sum of eigenvalues = sum of principle diagonal element.

Correct option is (d)

Question 10. What value of α will make the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & -\alpha \\ 0 & \alpha & \alpha \end{pmatrix}$ orthogonal

(a) $\sqrt{2}$ (b) 1 (c) $\frac{1}{\sqrt{2}}$ (d) $\frac{1}{2}$ **[H.C.U.-2015]**

Soln: If a matrix is orthogonal, then it's each row or column is a vector having unit magnitude.

Let us consider vector associated with second column.

$$\vec{c}_2 = 0\hat{i} + \alpha\hat{j} + \alpha\hat{k}$$

$$\therefore\ |\vec{c}_2| = \sqrt{d^2 + \alpha^2} = 1 \Rightarrow \alpha = \frac{1}{\sqrt{2}}$$

Correct option is (d)

Question 11. If $abc = P$ and $A = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}$, where A is an orthogonal matrix, then the value of $a + b + c$ is

(a) 2 (b) P (c) 2P (d) ±1 **[H.C.U.-2015]**

Soln: $A = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}$ is an orthogonal matrix.

Hence each row or column is a unit vector and every rows and columns are orthogonal to each.

$\therefore \;\; \vec{R}_1 = a\hat{i} + c\hat{j} + b\hat{k}$ and $\vec{R}_2 = b\hat{i} + a\hat{j} + c\hat{k}$

$\therefore \;\; \sqrt{a^2 + b^2 + c^2} = 1$ and $\vec{R}_1 \cdot \vec{R}_2 = 0$ or $a^2 + b^2 + c^2 = 1$ ... (1)

$ab + ac + bc = 0$

$ab + bc + ca = 0$ ... (2)

Now, $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca = 1 \neq 0$

$\therefore \;\; a + b + c = \pm 1.$

Correct option is (d)

Question 12. The eigenvalues of the matrix, $A = \begin{pmatrix} 5 & -2 \\ 0 & 0 \end{pmatrix}$ are

(a) 0, 0 (b) 5, 0 (c) 0, 6 (d) 5, 6 **[H.C.U.-2016]**

Soln: Sum of eigenvalues = sum of principle diagonal element.

Correct option is (b)

Question 13. The determinant of the matrix $A = \begin{pmatrix} e^{i\pi} & -1 \\ 1 & 1 \end{pmatrix}$ is

(a) 1 (b) π (c) $i\pi$ (d) 0 **[H.C.U.-2016]**

Soln: $A = \begin{pmatrix} e^{i\pi} & -1 \\ 1 & 1 \end{pmatrix}$

$A = \begin{vmatrix} e^{i\pi} & -1 \\ 1 & 1 \end{vmatrix} = \begin{vmatrix} -1 & -1 \\ 1 & 1 \end{vmatrix} = 0$

Correct option is (d)

Question 14. The eigenvalues of the matrix $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ are **[H.C.U.-2017]**

(a) $e^{\pm i\theta}$ (b) $e^{\pm 2i\theta}$ (c) $e^{\pm 3i\theta}$ (d) $e^{\pm i\theta/2}$

Soln: If eigenvalues are $e^{\pm i\theta}$, i.e. $\lambda_1 = e^{i\theta}$ and $\lambda_2 = e^{-i\theta}$

$\therefore \;\; \lambda_1 + \lambda_2 = e^{i\theta} + e^{-i\theta} = 2\cos\theta =$ sum of principle diagonal element.

Correct option is (a)

Question 15. The eigenvectors of the matrix $A=\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}$ are **[H.C.U.-2017]**

(a) $a_1=\begin{bmatrix}1\\0\end{bmatrix},\quad a_2=\begin{bmatrix}-1\\0\end{bmatrix}$ (b) $a_1=\begin{bmatrix}1\\1\end{bmatrix},\quad a_2=\begin{bmatrix}1\\-1\end{bmatrix}$

(c) $a_1=\begin{bmatrix}2\\0\end{bmatrix},\quad a_2=\begin{bmatrix}0\\2\end{bmatrix}$ (d) $a_1=\begin{bmatrix}1\\0\end{bmatrix},\quad a_2=\begin{bmatrix}0\\1\end{bmatrix}$

Soln: $A=\begin{pmatrix}2 & -1\\ -1 & 2\end{pmatrix}$. Therefore, characteristic equation, $|A-\lambda I|=0 \Rightarrow \begin{vmatrix}2-\lambda & -1\\ -1 & 2-\lambda\end{vmatrix}=0 \Rightarrow (2-\lambda)^2-1=0$

$4+\lambda^2-4\lambda-1=0 \Rightarrow \lambda^2-4\lambda+3=0$

$\therefore\ \lambda=3,1$

$\therefore$ eigenvector corresponding to $\lambda=3$

$\Rightarrow \begin{pmatrix}2-3 & -1\\ -1 & 2-3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=0 \Rightarrow -x-y=0\,;\ x=-y$

Hence, if $x=1,\ y=-1$. Therefore, eigenvector $\begin{pmatrix}1\\-1\end{pmatrix}$

Eigenvector corresponding to $\lambda=1$.

$\begin{pmatrix}2-1 & -1\\ -1 & 2-1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} \Rightarrow x-y=0\,;\ x=y$

Hence, if $x=1,\ y=1$. Therefore, eigenvector $\begin{pmatrix}1\\1\end{pmatrix}$

Correct option is (b)

BHU PREVIOUS YEAR QUESTIONS & SOLUTINS

Question 1. If a matrix is $A=\begin{pmatrix}1 & 3\\ 2 & 1\end{pmatrix}$, then A^{-1} will be **[B.H.U-2010]**

(a) $-\frac{1}{5}\begin{pmatrix}1 & -3\\ -2 & 1\end{pmatrix}$ (b) $\frac{1}{5}\begin{pmatrix}1 & 3\\ 2 & 1\end{pmatrix}$ (c) $\frac{1}{4}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$ (d) $-\frac{1}{3}\begin{pmatrix}-1 & 3\\ 2 & -1\end{pmatrix}$

Soln: $A=\begin{pmatrix}1 & 3\\ 2 & 1\end{pmatrix}$; It's characteristic equation, $|A-\lambda I|=0 \Rightarrow \begin{vmatrix}1-\lambda & 3\\ 2 & 1-\lambda\end{vmatrix}=0$

$\Rightarrow 1+\lambda^2-2\lambda-6=0 \Rightarrow \lambda^2-2\lambda-5=0$

Hence, from Caley Hamilton theorem, $A^2-2A-5=0$,

$A^2A^{-1}-2AA^{-1}-5A^{-1}=0\ ;\ A-2I=5A^{-1}$

$$A^{-1}=\frac{1}{5}\begin{pmatrix}1 & 3\\ 2 & 1\end{pmatrix}-\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}=\frac{1}{5}\begin{pmatrix}-1 & 3\\ 2 & -1\end{pmatrix}.$$

Correct option is (a)

Question 2. If a matrix is $A=\begin{pmatrix}3 & 1\\ 2 & 2\end{pmatrix}$, its eigenvalues are **[B.H.U-2010]**

(a) 1 and 2 (b) 2 and 4 (c) 1 and 3 (d) 1 and 4

Soln: Sum of eigenvalues = trace of $A = 5$. Therefore, only option (d) is satisfying this one.
Correct option is (d)

Question 3. The matrix **[B.H.U-2011]**

$$A=\begin{bmatrix}\cos\theta & i\sin\theta\\ i\sin\theta & \cos\theta\end{bmatrix} \text{ is}$$

(a) Hermitian (b) unitary (c) orthogonal (d) symmetric

Soln: $A=\begin{bmatrix}\cos\theta & i\sin\theta\\ i\sin\theta & \cos\theta\end{bmatrix}$

Vector corresponding to 1st and 2nd rows are $\vec{R}_1=\cos\theta\,\hat{i}+(i\sin\theta)\,\hat{j}$ and $\vec{R}_2=(i\sin\theta)\,\hat{i}+\cos\theta\,\hat{j}$.

Clearly, $|\vec{R}_1|=|\vec{R}_1|=1$ and $\vec{R}_1\cdot\vec{R}_2=0$.
Hence, unitary.
Correct option is (c)

Question 4. Indicate the statement which is not true **[B.H.U-2011]**

(a) the diagonalizing matrix of a unitary matrix is Hermitian
(b) the diagonalizing matrix of a real symmetric matrix is orthogonal
(c) the non-zero elements of a diagonal matrix are its eigenvalues
(d) the diagonalizing matrix of a Hermitian matrix is unitary

Soln: **Correct option is (a)**

Question 5. A matrix $C=\begin{pmatrix}5 & -1\\ 1 & 3\end{pmatrix}$ has **[B.H.U-2012]**

(a) no independent eigenvectors (b) 1 independent eigenvector
(c) 2 independent eigenvectors (d) 3 independent eigenvector

Soln: $C=\begin{pmatrix}5 & -1\\ 1 & 3\end{pmatrix}$; Eigenvalue equation is $\begin{vmatrix}5-\lambda & -1\\ 1 & 3-\lambda\end{vmatrix}=0\Rightarrow(5-\lambda)(3-\lambda)+1=0$

$15-8\lambda+\lambda^2+1=0\Rightarrow\lambda^2-8\lambda+16=0\Rightarrow\lambda=4,4.$

Since, eigenvalues are same, hence there would be only one independent eigenvectors.
Correct option is (b)

Question 6. The eigenvalues of the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ are **[B.H.U-2012]**

(a) 1, 0 (b) 1, 1 (c) 1, 2 (d) 0, 2

Soln: Trace A = sum of eigenvalues.

Only option (b) is satisfying this.

Correct option is (b)

Question 7. Eigenvalues of the matrix $\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$ are **[B.H.U-2013]**

(a) 1, 0, −1 (b) 1, 1, 0 (c) 2, 0, −2 (d) 2, 0, 2

Soln: Trace A = sum of eigenvalues. No options are satisfying this.

Question 8. Which of the following vectors is not an eigenvector of the matrix $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$? **[B.H.U-2013]**

(a) $\vec{r} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$ (b) $\vec{r} = (0, 0, 1)$

(c) $\vec{r} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ (d) $\vec{r} = (1, 0, -1)$

Soln: $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$; It's eigenvalue $\begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = 0$

$\Rightarrow -\lambda\left(\lambda^2\right) - 1\left(-\lambda\right) = 0 \Rightarrow -\lambda^3 + \lambda = 0 \Rightarrow \lambda\left(1 - \lambda^2\right) = 0 \Rightarrow \lambda = 0, 1, -1.$

Eigenvector corresponding to $\lambda = 0$

$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow \boxed{y = 0}\ \boxed{x = 0}$, z can be anything.

Therefore, option (b) is related to this.

Eigenvector corresponding to $\lambda = 1$.

$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$

$-x + y = 0 \Rightarrow \boxed{x = y}$; $x - y = 0$; $-z = 0 \Rightarrow \boxed{z = 0}$

Therefore, (a) is related to this.

Eigenvector corresponding to $\lambda = -1$.

$$\begin{pmatrix}1 & 1 & 0\\ 1 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=0\ ;\quad x+y=0\ \boxed{x=-y}\ \boxed{z=0}$$

Therefore, option (c) is related to this.

Correct option is (d)

Question 9. For any square matrix A, which of the following matrices is not Hermitian ?**[B.H.U-2013]**

(a) $A+A^{+}$ (b) AA^{+} (c) $A^{+}A$ (d) $A^{+}-A$

Soln: Check option (d) ; $\left(A^{+}-A\right)^{+}=\left(A^{+}\right)^{+}-A^{+}=A-A^{+}=-\left(A^{+}-A\right)$

Correct option is (d)

Question 10. A 2×2 matrix has determinant 1 and trace 2. Its eigen values are :

(a) ±1 (b) 0, 1 (c) 1, 1 (d) 0, 2 **[B.H.U-2014]**

Soln: Let eigenvalues are λ_1 and λ_2, then $\operatorname{tr}(A)=\lambda_1+\lambda_2 \Rightarrow \lambda_1+\lambda_2=2$ ... (1)

and $\det(A)=\lambda_1\lambda_2 \Rightarrow \lambda_1\lambda_2=1$

Correct option is (c)

Question 11. Eigenvalues of $\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 1\end{pmatrix}$ are **[B.H.U-2015]**

(a) −1, 0, 2 (b) 0, 1, 2 (c) −1, 1, 2 (d) −3, 1, 5

Soln: Sum of eigenvalues = trace A and product of eigenvalues = det (A)

Therefore, option (b) satisfies these two conditions.

Correct option is (b)

Question 12. If A is 3×3 matrix with Tr(A) = 3, det A = 0 and one of the eigenvalue 1, the other two eigenvalues are

(a) 2 and 0 (b) 1 and 1 (c) 0 and 1 (d) 3 and −1 **[B.H.U-2016]**

Soln: Sum of eigenvalues = trace A

Product of eigenvalues = det A

Therefore, option (a) satisfies these two conditions.

Correct option is (a)

Question 13. If A is a skew-symmetric matrix and R is its rank, then

(a) $R=1$ (b) $R\le 1$ (c) $R>1$ (d) $R\ge 1$ **[B.H.U-2016]**

Soln: Rank of a matrix is always greater than one.

Correct option is (c)

Question 15. If I and O are (2×2) identity and null matrices in $I=pA+qA^2=0$ then, p and q values for the matrix. $A=\begin{pmatrix}1 & 2\\ -2 & 1\end{pmatrix}$ are one of the following: **[B.H.U-2016]**

(a) $p=\frac{2}{5},q=\frac{1}{5}$ (b) $p=-\frac{2}{5},q=\frac{1}{5}$ (c) $p=\frac{2}{5},q=-\frac{1}{5}$ (d) $p=-\frac{2}{5},q=-\frac{1}{5}$

Soln: $A=\begin{pmatrix}1 & 2\\ -2 & 1\end{pmatrix}$; $I+pA+qA^2=0$

$$I+p\begin{pmatrix}1 & 2\\ -2 & 1\end{pmatrix}+q\begin{pmatrix}1 & 2\\ -2 & 1\end{pmatrix}\begin{pmatrix}1 & 2\\ -2 & 1\end{pmatrix}=0$$

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+p\begin{pmatrix}1 & 2\\ -2 & 1\end{pmatrix}+q\begin{pmatrix}-3 & 4\\ -4 & -3\end{pmatrix}=0$$

$$\begin{pmatrix}p-3q+1 & 2p+4q\\ -2p-4q & p-3q+1\end{pmatrix}=0$$

$p-3q+1=0 \quad \dots(1)$

$2p+4p=0 \quad \dots(2)$

From Eq. (1) and Eq. (2) ; $-2q-3q+1=0$

$\Rightarrow q=\frac{1}{5}\Rightarrow p=-\frac{2}{5}$.

Correct option is (b)

Question 16. The eigenvectors of the matrix $A=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$ are as follows: **[B.H.U-2016]**

(a) $\begin{pmatrix}1\\ i\end{pmatrix}$ and $\begin{pmatrix}1\\ -i\end{pmatrix}$ (b) $\begin{pmatrix}1\\ i\end{pmatrix}$ and $\begin{pmatrix}-1\\ i\end{pmatrix}$ (c) $\begin{pmatrix}i\\ 1\end{pmatrix}$ and $\begin{pmatrix}-i\\ 1\end{pmatrix}$ (d) $\begin{pmatrix}i\\ 1\end{pmatrix}$ and $\begin{pmatrix}i\\ -1\end{pmatrix}$

Soln: $A=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$; Eigenvalue equation is $|A-\lambda I|=0$

$$\begin{vmatrix}\cos\theta-\lambda & -\sin\theta\\ \sin\theta & \cos\theta-\lambda\end{vmatrix}=0$$

$$\cos^2\theta+\lambda^2-2\cos\theta\lambda+\sin^2\theta=0$$

$$\lambda^2-2\cos\theta\lambda+1=0$$

$$\lambda=\frac{2\cos\theta\pm\sqrt{4\cos^2\theta-4}}{2}=\frac{2\cos\theta\pm 2i\sin\theta}{2}$$

$$\lambda=\cos\theta\pm i\sin\theta\Rightarrow\lambda=e^{i\theta},e^{-i\theta}$$

Eigenvector corresponding to $\lambda=e^{i\theta}$.

$$(A-\lambda I)X=0\Rightarrow\begin{pmatrix}\cos\theta-e^{i\theta} & \sin\theta\\ -\sin\theta & \cos\theta-e^{i\theta}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=0\Rightarrow\begin{pmatrix}-i\sin\theta & \sin\theta\\ -\sin\theta & -i\sin\theta\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=0$$

$-ix+y=0$ $\boxed{x=-iy}$, if $y=1, x=-i$

Therefore, $X_{\lambda=e^{i\theta}}=\begin{pmatrix}-i\\1\end{pmatrix}$

Eigenvector corresponding to $\lambda=e^{i\theta}$; $(A-\lambda I)X=0$

$$\begin{pmatrix}\cos\theta-e^{-i\theta} & \sin\theta\\ -\sin\theta & \cos\theta-e^{-i\theta}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=0$$

$$\begin{pmatrix}i\sin\theta & \sin\theta\\ -\sin\theta & i\sin\theta\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=0$$

$ix+y=0$ $\boxed{x=iy}$; if $y=1$, $x=i$

$$X_{\lambda=e^{-i\theta}}=\begin{pmatrix}i\\1\end{pmatrix}$$

Hence, two eigenvectors are $\begin{pmatrix}-i\\1\end{pmatrix}$ and $\begin{pmatrix}i\\1\end{pmatrix}$.

Correct option is (c)

Question 17. Transpose conjugate of two matrices A and B i.e., $(AB)^+$ equals **[B.H.U-2017]**

(a) A^+B^+ (b) B^+A^+ (c) $B'A'$ (d) AB

Soln: $(AB)^+=B^+A^+$

Correct option is (b)

Question 18. The product of a singular matrix with its adjoint gives **[B.H.U-2017]**

(a) a unitary matrix (b) a null matrix (c) a diagonal matrix (d) None of these

Soln: We have $A(\text{adj}\,A)=|A|I$. Since, A is a singular matrix, $|A|=0$

Therefore, $A(\text{adj}\,A)=0$ null matrix.

Correct option is (b)

Chapter 3

DIFFERENTIAL EQUATIONS

3.1 Basic Review of Differential Equation

Definition:

An equation which involves differential co-efficient is called differential equation.

Example: $\frac{dy}{dx}=\frac{1-x^2}{1+y^2}$; $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+6y=0$

All differential equations can be classified into two types:

(i) Ordinary differential equation: This consists of derivatives w.r.t. a single independent variable.

(ii) Partial differential equation: This consists of partial derivatives w.r.t. more than one independent variable.

Order and Degree of a differential equation:

The order of the highest order derivative present in the given equation is known as order of the differential equation and the power of the highest order derivative term present in the given equation is known as degree of differential equation.

Example: $L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{c}=E\sin\omega t \Rightarrow$ order 2, degree 1

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3=\left(\frac{d^2y}{dx^2}\right)^2 \quad \Rightarrow \text{ order 2, degree 2}$$

Linear Differential Equation:

A differential equation is said to be linear if (i) every dependent variable and its derivative occurs in first degree only and (ii) no products of dependent variables and its derivatives occur.

Example: $\frac{dy}{dx}=(x+\sin x)$, $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+7y=0$

Family of Curves:

An n-parameter family of curves is a set of relations of the form

$$\{(x,y): f(x,y,c_1,c_2.........c_n)\}=0$$

where 'f' is a real valued function of $x, y, c_1, c_2.........c_n$ and each c_i ranges over an interval of real values.

Example: The set of concentric circles is defined by $x^2+y^2=c$ is a one parameter family if c takes all positive real values.

There will be n arbitary constants in n-parameter family of curves. Then, we can obtain n^{th} order differential equation whose solution is the given family.

Example 1: The differential equation of all circles passing through the origin and having their centres on the y-axis

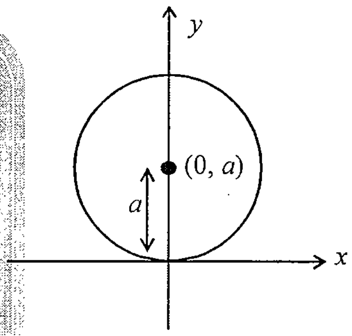
(a) $\left(x^2+y^2\right)dy/dx=2xy$ (b) $\left(x^2-y^2\right)dy/dx=2xy$

(c) $dy/dx=2xy\left(x^2+y^2\right)$ (d) $dy/dx=2xy\left(x^2-y^2\right)$

Soln: Equation of the circle passing through the origin and having their centers on the y-axis will be of the form

$$x^2+(y-a)^2=a^2$$

$$\Rightarrow 2x+2(y-a)\frac{dy}{dx}=0 \Rightarrow (y-a)=-\frac{x}{(dy/dx)} \Rightarrow a=y+\frac{x}{(dy/dx)}$$



Therefore, $x^2+\dfrac{x^2}{(dy/dx)^2}=\left(y+\dfrac{x}{(dy/dx)}\right)^2$

$$\Rightarrow x^2=y^2+\frac{2xy}{(dy/dx)} \Rightarrow \left(x^2-y^2\right)\frac{dy}{dx}=2xy$$

Correct option is (b)

Example 2: The function $y=\ln\left[sin(x+a)\right]+b$ is the solution of the following differential equation:

(a) $y''=-\left[1+\left(y'\right)^2\right]$ (b) $y''=y^2-\left(y'\right)^2$ (c) $y''=\left[1+\left(y'\right)^2\right]$ (d) $y''=y'+y^2$

Soln: $y=\ln\left[\sin(x+a)\right]+b \Rightarrow \dfrac{dy}{dx}=\cot(x+a) \Rightarrow \dfrac{d^2y}{dx^2}=-\operatorname{cosec}^2(x+a)=-\left[1+\cot^2(x+a)\right]$

Therefore, $\dfrac{d^2y}{dx^2}=-\left[1+\left(\dfrac{dy}{dx}\right)^2\right]$

Correct option is (a)

3.2 Differential Equations of the First Order and First Degree:

CASE I: Separation of Variables

Every Differential equation can be written in the form,

$$f(y)dy=\phi(x)dx \Rightarrow \int f(y)dy=\int \phi(x)dx+C$$

Example 3: Solve $e^{dy/dx} = (x+1)$; given $y = 3\ at\ x = 0$

Soln: Taking log of both sides we get, $\dfrac{dy}{dx} = \ln(x+1) \Rightarrow dy = \ln(x+1)\,dx$

on integration, $\int dy = \int 1.\ln(x+1)dx$

$$\Rightarrow \quad y = x\ln(x+1) - \int \frac{x}{(x+1)}dx + C \quad \Rightarrow \quad y = x\ln(x+1) - \int \frac{(x+1)-1}{(x+1)}dx + C$$

$$\Rightarrow \quad y = x\ln(x+1) - \int \frac{(x+1)}{(x+1)}dx + \int \frac{1}{(x+1)}dx + C \quad = x\ln(x+1) - x + \ln(x+1) + C$$

$$\Rightarrow \quad y = (x+1)\ln(x+1) - x + C$$

Given: at $x = 0 \quad y = 3 \quad \Rightarrow \quad C = 3$

Therefore, $y = (x+1)\ln(x+1) - x + 3$

Example 4: Solve $\dfrac{dy}{dx} = e^{y}\left(e^{x} + x^{2}e^{x^{3}}\right)$

Soln: $\dfrac{dy}{dx} = e^{y}\left(e^{x} + x^{2}e^{x^{3}}\right) \quad \Rightarrow \int e^{-y}dy = \int\left(e^{x} + x^{2}e^{x^{3}}\right)dx \quad \Rightarrow -e^{-y} = e^{x} + (1/3)e^{x^{3}} + C$

Example 5: Solve $(x+y)(dx - dy) = dx + dy$

Soln: Re-writting the given equation, we get $(x+y-1)dx = (x+y+1)dy \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{x+y-1}{x+y+1}$

Let $x + y = v \Rightarrow \dfrac{dy}{dx} = (dv/dx) - 1$

Therfore the given equation becomes,

$$\frac{dv}{dx} - 1 = \frac{v-1}{v+1} \quad \Rightarrow \quad \frac{dv}{dx} = \frac{2v}{v+1} \quad \Rightarrow \int 2dx = \int\left(1 + \frac{1}{v}\right)dv$$

$$\Rightarrow 2x + c = v + \ln v \quad \Rightarrow -y + c = \ln(x+y)$$

Example 6: Solve $\dfrac{dy}{dx} = \dfrac{4x+6y+5}{3y+2x+4}$

Soln: The given equation may be re-written as $\dfrac{dy}{dx} = \dfrac{2(2x+3y)+5}{(2x+3y)+4}$

Taking $2x + 3y = v \Rightarrow 2 + 3\dfrac{dy}{dx} = \dfrac{dv}{dx} \quad \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{3}\left(\dfrac{dv}{dx} - 2\right)$

Therefore, the given equation becomes,

$$\frac{1}{3}\left(\frac{dv}{dx} - 2\right) = \frac{2v+5}{v+4} \quad \Rightarrow \quad \frac{dv}{dx} = \frac{3(2v+5)}{v+4} + 2 = \frac{8v+23}{v+4}$$

$$\Rightarrow \frac{dx}{dv}=\frac{v+4}{8v+23}=\frac{\frac{1}{8}(8v+23)+\left(4-\frac{23}{8}\right)}{8v+23}=\left[\frac{1}{8}+\frac{9}{8(8v+23)}\right] \Rightarrow \int dx=\int\left[\frac{1}{8}+\frac{9}{8(8v+23)}\right]dv$$

$$\Rightarrow x+c=\left(\frac{v}{8}\right)+\left(\frac{9}{64}\right)\log(16x+24y+23) \quad \Rightarrow 3y-6x+\left(\frac{9}{8}\right)\log(16x+24y+23)=8c$$

$\Rightarrow y-2x+\left(\frac{3}{8}\right)\log(16x+24y+23)=c'$ where $c'(=8c/3)$ is an arbitrary constant.

Example 7: Solution of the differential equation $\frac{dy}{dx}=\sin(x+y)+\cos(x+y)$ is

(a) $x+c=\log\left[1+\tan\left(\frac{x+y}{2}\right)\right]$

(b) $x+c=\log\left[1-\tan\left(\frac{x+y}{2}\right)\right]$

(c) $x+c=\log\left[1+\cot\left(\frac{x+y}{2}\right)\right]$

(d) $x+c=\log\left[1-\cot\left(\frac{x+y}{2}\right)\right]$

Soln: Put $x+y=z \Rightarrow 1+\frac{dy}{dx}=\frac{dz}{dx} \Rightarrow \frac{dy}{dx}=\frac{dz}{dx}-1$

Hence, given differential equation can be written as,

$$\frac{dz}{dx}-1=\sin z+\cos z \qquad \Rightarrow \int\frac{dz}{1+\sin z+\cos z}=\int dx$$

$$\Rightarrow \int\frac{dz}{1+\frac{2\tan\frac{z}{2}}{1+\tan^2\frac{z}{2}}+\frac{1-\tan^2\frac{z}{2}}{1+\tan^2\frac{z}{2}}}=x+c$$

$$\Rightarrow \int\frac{\sec^2\frac{z}{2}dz}{1+\tan^2\frac{z}{2}+2\tan\frac{z}{2}+1-\tan^2\frac{z}{2}}=x+c \Rightarrow \int\frac{\sec^2\frac{z}{2}dz}{1+2\tan\frac{z}{2}}=x+c$$

Put $1+2\tan\frac{z}{2}=t \Rightarrow 2\times\sec^2\frac{z}{2}\frac{dz}{2}=dt$

$$\therefore \quad \int\frac{dt}{t}=x+c \Rightarrow \log(t)=x+c \Rightarrow x+c=\log\left[1+2\tan\left(\frac{x+y}{2}\right)\right].$$

Correct option is (a).

Example 8: Which of the following statement(s) about the solution $y(x)$ of the differential equation $\frac{1}{x^2-3}\frac{dy}{dz}=1$ is correct (with $y(0)=0$)

(a) $y(x)$ is given by $\frac{x^3}{3}-3x$

(b) $y(x)$ is given by $\frac{x^2}{2}-2x$

(c) $y(x)$ has extremum points at $\sqrt{3}$ and $-\sqrt{3}$

(d) $y(x)$ is increasing in the interval $\left(-\infty,-\sqrt{3}\right)\cup\left(\sqrt{3},\infty\right)$

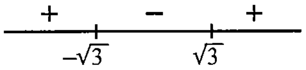
Soln: We have $\int dy=\int (x^2-3)\,dx \Rightarrow y=\frac{x^3}{3}-3x+c$

Since, $y(0)=0$, Put $x=0,\ y=0 \Rightarrow c=0$. Therefore, $y(x)=\frac{x^3}{3}-3x$

Now for extremum points, $y'(x)=0 \Rightarrow x^2-3=0,\ x=\pm\sqrt{3}$

Now using sign scheme for $y'(x)$,

We have



So, $y(x)$ is increasing in the interval $\left(-\infty,-\sqrt{3}\right)\cup\left(\sqrt{3},0\right)$

Correct options are (a), (c) and (d)

Example 9: The solution of the differential equation $\frac{dx}{dt}=x^2$ with the initial condition x(0) =1 will blow up as t tend to

(a) 1 (b) 2 (c) 1/2 (d) ∞

Soln: Given: $\frac{dx}{dt}=x^2 \Rightarrow \int\frac{dx}{x^2}=\int dt \Rightarrow -\frac{1}{x}=t+C$

Applying $x(0)=1$, we get $C=1 \Rightarrow -\frac{1}{x}=t-1 \Rightarrow x=\frac{1}{1-t}$

Therefore, the solution of the given differntial equation will blow up as t tends to 1.

Correct answer is (a)

Example 10: The solutions to the differential equation $\frac{dy}{dx}=-\frac{x}{y+1}$ are a family of

(a) Circles with different radii (b) Circles with different centres.

(c) Straight lines with different slopes (d) Straight lines with different intercepts on the y-axis

Soln: $\frac{dy}{dx}=-\frac{x}{y+1} \Rightarrow \int (y+1)\,dy=-\int x\,dx$

$$\Rightarrow \frac{y^2}{2}+y=-\frac{x^2}{2}+c$$

$$\Rightarrow x^2+y^2+2y=2c$$

$$\Rightarrow x^2+(y+1)^2=2c+1$$

This represents a family of circles having $(0,-1)$ and radius of $\sqrt{2c+1}$ units.

Correct option is (a)

Example 11: Consider the linear differential equation $\frac{dy}{dx} = xy$. If $y = 2$ at $x = 0$, then the value of y at $x =$ 2 is

(a) e^{-2} (b) $2e^{-2}$ (c) e^{2} (d) $2e^{2}$

Soln: We have, $\frac{dy}{dx} = xy \Rightarrow \frac{dy}{y} = xdx \Rightarrow \ln y = \frac{x^2}{2} + c$

Given, $x = 0, y = 2, \therefore c = \ln 2$

$$\Rightarrow \quad \ln y = \frac{4}{2} + \ln 2 \quad \Rightarrow \quad \ln\left(\frac{y}{2}\right) = 2$$

$$\Rightarrow \quad y = 2e^2$$

Correct option is (d)

CASE II: Homogeneous Differential equation:

A differential equation $\frac{dy}{dx} = \frac{f(x,y)}{\phi(x,y)}$ is said to homogeneous if $f(x, y)$ and $\phi(x, y)$ are of same degree.

In such equations, put $y = vx$ & $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Example 12: Solve $xdy - ydx = \sqrt{(x^2 + y^2)}dx$

Soln: $\frac{dy}{dx} = \frac{y + \sqrt{(x^2 + y^2)}}{x} = \frac{y}{x} + \sqrt{\left\{1 + \left(\frac{y}{x}\right)^2\right\}}$

Take, $\frac{y}{x} = v$, *i.e.* $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\Rightarrow v + x\frac{dv}{dx} = v + \sqrt{(1+v^2)} \Rightarrow \int\frac{dx}{x} = \int\frac{dv}{\sqrt{(1+v^2)}} \quad \Rightarrow \log x + \log c = \log\left[v + \sqrt{(v^2+1)}\right]$$

$$\Rightarrow xc = v + \sqrt{(v^2+1)} \quad \Rightarrow x^2c = y + \sqrt{(y^2 + x^2)}$$

Example 13: Solve $x\cos\left(\frac{y}{x}\right)(ydx + xdy) = y\sin\left(\frac{y}{x}\right)(xdy - ydx)$

Soln: $\frac{dy}{dx} = \frac{\left\{x\cos\left(\frac{y}{x}\right) + y\sin\left(\frac{y}{x}\right)\right\}y}{\left\{y\sin\left(\frac{y}{x}\right) - x\cos\left(\frac{y}{x}\right)\right\}x} \quad \Rightarrow \frac{dy}{dx} = \frac{\left[\cos\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)\sin\left(\frac{y}{x}\right)\right]\left(\frac{y}{x}\right)}{\left[\left(\frac{y}{x}\right)\sin\left(\frac{y}{x}\right) - \cos\left(\frac{y}{x}\right)\right]}$

Take, $\frac{y}{x} = v$ *i.e.* $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\Rightarrow v+x\frac{dv}{dx}=\frac{v(\cos v+v\sin v)}{v\sin v-\cos v} \quad \Rightarrow x\frac{dv}{dx}=\frac{v(\cos v+v\sin v)}{v\sin v-\cos v}-v=\frac{2v\cos v}{v\sin v-\cos v}$$

$$\Rightarrow \int 2\frac{dx}{x}=\int\frac{v\sin v-\cos v}{v\cos v}dv=\int\left[\frac{\sin v}{\cos v}-\frac{1}{v}\right]dv \Rightarrow 2\log x=-\log\cos v-\log v+\log c$$

$$\Rightarrow \log x^2=\log\left(c/v\cos v\right) \Rightarrow xy\cos\frac{y}{x}=c$$

Example 14: Solve $\frac{dy}{dx}=\frac{x+2y-3}{2x+y-3}$

Soln: Take $x=X+h, y=Y+k$, so that $\frac{dy}{dx}=\frac{dY}{dX}$ ... (1)

Therefore, given equation becomes $\frac{dy}{dx}=\frac{X+2Y+(h+2k-3)}{2X+Y+3+(2h+k-3)}$ ... (2)

Choose h, k so that $h+2k-3=0$ and $2h+k-3=0$ ... (3)
Solving (3) we get h = 1, k = 1 so that from (1), we have $X=x-1, Y=y-1$

Using (3) in (2), we get $\frac{dY}{dX}=\frac{X+2Y}{2X+Y}=\frac{1+(2Y/X)}{2+(Y/X)}$

Take $\frac{Y}{X}=v, i.e. Y=vX$. $\Rightarrow \frac{dY}{dX}=v+X\frac{dv}{dX}$

$$\Rightarrow v+X\frac{dv}{dX}=\frac{1+2v}{2+v} \Rightarrow \mathrm{X}\frac{dv}{dX}=\frac{1+2v}{2+v}-v=\frac{1-v^2}{2+v}$$

$$\Rightarrow \int\frac{dX}{X}=\int\frac{(2+v)\,dv}{(1-v)(1+v)}=\int\left[\frac{1}{2}\left(\frac{1}{1+v}\right)+\frac{3}{2}\left(\frac{1}{1-v}\right)\right]dv$$ [Resolving into partial fractions]

$$\Rightarrow \log X+\log c=\left(\frac{1}{2}\right)\left[\log(1+v)-3\log(1-v)\right] \Rightarrow 2\log(cX)=\log\frac{1+v}{(1-v)^3} \Rightarrow \mathrm{X}^2c^2=\frac{1+v}{(1-v)^3}$$

$$\Rightarrow X^2c^2\left(1-\frac{Y}{X}\right)^3=1+\frac{Y}{X} \Rightarrow c^2(X-Y)^3=X+Y \quad \Rightarrow c^2\{x-y\}^2=x+y-2$$

Example 15: If $\frac{dy}{dx}=f\left(\frac{y}{x}\right)$, then the value of the integral $\int\frac{dv}{f(v)-v}$ is equal to (where $v=\frac{y}{x}$).

(a) $\log x+c$ (b) e^x+c (c) $\frac{y}{x}+c$ (d) None of these

(Here c is an integration constant)

Soln: Since, $v=\frac{y}{x}$ or $y=vx \Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$

Now, $\frac{dy}{dx}=f\left(\frac{y}{x}\right) \Rightarrow v+x\frac{dv}{dx}=f(v) \Rightarrow \int\frac{dv}{f(v)-v}=\int\frac{dx}{x}=\log x+c$

Correct option is (a)

CASE III: Linear differential equations of first order:

$$\frac{dy}{dx}+P(x)y=Q(x)$$

where P and Q are the functions of x only or constant.

For the solution of these equations we have to calculate integral factor $(I.F)=e^{\int P(x)dx}$ and the solution of differential equation will be of the form

$$I.F.\times y=\int Q(x)\times I.F.dx+C$$

Example 16: Solve $(x+1)\frac{dy}{dx}-y=e^x(x+1)^2$

Soln: The equation can be rewritten as $\frac{dy}{dx}-\frac{y}{x+1}=e^x(x+1)$

Compare with the standard form of first order linear differential equation we get,

$$P=-\frac{1}{x+1}\text{ and }Q=e^x(x+1)$$

Integrating factor, $I.F.=e^{\int Pdx}=e^{-\int\frac{dx}{x+1}}=e^{-\log(x+1)}=\frac{1}{x+1}$

Therefore, the solution : $y\frac{1}{x+1}=\int e^x(x+1)\frac{1}{(x+1)}dx=\int e^x dx\Rightarrow y=e^x(x+1)+C$

Example 17: Solve $(1-x^2)\left(\frac{dy}{dx}\right)+2xy=x\sqrt{(1-x^2)}$

Soln: The given equation is $\frac{dy}{dx}+\frac{2x}{1-x^2}y=\frac{x}{\sqrt{(1-x^2)}}$

I.F. $=e^{\int Pdx}=e^{\int\frac{2x}{1-x^2}dx}=e^{-\ln(1-x^2)}=\frac{1}{1-x^2}$.

So the required solution is

$$\frac{y}{1-x^2}=\int\frac{x}{\sqrt{(1-x^2)}}\times\frac{1}{1-x^2}dx=\int\frac{x}{(1-x^2)^{3/2}}dx$$

Putting $1-x^2=t$ and $-2xdx=dt$, we get

$$\Rightarrow\frac{y}{1-x^2}=-\frac{1}{2}\int t^{-3/2}dt=t^{-1/2}+c$$

The solution is $\frac{y}{1-x^2}=\frac{1}{\sqrt{(1-x^2)}}+c$

Example 18: Integrate $\left(1+x^2\right)\left(\frac{dy}{dx}\right)+2xy-4x^2=0$. Obtain equation of the curve satisfying this equation and passing through origin.

Soln: Re-writting the given equation, $\frac{dy}{dx}+\frac{2x}{1+x^2}y=\frac{4x^2}{1+x^2}$

Here, I.F. $=e^{\int Pdx}=e^{\int\frac{2x}{1+x^2}dx}=e^{\ln\left(1+x^2\right)}=\left(1+x^2\right)$

Hence the requied solution,

$$y\left(1+x^2\right)=\int\frac{4x^2}{1+x^2}\left(1+x^2\right)dx+c \;\Rightarrow\; y\left(1+x^2\right)=\left(\frac{4}{3}\right)x^3+c$$

Since the required curve passes through origin i.e. for $x = 0, y = 0 \Rightarrow c = 0$

Hence, the equation of the required curve is $4x^3=3y\left(1+x^2\right)$

Example 19: The particular solution of the differential equation $3xy'-y=\ln x+1,\ x>0$ with the condition $y(1)=-2$, is

(a) $y=2x^{1/3}-\ln x-4$ (b) $y=x^{1/3}+\ln x-4$

(c) $y=2x^{2/3}-\ln x-4$ (d) $y=x^{2/3}+\ln x-4$

Soln:
$$y'-\frac{1}{3x}y=\frac{\ln x+1}{3x}$$

Here, I.F.= $e^{\int-\frac{1}{3}x\,dx}=x^{-1/3}$

Therefore, the required solution will be

$$yx^{-1/3}=\int\frac{\ln x+1}{3x}\times x^{-1/3}dx+c=-x^{-1/3}(\ln x+1)-3x^{-1/3}dx+c$$

$\Rightarrow y=-(\ln x+4)+c\,x^{1/3}$

Since, $y(1)=-2$, putting $x=1,\ y=-2$, we have $c=2$

Therefore, $y=2x^{1/3}-(\ln x+4)$.

Correct option is (a)

Example 20: Solution of the differential equation $x^2\frac{dy}{dx}+2xy=y^3$ is

(a) $\frac{1}{5x}+cx^4$ (b) $-\frac{1}{5x}+cx^{-4}$ (c) $5x+cx^4$ (d) $5x-cx^{-4}$

Soln: $x^2\frac{dy}{dx}+2xy=y^3 \Rightarrow \frac{1}{y^3}\frac{dy}{dx}+\frac{2}{x}\times\frac{1}{y^2}=\frac{1}{x^2}$

Let $\frac{1}{y^2}=z \Rightarrow -\frac{2}{y^3}\frac{dy}{dx}=\frac{dz}{dx}$

Therefore, differential equation now becomes,

$$-\frac{1}{2}\frac{dz}{dx}+\frac{2}{x}z=\frac{1}{x^2} \Rightarrow \frac{dz}{dx}-\frac{4z}{x}=-\frac{2}{x^2}$$

Hence, I.F.$= e^{\int -\frac{4}{x}dx} = x^{-4}$

Therefore, the required solution will be

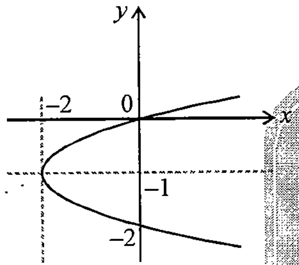
$$z \times x^{-4} = \int \frac{1}{x^2} x^{-4}\, dx + c$$

$$\Rightarrow \frac{1}{y^2} x^{-4} = -\frac{1}{5} x^{-5} + c \Rightarrow \frac{1}{y^2} = -\frac{1}{5x} + c x^4.$$

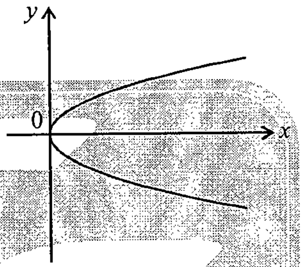
Correct option is (b)

Example 21: If $y(x)$ is the solution of the differential equation $y\frac{dx}{dy} - x = 2y^2,\ y(6) = 1$, then which of the following graph best represents $y(x)$.

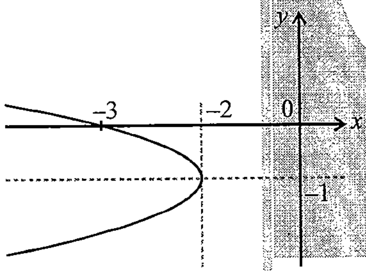
(a)



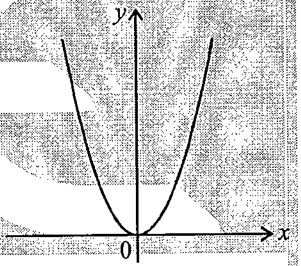
(b)



(c)



(d)



Soln: $y\frac{dx}{dy} - x = 2y^2 \Rightarrow \frac{dx}{dy} - \frac{1}{y}x = 2y$

Hence, I.F. $= e^{\int -\frac{1}{y}dy} = \frac{1}{y}.$

Therefore, the requires solution will be,

$$x \times \frac{1}{y} = \int 2y \times \frac{1}{y} dy + c \quad \Rightarrow x \times \frac{1}{y} = 2y + c \Rightarrow x = 2y^2 + c\,y$$

Since, $y(6) = 1$, put $x = 6,\ y = 1 \Rightarrow c = 4$

Therefore, $x = 2y^2 + 4y \Rightarrow x = 2(y^2 + 2y + 1) - 2$

$(y+1)^2 = \frac{1}{2}(x+2)$, it's a parabola with vertex at $x = -2,\ y = -1$ and symmetric about the axis $y = -1$.

Also, parabola cuts the y-axis (put $x = 0$) at $y = 0, -2$.

Correct option is (a)

Example 22: Which of the following statement(s) is/are correct for the differential equation $y' - y = 1 - e^{-x}$; $y(0) = y_0$?

(a) I.F. for this differential equation is e^{-x}

(b) I.F. for this differential equation e^{x}

(c) Value of y_0 is –1/2, if the given differential equation has infinite limit as $x \to \infty$

(d) Value of y_0 is –1/2, if the given differential equation has finite limit as $x \to \infty$

Soln: We have $y' - y = 1 - e^{-x}$

Here, I.F.$= e^{\int -dx} = e^{-x}$

Therefore, the required solution will be

$$\Rightarrow\ ye^{-x} = \int (1 - e^{-x}) e^{-x}\, dx + c\ = -e^{-x} + \frac{e^{-2x}}{2} + c$$

$$\because\ y(0) = y_0 \Rightarrow y_0 = -1 + \frac{1}{2} + c\ \ \boxed{c = y_0 + \frac{1}{2}}$$

$$\therefore\ ye^{-x} = -e^{-x} + \frac{1}{2}e^{-2x} + \left(y_0 + \frac{1}{2}\right)$$

$$y = -1 + \frac{e^{-x}}{2} + e^{x}\left(y_0 + \frac{1}{2}\right)$$

So, as $x \to \infty$, y will be finite, if $y_0 + \frac{1}{2} = 0 \Rightarrow y_0 = -\frac{1}{2}$.

Correct options are (a) and (c)

Example 23: If $y(x)$ is the solution of the differential equation $-x\frac{dy}{dx} + y = y^2 \log x$ with $y(1) = -1$, then

(a) $y(x)$ is defined and finite in the range $-\infty < x \le 0$

(b) $y(x)$ is defined and finite in the range $0 < x < 3$

(c) $y(x)$ is defined and finite in the range $x \ge 3$

(d) $y(x)$ blows up at $x = e$

Soln: $-x\frac{dy}{dx} + y = y^2 \log x \Rightarrow -\frac{dy}{dx} + \frac{y}{x} = y^2 \frac{\log x}{x} \Rightarrow -\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{x} \times \frac{1}{y} = \frac{\log}{x}$

Let $\frac{1}{y} = z \Rightarrow -\frac{1}{y^2}\frac{dy}{dx} = \frac{dz}{dx}$

Therefore, $\frac{dz}{dx} + \frac{z}{x} = \frac{\log x}{x} \Rightarrow \frac{dz}{dx} + \frac{1}{x}z = \frac{\log x}{x}$

IHere, I.F.$= e^{\int \frac{1}{x}dx} = e^{\log x} = x$

Therefore, required solution will be

$$zx=\int\frac{\log x}{x}\times x\,dx+c\ \Rightarrow\ \frac{x}{y}=\int\log x+c\ \Rightarrow\ \frac{x}{y}=(x\log x-x)+c$$

Now, $y(1)=-1$, put $x=1,\ y=-1\ \ \Rightarrow -1=0-1+c\ \Rightarrow\ \boxed{c=0}$

Therefore, $\frac{x}{y}=x\log x-x\ \Rightarrow\ y=\frac{1}{\log x-1}$

Therefore, for y to be defined $\log x$ must be defined i.e., $x>0$

Also the solution will not be defined when $\log x-1=0\Rightarrow\ \log x=1\ \Rightarrow\ \boxed{x=e\simeq 2.73}$

So in the range $0<x<3$ [option (b)], there will be a point $x=2.73$ at which y is not defined.

Correct options are (c) and (d)

Example 24: If $y(x)$ is solution for first order differential equation $xy'+(x+1)y=6$, then value of $y(x)$ as $x\to\infty$ is ___________.

Soln: $xy'+(x+1)y=6\Rightarrow y'+\frac{x+1}{x}y=\frac{6}{x}$

Here, $\text{I.F.}=e^{\int\frac{x+1}{x}dx}=e^{x+\log x}=xe^{x}$

Therefore, $yxe^{x}=\int\frac{6}{x}xe^{x}dx+c=6e^{x}+c\Rightarrow y=\frac{6}{x}+\frac{c}{xe^{x}}$

So, as $x\to\infty,\ y\to 0$

Correct answer is (0)

Example 25: Solution of the differential equation $x\frac{dy}{dx}+y=x^{4}$, with the boundary condition that $y=1$ at $x=1$, is

(a) $y=5x^{4}-4$ (b) $y=\frac{x^{4}}{5}+\frac{4x}{5}$ (c) $y=\frac{4x^{4}}{5}+\frac{1}{5x}$ (d) $y=\frac{x^{4}}{5}+\frac{4}{5x}$

Soln: $x\frac{dy}{dx}+y=x^{4}\ \ \Rightarrow\frac{dy}{dx}+\frac{y}{x}=x^{3}$

$I.F.=e^{\int dx/x}=e^{\ln x}=x$

Solution will be $xy=\int x^{3}x\,dx\ \ \Rightarrow xy=\frac{x^{5}}{5}+c$

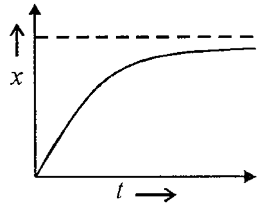
Putting the condition, at $x=1,\ y=1$, we get $c=\frac{4}{5}$

So, $xy=\frac{x^{5}}{5}+\frac{4}{5}\ \ \ \Rightarrow y=\frac{x^{4}}{5}+\frac{4}{5x}$

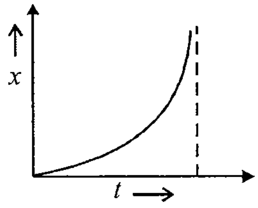
Correct option is (d)

Example 26: Which one of the following curves gives the solution of the differential equation $k_1\frac{dx}{dt}+k_2x=k_3$, where k_1, k_2 and k_3 are positive constant with initial conditions $x=0$ and $t=0$?

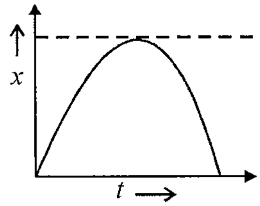
(a)



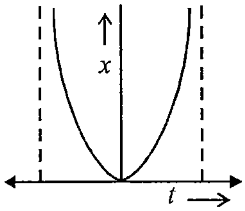
(b)



(c)



(d)



Soln: $k_1\frac{dx}{dt}+k_2x=k_3 \Rightarrow \frac{dx}{dt}+\frac{k_2}{k_1}x=\frac{k_3}{k_1}$

$I.F.=e^{\int\frac{k_2}{k_1}.dt}=e^{(k_2/k_1)dt}$

Therefore, the solution will be

$$x\cdot e^{(k_2/k_1)t}=\int\frac{k_3}{k_1}\exp\left(\frac{k_2}{k_1}t\right)dt$$

$$\Rightarrow \quad x\cdot e^{(k_2/k_1)t}=\frac{k_3}{k_2}\exp\left(\frac{k_2}{k_1}t\right)+c$$

$$\Rightarrow \quad x=\frac{k_3}{k_2}+c\cdot\exp\left(-\frac{k_2}{k_1}t\right)$$

Putting the condition, at $t=0, x=0$, we get $C=-\frac{k_3}{k_2}$

Therefore, $x=\frac{k_3}{k_2}\left[1-\exp\left(-\frac{k_2}{k_1}t\right)\right]$

Correct option is (a)

CASE IV: Exact Differential equations:

$$M(x,y)dx+N(x,y)dy=0$$

The above equation will be exact in nature if $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

Solution will be of the form:

$$\underset{\text{[Taking y in M as constant]}}{\int M dx} \quad + \quad \underset{\substack{\text{[Take only those terms in N}\\ \text{which do not contain x]}}}{\int N dy} \quad = C$$

Or

$$\underset{\substack{\text{[Take only those terms in N}\\ \text{which do not contain y]}}}{\int M dx} \quad + \quad \underset{\text{[Taking x in N as constant]}}{\int N dy} \quad = C$$

Example 27: Solve $\left(x^2+2xy\right)dx+\left(x^2+y^2\right)dy=0$

Soln: Here, $M=(x^2+2xy)$ and $N=(x^2+y^2) \Rightarrow \frac{\partial M}{\partial y}=2x$ and $\frac{\partial N}{\partial x}=2x$

Hence , the given equation is exact.

The solution is: $\int (x^2+2xy)dx+\int y^2 dy=C \Rightarrow \frac{x^3}{3}+x^2y+\frac{y^3}{3}=C$

Example 28: Solve $\{y(1+1/x)\cos y\}dx+(x+\log x-x\sin y)dy=0$

Soln: Here $M=y(1+1/x)+\cos y$, $N=x+\log x-x\sin y$ and $\frac{\partial M}{\partial y}=1+\frac{1}{x}-\sin y=\frac{\partial N}{\partial x}$

Solution is $\int (y+y/x+\cos y)dx+\int 0.dy=c \Rightarrow yx+y\log x+x\cos y=c$

Example 29: If $y(x)$ is the solution of first order differential equation $(e^{3y}-5)\,dx+3x\,e^{3y}\,dy=0$, $x>0$, then for $y(x)$ to be defined, x must be $[y(1)=0]$

(a) $x>4$ (b) $4 \le x \le 5$ (c) $x>\frac{4}{5}$ (d) $x>5$

Soln: $(e^{3y}-5)\,dx+3x\,e^{3y}\,dy=0$

Compare this with the form $M\,dx+N{\cdot}dy=0$

$M=e^{3y}-5 \Rightarrow \frac{\partial M}{\partial y}=3e^{3y}$ and $N=3x\,e^{3y} \Rightarrow \frac{\partial N}{\partial x}=3e^{3y}$

Since, $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$; Hence, given differential equation is exact.

Hence solution is $\underset{\substack{\text{Term of } M\\ \text{not contain-}\\ \text{ing } y}}{\int M\,dx}+\underset{x\,=\,\text{constant}}{\int N\,dy}=c$

$\int -5\,dx+\int 3x\,e^{3y}\,dy=c \Rightarrow x\,e^{3y}-5x=c$

Since, $y(1)=0$, put $x=1$, $y=0$

We get $c=-4$

Therefore, $x\,e^{3y}-5x=-4$

$$e^{3y} = \frac{-4+5x}{x} \Rightarrow y = \frac{1}{3}\ln\left(\frac{5x-4}{x}\right)$$

Thus, for $y(x)$ to be defined, $\frac{5x-4}{x} > 0 \Rightarrow x > \frac{4}{5}$.

Correct option is (c)

Example 30: Consider the differential equation $(3x^2+2xy^2)\,dx+(2x^2y)\,dy=0$, where $y(2)=-3$. Now, choose the incorrect statement(s) :

(a) Solution $y(x)$ is given by $\sqrt{\frac{44-x^3}{x^2}}$

(b) Solution $y(x)$ is given by $-\sqrt{\frac{44-x^3}{x^2}}$

(c) Line integral of $y(x)$ over the closed curve 'c' will not be zero (where $c: x^2+y^2=4$)

(d) Domain of x for which $y(x)$ is real is $R-\left\{-\sqrt[3]{44}, \sqrt[3]{44}\right\}-\{0\}$

Soln: We have $(3x^2+2xy^2)\,dx+(2x^2y)\,dy=0$

Comparing this with $M\,dx+N\,dy=0$

$$M = 3x^2+2xy^2 \Rightarrow \frac{\partial M}{\partial y} = 4xy \text{ and } N = 2x^2y \Rightarrow \frac{\partial N}{\partial x} = 4xy$$

Since, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$; Hence, it is an exact differential equation.

Hence, line integral of y over any closed path is zero.

Also, solution is given as $\underset{\substack{\text{Term is } M \\ \text{is not contain-} \\ \text{ing } y}}{\int 3x^2\,dx} + \underset{x=\text{constant}}{\int 2x^2y\,dy} = c \Rightarrow x^3+x^2y^2=c$

Since, $y(0)=-3$, put $x=2$, $y=-3$ $\boxed{c=44}$

Therefore, $x^3+x^2y^2=44 \Rightarrow y = \pm\sqrt{\frac{44-x^3}{x^2}}$

Since, $y(2)$ is negative.

Hence, $y(x) = -\sqrt{\frac{44-x^3}{x^2}}$.

Also, y is defined for $x \neq 0$ and $44-x^3 \geq 0$ or $x \neq 0$ and $x \in \left\{-\sqrt[3]{44}, \sqrt[3]{44}\right\}$.

Correct options are (a), (c) and (d)

List of Exact differentials:

(i) $d\left(\frac{y}{x}\right) = \frac{xdy-ydx}{x^2}$ (ii) $d\left(\frac{x}{y}\right) = \frac{ydx-xdy}{y^2}$

(iii) $d\left(\frac{y^2}{x}\right)=\frac{2xydx-y^2dx}{x^2}$ (iv) $d\left(\frac{x^2}{y}\right)=\frac{2yxdx-x^2dy}{y^2}$

(v) $d\left(\frac{y^2}{x^2}\right)=\frac{2x^2ydx-2xy^2dx}{x^4}$ (vi) $d\left(\frac{x^2}{y^2}\right)=\frac{2y^2xdx-2yx^2dy}{y^4}$

(vii) $d\left[\log(xy)\right]=\frac{xdy+ydx}{xy}$ (viii) $d(xy)=xdy+ydx$

(ix) $d\left[\tan^{-1}\frac{y}{x}\right]=\frac{xdy-ydx}{x^2+y^2}$ (x) $d\left[\tan^{-1}\frac{x}{y}\right]=\frac{ydx-xdy}{x^2+y^2}$

(xi) $d\left[\log\left(\frac{x}{y}\right)\right]=\frac{xdy-ydx}{xy}$ (xii) $d\left[\log\left(\frac{x}{y}\right)\right]=\frac{ydx-xdy}{xy}$

(xiii) $d\left[\frac{1}{2}\log\left(x^2+y^2\right)\right]=\frac{xdx+ydy}{x^2+y^2}$ (xiv) $d\left[-\frac{1}{xy}\right]=\frac{xdy+ydx}{x^2y^2}$

Example 31: Solve $ydx-xdy+\left(1+x^2\right)dx+x^2\sin ydy=0$

Soln: Dividing each term of the given equation by x^2, we get

$$\frac{ydx-xdy}{x^2}+\frac{1+x^2}{x^2}dx+\sin ydy=0 \Rightarrow -\frac{xdy-ydx}{x^2}+\left(\frac{1}{x^2}+1\right)dx+\sin ydy=0$$

$$\Rightarrow -d\left(\frac{y}{x}\right)+\left(1+\frac{1}{x^2}\right)dx+\sin ydy=0$$

Integrating, $-\left(\frac{y}{x}\right)+x-\left(\frac{1}{x}\right)-\cos y=c \Rightarrow -y+x^2-1-x\cos y=cx$

Orthogonal Trajectories:

Two family of curves are such that every curve of either family cuts each curve of the other family at right angles, then they are called orthogonal trajectories of each other.

Example: (i) The path of a electric field is perpendicular to equipotential surface.

(ii) The line of heat flow is perpendicular to isothermal curves.

Example 32: Find the orthogonal trajectories of the family of curves xy = c

Soln: Given: xy = c $\Rightarrow y+x\frac{dy}{dx}=0 \quad \Rightarrow \quad \frac{dy}{dx}=-\frac{y}{x}$

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ we get, $-\frac{dx}{dy}=-\frac{y}{x}$

$$\Rightarrow \quad ydy=xdx \quad \Rightarrow \quad \frac{y^2}{2}=\frac{x^2}{2}+c \Rightarrow \quad y^2-x^2=2c$$

Example 33: Find the orthogonal trajectories of the family of curves $x^2+y^2=C$

Soln: Given: $x^2+y^2=C \Rightarrow 2x+2y\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=-\frac{x}{y}$

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ we get, $-\frac{dx}{dy}=-\frac{x}{y} \Rightarrow \int\frac{dx}{x}=\int\frac{dy}{y} \Rightarrow \ln x+\ln m=\ln y \Rightarrow y=mx$

Example 34: If y(x) satisfies the differential equation $\frac{dy}{dx}+2y=2+e^{-x^2}$ with y(0) = 0, then $\underset{x\to\infty}{Lt}\, y(x)$ is equal to

(a) 0 (b) 1 (c) 2 (d) -1

Soln: Given differential equation is a first order linear differential equation.

I.F. = $e^{\int 2dx}=e^{2x}$;

Therefore the solution is

$$y.e^{2x}=\int\left(2e^{2x}+e^{2x-x^2}\right)dx \Rightarrow y.e^{2x}=e^{2x}+\int e^{2x-x^2}dx \Rightarrow y=1+e^{-2x}\int e^{2x-x^2}dx \Rightarrow \underset{x\to\infty}{Lt}\, y(x)=1$$

3.3 Linear Second Order Differential Equations

CASE I: Homogeneous Differential equation:

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+QY=0 \qquad (P, Q, \text{ are constant coefficients})$$

Say, $y=ce^{mx}$ be the trial solution

$$\Rightarrow ce^{mx}(m^2+Pm+Q)=0 \Rightarrow (m^2+Pm+Q)=0$$

The above equation is known as auxilliary equation corresponding to the differential equation.

Suppose m_1, m_2 are two solutions of auxillary equation. Depending on m_1, m_2 there will be varoius forms of solution possible. Now,

(i) If m_1, m_2 are real and unequal, then solution will be of the form:

$$y=c_1e^{m_1x}+c_2e^{m_2x}$$

(ii) If m_1, m_2 are real and equal ($m_1=m_2=m$), then solution will be of the form:

$$y=(c_1+c_2x)e^{mx}$$

(iii) If m_1, m_2 are complex numbers, then solution will be of the form:

$$y=c_1e^{m_1x}+c_2e^{m_2x}$$

(iv) If m_1, m_2 are complex conjugate of each other i.e. $m_1, m_2=\alpha\pm i\beta$, then solution will be of the form:

$$y=e^{\alpha x}(A\cos\beta x+B\sin\beta x)$$

Example 35: Solve $\frac{d^2y}{dx^2}-7\frac{dy}{dx}+12y=0$

Soln: The given differential equation is $\left(D^2-7D+12\right)y=0$ [where $D\equiv\dfrac{d}{dx}$]

The auxilary equation is $m^2-7m+12=0 \quad \Rightarrow \left(m-3\right)\left(m-4\right)=0 \;\Rightarrow m=3,4$

The complete solution is $y\left(x\right)=C_1e^{3x}+C_2e^{4x}$, where C_1 & C_2 are constants.

Example 36: Solve $\dfrac{d^3y}{dx^3}-8y=0$

Soln: The auxilary equation is $m^3-8=0 \Rightarrow \left(m-2\right)\left(m^2+2m+4\right)=0 \Rightarrow m=2,-1+i\sqrt{3},-1-i\sqrt{3}$

The complete solution is $y\left(x\right)=C_1e^{2x}+e^{-x}\left[C_2\cos\sqrt{3}\,x+C_3\sin\sqrt{3}\,x\right]$

Example 37: Consider the differential equation.

$$\frac{d^2x}{dt^2}+2\frac{dx}{dt}+x=0; \quad at\ t=0,\ x=1\ \&\ \frac{dx}{dt}=0$$

Find the value of x at $t=1$.

Soln: The auxilary equation is $m^2+2m+1=0 \Rightarrow m=-1,-1$

Therefore, the solution is $x\left(t\right)=C_1e^{-t}+C_2te^{-t} \Rightarrow \dfrac{dx}{dt}=-C_1e^{-t}+C_2e^{-t}-C_2t\,e^{-t}$

At $t=0,\ x=1 \Rightarrow C_1=1$

At $t=0$, $\dfrac{dx}{dt}=0 \Rightarrow 0=-C_1+C_2 \;\Rightarrow C_1=C_2=1 \Rightarrow x\left(t\right)=e^{-t}+t\,e^{-t}$

At t = 1, the value of x will be $e^{-1}+e^{-1}=\dfrac{2}{e}$

Example 38: Solution of the differential equation $\dfrac{d^2y}{dx^2}+2p\dfrac{dy}{dx}+(p^2+q^2)y=0$ is

(a) $y=e^{px}(c_1\cos qx+c_2\sin qx)$ (b) $y=e^{-px}(c_1+c_2x)$

(c) $y=e^{-px}(c_1\cos qx+c_2\sin qx)$ (d) None of these

Soln: Auxillary equation is $\{m^2+2pm+(p^2+q^2)\}y=0$

For non-trivial solution,

$$m^2+(p+q)m+(p-q)m+(p^2+q^2)=0$$

$$\Rightarrow m\left[m+(p+q)\right]+(p-q)\left[m+(p+q)\right]=0$$

$$\Rightarrow \left[m+(p+q)\right]\left[m+(p-q)\right]=0$$

Hence, $m=-(p+q),\ -(p-q)$

Therefore, C.F. $y=c_1\,e^{-(p+q)x}+c_2\,e^{-(p-q)x}=e^{-px}\left[c_1\,e^{-qx}+c_2\,e^{qx}\right]$

Since, $\cosh qx = \dfrac{e^{qx}+e^{-qx}}{2}$, $\sinh qx = \dfrac{e^{qx}-e^{-qx}}{2}$

On putting the values of e^{qx} and e^{-qx}, we get

$y(x) = e^{-px}\left[c_1 \cosh qx + c_2 \sinh qx\right]$

Correct option is (d)

Example 39: Value of $y(x=-1)$, where $y(x)$ is the solution of the differential equation $y'' + 2y' - 8y = 0$ with $y(0) = 3$, $y'(0) = -12$, is __________.

Soln: Auxillary equation,

$$(m^2 + 2m - 8)\,y = 0 \Rightarrow m^2 + 2m - 8 = 0 \Rightarrow (m+4)(m-2) = 0 \Rightarrow m = -4, 2$$

Therefore, C.F. $y(x) = Ae^{-4x} + Be^{2x}$

Since, $y(0) = 3$, put $x = 0$, $y = 3$, we get $3 = A + B$ ... (1)

and $y'(x) = -4Ae^{-4x} + 2Be^{2x}$

Therefore, $y'(x=0) = -12 \Rightarrow -4A + 2B = -12$ ... (2)

Solving eqn. (1) and eqn. (2) ; $A = 3$, $B = 0$

Therefore, $y(x) = 3e^{-4x} \Rightarrow y(x=-1) = 3e^4 = 161.80.$

Correct answer is (161.80)

Example 40: A function $f(x)$ obeys the differential equation $\dfrac{d^2f}{dx^2} - (3-4i)f = 0$ and satisfies the conditions $f(0) = 1$ and $f(x) \to 0$ as $x \to \infty$. The value of $f(\pi)$ is :

(a) $e^{2\pi}$ (b) $e^{-2\pi}$ (c) $-e^{-2\pi}$ (d) $-e^{2\pi i}$

Soln: Given differential equation: $\dfrac{d^2f}{dx^2} - (3-4i)f = 0$

Let, $f = c.e^{mx}$ be trial solution.

$$\Rightarrow \quad \frac{df}{dx} = c.m\, e^{mx} \quad \Rightarrow \quad \frac{d^2f}{dx^2} = c.m^2\, e^{mx}$$

Substituting in the given equation, we get

$$m^2 - (3-4i) = 0 \quad (e^{mx} \neq 0)$$

$$\Rightarrow \quad m = \sqrt{3-4i} = \sqrt{4-1-4i} = \sqrt{(2)^2 + (i)^2 - 2.2.i} = \sqrt{(2-i)^2} = \pm(2-i)$$

Therefore, $f(x) = A\,e^{(2-i)x} + B\,e^{-(2-i)x}$

Since, $f(0) = 1 \quad \Rightarrow A + B = 1$

Since, $f(x) \to 0$ when $x \to \infty \Rightarrow A = 0$, then $B = 1$

$$\therefore \quad f(x) = e^{-(2-i)x} \quad \Rightarrow f(\pi) = e^{-(2-i)\pi} = e^{-2\pi}\, e^{i\pi} \quad \Rightarrow \boxed{f(\pi) = -e^{-2\pi}}$$

Correct answer is (c)

Example 41: Consider the differential equation

$$\frac{d^2x}{dt^2}+2\frac{dx}{dt}+x=0$$

with the initial conditions $x(0)=0$ and $\dot{x}(0)=1$. The solution $x(t)$ attains its maximum value when 't' is

(a) 1/2 (b) 1 (c) 2 (d) ∞

Soln: $\frac{d^2x}{dt^2}+2\frac{dx}{dt}+x=0$

Assume the trial solution : $x=ce^{mt}$

$\Rightarrow\left(m^2+2m+1\right)c.e^{mt}=0 \Rightarrow\left(m^2+2m+1\right)=0 \quad \Rightarrow(m+1)^2=0 \quad \Rightarrow m=-1,-1$

$\therefore \quad x(t)=\left(c_1+c_2t\right)e^{-t}$

Putting $x(0)=0 \quad \Rightarrow c_1=0$

$\therefore \quad x(t)=c_2t\,e^{-t} \qquad \Rightarrow \dot{x}(t)=\frac{dx}{dt}=c_2e^{-t}-c_2te^{-t}$

putting $\dot{x}(0)=1 \qquad \Rightarrow c_2=1$

$\therefore \quad x(t)=t\,e^{-t} \qquad \Rightarrow \frac{dx}{dt}=e^{-t}-te^{-t} \qquad \Rightarrow \frac{d^2x}{dt^2}=te^{-t}-2e^{-t}$

For maxima or minima, $\frac{dx}{dt}=0 \quad \Rightarrow t=1$

At $t=1, \frac{d^2x}{dt^2}=-e^{-1}=-ve$

Therefore, $x(t)$ attains its maximum value at $t=1$

Correct answer is (b)

Example 42: Consider the differential equation $\frac{d^2x}{dt^2}+\frac{2dx}{dt}+x=0$. At time $t=0$, it is given that $x=1$ and $\frac{dx}{dt}=0$. At $t=1$, the value of x is given by

(a) $\frac{1}{e}$ (b) $\frac{2}{e}$ (c) 1 (d) $\frac{3}{e}$

Soln: $\frac{d^2x}{dt^2}+2\frac{dx}{dt}+x=0$

Let, $x=C.e^{mt}$ be the trial solution.

So, $\left(m^2+2m+1\right)C.e^{mt}=0 \Rightarrow m=-1,-1$

Therefore, $x=\left(C_1+C_2t\right)e^{-t}$

At $t=0$, $x=1$ then $C_1=1$; so, $x=\left(1+C_2t\right)e^{-t}$

Now, $\frac{dx}{dt}=C_2e^{-t}+\left(1+C_2t\right)\left(-e^{-t}\right)$

At $t=0, \frac{dx}{dt}=0$, then $C_2=-1=0 \Rightarrow C_2=1$

So, $C_2 x=(1+t)e^{-t}$; At $t=1, x=\frac{2}{e}$

Correct option is (b)

Example 43: The solution of the differential equation $\frac{d^2y}{dt^2}-y=0$ subject to the boundary conditions $y(0)=1$ and $y(\infty)=0$, is

(a) $\cos t+\sin t$ (b) $\cosh t+\sinh t$ (c) $\cos t-\sin t$ (d) $\cosh t-\sinh t$

Soln: $\frac{d^2y}{dt^2}-y=0$

Assume, the trial solution to be $y=c\cdot e^{mx}$

$\Rightarrow m^2-1=0 \Rightarrow m=\pm 1$

So, $y=Ae^{t}+Be^{-t}$

Putting $y(t=0)=1$, we get $A+B=1$

Putting $y(t=\infty)=0$, we get $A=0, B=1$

Therefore, $y(t)=e^{-t}=\cosh t-\sinh t$

Correct option is (d)

CASE II: Non-Homogeneous Differential equation:

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R(x)$$

The above equation can be written as

$$\left(D^2+P.D+Q\right)y=R(x) \Rightarrow f(D)y=R(x) \quad \left[\text{where } D\equiv\frac{d}{dx}\right]$$

The complete solution of the above differential equation consists of two parts:

(i) Complementary function (C.F.): same as the solution of the equivalent homogenous differential equation,

(ii) Particular integral (P.I): depends on the form of $R(x)$

Rules to get Particular Integral:

(1) $R(x)=e^{ax}$, then particular integral $=\frac{1}{f(D)}e^{ax}=\frac{1}{f(a)}e^{ax}$

(i) If $f(a)=0$, then particular integral $=x\frac{1}{f'(a)}e^{ax}$

(ii) If $f'(a)=0$ then particular integral $=x^2\frac{1}{f''(a)}e^{ax}$

(2) $R(x) =$ polynomial of x; then particular integral $= \frac{1}{f(D)}x^n$ [Expand $\frac{1}{f(D)}$ in series containing poisitive powers of D]

(3) $R(x) = \cos ax$ or $\sin ax$; then particular integral $= \frac{1}{f(D^2)}\sin ax = \frac{1}{f(-a^2)}\sin ax$;

(i) if $f(-a^2) = 0$, then particular integral $= \frac{x}{f'(-a^2)}\sin ax$

(4) $R(x) = e^{\alpha x}\phi(x)$; then particular integral $= \frac{1}{f(D)}e^{ax}\cdot\phi(x) = e^{ax}\frac{1}{f(D+a)}\phi(x)$

Example 44: Solve $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{5x}$

Soln: Auxillary equation: $m^2 - 3m + 2 = 0 \Rightarrow (m-1)(m-2) = 0 \Rightarrow m = 1, 2 \Rightarrow$ C.F. $= C_1e^x + C_2e^{2x}$

Particular integral (P.I) $= \frac{e^{5x}}{D^2 - 3D + 2} = \frac{e^{5x}}{5^2 - 3.5 + 2} = \frac{e^{5x}}{12}$

Therefore, Complete solution is $y = C.F. + P.I. = C_1e^x + C_2e^{2x} + \frac{e^{5x}}{12}$

Example 45: $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = \sin 2x$

Soln: Auxillary equation is $m^2 - m - 2 = 0 \Rightarrow m = -1, 2 \Rightarrow C.F. = C_1e^{-x} + C_2e^{2x}$

$$P.I. = \frac{\sin 2x}{D^2 - D - 2} = \frac{\sin 2x}{-(2)^2 - D - 2} = \frac{-\sin 2x}{D+6}$$

$$= -\frac{(D-6)\sin 2x}{D^2 - 36} = -\frac{(D-6)\sin 2x}{-(2)^2 - 36} = \frac{1}{40}(D-6)\sin 2x = \frac{2\cos 2x - 6\sin 2x}{40}$$

Therefore, Complete solution is $y = C.F. + P.I. = C_1e^{-x} + C_2e^{2x} + \frac{1}{20}(\cos 2x - 3\sin 2x)$

Example 46: Solve $(D^2 - 2D + 1)y = x^2e^{3x}$

Soln: Auxillary equation is $m^2 - 2m + 1 = 0 \Rightarrow m = 1, 1 \Rightarrow C.F. = C_1e^x + C_2xe^x = (C_1 + C_2x)e^x$

$$P.I. = \frac{1(x^2e^{3x})}{D^2 - 2D + 1} = \frac{1}{(D-1)^2}e^{3x}x^2 = e^{3x}\frac{1}{\{(D+3)-1\}^2}x^2 = e^{3x}.\frac{1}{(D+2)^2}x^2$$

$$= \frac{1}{4}e^{3x}\left(1 + \frac{D}{2}\right)^{-2}x^2 = \frac{1}{4}e^{3x}\left[1 - D + \frac{3}{4}D^2 + ...\right]x^2 \quad \Rightarrow \quad P.I. = \frac{e^{3x}}{4}\left[x^2 - 2x + \frac{3}{2}\right]$$

Hence complete solution is y = C.F. + P.I. $\Rightarrow y = (C_1 + C_2x)e^x + \frac{e^{3x}}{4}\left[x^2 - 2x + \frac{3}{2}\right]$

Example 47: Solve $\left(D^3-D^2-6D\right)y=x^2+1$

Soln: Auxiliary equation: $m\left(m^2-m-6\right)=0 \Rightarrow$ m = 0,3,-2 $\Rightarrow C.F.=c_1+c_2e^{3x}+c_3e^{-2x}$

$$P.I.=\frac{1}{D^3-D^2-6D}\left(x^2+1\right)=\frac{1}{-6D\left(1+\frac{D}{6}-\frac{D^2}{6}\right)}\left(x^2+1\right)=-\frac{1}{6D}\left[1+\left(\frac{D}{6}-\frac{D^2}{6}\right)\right]^{-1}\left(x^2+1\right)$$

$$=-\frac{1}{6D}\left[1-\left(\frac{D}{6}-\frac{D^2}{6}\right)+\left(\frac{D}{6}-\frac{D^2}{6}\right)^2-\ldots\ldots\right]\left(x^2+1\right)$$

$$=-\frac{1}{6D}\left[x^2+1-\frac{1}{6}D\left(x^2-1\right)+\frac{7}{36}D^2\left(x^2+1\right)+\ldots..\right]=-\frac{1}{6D}\left[x^2+1-\frac{1}{6}(2x+0)+\frac{7}{36}(2+0)+\ldots\right]$$

$$=-\frac{1}{6D}\left[x^2-\frac{1}{3}x+\frac{25}{18}\right]=-\frac{1}{6}\left[\frac{x^3}{3}-\frac{x^2}{6}+\frac{25}{18}x\right]$$

Hence, the required solution is $y=C.F.+P.I.\Rightarrow y=c_1+c_2e^{3x}+c_3e^{-2x}-\left(\frac{1}{18}\right)\left(x^3-\frac{x^2}{2}+\frac{25x}{6}\right)$

Example 48: Solve $\left(D^3+8\right)y=x^4+2x+1$

Soln: The auxialiary equation is $(m+2)\left(m^2-2m+4\right)=0 \Rightarrow m=-2, 1\pm i\sqrt{3}$

$$\Rightarrow C.F.=c_1e^{-2x}+e^x\left(c_2\cos\sqrt{3}x+c_3\sin\sqrt{3}x\right)$$

$$P.I.=\frac{1}{D^3+8}\left(x^4+2x+1\right)=\frac{1}{8\left(1+\frac{1}{8}D^3\right)}\left(x^4+2x+1\right)$$

$$=\frac{1}{8}\left(1+\frac{1}{8}D^3\right)^{-1}\left(x^4+2x+1\right)=\frac{1}{8}\left(1-\frac{1}{8}D^3+\ldots.\right)\left(x^4+2x+1\right)$$

$$=\frac{1}{8}\left[\left(x^4+2x+1\right)-\frac{1}{8}D^3\left(x^4+2x+1\right)\right] \qquad =\frac{1}{8}\left[\left(x^4+2x+1\right)-3x\right]=\frac{1}{8}\left(x^4-x+1\right)$$

$$\Rightarrow y=c_1e^{-2x}+e^x\left(c_2\cos\sqrt{3}x+c_3\sin\sqrt{3}x\right)+\left(\frac{1}{8}\right)\left(x^4-x+1\right)$$

Example 49: Solve $\left(D^2-1\right)y=\cosh x\cos x=\frac{1}{2}\left(e^x+e^{-x}\right)\cos x$

Soln: Auxiliary equation is $m^2-1=0 \Rightarrow$ m = -1,1 $\Rightarrow$ C.F. = $c_1e^x+c_2e^{-x}$

$$\text{P.I.}=\frac{1}{2}\frac{1}{D^2-1}e^x\cos x+\frac{1}{2}\frac{1}{D^2-1}e^{-x}\cos x=\frac{1}{2}e^x\frac{1}{(D+1)^2-1}\cos x+\frac{1}{2}e^{-x}\frac{1}{(D-1)^2-1}\cos x$$

$$=\frac{1}{2}e^{x}\frac{1}{D^2+2D}\cos x+\frac{e^{-x}}{2}\frac{1}{D^2-2D}\cos x=\frac{1}{2}e^{x}\frac{1}{-1+2D}\cos x+\frac{e^{-x}}{2}\frac{1}{-1-2D}\cos x$$

$$=\frac{e^{x}}{2}(2D+1).\frac{1}{(2D+1)(2D-1)}\cos x-\frac{e^{-x}}{2}(2D-1)\frac{1}{(2D-1)(2D+1)}\cos x$$

$$=\frac{e^{x}}{2}(2D+1)\frac{1}{(4D^2-1)}\cos x-\frac{e^{-x}}{10}(2D-1)\frac{1}{4D^2-1}\cos x$$

$$=\frac{e^{x}}{2}(2D+1)\frac{1}{-4-1}\cos x-\frac{e^{-x}}{2}(2D-1)\frac{1}{-4-1}\cos x$$

$$=-\frac{e^{x}}{10}(2D+1)\cos x+\frac{e^{-x}}{10}(2D-1)\cos x=-\frac{e^{x}}{10}(-2\sin x+\cos x)+\frac{e^{-x}}{10}(-2\sin x-\cos x)$$

$$\therefore \quad y=c_1e^{x}+c_2e^{-x}-\left(\frac{1}{10}\right)e^{x}(-2\sin x+\cos x)+\left(\frac{1}{10}\right)e^{-x}(-2\sin x-\cos x)$$

$$\Rightarrow y=c_1e^{x}+c_2e^{-x}+\frac{2}{5}\sin x\left(\frac{e^{x}-e^{-x}}{2}\right)-\frac{1}{5}\cos x\left(\frac{e^{x}+e^{-x}}{2}\right)$$

$$\Rightarrow y=c_1e^{x}+c_2e^{-x}+\left(\frac{2}{5}\right)\sin x\sinh x-\left(\frac{1}{5}\right)\cos x\cos hx$$

Example 50: Solve $(D^2+2D+1)y=x\cos x$

Soln: Auxiliary equation is $(m+1)^2=0\Rightarrow m=-1,-1$

$$P.I.=\frac{1}{(D+1)^2}x\cos x=\text{real part of }\frac{1}{(D+1)^2}xe^{ix}$$

$$\text{Now,}\quad \frac{1}{(D+1)^2}xe^{ix}=e^{ix}\frac{1}{(D+i+1)^2}x=\frac{e^{iax}}{(1+i)^2}\frac{1}{\left[1+\frac{D}{1+i}\right]^2}.x$$

$$=\frac{e^{ix}}{2i}\left[1+\frac{D}{1+i}\right]^{-2}x=\frac{e^{ix}}{2i}\left[1-\frac{2D}{1+i}+.....\right]x=-\frac{1}{2}ie^{ix}\left[1-\frac{1-i}{2}.2\right]$$

$$=-\frac{1}{2}ie^{ix}\left[(x-1)+i\right]=\frac{1}{2}e^{ix}\left[(x-1)i-1\right]=-\frac{1}{2}(\cos x+i\sin x)\left[(x-1)i-1\right]$$

$$\text{P.I.}=-\frac{1}{2}\left[-\cos x-(x-1)\sin x\right]$$

$$\therefore \qquad y=(c_1+c_2x)e^{-x}+\left(\frac{1}{2}\right)\left[\cos x+(x-1)\sin x\right]$$

Example 51: Solution of the differential equation $\frac{d^3y}{dx^3}-5\frac{d^2y}{dx^2}+7\frac{dy}{dx}-3y=e^{2x}\cosh x$ is

(a) $(c_1+c_2x)e^x+\frac{x}{8}e^{3x}-\frac{x^2}{8}e^x+c_3e^{3x}$ (b) $2c_1e^x+c_2e^{3x}+\frac{x}{8}e^{3x}-\frac{x}{8}e^{3x}$

(c) $c_1e^x+c_2e^{3x}+\frac{x}{8}e^{3x}$ (d) $(c_1+c_2x)e^x+c_3e^{3x}+\frac{x}{8}e^{3x}-\frac{x}{8}e^x$

Soln: Auxillary equation $(m^3-5m^2+7m-3)y=0$

For non-trivial solution, $m^2-5m^2+7m-3=0\Rightarrow(m-1)(m^2-4m+3)=0$

$\Rightarrow(m-1)(m-3)(m-1)=1\Rightarrow m=1,1,3.$

Therefore, C.F. $(c_1+c_2x)e^x+c_3e^{3x}$

Now, P.I. $\frac{1}{D^3-5D^2+7D-3}\times e^{2x}\cosh x=\frac{1}{D^3-5D^2+7D-3}\times e^{2x}\times\frac{e^x+e^{-x}}{2}$

$$\Rightarrow\frac{1}{2}\frac{1}{D^3-5D^2+7D-3}e^{3x}+\frac{1}{2}\frac{1}{D^3-5D^2+7D-3}\times e^x$$

$$=\frac{1}{2}\frac{x}{3D^2-10D+7}e^{3x}+\frac{1}{2}\frac{x}{3D^2-10D+7}e^x$$

$$=\frac{x}{8}e^{3x}+\frac{1}{2}\frac{x^2}{6D-10}e^x=\frac{x}{8}e^{3x}-\frac{x^2}{8}e^x$$

Therefore, $y(x)=(c_1+c_2x)e^x+c_3e^{3x}+\frac{x}{8}e^{3x}-\frac{x^2}{8}e^x.$

Correct option is (a)

Example 52: Particular integral of the differential equation $(D^3+8)y=x^4+x+1$, where $D=\frac{d}{dx}$ is

(a) $\frac{1}{8}(x^4+x+1)$ (b) $\frac{1}{8}(x^3+4x+1)$ (c) $\frac{1}{8}(x^4+x-1)$ (d) $\frac{1}{8}(x^4+4x+1)$

Soln: P.I. $=\frac{1}{D^3+8}(x^4+x+1)=\frac{1}{8}\left(1+\frac{D^3}{8}\right)^{-1}(x^4+x+1)$

$$=\frac{1}{8}\left(1+\frac{D^3}{8}+\cdots\right)(x^4+x+1)$$

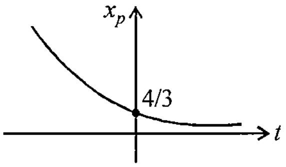
$$=\frac{1}{8}(x^4+x+1+\frac{1}{8}\times 24x)$$

$$=\frac{1}{8}(x^4+4x+1).$$

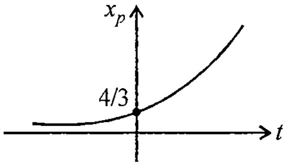
Correct option is (d)

Example 53: If x_p is the particular integral of second order differential equation $\frac{d^2x}{dt^2} - x = 4e^{-2t}$, then which of the following graph best represent the x_p with t?

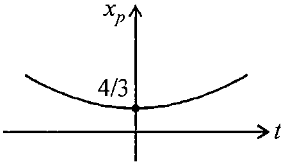
(a)



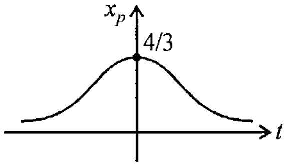
(b)



(c)



(d)



Soln: $\text{P.I.} = \frac{1}{D^2 - 1} 4e^{-2t} = \frac{1}{4-1} \times 4e^{-2t}$

Hence, P.I. $x_p = \frac{4}{3} e^{-2t}$.

Correct option is (a)

Example 54: Which of the following statements is/are correct about the differential equation $(D^2 + 2D + 1)y = x\cos x$

(a) C.F. of this equation is $(c_1 + c_2 x)e^{-x}$

(b) C.F. for this equation is $c_1 e^{-x} + c_2 e^{2x}$

(c) P.I. for this equation is $\frac{1}{2}[(x-1)\sin x + \cos x]$

(d) P.I. for this equation is $\frac{1}{2}[(x-1)\sin x - \cos x]$

Soln: Auxillary equation $(m^2 + 2m + 1)\,y = 0$

For non trivial solution $m^2 + 2m + 1 = 0 \Rightarrow \boxed{m = -1, -1}$

Therefore, $\text{C.F.} = (c_1 + c_2 x)\,e^{-x}$

$$\text{P.I.} = \frac{1}{(D+1)^2} x\cos x \Rightarrow \text{R.P. of } \frac{1}{(D+1)^2} x e^{ix}$$

$$= e^{ix} \frac{1}{(D+i+1)^2} x = \frac{e^{ix}}{(1+i)^2} \frac{1}{\left[1 + \frac{D}{1+i}\right]^2} x$$

$$= \frac{e^{ix}}{2i}\left[1 + \frac{D}{1+i}\right]^{-2} x = \frac{e^{ix}}{2i}\left[1 - \frac{2D}{1+i}\right] x$$

$$= \frac{e^{ix}}{2i}\left[x - \frac{2}{1+i}\times\frac{1-i}{1-i}\right] = \frac{e^{ix}}{2i}\left[x-(1-i)\right]$$

Hence real part $\frac{1}{2}\left[(x-1)\sin x + \cos x\right]$

Correct options are (a) and (c)

Example 55: If $x(t)$ and $y(t)$ are differential functions on R such that $x(0)=2$ and $y(0)=1$. Also $x(t)+y'(t)=1$ and $x'(t)+y(t)=1\ \forall\ t\in[0,1]$, then the value of $x(\log 3)$ is ___________.

Soln: We have $x(t)+y'(t)=1$ ... (1)

and $x'(t)+y(t)=1$ ... (2)

Hence, from Eq. (1) and Eq. (2) we get

$x''(t)+1-x(t)=0 \Rightarrow x''(t)-x(t)=1$

Auxillary equation $(m^2-1)\,x(t)=0$

For non trivial solution, $m^2-1=0 \Rightarrow m=\pm 1$

Therefore, C.F. $= Ae^{t}+Be^{-t}$

Now, P.I. $= \frac{1}{D^2-1}1 = \frac{1}{D^2-1}e^{0} = -1$

Hence, $x(t) = Ae^{t}+Be^{-t}-1$

Since, $x(0)=2$, put $t=0$, $x=2$

$\Rightarrow 2 = A+B-1 \Rightarrow A+B=3$ ... (3)

and $y(0)=1$, hence from Eq. (2) ; $x'(t=0)=0$

Here, $x'(t)=Ae^{t}-Be^{-t}$

Therefore, $x'(t=0)=A-B=0 \Rightarrow A=B$ ... (4)

Therefore, from Eq. (3) and Eq. (4), we get $A=B=\frac{3}{2}$

Therefore, $x(t)=\frac{3}{2}\left(e^{t}+e^{-t}\right)-1$

$$\Rightarrow x(\log 3) = \frac{3}{2}\left(e^{\log 3}+e^{-\log 3}\right)-1 = \frac{3}{2}\left(3+\frac{1}{3}\right)-1 = \frac{3}{2}\times\frac{10}{3}-1 = 4.$$

Correct answer is (4)

Example 56: Value of the particular integral of the differential equation

$\frac{d^2y}{dx^2}-6\frac{dy}{dx}+8y=3\cos x$ at $x=\frac{\pi}{2}$ is___________.

Soln: Particular integral $= \frac{1}{D^2-6D+8}3\cos x$ (put $D^2=-1$)

$$= 3\times\frac{1}{-1-6D+8}\cos x = -3\times\frac{1}{6D-7}\cos x = -3\times\frac{1}{6D-7}\times\left(\frac{6D+7}{6D+7}\right)\cos x$$

$$= -3\times\frac{(6D+7)}{36D^2-49}\cos x = \frac{-3\times(-6\sin x+7\cos x)}{36-49}$$

Hence, P.I. at $x=\frac{\pi}{2}=\frac{-3(-6+0)}{-13}=\frac{-18}{13}=1.38.$

Correct answer is (1.38)

Example 57: If y_1 and y_2 are the particular solution of the differential equation $y''-y'-2y=4x^2$ and $y''-y'-2y=\sin 2x$ respectively, then value of y_1/y_2 at $x=0$ is ___________.

Soln: Particular solution of 1st differential equation,

$$y_1=\frac{1}{D^2-D-2}4x^2=-\frac{1}{2}\left(1-\frac{D^2-D}{2}\right)^{-1}(4x^2)$$

$$=-\frac{1}{2}\left(1+\frac{D^2-D}{2}+\left(\frac{D^2-D}{2}\right)^2\cdots\right)(4x^2)=-2x^2+2x-3$$

Particular solution of 2nd differential equation,

$$y_2=\frac{1}{D^2-D-2}\sin 2x,\ (\text{put } D^2=-4)$$

$$=\frac{1}{-D-6}\sin 2x=-\frac{1}{D+6}\times\frac{D-6}{D-6}\sin 2x$$

$$=\frac{(D-6)\sin 2x}{D^2-36}=-\frac{2\cos 2x-6\sin 2x}{-40}=-\frac{3}{20}\sin 2x+\frac{1}{20}\cos 2x$$

Therefore, $\left(\frac{y_1}{y_2}\right)_{x=0}=-\frac{3}{1/20}=-60.$

Correct answer is (- 60)

Example 58: The solution of the differential equation for $y(t):\frac{d^2y}{dt^2}-y=2\cosh(t)$, subject to the initial conditions $y(0)=0$ and $\left.\frac{dy}{dt}\right|_{t=0}=0$ is

(a) $\frac{1}{2}\cosh(t)+t\sinh(t)$

(b) $-\sinh(t)+t\cosh(t)$

(c) $t\cosh(t)$

(d) $t\sinh(t)$

Soln: $\frac{d^2y}{dt^2}-y=2\cdot\cosh t$

$C.F.=Ae^t+Be^{-t}$

$$P.I.=\frac{1}{D^2-1}(2\cdot\cosh t)=\frac{1}{(D^2-1)}\left(e^t+e^{-t}\right)$$

$$=t\frac{1}{2D}\left(e^t+e^{-t}\right)=\frac{t}{2}\left(e^t-e^{-t}\right)=t\sinh t$$

Total solution, $y = Ae^{t} + Be^{-t} + t\sinh t$

Putting the condition, $\left.\frac{dy}{dt}\right|_{t=0} = 0 \Rightarrow A - B = 0$

Solving above two equations, $A = 0, B = 0$

Therefore, $y = t\sin t$

Correct option is (d)

Wronskian of the differential equation:

If $y_1(x)$ and $y_2(x)$ are two soultions of a second order linear homogeneous differential equation, then Wronskian of the differential equation is defined as

$$W(y_1, y_2, x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

(i) if $W(y_1, y_2, x) = 0$, then $y_1(x), y_2(x)$ are linearly dependent.

(ii) if $W(y_1, y_2, x) \neq 0$, then $y_1(x), y_2(x)$ are linearly independent.

Note: (i) If y_1 and y_2 are two linearly independent solution of a second order linear homogeneous differential equation then their linear combination $c_1y_1 + c_2y_2$ will also be a solution of that differential equation.

(ii) The value of the Wronskian at different values of x can be related by the following relation:

$$W(x = x_1) = W(x = x_0)e^{-\int_{x_0}^{x_1} P(x)dx}$$

(iii) A differential equation is said to have dimension 'k' if the differential equation has 'k' number of linearly independent solutions.

Cauchy-Euler differential equation of second order:

$$a_2x^2\frac{d^2y}{dx^2} + a_1x\frac{dy}{dx} + a_0y = Q(x)$$

Put $x = e^z \Rightarrow ln\,x = z \Rightarrow \frac{1}{x} = \frac{dz}{dx}$

$$\frac{dy}{dx} = \frac{dy}{dx}\cdot\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz} \Rightarrow x\frac{dy}{dx} = \frac{dy}{dz}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2} \Rightarrow x^2\frac{dy}{dx^2} = -\frac{dy}{dz} + \frac{d^2y}{dz^2}$$

Then the given equation becomes

$$a_2\frac{d^2y}{dz^2} + (a_1 - a_2)\frac{dy}{dz} + a_0y = Q(z)$$

The above equation is nothing but a linear second order non-homogeneous differential equation.

Example 59: $x^2\dfrac{d^2y}{dx^2}-2x\dfrac{dy}{dx}-4y=x^4$

Soln: $x=e^z \Rightarrow \log x=z \Rightarrow \dfrac{1}{x}=\dfrac{dz}{dx}$

Putting $x\dfrac{dy}{dx}=\dfrac{dy}{dz}$ and $x^2\dfrac{d^2y}{dx^2}=\dfrac{d^2y}{dz^2}-\dfrac{dy}{dz}$, we get $\Rightarrow \dfrac{d^2y}{dz^2}-3\dfrac{dy}{dz}-4y=\mathrm{e}^{4z}$

Auxillary equation: $m^2-3m-4=0 \Rightarrow m=4,-1 \Rightarrow \text{C.F.}=c_1e^{4z}+c_2e^{-z}$

Particular integral $=\dfrac{1}{D^2-3D-4}e^{4z}=z\cdot\dfrac{1}{2D-3}e^{4z}=\dfrac{z}{5}e^{4z}$

Total solution is $c_1e^{4z}+c_2e^{-z}+\dfrac{z}{5}e^{4z}=c_1x^4+c_2\dfrac{1}{x}+\dfrac{1}{5}x^4\log x$

Example 60: $x^2\dfrac{d^2y}{dx^2}+2x\dfrac{dy}{dx}-20y=(x+1)^2$

Soln: $x=e^z \Rightarrow \log x=z \Rightarrow \dfrac{1}{x}=\dfrac{dz}{dx}$

Putting $x\dfrac{dy}{dx}=\dfrac{dy}{dz}$ and $x^2\dfrac{d^2y}{dx^2}=\dfrac{d^2y}{dz^2}-\dfrac{dy}{dz}$, we get $\Rightarrow \dfrac{d^2y}{dz^2}+\dfrac{dy}{dz}-20y=\left(e^z+1\right)^2$

$$C.F.=c_1e^{4z}+c_2e^{-5z}=c_1x^4+c_2x^{-5}$$

$$P.I.=\frac{1}{D^2+D-20}\left(e^z+1\right)^2=\frac{1}{D^2+D-20}\left(e^{2z}+2e^z+1\right)=-\frac{1}{14}e^{2z}-\frac{1}{9}e^z-\frac{1}{20}$$

Total solution $y=c_1x^4+c_2x^{-5}-\dfrac{1}{14}x^2-\dfrac{1}{9}x-\dfrac{1}{20}$

Example 61: A function $y(z)$ satisfies the ordinary differential equation $y''+\dfrac{1}{z}y'-\dfrac{m^2}{z^2}y=0$, where $m=0,1,2,3,\ldots$. Consider the four statements P, Q, R, S as given below.

P : z^m and z^{-m} are linearly independent solutions for all values of m

Q : z^m and z^{-m} are linearly independent solutions for all values of $m>0$

R : $\ell n\ z$ and 1 are linearly independent solutions for $m=0$

S : z^m and $\ell n\ z$ are linearly independent solutions for all values of m

The correct option for the combination of valid statement is

(a) P, R and S only (b) P and R only (c) Q and R only (d) R and S only

Soln: $y''+\frac{1}{z}y'-\frac{m^2}{z^2}y=0 \Rightarrow \quad z^2\frac{d^2y}{dz^2}+z\frac{dy}{dz}-m^2y=0 \rightarrow$ Cauchy Euler equation.

Putting $z=e^x$ and replacing

$$z\frac{dy}{dz} \text{ by } \frac{dy}{dx} \text{ and } z^2\frac{d^2y}{dz^2} \text{ by } \frac{d^2y}{dx^2}-\frac{dy}{dx}$$

We get,

$$\frac{d^2y}{dx^2}-\frac{dy}{dx}+\frac{dy}{dx}-m^2y=0$$

Trial soln : $y=C\cdot e^{kx}$

$\Rightarrow \quad (k^2-m^2)=0 \Rightarrow k=\pm m$

So, $\quad y=C_1e^{mx}+C_2e^{-mx}=C_1z^m+C_1z^{-m}$

Therefore, z^m and z^{-m} are linearly independent solutions for $m>0$.

For $m=0, k=0 \Rightarrow y=(C_1+C_2x)e^{0.x}=(C_1+C_2\ell nz)$

Therefore, 1 and *l*nz are linearly independent solutions for $m=0$

Correct option is (c)

Method of Variation of parameters:

Consider the following second order non-homogeneous linear differential equation:

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R(x)$$

Suppose $y_1=e^{m_1x}$ and $y_2=e^{m_2x}$ are two linearly independent solutions of the given differential equation, then

Complementary function = C.F. = Ay_1+By_2

Particular integral = P.I. = uy_1+vy_2

where $u=\int-\frac{y_2R(x)}{y_1y_2'-y_2y_1'}dx$ and $v=\int\frac{y_1R(x)}{y_1y_2'-y_2y_1'}dx$

Example 62: Solve: $\frac{d^2y}{dx^2}+y=\tan x$

Soln: Auxilliary equation: $m^2+1=0 \Rightarrow m=\pm i \Rightarrow C.F.=A\cos x+B\sin x$

In this case, $y_1=\cos x, y_2=\sin x$ and $y_1y_2'-y_1'y_2=1$

Particular solution = $u\cos x+v\sin x$

$$u=-\int y_2\tan x\,dx=-\int\frac{\sin^2x}{\cos x}dx=-\int\frac{1-\cos^2x}{\cos x}dx$$

$$=-\int(\sec x-\cos x)dx=-\ln(\sec x+\tan x)+\sin x$$

$$v=\int y_1\tan x\,dx=\int\sin x\,dx=-\cos x$$

Particular solution = $-\cos x.\ln(\sec x+\tan x)$

Total Solution is $y = A\cos x + B\sin x - \cos x.\ln\left(\sec x + \tan x\right)$

3.4 Special Functions

[Note: Question from this topic is only asked in BHU entrance examination]

(A) Legendre Differential Equation

$$\left(1-x^2\right)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n\left(n+1\right)y = 0$$

where n is a positive real integer.

Note: $x = \pm 1$ are regular singular points of the Legendre differential equation.

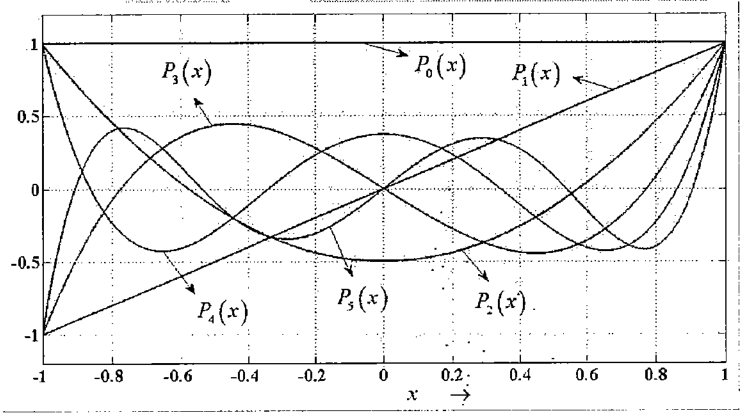
Solution:

Legendre polynomial of degree n and it is given by Rodrigue's formula as follows:

$$P_n\left(x\right) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left(x^2-1\right)^n$$

The first few Legendre polynomials are

$$P_0\left(x\right) = 1; \qquad P_1\left(x\right) = x; \qquad P_2\left(x\right) = \frac{1}{2}\left(3x^2-1\right); \qquad P_3\left(x\right) = \frac{1}{2}\left(5x^3-3x\right)$$



Properties of Legendre polynomials:

(1) $P_n\left(x=1\right) = 1$ (2) $P_n\left(x=-1\right) = \left(-1\right)^n$

(3) $P_n\left(x\right)$ is an even function of x for even n and is an odd function of x for odd n.

(4) $P_n\left(-x\right) = \left(-1\right)^n P_n\left(x\right)$ (5) $P_n'\left(x=1\right) = \frac{1}{2}n\left(n+1\right)$

$$(6)\ P_n\left(x=0\right) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \left(-1\right)^{n/2}\dfrac{1.3.5........\left(n-1\right)}{2.4.6........n} & \text{if } n \text{ is even} \end{cases}$$

Generating function of Legendre polynomial:

$$\left(1-2xz+z^2\right)^{-1/2}=\sum_{n=0}^{\infty} z^n P_n(x)$$

Co-efficient of z^n in the exapansion of the generating function is the Legendre polynomial of order n.

Orthonormal property of Legendre polynomial:

$$\int_{-1}^{+1} P_m(x) P_n(x)\, dx=\frac{2}{2n+1}\delta_{mn}$$

Recurrence relations for Legendre polynomials:

(i) $n P_n=(2n-1)xP_{n-1}-(n-1)P_{n-2}$ (ii) $n P_n=xP_n'-P_{n-1}'$

(iii) $(2n+1)P_n=P_{n+1}'-P_{n-1}'$ (iv) $(n+1)P_n=P_{n+1}'-xP_n'$

(v) $\left(1-x^2\right)P_n'=n\left(P_{n-1}-xP_n\right)$ (vi) $\left(1-x^2\right)P_n'=(n+1)\left(xP_n-P_{n+1}\right)$

(B) Bessel Differential Equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+\left(x^2-n^2\right)y=0$$

Note: $x=0$ is a regular singular point of the Bessel differential equation.

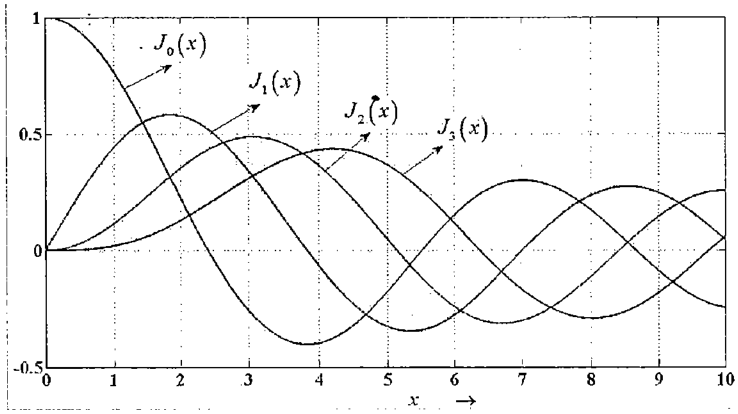
Solution:

Bessel function order n:

$$J_n(x)=\sum_{r=0}^{\infty}\frac{(-1)^r}{r!(n+r)!}\left(\frac{x}{2}\right)^{n+2r}$$

First few order of Bessel functions are

$$J_0(x)=1-\frac{x^2}{2^2}+\frac{x^4}{2^2\cdot 4^2}-\frac{x^6}{2^2.4^2.6^2}+\ldots..;\quad J_1(x)=\frac{x}{2}-\frac{x^3}{2^2 4}+\frac{x^5}{2^2\cdot 4^2\cdot 6}-\ldots..\text{ and so on}$$



Properties of Bessel functions:

(i) $J_n(x)$ are oscillatory function with varying period and decreasing amplitude.

(ii) $J_n(x)$ is an even function for even n and $J_n(x)$ is an odd function for odd n.

(iii) $J_0(x)$ has nodes at $x = 2.4048, 5.5201$ .... and $J_1(x)$ has nodes at $x = 3.8317, 7.015$ .... and so on.

(iv) $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x$ and $J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x$

Note:

(1) Since the Bessel differential equation involves only n^2, then

$$J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!(-n+r)!}\left(\frac{x}{2}\right)^{-n+2r}$$

is also a solution of the Bessel differential equation.

(2) If n is not an integer, then $J_n(x)$ and $J_{-n}(x)$ are linearly independent solutions.

(3) If n is an integer, then $J_n(x)$ and $J_{-n}(x)$ are linearly dependent and are related by

$$J_{-n}(x) = (-1)^n J_n(x)$$

Generating function of Bessel function:

$$e^{\frac{x}{2}\left(z-\frac{1}{z}\right)} = \sum_{n=-\infty}^{\infty} z^n J_n(x)$$

The coefficient of z^n in the expansion of the generating function is the Bessel function of order n.

Orthonormal property of Bessel Fucntion:

$$\int_0^1 xJ_n(\alpha x)J_n(\beta x)\,dx = \frac{\delta_{\alpha\beta}}{2}\left[J_{n+1}(\alpha)\right]^2$$

Recurrence relations for Bessel Fucntion:

(i) $x\,J_n'(x) = nJ_n - xJ_{n+1}$

(ii) $x\,J_n' = -nJ_n x\,J_{n-1}$

(iii) $2J_n' = J_{n-1} - J_{n+1}$

(iv) $2n\,J_n = x\left[J_{n-1} + J_{n+1}\right]$

(v) $\frac{d}{dx}\left[x^{-n}J_n\right] = -x^{-n}J_{n+1}$

(vi) $\frac{d}{dx}\left[x^{n}J_n\right] = x^{n}J_{n-1}$

(C) Hermite Differential Equation

$$\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2ny=0$$

Note: $x=0$ is an ordinary point of the Hermite differential equation

Solution:

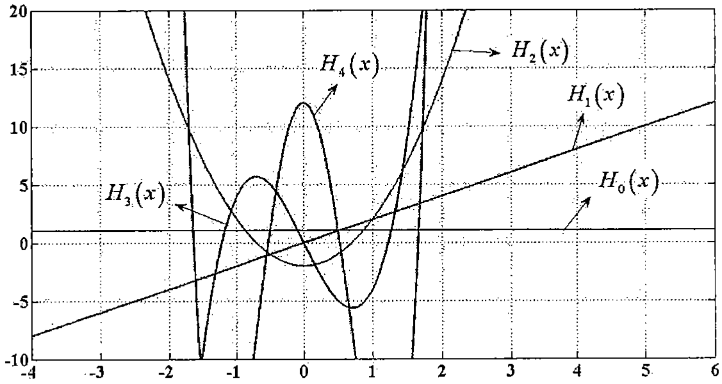
Hermite polynomial of order n:

$$H_n(x)=(-1)^n e^{x^2}\frac{d^n}{dx^n}\left(e^{-x^2}\right)$$

First few orders of Hermite polynomials are:

$$H_0(x)=1 \qquad H_3(x)=8x^3-12x$$
$$H_1(x)=2x \qquad H_4(x)=16x^4-48x^2+12$$
$$H_2(x)=4x^2-2$$

Note: $H_n(x)$ will be even function if n is even and will be odd function if n is odd.



Generating function for Hermite polynomials:

$$e^{2zx-z^2}=\sum_{n=0}^{\infty}\frac{H_n(x)}{n!}z^n$$

Coefficient of z^n in the expansion of e^{2zx-z^2} is $\dfrac{H_n(x)}{n!}$

Orthonormal property of Hermite polynomials:

$$\int_{-\infty}^{\infty}e^{-x^2}H_n(x)H_m(x)\,dx=2^n n!\sqrt{\pi}\,\delta_{mn}$$

Recurrence relation for Hermite polynomials:

(i) $H_n'(x) = 2n\,H_{n-1}(x)$ (ii) $2xH_n(x) = 2nH_{n-1}(x) + H_{n+1}(x)$

(iii) $H_n'(x) = 2xH_n(x) - H_{n+1}(x)$ (iv) $H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0$

(v) $H_n(-x) = (-1)^n H_n(x)$

(D) Laguerre Differential Equation

$$x\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} + ny = 0$$

Note: $x = 0$ is a regular singular point of the Laguerre differential equation.

Solution:

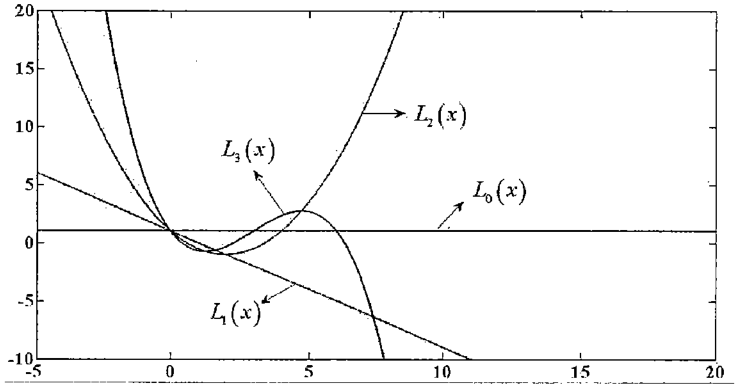
Laguerre polynomial of order n: $L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}\left(x^n e^{-x}\right)$

First few orders of Laguerre polynomials are

$$L_0(x) = 1 \qquad L_1(x) = 1 - x \qquad L_2(x) = \frac{1}{2}\left(x^2 - 4x + 2\right)$$

$$L_3(x) = \frac{1}{6}\left(-x^3 + 9x^2 - 18x + 6\right)$$

Note: Laguerre polynomials are neither even nor odd functions of x.



Generating function for Laguerre polynomials:

$$\frac{e^{-xz/(1-z)}}{(1-z)} = \sum_{n=0}^{\infty} z^n L_n(x)$$

Orthogonal property of Laguerre polynomials:

$$\int_0^{\infty} e^{-x} L_n(x) L_m(x)\,dx = \delta_{mn}$$

TIFR PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. The differential equation $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=0$ has the complete solution, in terms of arbitrary constants A and B as **[TIFR 2013]**

(a) $A\exp x+Bx\exp x$

(b) $A\exp x+B\exp(-x)$

(c) $A\exp x+Bx\exp(-x)$

(d) $x\{A\exp x+B\exp(-x)\}$

Soln: $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=0$

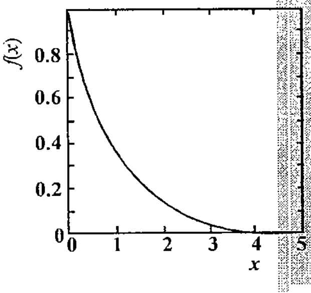
Auxillary equation : $m^2-2m+1=0 \Rightarrow (m-1)^2=0 \Rightarrow m=1,1$

Hence, $y(x)=(A+Bx)e^x$

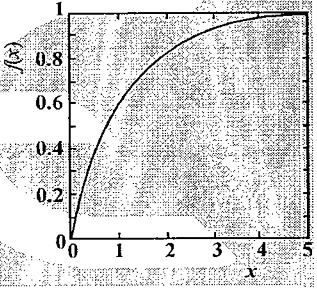
Correct option is (a)

Question 2. The solution of the integral equation $f(x)=x-\int_0^x dt\, f(t)$ has the graphical form **[TIFR 2014]**

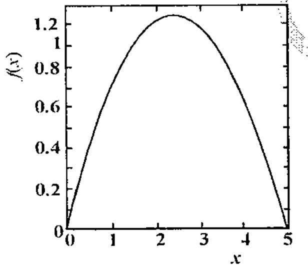
(a)



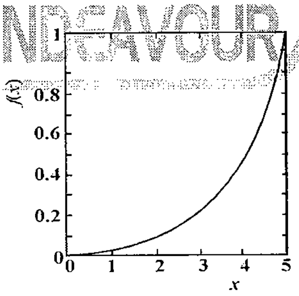
(b)



(c)



(d)



Soln: $f(x)=x-\int_0^x dt\, f(t) \Rightarrow f'(x)=1-f(x) \Rightarrow f'(x)+f(x)=1 \;\Rightarrow \frac{d\,f(x)}{dx}+f(x)=1.$

Therefore, I.F. $=e^{\int 1dx}=e^x$

Therefore, solution will be $e^x f(x)=\int 1\times e^x\,dx+C \Rightarrow f(x)=\frac{e^x+C}{e^x} \Rightarrow f(x)=1+Ce^{-x}$

Hence, as $x\to\infty$; $f(x)\to 1$. Here, only option (b) is satisfying this condition.

Correct option is (b)

Question 3. Consider the differential equation $\frac{d^2y}{dx^2} = -4\left(y + \frac{dy}{dx}\right)$ with the boundary conditions that $y(x) = 0$ at $x = \frac{1}{5}$. When plotted as a function of x , for $x \geq 0$, we can say with certainity that the value of y

(a) first increases, then decreases to zero. **[TIFR 2015]**

(b) first decreases, then increases to zero.

(c) has an extremum in the range $0 < x < 1$.

(d) oscillates from positive to neagtive with amplitude decreasing to zero.

Soln: $\frac{d^2y}{dx^2} = -4\left(y + \frac{dy}{dx}\right) \Rightarrow \frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$

Auxillary equation: $m^2 + 4m + 4 = 0 \Rightarrow (m+2)^2 = 0 \Rightarrow m = -2, -2$

Hence, $y(x) = (c_1 + c_2 x)e^{-2x}$.

Also, $y(x) = 0$ at $x = \frac{1}{5}$

Therefore, $0 = \left(c_1 + c_2 \times \frac{1}{5}\right)e^{-2/5} \Rightarrow c_1 + \frac{c_2}{5} = 0 \Rightarrow c_1 = -\frac{c_2}{5}$

Therefore, $y(x) = c_2\left(\frac{1}{5} - x\right)e^{-2x}$

Now for maxima and minima $y'(x) = c_2(-1)e^{-2x} + c_2\left(\frac{1}{5} - x\right) \times -2e^{-2x} = 0$

$\Rightarrow c_2\left[-1 - \frac{2}{5} + 2x\right]e^{-2x} = 0 \qquad \Rightarrow 2x = \frac{7}{5} \Rightarrow x = \frac{7}{10}$

Hence, $y(x)$ has an extremum (either maxima or minima) at the point 7/10. We can not claim it to have maxima or minima at $x = \frac{7}{10}$, becuase we don't know the sign of c_2.

Correct option is (c)

Question 4. The function $y(x)$ satisfies the differential equation $x\frac{dy}{dx} = y(\ln y - \ln x + 1)$ with the initial condition $y(1) = 3$. What will be the value of $y(3)$? **[TIFR 2016]**

Soln: We have, $x\frac{dy}{dx} = y(\ln y - \ln x + 1) \Rightarrow \frac{dy}{dx} = \left(\frac{y}{x}\right)\left[\ln\left(\frac{y}{x}\right) + 1\right]$

Let, $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\Rightarrow \quad v + x\frac{dv}{dx} = v[\ln v + 1] \quad \Rightarrow \quad v + x\frac{dv}{dx} = v\ln v + v \quad \Rightarrow \quad \int\frac{dv}{v\ln v} = \int\frac{dx}{x}$

$\Rightarrow \quad \ln(\ln v) = \ln x + c \quad \Rightarrow \quad \ln\left(\ln\frac{y}{x}\right) = \ln x + c$ (puting $v = \frac{y}{x}$)

Given: $y(1)=3 \Rightarrow \quad \ln(\ln(3))=c$

$\therefore \quad \ln\left[\ln\left(\frac{y}{x}\right)\right]=\ln x+\ln(\ln 3) \Rightarrow \quad \ln\left(\frac{y}{x}\right)=x\cdot\ln 3 \Rightarrow y=xe^{(x\ln 3)}$

$\therefore \quad y(3)=3\times e^{(3\ln 3)}=3\times 27=81$

Correct answer is (81)

Question 5. Write down $x(t)$, where $x(t)$ is the solution of the following differential equation

$$\left(\frac{d}{dt}+2\right)\left(\frac{d}{dt}+1\right)x=1,$$

with the boundary conditions $\left.\frac{dx}{dt}\right|_{t=0}=0, \quad x(t)|_{t=0}=-\frac{1}{2}.$ **[TIFR 2017]**

Soln: $\left(\frac{d}{dt}+2\right)\left(\frac{d}{dt}+1\right)x=1 \Rightarrow \frac{d^2x}{dt^2}+\frac{dx}{dt}+2\frac{dx}{dt}+2x=1 \Rightarrow \frac{d^2x}{dt^2}+3\frac{dx}{dt}+2x=1$

Auxillary equation : $m^2+3m+2=0 \Rightarrow (m+1)(m+2)=0 \Rightarrow m=-1,-2$

Therefore, C.F. $=c_1e^{-x}+c_2e^{-2x}$

P.I. $=\frac{1}{D^2+3D+2}\times 1\, e^0=\frac{1}{2}$ (Put $D=0$)

Therefore, $x(t)=c_1e^{-x}+c_2e^{-2x}+\frac{1}{2}$

Now, $x(t)|_{t=0}=c_1+c_2+\frac{1}{2}=-\frac{1}{2} \Rightarrow c_1+c_2=1$ ...(1)

And $\frac{d\,x(t)}{dt}=-1c_1e^{-x}-2c_2e^{-2x}$

$\left.\frac{d\,x(t)}{dt}\right|_{t=0}=-c_1-2c_2=0 \Rightarrow c_1=-2c_2$ ...(2)

From Eq. (1) and Eq. (2) ; $c_2=-1$ and $c_1=2$.

Therefore, $x(t)=-e^{-x}+2e^{-2x}+\frac{1}{2}$.

Question 6. If $y(x)$ satisfies the differential equation $y''-4y'+4y=0$ with boundry conditions $y(0)=1, y'(0)=0$, then $y\left(-\frac{1}{2}\right)=$ **[TIFR 2018]**

(a) $\frac{2}{e}$ (b) $\frac{1}{2}\left(e+\frac{1}{e}\right)$ (c) $\frac{1}{e}$ (d) $-\frac{e}{2}$

Soln: Auxillary equation : $m^2-4m+4=0 \Rightarrow (m-2)^2=0 \Rightarrow m=2,2$

Therefore, $y=(c_1+c_2x)e^{2x}$

Given $y(0)=1 \Rightarrow (c_1+c_2.0)e^{2.0}=1 \Rightarrow c_1=1$

Here, $y'(x)=c_2e^{2x}+(c_1+c_2x)(2e^{2x})$

Given $y'(0)=0 \Rightarrow c_2.e^{2.0}+(c_1+c_2.0)(2.e^{2.0})=0 \Rightarrow c_2+2c_1=0 \Rightarrow c_2=-2$

Therefore, the solution will be $y(x)=(1-2x)e^{2x}$

So, $y\left(x=-\frac{1}{2}\right)=\left[1-2.\left(-\frac{1}{2}\right)\right]e^{2\left(-\frac{1}{2}\right)}=\frac{2}{e}$

Correct option is (a)

JEST PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. Consider the differential equation $\frac{dG(x)}{dx}+kG(x)=\delta(x)$

where k is a constant. Which of the following statement is true? **[JEST 2013, 2015]**

(a) Both G(x) and G'(x) are continuous at $x=0$

(b) G(x) is continuous at $x=0$ but G'(x) is not.

(c) G(x) is discontinuous at $x=0$

(d) The continuity properties of G(x) and G'(x) at $x=0$ depend on the value of k.

Soln: $\frac{dG(x)}{dx}+kG(x)=\delta(x)$

I.F. $= e^{\int k\,dx}=e^{kx}$

Therefore, the required solution will be

$$G(x)e^{kx}=\int \delta(x)e^{kx}+c$$

Continuity check at x = 0:

L.H.L. $= \lim_{x\to 0^-} G(x)e^{kx} = \lim_{x\to 0^-}\int \delta(x)e^{kx}+c=c$

R.H.L. $= \lim_{x\to 0^+} G(x)e^{kx} = \lim_{x\to 0^+}\int \delta(x)e^{kx}+c=c$

At $x=0$, $G(0)e^{kx}\Big|_{x=0}=e^0+c=1+c$

Since, L.H.L. = R.H.L. $\neq G(0)$. Hence, $G(x)$ is not continuous at $x=0$.

Correct option is (c)

Question 2. What are the solutions to $f''(x)-2f'(x)+f(x)=0$? **[JEST 2014]**

(a) c_1e^x/x (b) c_1x+c_2/x (c) $c_1xe^x+c_2$ (d) $c_1e^x+c_2xe^x$

Soln: $f''(x)-2f'(x)+f(x)=0$

Auxillary equation $m^2-2m+1=0 \Rightarrow (m-1)^2=0 \Rightarrow m=1,1$

Therefore, $f(x)=(c_1+c_2x)e^x$

Correct option is (d)

JNU PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. The general solution of the differential equation, $x^2\dfrac{d^2y}{dx^2}-2x\dfrac{dy}{dx}+2y=0$ in terms of two arbitrary constants A and B, is **[JNU 2012]**

(a) $e^{1/x}\left(A\cos\left(\dfrac{1}{x}\right)+B\sin\left(\dfrac{1}{x}\right)\right)$ (b) $Ax+\dfrac{B}{x}$

(c) $Ax + Bxe^x$ (d) $Ax + Bx^2$

Soln: Given equation is Cauchy Euler equation of second order.

$$x=e^z \Rightarrow \log x = z \Rightarrow \frac{1}{x}=\frac{dz}{dx}$$

Putting $x\dfrac{dy}{dx}=\dfrac{dy}{dz}$ and $x^2\dfrac{d^2y}{dx^2}=\dfrac{d^2y}{dz^2}-\dfrac{dy}{dz}$, we get $\Rightarrow \dfrac{d^2y}{dz^2}-3\dfrac{dy}{dz}+2y=0$

Auxilliary equation: $m^2-3m+2=0 \Rightarrow (m-2)(m-1)=0 \Rightarrow m=1,2$

Therefore, solution will be $y=c_1e^z+c_2e^{2z} \Rightarrow y=c_1x+c_2x^2$.

Correct option is (d)

Question 2. For $x > 0$, the solution of the differential equation

$$\frac{dy}{dx}+2xy-\frac{1}{x}e^{-x^2}=0$$

(with the condition that y = 0 at x = 1) is (In the following: $Erf(x)=\dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^x e^{-\zeta^2}d\zeta$) **[JNU 2015]**

(a) $e^{-x^2}\ln x$ (b) $\ln x . Erf(x)$ (c) $(x-1)e^{-x^2}$ (d) $1-\dfrac{1}{x}$

Soln: $\dfrac{dy}{dx}+2xy-\dfrac{1}{x}e^{-x^2}=0$

$$\text{I.F.}=e^{\int 2x\,dx}=e^{x^2}$$

Therefore, the required solution will be $y(x)e^{x^2}=\displaystyle\int\frac{1}{x}e^{-x^2}e^{x^2}+c$

$$\Rightarrow y(x)e^{x^2}=\ln x+c \Rightarrow y(x)=e^{-x^2}\ln x+c\,e^{-x^2}$$

Also, $y=0$ at $x=1 \Rightarrow 0=0+ce^{-1} \Rightarrow \boxed{c=0}$

Therefore, $y(x)=\ln e^{-x^2}$.

Correct option is (a)

Question 3. Consider the following differential equation $\frac{d^3y}{dx^3}-3\frac{d^2y}{dx^2}+2\frac{dy}{dx}=0$. It is given that $\frac{d^2y}{dx^2}=1,\ \frac{dy}{dx}=0$ and $y=1$ at $x=0$. Then the value of y at $x=2$ is nearest to **[JNU 2016]**

(a) 30 (b) 42 (c) 21 (d) 63 (d) 12

Soln: $\frac{d^3y}{dx^3}-3\frac{d^2y}{dx^2}+2\frac{dy}{dx}=0$

Auxilliary equation is $\left(D^3-3D+2D\right)y=0$

For non-trivial solution $D^3-3D+2D=0 \Rightarrow D\left(D^2-3D+2\right)=0$

Hence, $\boxed{D=0,1,2}$

$\therefore \quad y=C_1e^{0x}+C_3e^x+C_3e^{2x}$

$y=C_1+C_2e^x+C_3e^{2x} \quad \dots(1)$

$\frac{dy}{dx}=C_2e^x+2C_3e^{2x} \quad \dots(2)$

And $\frac{d^2y}{dx^2}=C_2e^x+4C_3e^{2x} \quad \dots(3)$

$\because \quad y=1$ at $x=0$, from (1), $1=C_1+C_2+C_3 \quad \dots(4)$

$\frac{dy}{dx}=0$ at $x=0$, from (2), $0=C_2+2C_3 \quad \dots(5)$

And $\frac{d^2y}{dx^2}=1$ at $x=0$, from (3), $1=C_2+4C_3 \quad \dots(6)$

Solving equations (4), (5) and (6), we get

$$\boxed{C_1=\frac{3}{2},\ C_2=-1,\ C_3=\frac{1}{2}}$$

Hence, $y(x)=\frac{3}{2}-e^x+\frac{1}{2}e^{2x}$

Therefore, at $x=2$,

$$y(x=2)=\frac{3}{2}-e^2+\frac{1}{2}e^4\simeq 21$$

Correct option is (c)

Question 4. The general solution of the ordinary differential equation, $\frac{d^2y}{dx^2}+y+1=0$ is (A and B being constants). **[JNU 2017]**

(a) x^2+Ax+B (b) $A(x-1)^3+B$ (c) $A\sin x$

(d) $A\sin(x+B)$ (e) $A\cos x+B\sin x-1$

Soln: $\frac{d^2y}{dx^2}+y+1=0$ Let $z=y+1$ $\therefore$ $\frac{d^2y}{dx^2}=\frac{d^2z}{dx^2}$

$\therefore \quad \frac{d^2z}{dx^2}+z=0$

Auxillary equation $m^2+1=0 \Rightarrow m=\pm i$

Therefore, the solution will be $z=A\cos x+B\sin x$

$\therefore \quad y=A\cos x+B\sin x-1$

Correct option is (d)

DU PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. The general solution of the differential equation $\frac{d^3y}{dx^3}+\frac{1}{x}\frac{d^2y}{dx^2}+\frac{1}{x^2}\frac{dy}{dx}=0$ can be expressed as (C_1, C_2 and C_3 are constants): **[DU 2014]**

(a) $C_1+C_2x^{1+i}+C_3x^{1-i}$
(b) $C_1+C_2x+C_3x^i$
(c) $C_1x+C_2x^2+C_3x^3$
(d) $C_1x+C_2x^{1+i}+C_3x^{1-i}$

Soln: $\frac{d^3y}{dx^3}+\frac{1}{x}\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}=0 \Rightarrow x^3\frac{d^3y}{dx^3}+x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}=0$

Given equation is Cauchy Euler equation of third order $x=e^z \Rightarrow \log x=z \Rightarrow \frac{1}{x}=\frac{dz}{dx}$

Putting $x\frac{dy}{dx}=\frac{dy}{dz}$, $x^2\frac{d^2y}{dx^2}=\frac{d^2y}{dz^2}-\frac{dy}{dz}$ and $x^3\frac{d^3y}{dx^3}=\frac{d^3y}{dz^3}-3\frac{d^2y}{dz^2}+2\frac{dy}{dz}$,

we get $\Rightarrow \frac{d^3y}{dz^3}-2\frac{d^2y}{dz^2}+2\frac{dy}{dz}=0$

Auxilliary equation: $m^3-2m^2+2m=0 \Rightarrow m\left(m^2-2m+2\right)=0 \Rightarrow m=0,1+i,1-i$

Therefore, the solution will be $y=c_1e^{0z}+c_2e^{(1+i)z}+c_3e^{(1-i)z} \Rightarrow y=c_1+c_2x^{1+i}+c_3x^{1-i}$

Correct option is (a)

Question 2. The most general solution of the differential equation $\frac{d^2y}{dx^2}+\frac{4dy}{dx}+4y=0$ is (here 'a' and 'b' are constants)

[DU 2016]

(a) $ae^{-2x}+be^{2x}$ (b) $ae^{-2x}+be^{-2x}$ (c) $ae^{-2x}+bxe^{-2x}$ (d) $ae^{2x}+bxe^{2x}$

Soln: Auxilliary equation: $m^2+4m+4=0 \Rightarrow (m+2)^2=0 \Rightarrow m=-2,-2$

Hence, $y=(a+bx)e^{-2x}$.

Correct option is (c)

Question 3. The solution to the differential equation $\frac{dy}{dx}+2xy=x$ given that $y(x=0)=0$ is **[DU 2017]**

(a) $y=(1-e^{-x^3})$ (b) $y=\frac{1}{4}(1-e^{-x^2})$ (c) $y=\frac{1}{2}(1-e^{-x^2})$ (d) $y=\frac{1}{2}(1-e^{-x})$

Soln: $\text{I.F.}=e^{\int 2x\,dx}=e^{x^2}$

Therefore, the solution will be $y\times e^{x^2}=\int xe^{x^2}dx+c$

Putting $x^2=t\Rightarrow 2x\,dx=dt$, we get

$$ye^{x^2}=\int\frac{dt}{2}e^t+c=\frac{1}{2}e^t+c \quad\Rightarrow ye^{x^2}=\frac{1}{2}e^{x^2}+c\Rightarrow y=\frac{1}{2}+ce^{-x^2}$$

Since, $y(0)=0$, put $x=0,\ y=0$ $\Rightarrow 0=\frac{1}{2}+c\Rightarrow c=-\frac{1}{2}$

Therefore, $y=\frac{1}{2}\left(1-e^{-x^2}\right)$.

Correct option is (c)

HCU PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. The most general solution of the equation $\frac{d^2x}{dt^2}+\omega^2x=ce^{-\alpha t}$ can be written as

(a) $A\sin\omega t+B\cos\omega t$ (b) $A\sin\omega t+B\cos\omega t+Ce^{-\alpha t}$ **[H.C.U.-2010]**

(c) $A\sin\omega t+B\cos\omega t+C(\alpha^2+\omega^2)e^{-\alpha t}$ (d) $A\sin\omega t+B\cos\omega t+Ce^{-\alpha t}/(\alpha^2+\omega^2)$

Soln: $\frac{d^2x}{dt^2}+\omega^2x=ce^{-\alpha t}$

Auxilliary equation: $m^2+\omega^2=0\Rightarrow m=\pm i\omega$

Therefore, C.F. = $A\cos\omega t+B\sin\omega t$

$$\text{P.I.}=\frac{1}{D^2+\omega^2}ce^{-\alpha t}=\frac{ce^{-\alpha t}}{\alpha^2+\omega^2}$$

Therefore, solution will be $x=A\cos\omega t+B\sin\omega t+\frac{ce^{-\alpha t}}{\alpha^2+\omega^2}$

Correct option is (a)

Question 2. The solution of the differential equation $\frac{dy}{dx}=x+xy$ is to be obtained under the condition that $y=0$ at $x=0$. The solution is given by **[H.C.U.-2011]**

(a) $y=\frac{x^2}{2}$ (b) $y=e^{-x^2/2}-1$ (c) $y=e^{x^2/2}-1$ (d) $y=\frac{e^x}{2}-1$

Soln: $\frac{dy}{dx}=x+xy \quad\Rightarrow\int\frac{dy}{1+y}=\int x\,dx \quad\Rightarrow\ln(1+y)=\frac{x^2}{2}+c$

$\because\ y = 0$ at $x = 0 \quad \Rightarrow \ln(1) = 0 + c \ \Rightarrow c = 0$

Therefore, the solution will be, $\ln(1+y) = \dfrac{x^2}{2} \Rightarrow y = e^{x^2/2} - 1$

Correct option is (c)

Question 3. Complete solution of the differential equation

$$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 6y = e^{2x}$$

is given by (where c_1 and c_2 are arbitrary constants) **[H.C.U.-2012]**

(a) $c_1e^x + c_2e^{3x} - e^{2x}$ (b) $c_1e^x + c_2e^{6x} - e^{2x}$ (c) $c_1e^x + c_2e^{6x} - \frac{1}{4}e^{2x}$ (d) $c_1e^x + c_2e^{3x} - \frac{1}{4}e^{2x}$

Soln: Auxilliary equation: $m^2 - 7m + 6 = 0 \ \Rightarrow m = 1, 6$

$\text{C.F.} = c_1\, e^x + c_2\, e^{6x}$

$$\text{P.I.} = \frac{1}{D^2 - 7D + 6} \times e^{2x} = \frac{e^{2x}}{4 - 14 + 6} = \frac{e^{2x}}{-4}$$

$\therefore\ y = c_1\, e^x + c_2\, e^{6x} - \frac{1}{4}\, e^{2x}$

Correct option is (c)

Question 4. Two independent solutions of the differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$$

may be chosen to be

(a) $e^{-2x} + e^{-x}, 2\left(e^{-2x} - e^{-x}\right)$ (b) $e^{-2x} + e^{-x}, 2\left(e^{-2x} + e^{-x}\right)$

(c) $e^{-2x} + e^{-x}, x\left(e^{-2x} - e^{-x}\right)$ (d) e^{-2x}, e^{x} **[H.C.U.-2012]**

Soln: Auxilliary equation: $m^2 + 3m + 2 = 0 \Rightarrow m = -1, -2$

$\therefore\ y = c_1\, e^{-x} + c_2\, e^{-2x}$ where c_1 and c_2 are arbitary constants.

Correct option is (a)

Question 5. The solution to the differential equation $\dfrac{d^2x}{dt^2} = t^2$ is given by **[H.C.U.-2015]**

(a) $x = \dfrac{t^3}{3} + A$ (b) $x = \dfrac{t^3}{3} + At + B$ (c) $x = \dfrac{t^4}{12} + At + B$ (d) $x = \dfrac{t^4}{3} + \dfrac{t^2}{2} + A$

where A and B are constants.

Soln: $\dfrac{d^2x}{dt^2} = t^2 \quad \Rightarrow x = \dfrac{t^4}{12} + At + B$

Correct option is (c)

Question 6. The general soution to the differential equation $\frac{d^2y\cdot}{dx}-\frac{2dy}{dx}+2y=0$, is

(a) $e^x\left[A\cos x+iB\sin x\right]$ (b) $e^{ix}\left[A\cos x+iB\sin x\right]$

(c) $e^{ix}\left[A\cos x-iB\sin x\right]$ (d) $e^x\left[A\cos x-iB\sin x\right]$ **[H.C.U.-2015]**

where A and B are arbitrary constants.

Soln: Auxilliary equation: $m^2-2m+2=0 \Rightarrow m=1+i, 1-i$

$\therefore \quad y=e^x\,(A\cos x+B\sin x)$

Correct option is ( )

Question 7. On changing variable from x to t where $t=\log x$, the differential equation $x^2\frac{d^2y}{dx^2}+\alpha x\frac{dy}{dx}+\beta y=0$ becomes

(a) $e^{2t}\frac{d^2y}{dx^2}+\alpha e^t\frac{dy}{dx}+\beta y=0$ (b) $\frac{d^2y}{dx^2}+\alpha\frac{dy}{dx}+\beta y=0$

(c) $\frac{d^2y}{dx^2}+\alpha\frac{dy}{dx}+(\alpha+\beta)y=0$ (d) $\frac{d^2y}{dx^2}+(\alpha-1)\frac{dy}{dx}+\beta y=0$ **[H.C.U.-2015]**

Soln: $t=\log x$ or $x=e^t$; $\frac{dx}{dt}=e^t$

If $\frac{dy}{dx}=D$ and $\frac{dy}{dt}=D_1$, then $x^2\frac{d^2y}{dx^2}=D_1(D_1-1)y$ and $x\frac{dy}{dx}=D_1y$

Therefore, given differential equation $x^2\frac{d^2y}{dx^2}+\alpha x\frac{dy}{dx}+\beta y=0$

is now $D_1(D_1-1)y+\alpha D_1y+\beta y=0$

$\Rightarrow D_1^2y+(\alpha-1)D_1+\beta y=0 \quad \Rightarrow \frac{d^2y}{dt^2}+(\alpha-1)\frac{dy}{dt}+\beta y=0$

Correct option is (d)

Question 8. For $\lambda>0$ and $\lambda\in\mathbb{R}$, what are the eigenfunctions of the following differential equation $y''+\lambda y=0$, for the boundary conditions $y(0)=0,\ y'(\pi)=0$? **[H.C.U.-2017]**

(a) $C_1\cos\left(\frac{2n-1}{2}x\right)$ (b) $C_2\sin\left(\frac{2n-1}{2}x\right)$

(c) $C_1\cos(nx)+C_2\sin(nx)$ (d) $C_1\cos(2n-1)x+C_2\sin(2n-1)x$

Soln: Auxilliary equation: $m^2+\lambda=0 \Rightarrow m=+i\sqrt{\lambda},-i\sqrt{\lambda}$

Therefore, C.F. $=A\cos\sqrt{\lambda}x+B\sin\sqrt{\lambda}x$

Now given condition $y(0)=0 \Rightarrow \boxed{A=0}$

Therefore, $y = B\sin\sqrt{\lambda}x$

Here, $\frac{dy}{dx} = B\cos\sqrt{\lambda}x \times \sqrt{\lambda}$

Put $\frac{dy}{dx}=0$ and $x=\pi \Rightarrow \cos\sqrt{\lambda}\pi = 0 \;\Rightarrow \sqrt{\lambda}\pi = \left(\frac{2n-1}{2}\right)\pi \Rightarrow \sqrt{\lambda} = \left(\frac{2n-1}{2}\right)$

$\therefore\; y = A\sin\left(\frac{2n-1}{2}x\right)$

Correct option is (a)

BHU PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. Consider the solution $y(t)$ of the equation **[B.H.U-2010]**

$$\frac{d^2y}{dt^2} + 2k\frac{dy}{dt} + k^2y = \exp(-kt)$$

where k is a constant and whose value and derivative with respect to t are zero at $t=0$. Then $y(t)$ is

(a) $\frac{1}{2}\exp(-kt)$ (b) $\frac{1}{2}t\exp(-kt)$ (c) $\frac{1}{2}t^2\exp(-kt)$ (d) $\frac{1}{2}\exp(-kt^2)$

Soln: $\frac{d^2y}{dt^2} + 2k\frac{dy}{dt} + k^2y = \exp(-kt)$

Auxillary equation, $(D^2+2kD+k^2)y=0$

For non-trivial solution, $D^2+2kD+k^2=0$

$(D+k)^2=0 \Rightarrow D=-k,-k$

Therefore, C.F. $=(c_1+c_2t)e^{-kt}$

$$\text{P.I.} = \frac{1}{D^2+2kD+k^2}e^{-kt} = t\times\frac{1}{2D+2k}e^{-kt} = t^2\times\frac{1}{2}e^{-kt}$$

Hence, $y(t) = (c_1+c_2t)e^{-kt} + \frac{t^2e^{-kt}}{2}$

Since, $y(t=0)=0$. Hence, $0=c_1$

$$y'(t) = c_2te^{-kt}\times -k + c_2e^{-kt} + \frac{t^2}{2}e^{-kt}\times -k + \frac{2te^{-kt}}{2}$$

Since, $y'(t=0)=0 \Rightarrow 0=c_2$

Hence, $y(t) = \frac{t^2}{2}e^{-kt}$.

Correct option is (c)

Question 2. The generating function for Legendre polynomial $P_n(x)$ is **[B.H.U-2011]**

(a) $\left(1-2xt+t^2\right)^{-1/2}$ (b) $\left(1+2xt-t^2\right)^{-1/2}$ (c) $\left(1-2xt+t^2\right)^{1/2}$ (d) $\left(1+2xt-t^2\right)^{1/2}$

Soln: **Correct option is (a)**

Question 3. The value of $\frac{d}{dx}\left[x^n J_n(x)\right]$ is **[B.H.U-2011]**

(a) $x^n J_{n+1}(x)$ (b) $-x^{-n} J_{n+1}(x)$ (c) $x^n J_{n-1}(x)$ (d) $x^{n-1} J_{n-1}(x)$

Soln: **Correct option is (b)**

Question 4. If the Rodrigue's formula for a polynomial is **[B.H.U-2011]**

$$(-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right)$$

then the polynomial is

(a) Legendre (b) Hermite (c) Lagure (d) Bessel

Soln: **Correct option is (b)**

Question 5. $a_1 = 1$ and $a_n = n(1+a_{n-1})$ for $n = 2,3$. Define $P_n = \prod_{k=1}^{n}\left(1+\frac{1}{a_k}\right)$. Then $\lim_{n\to\infty} P_n$ is

(a) 1 (b) ∞ (c) e (d) $1+e$ **[B.H.U-2012]**

Soln: **Correct option is (a)**

Question 6. The differential equation $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0$ has solution as **[B.H.U-2013]**

(a) Bessel function $J_n(x)$ (b) Hermite polynomial $H_n(x)$

(c) Legendre polynomial $P_n(x)$ (d) Laguerre polynomial $L_n(x)$

Soln: **Correct option is (a)**

Question 7. The spherical Bessel function $j_n(x)$ is related to the Bessel function $J_n(x)$ by the relation

(a) $j_n(x)=\sqrt{\frac{\pi}{2x}}J_{n+\frac{1}{2}}(x)$ (b) $j_n(x)=\sqrt{\frac{\pi}{2x}}J_n(x)$ **[B.H.U-2013]**

(c) $j_n(x)=\sqrt{\frac{2x}{\pi}}J_{n+\frac{1}{2}}(x)$ (d) $j_n(x)=\sqrt{\frac{2x}{\pi}}J_n(x)$

Soln: **Correct option is (a)**

Question 8. The value of $J_0(x)$ at $x=0$ is given by **[B.H.U-2013]**

(a) 1 (b) 0 (c) ∞ (d) $\sqrt{\pi}$

Soln: $J_0(x)$ at $x=0$ is equal to 1.

Correct option is (a)

Question 9. The Legendre series expansion for the function $f(\theta) = \sin^2\left(\frac{\theta}{2}\right)$ can be given as :

(a) $\frac{1}{2}\left[P_0(\cos\theta) - P_1(\cos\theta)\right]$ (b) $\frac{1}{2}\left[P_2(\cos\theta) - P_3(\cos\theta)\right]$ **[B.H.U-2014]**

(c) $\left[P_1(\cos\theta) - P_2(\cos\theta)\right]$ (d) $\left[P_0(\cos\theta) + P_1(\cos\theta)\right]$

Soln: Since, $P_0(x) = 1$; $P_1(x) = x$; $P_2(x) = \frac{1}{2}(3x^2 - 1)$

Now, $f(\theta) = \sin^2\left(\frac{\theta}{2}\right) = \frac{1 - \cos\theta}{2}$

Put $x = \cos\theta$, $f(\theta) = \frac{1}{2}\left[P_0(\cos\theta) - P_1(\cos\theta)\right]$.

Correct option is (a)

Question 10. The generating function for Hermite polynomial is $g(t, x) = e^{-t_2+2tx}$. The zero of the Hermie polynomial $H_2(x)$ on the positive x-axis is **[B.H.U-2014]**

(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\frac{1}{4}$ (d) $\frac{\sqrt{3}}{2}$

Soln: We know, Hermite polynomial $H_2(x) = 4x^2 - 2$

$H_2(x) = 0 \Rightarrow 4x^2 - 2 = 0$

$x^2 = \frac{1}{2} \Rightarrow x = \pm\frac{1}{\sqrt{2}}$

So, on positive x-axis, $x = \frac{1}{\sqrt{2}}$.

Correct option is (b)

Question 11. The function $e^{\frac{x(t^2-1)}{2t}}$ is a generating function of **[B.H.U-2015]**

(a) Hermite polynomials (b) Legendre polynomials

(c) Laguerre polynomials (d) Bessel functions

Soln: **Correct option is (d)**

Question 12. The value of $J_{1/2}(x)$ is **[B.H.U-2015]**

(a) $\sqrt{\frac{2}{\pi x}}\cos x$ (b) $\sqrt{\frac{2}{\pi x}}\sin x$ (c) $\sqrt{\frac{\pi x}{2}}\cos x$ (d) $\sqrt{\frac{\pi x}{2}}\sin x$

Soln: $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x$.

Correct option is (b)

Question 13. Which of the following is true ?

(a) $J_{5/2}(x)=\frac{3}{x}J_{3/2}(x)-J_{1/2}(x)$ (b) $J_{5/2}(x)=\frac{x}{3}J_{3/2}(x)-J_{1/2}(x)$ **[B.H.U-2015]**

(c) $J_{5/2}(x)=\frac{3}{x}J_{3/2}(x)+J_{1/2}(x)$ (d) $J_{5/2}(x)=\frac{x}{3}J_{3/2}(x)+J_{1/2}(x)$

Soln: Using the recurrence relation, $2nJ_n=x\left[J_{n+1}(x)+J_{n-1}(x)\right]$

Put $n=\frac{3}{2}$, $2\times\frac{3}{2}J_{3/2}(x)=x\left[J_{5/2}(x)+J_{1/2}(x)\right]$

$J_{5/2}(x)=\frac{3}{x}J_{3/2}(x)-J_{1/2}(x)$.

Correct option is (a)

Question 14. Legendre polynomial $P_n(x)$ for order $n=0$ has, for any value of x, value **[B.H.U-2015**

(a) 0 (b) ∞ (c) 1 (d) none of these

Soln: For $n=0$, $P_0(x)=1$

Correct option is (c)

Question 15. For $P_n(x)$, which of the following is true ? **[B.H.U-2015]**

(a) $P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2+1)^n$ (b) $P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}(1-x^2)^n$

(c) $P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$ (d) $P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}(x-1)^n$

Soln: **Correct option is (c)**

Question 16. The value of the Legendre polynomial $P_n(x)$ for $n=3$, is **[B.H.U-2016]**

(a) $\left(\frac{5x^3+3x}{2}\right)$ (b) $\left(\frac{5x^3+3x}{1}\right)$ (c) $\left(\frac{5x^3-3x}{2}\right)$ (d) $\left(\frac{5x^3-3x}{1}\right)$

Soln: $P_3(x)=\frac{1}{2}(5x^3-3x)$.

Correct option is (c)

Question 17. One of the following recurrence relations is correct for the Hermite polynomial $H_n(x)$:

(a) $H_n'(x)=2(n+1)H_{n+1}(x)$ (b) $H_n'(x)=2(n-1)H_{n-1}(x)$

(c) $H_n'(x)=2nH_{n+1}(x)$ (d) $H_n'(x)=2nH_{n-1}(x)$ **[B.H.U-2016]**

Soln: **Correct option is (d)**

Question 18. The value of the Bessal functions $[J_{1/2}(x)]^2+[J_{-1/2}(x)]^2$ is one of the following:

(a) $\frac{2}{\pi x}$ (b) $\frac{\pi x}{2}$ (c) $\sqrt{\frac{2}{\pi x}}$ (d) $\sqrt{\frac{\pi x}{2}}$ **[B.H.U-2016]**

Soln: We have, $J_{1/2}(x)=\sqrt{\frac{2}{\pi x}}\sin x$; $J_{-1/2}(x)=\sqrt{\frac{2}{\pi x}}\cos x$

Hence, $\left[J_{1/2}(x)\right]^2+\left[J_{-1/2}(x)\right]^2=\frac{2}{\pi x}$.

Correct option is (a)

Question 19. The generating functional for the Bessel function is :

(a) $e^{x(z+z^{-1})}=\sum\limits_{n=-\infty}^{+\infty}J_n(x)z^n$ (b) $e^{x(z-z^{-1})}=\sum\limits_{n=-\infty}^{+\infty}J_n(x)z^n$

(c) $e^{\frac{x(z+z^{-1})}{2}}=\sum\limits_{n=-\infty}^{+\infty}J_n(x)z^n$ (d) $e^{\frac{x(z-z^{-1})}{2}}=\sum\limits_{n=-\infty}^{+\infty}J_n(x)z^n$ **[B.H.U-2016]**

Soln: **Correct option is (d)**

Question 20. One of the following is true :

(a) $\frac{d}{dx}\left[x^nJ_n(x)\right]=x^{n-1}J_{n-1}(x)$ (b) $\frac{d}{dx}\left[x^nJ_n(x)\right]=x^{n+1}J_{n-1}(x)$

(c) $\frac{d}{dx}\left[x^nJ_n(x)\right]=x^nJ_{n+1}(x)$ (d) $\frac{d}{dx}\left[x^nJ_n(x)\right]=x^nJ_{n-1}(x)$ **[B.H.U-2016]**

Soln: **Correct option is (d)**

Question 21. In the Bessel equation $x^2\frac{d^2y}{dx^2}+(x^2-n^2)y+x\frac{dy}{dx}=0$ **[B.H.U-2016]**

(a) $x=1$ is a regular singular point (b) $x=1$ is an irregular singular point

(c) $x=0$ is a regular singular point (d) $x=0$ is an irregular singular point

Soln: $x=0$ is a regular singular point.

Correct option is (c)

Question 22. The Wronskian $W(y_1,y_2)$ for the two solutions $y_1(x)$ and $y_2(x)$ of the second order differential equation is

(a) $W=y_1y_2'-y_2y_1'\left(y_1'=\frac{dy_1}{dx}\right)$ (b) $W=y_1y_2'+y_2y_1'\left(y_2'=\frac{dy_2}{dx}\right)$ **[B.H.U-2016]**

(c) $W=y_1y_2'-y_1'y_2'$ (d) $W=y_1y_2'+y_1'y_2'$

Soln: We have, $W(y_1\,y_2\,x)=\begin{vmatrix} y_1(x) & y_2(x)\\ y_1'(x) & y_2'(x)\end{vmatrix}$

Correct option is (a)

Question 23. The coefficient of t^n in the expansion of the function $e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}$ is called **[B.H.U-2017]**

(a) The Legendre function (b) The Bessel function of first kind of order n

(c) Laugurre function (d) Hermite polynomial of order n

Soln: **Correct option is (b)**

Question 24. $H_{n-1}(x)+H_{n+1}(x)$ equals (where terms have their usual meaning) **[B.H.U-2017]**

(a) $\frac{2n}{x}H_n(x)$ (b) $2n\,H_n(x)$ (c) $2H_n'(x)$ (d) $H_{n+2}(x)$

Soln: **None of the options are matching.**

Chapter 4

FOURIER SERIES

4.1 Periodic Function

If the value of function $f(x)$ repeats itself after equal interval of x, then $f(x)$ is said to be periodic function of x.

$$f(x) = f(x+T) = f(x+2T) = \ldots\ldots$$

then $f(x)$ is said to be periodic function of x with period T.

Example: (i) $\sin x$, $\cos x$, $\sec x$, $\operatorname{cosec} x$: $T = 2\pi$

(ii) $\tan x$, $\cot x$: $T = \pi$

(iii) $\sin(5x + 2\pi) = \sin 5x = \sin 5\left(x + \frac{2\pi}{5}\right)$; $T = \frac{2\pi}{5}$

4.2 Fourier Series Expansion

A non-sinusoidal periodic function can be expressed by a sum of a set of sinusoidal oscillating functions or periodic functions i.e. mainly sine and cosine functions (or complex exponentials). This series is known as 'Fourier Series' expansion of the non-sinisoidal periodic function.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

a_0, a_n, b_n are constants, known as Fourier co-efficients.

A periodic function $f(x)$ can be expanded into fourier series which consists of the following terms:

(i) a_0 : a constant term or D. C. term

(ii) $a_1 \cos\frac{\pi x}{L}$ and $b_1 \sin\frac{\pi x}{L}$: Fundamental frequency term i.e term having lowest frequency

(iii) $a_2 \cos\frac{2\pi x}{L}, a_3 \cos\frac{3\pi x}{L} \ldots\ldots.,\ b_2 \sin\frac{2\pi x}{L}, b_3 \sin\frac{3\pi x}{L} \ldots\ldots.$: Overtones i.e terms having frequencies integer multiple of fundamental frequency term.

Note: When a function and its derivative are continuous, then the function can be expanded by powers of x by Taylor series. But for both continuous and discontinuous functions, fourier series expansion is possible under certain conditions.

Dirichlet's conditions:

Suppose a function $f(x)$ is defined in the interval (a,b) and if

(i) it is single-valued in the interval (a,b)

(ii) it is bounded in the interval (a,b)

(iii) it has at most a finite number of maxima and minima in the interval (a,b)

(iv) it has only a finite number of discontinuity in the interval (a,b)

(v) $f(x+T)=f(x)$ where $[T=b-a]$, for all values of x outside $[a,b]$ then,

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{L}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{L}$$

Useful Integrals

(1) $\int_a^b \sin\frac{n\pi x}{L}dx=\int_a^b\cos\frac{n\pi x}{L}dx=0$

(2) $\int_a^b \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx=\int_a^b\cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx=L\,\delta_{mn}$

(3) $\int_a^b \sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx=0$ (4) $\sin n\pi=0,\ \cos n\pi=(-1)^n$

Note: In all of these integrals, $(b-a)$ should be $2L$.

Evaluation of Fourier co-efficients:

CASE 1:

Suppose a function $f(x)$ is defined in the interval $(-L,L)$, then the Fourier series expansion of $f(x)$ is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{L}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{L}$$

where $a_0=\frac{1}{L}\int_{-L}^{L}f(x)dx$; $a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\frac{n\pi x}{L}dx$; $b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\frac{n\pi x}{L}dx$

CASE 2:

Suppose a function $f(x)$ is defined in the interval $(0,2L)$, then the Fourier series expansion of $f(x)$ is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{L}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{L}$$

where $a_0=\frac{1}{L}\int_{0}^{2L}f(x)dx$; $a_n=\frac{1}{L}\int_{0}^{2L}f(x)\cos\frac{n\pi x}{L}dx$; $b_n=\frac{1}{L}\int_{0}^{2L}f(x)\sin\frac{n\pi x}{L}dx$

CASE 3:

Suppose a function $f(t)$ is defined in the interval $\left(0, \frac{2\pi}{\omega}\right)$, then the Fourier series expansion of $f(x)$ is

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t$$

$$a_0 = \frac{\omega}{\pi}\int_0^{2\pi/\omega} f(t)dt\,;\; a_n = \frac{\omega}{\pi}\int_0^{2\pi/\omega} f(t)\cos n\omega t\, dt\,;\; b_n = \frac{\omega}{\pi}\int_0^{2\pi/\omega} f(t)\sin n\omega t\, dt$$

CASE 4:

Suppose a function $f(t)$ is defined in the interval $\left(-\frac{\pi}{\omega}, \frac{\pi}{\omega}\right)$, then the Fourier series expansion of $f(x)$ is

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t$$

$$a_0 = \frac{\omega}{\pi}\int_{-\pi/\omega}^{\pi/\omega} f(t)dt\,;\; a_n = \frac{\omega}{\pi}\int_{-\pi/\omega}^{\pi/\omega} f(t)\cos n\omega t\, dt\,;\; b_n = \frac{\omega}{\pi}\int_{-\pi/\omega}^{\pi/\omega} f(t)\sin n\omega t\, dt$$

Nature of the fourier series for even function and odd function:

CASE 1: EVEN FUNCTION

A function is said to be even if $f(-x) = f(x)$ i.e. nature of the function is symmetric about $x = 0$.

We know that

$$\int_{-L}^{L} F(x)\,dx = \begin{cases} 2\int_0^L F(x)\,dx & \text{if } F(x) \text{ is an even function} \\ 0 & \text{if } F(x) \text{ is an odd function} \end{cases}$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{2}{L}\int_0^L f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx = 0$$

This series will contain only cosine terms and the fourier series expansion can be written as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

CASE 2: ODD FUNCTION

A function is said to be odd if $f(-x) = -f(x)$ i.e the nature of the function is anti-symmetric about $x = 0$.

We know that

$$\int_{-L}^{L} F(x)\,dx = \begin{cases} 2\int_{0}^{L} F(x)\,dx & \text{if } F(x) \text{ is an even function} \\ 0 & \text{if } F(x) \text{ is an odd function} \end{cases}$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = 0$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

This series will contain only sine terms and the fourier series expansion can be written as

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

Example 1: Find the fourier series representing, $f(x) = x,\ 0 < x < 2\pi$ and sketch its graph from $x = -4\pi$ to $x = 4\pi$.

Soln: $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x\,dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{2\pi} = 2\pi$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\cos nx\,dx$$

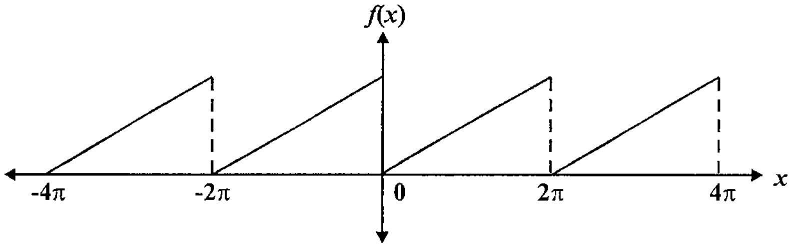
$$= \frac{1}{\pi}\left[x\frac{\sin nx}{n} - 1.\left(-\frac{\cos nx}{n^2}\right)\right]_0^{2\pi} = \frac{1}{\pi}\left[\frac{\cos 2n\pi}{n^2} - \frac{1}{n^2}\right] = \frac{1}{n^2\pi}(1-1) = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin nx\,dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - 1.\left(\frac{-\sin nx}{n^2}\right)\right]_0^{2\pi} = \frac{1}{\pi}\left[\frac{-2\pi\cos 2n\pi}{n}\right] = -\frac{2}{\pi}$$

The required fourier series expansion of the function:

$$f(x) = \pi - 2\left[\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + ...\right]$$



Example 2: Given that $f(x) = x + x^2$ for $-\pi < x < \pi$, find the Fourier expression of $f(x)$. Deduce that $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ...$

Soln: $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\,dx$

$$= \frac{1}{\pi}\left[\frac{x^2}{2} + \frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left[\frac{\pi^2}{2} + \frac{\pi^3}{3} - \frac{\pi^2}{2} + \frac{\pi^3}{3}\right] = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \int_{-\pi}^{\pi}(x+x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[(x+x^2)\frac{\sin nx}{n} - (2x+1)\frac{(-\cos nx)}{n^2} + (2)\left(-\frac{\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(2\pi+1)\frac{\cos n\pi}{n^2} - (-2\pi+1)\frac{\cos(-n\pi)}{n^2}\right] = \frac{1}{\pi}\left[4\pi\frac{\cos n\pi}{n^2}\right] = \frac{4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,x\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x+x^2)\left(-\frac{\cos nx}{n}\right) - (2x+1)\left(\frac{-\sin nx}{n^2}\right) + 2\frac{\cos nx}{n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-(\pi+\pi^2)\frac{\cos n\pi}{n} + 2\frac{\cos n\pi}{n^3} + (-\pi+\pi^2)\frac{\cos n\pi}{n} - 2\frac{\cos n\pi}{n^3}\right]$$

$$= \frac{1}{\pi}\left[-\frac{2\pi}{n}\cos n\pi\right] = -\frac{2}{n}(-1)^n$$

The required fourier series series expansion of the function:

$$x + x^2 = \frac{\pi^3}{3} + 4\left[-\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + ...\right] - 2\left[-\sin x + \frac{1}{2}\sin 2x - \frac{1}{3}\sin 3x + ...\right]$$

Putting $x = \pi$ we get, $\pi + \pi^2 = \frac{\pi^2}{3} + 4\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ...\right]$

Putting $x = -\pi$ we get, $-\pi + \pi^2 = \frac{\pi^2}{3} + 4\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ...\right]$

$$\Rightarrow \frac{4\pi^2}{3} = 8\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ...\right] \Rightarrow \frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ... = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

Example 3: Find the fourier series to represent the function $f(x)$ given by

$$f(x)=\begin{cases}-k & \text{for } -\pi<x<0\\ k & \text{for } 0<x<\pi\end{cases}$$

hence show that $1-\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+......=\frac{\pi}{4}$

Soln: Given function is odd in nature, so $a_0=0$ and $a_n=0$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx dx=\frac{1}{\pi}\left[\int_{-\pi}^{0}-k\sin nx dx+\int_{0}^{\pi}k\sin nx dx\right]$$

$$=\frac{1}{\pi}k\left[\left\{\frac{\cos nx}{n}\right\}_{-\pi}^{0}-\left\{\frac{\cos nx}{n}\right\}_{0}^{\pi}\right]$$

$$=\frac{1}{\pi}k\left[\frac{1}{n}-\frac{(-1)^n}{n}-\frac{(-1)^n}{n}+\frac{1}{n}\right]=\frac{1}{\pi}k\left[\frac{2}{n}-\frac{2(-1)^n}{n}\right]$$

If n is even $b_n=0$

If n is odd $b_n=\frac{4k}{n\pi}$

Thus required fourier sine series is

$$f(x)=\frac{4k}{\pi}\sin x+\frac{4k}{3\pi}\sin 3x+\frac{4k}{5\pi}\sin 5x+.....$$

$$\Rightarrow \quad f(x)=\frac{4k}{\pi}\left[\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+....\right]$$

Putting $x=\frac{\pi}{2}$, we get

$$k=\frac{4k}{\pi}\left[\sin\frac{\pi}{2}+\frac{1}{3}\sin\frac{3\pi}{2}+\frac{1}{5}\sin\frac{5\pi}{2}+....\right]$$

$$\Rightarrow 1=\frac{4}{\pi}\left[1+\frac{1}{3}(-1)+\frac{1}{5}(1)+\frac{1}{7}(-1)+......\right]=\frac{4}{\pi}\left[1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+....\right]\Rightarrow\ \frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+......$$

Example 4: Find the fourier series of the function:

$$f(x)=\begin{cases}-1 & \text{for } -\pi<x<-\frac{\pi}{2}\\ 0 & \text{for } -\frac{\pi}{2}<x<\frac{\pi}{2}\\ +1 & \text{for } \frac{\pi}{2}<x<\pi\end{cases}$$

Soln: $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\int_{-\pi}^{\pi/2}(-1)dx+\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}0\,dx+\frac{1}{\pi}\int_{\pi/2}^{\pi}1\,dx$

$$=\frac{1}{\pi}\left[-x\right]_{-\pi}^{-\pi/2}+\frac{1}{\pi}\left[x\right]_{\pi/2}^{\pi}=\frac{1}{\pi}\left[\frac{\pi}{2}-\pi+\pi-\frac{\pi}{2}\right]=0$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx\quad=\frac{1}{\pi}\int_{-\pi}^{\pi/2}(-1)\cos nxdx+\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}(0)\cos nxdx+\frac{1}{\pi}\int_{\pi/2}^{\pi}(1)\cos nxdx$$

$$=-\frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_{-\pi}^{-\pi/2}+\frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_{\pi/2}^{\pi}\qquad=-\frac{1}{\pi}\left[-\frac{\sin\frac{n\pi}{2}}{n}\right]+\frac{1}{\pi}\left[-\frac{\sin\frac{n\pi}{2}}{n}\right]=0$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx$$

$$=\frac{1}{\pi}\int_{-\pi}^{-\pi/2}(-1)\sin nxdx+\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}(0)\sin nxdx+\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}(1)\sin nxdx$$

$$=\frac{1}{\pi}\left[\frac{\cos nx}{n}\right]_{-\pi}^{-\pi/2}-\frac{1}{\pi}\left[\frac{\cos nx}{n}\right]_{\pi/2}^{\pi}$$

$$=\frac{1}{n\pi}\left[\cos\frac{n\pi}{2}-\cos n\pi\right]-\frac{1}{n\pi}\left(\cos n\pi-\cos\frac{n\pi}{2}\right)=\frac{2}{n\pi}\left[\cos\frac{n\pi}{2}-\cos n\pi\right]$$

$b_1=\frac{2}{\pi},\quad b_2=-\frac{2}{\pi},\quad b_3=\frac{2}{3\pi}$, therefore the fourier series expansion becomes

$$f(x)=\frac{1}{\pi}\left[2\sin x-2\sin 2x+\frac{2}{3}\sin 3x+.....\right]$$

Example 5: Find the fourier seires of the function defined as

$$f(x)=\begin{cases}x+\pi, & \text{for } 0\le x\le\pi\\ -x-\pi, & \text{for } -\pi\le x<0\end{cases}\text{ and } f(x+2\pi)=f(x)$$

Soln: $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx=\frac{1}{\pi}\int_{-\pi}^{0}f(x)\,dx+\frac{1}{\pi}\int_{0}^{\pi}f(x)\,dx$

$$=\frac{1}{\pi}\int_{-\pi}^{0}(-x-\pi)dx+\frac{1}{\pi}\int_{0}^{\pi}(x+\pi)\,dx=\frac{1}{\pi}\left(-\frac{x^2}{2}-\pi x\right)_{-\pi}^{0}+\frac{1}{\pi}\left(\frac{x^2}{2}+\pi x\right)_{0}^{\pi}$$

$$=\frac{1}{\pi}\left(\frac{\pi^2}{2}-\pi^2\right)+\frac{1}{\pi}\left(\frac{\pi^2}{2}+\pi^2\right)=\pi\left(\frac{1}{2}-1\right)+\pi\left(\frac{1}{2}+1\right)=\pi$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{0}f(x)\cos nxdx=\frac{1}{\pi}\int_{-\pi}^{0}f(x)\cos nxdx+\frac{1}{\pi}\int_{0}^{\pi}f(x)\cos nxdx$$

$$=\frac{1}{\pi}\int_{-\pi}^{0}(-x-\pi)\cos nxdx+\frac{1}{\pi}\int_{0}^{\pi}(x+\pi)\cos nxdx$$

$$=\frac{1}{\pi}\left[(-x-\pi)\frac{\sin nx}{n}-(-1)\left\{-\frac{\cos nx}{n^2}\right\}\right]_{-\pi}^{0}+\frac{1}{\pi}\left[(x+\pi)\frac{\sin nx}{n}-(-1)\left\{-\frac{\cos nx}{n^2}\right\}\right]_{0}^{\pi}$$

$$=\frac{1}{\pi}\left[-\frac{1}{n^2}+\frac{(-1)^n}{n^2}\right]+\frac{1}{\pi}\left[\frac{(-1)^n}{n^2}-\frac{1}{n^2}\right]=\frac{2}{n^2\pi}\left[(-1)^n-1\right]$$

$$=\frac{-4}{n^2\pi},\ \text{if n is odd}$$

$$=0,\ \text{if n is even}$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx dx=\frac{1}{\pi}\int_{-\pi}^{0}f(x)\sin nx dx+\frac{1}{\pi}\int_{0}^{\pi}f(x)\sin nx dx$$

$$=\frac{1}{\pi}\int_{-\pi}^{0}(-x-\pi)\sin nx dx+\frac{1}{\pi}\int_{0}^{\pi}(x+\pi)\sin nx dx$$

$$=\frac{1}{\pi}\left[(-x-\pi)\left(\frac{-\cos nx}{n}\right)-(-1)\left(-\frac{\sin nx}{n^2}\right)\right]_{-\pi}^{0}+\frac{1}{\pi}\left[(x+\pi)\left(-\frac{\cos nx}{n}\right)-(-1)\left(-\frac{\sin nx}{n^2}\right)\right]_{0}^{\pi}$$

$$=\frac{1}{\pi}\left[\frac{\pi}{n}\right]+\frac{1}{\pi}\left[-\frac{2\pi}{n}(-1)^n+\frac{\pi}{n}\right]=\frac{1}{n}\left[(1)-2(-1)^n\right]=\frac{2}{n}\left[1-(-1)^n\right]$$

$$=\frac{4}{n},\ \text{if n is odd}$$

$$=0\ ,\ \text{if n is even}$$

The fourier series expansion of the function is

$$f(x)=\frac{a_0}{2}+a_1\cos x+a_2\cos 2x+\ldots+b_1\sin x+b_2\sin 2x+\ldots..$$

$$f(x)=\frac{\pi}{2}-\frac{4}{\pi}\left(\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\ldots..\right)+4\left(\frac{\sin x}{1}+\frac{\sin 3x}{3}+\ldots..\right)$$

Example 6: Find the Fourier series expansion of the periodic function of period 2π

$$f(x)=x^2,\qquad -\pi\le x\le\pi$$

Hence, find the sum of the series $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots..$

Soln: $f(x)=x^2,\qquad -\pi\le x\le\pi$

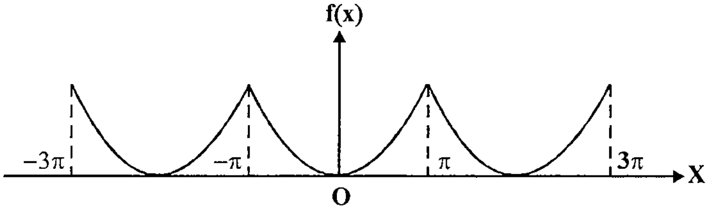
This is an even function, $\therefore\ b_n=0$

$$a_0=\frac{2}{\pi}\int_0^{\pi}f(x)dx=\frac{2}{\pi}\int_0^{\pi}x^2dx=\frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}=\frac{2\pi^2}{3}$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nx dx=\frac{2}{\pi}\int_0^{\pi}x^2\cos nx dx$$

$$=\frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right)-(2x)\left(-\frac{\cos nx}{n^2}\right)+(2)\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$=\frac{2}{\pi}\left[\frac{\pi^2\sin n\pi}{n}+\frac{2\pi\cos n\pi}{n^2}-\frac{2\sin n\pi}{n^3}\right]=\frac{4(-1)^n}{n^2}$$



Fourier series expansion: $x^2 = \frac{\pi^2}{3} - 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + ...\right]$

On putting $x = 0$, we have $0 = \frac{\pi^2}{3} - 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} .....\right]$

$$\Rightarrow \quad \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} .... = \frac{\pi^2}{12}$$

Example 7: Find the Fourier Series expansion for the function

$$f(x) = x\cos x, -\pi < x < \pi$$

Soln: Since $x\cos x$ is an odd function therefore, $a_0 = a_n = 0$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\sin(n+1)x dx + \frac{1}{\pi}\int_0^{\pi} x\sin(n-1)x dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\sin(n+1)x dx + \frac{1}{\pi}\int_0^{\pi} x\sin(n-1)dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos(n+1)x}{n+1}\right) + \frac{\sin(n+1)x}{(n+1)^2}\right]_0^{\pi} + \frac{1}{\pi}\left[-x\frac{\cos(n-1)x}{n-1} + \frac{\sin(n-1)x}{(n-1)^2}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[x\left\{-\frac{\cos(n+1)x}{(n+1)} - \frac{\cos(n-1)x}{(n-1)}\right\} + 1.\left\{\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\pi.\left\{-\frac{\cos(n+1)\pi}{(n+1)} - \frac{\cos(n-1)\pi}{(n-1)}\right\}\right] \Rightarrow b_n = \left\{-\frac{(-1)^{n+1}}{n+1} - \frac{(-1)^{n-1}}{n-1}\right\}, \ n \neq 1$$

$$b_n = -(-1)^{n+1}\left[\frac{1}{n+1} + \frac{1}{n-1}\right]$$

$$= -\left\{\frac{1}{(n+1)} + \frac{1}{(n-1)}\right\} = \frac{-2n}{n^2-1} \qquad \text{If n is odd, } n \neq 1$$

But $\quad b_n = \left\{\frac{1}{n+1} + \frac{1}{n-1}\right\} = \frac{2n}{n^2-1}$, $\qquad$ If n is even, $n \neq 1$

If n = 1, then $b_1 = \frac{2}{\pi}\int_0^{\pi} x\cos x.\sin x dx = \frac{1}{\pi}\int_0^{\pi} x.\sin 2x dx$

$$= \frac{1}{\pi}\left[x.\left(-\frac{\cos 2x}{2}\right) - 1\left(-\frac{\sin 2x}{4}\right)\right]_0^{\pi} = \frac{1}{\pi}\left[\pi\left(-\frac{1}{2}\right)\right] = -\frac{1}{2}$$

$$\therefore \quad x\cos x = b_1\sin x + b_2\sin 2x + b_3\sin 3x + \ldots\ldots = -\frac{1}{2}\sin x + \frac{4\sin 2x}{2-1} - \frac{6\sin 3x}{3^2-1} + \ldots..$$

Example 8: Represent the following function by fourier sine series

$$f(t) = \begin{cases} t, & 0 < t \le \dfrac{\pi}{2} \\ \dfrac{\pi}{2}, & \dfrac{\pi}{2} < t \le \pi \end{cases}$$

Soln: $b_n = \frac{2}{\pi}\int_0^{\pi} f(t)\sin nt\,dt \quad = \frac{2}{\pi}\int_0^{\pi/2} t\sin nt\,dt + \frac{2}{\pi}\int_{\pi/2}^{\pi}\frac{\pi}{2}\sin nt\,dt$

$$= \frac{2}{\pi}\left[t\left(-\frac{\cos nt}{n}\right) - (1)\left(-\frac{\sin nt}{n^2}\right)\right]_0^{\pi/2} + \frac{2}{\pi}\frac{\pi}{2}\left[-\frac{\cos nt}{n}\right]_{\pi/2}^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{2}\frac{\cos\frac{n\pi}{2}}{n} + \frac{\sin\frac{n\pi}{2}}{n^2}\right] + \left[-\frac{\cos n\pi}{n} + \frac{\cos\frac{n\pi}{2}}{n}\right]$$

$$b_1 = \frac{2}{\pi}\left[-\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right] + \left[-\cos\pi + \cos\frac{\pi}{2}\right] = \frac{2}{\pi}[0+1] + [1] = \frac{2}{\pi} + 1$$

$$b_2 = \frac{2}{\pi}\left[-\frac{\pi}{2}\frac{\cos\pi}{2} + \frac{\sin\pi}{2}\right] + \left[-\frac{\cos 2\pi}{2} + \frac{\cos\pi}{2}\right] = \frac{2}{\pi}\left[-\frac{\pi}{2}\frac{(-1)}{2} + 0\right] + \left[-\frac{1}{2} - \frac{1}{2}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi}{4}\right] - 1 = \frac{1}{2} - 1 = -\frac{1}{2}$$

$$b_3 = \frac{2}{\pi}\left[-\frac{\pi}{2}\frac{\cos\frac{3\pi}{2}}{3} + \frac{\sin\frac{3\pi}{2}}{3^2}\right] + \left[-\frac{\cos 3\pi}{3} + \frac{\cos\frac{3\pi}{2}}{3}\right]$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{2}(0) - \frac{1}{9}\right] + \left[\frac{1}{3} + 0\right] = -\frac{2}{9\pi} + \frac{1}{3}$$

$$f(t) = \left(\frac{2}{\pi} + 1\right)\sin t - \frac{1}{2}\sin 2t + \left(-\frac{2}{9\pi} + \frac{1}{3}\right)\sin 3t + \ldots..$$

Example 9: A periodic function of period 4 is defined as

$$f(x) = |x|. \quad -2 < x < 2$$

Find its Fourier series expansion.

Soln: $f(x) = |x|. \quad -2 < x < 2$

$$\Rightarrow \quad f(x) = \begin{cases} x, & 0 < x < 2 \\ -x, & -2 < x < 0 \end{cases}$$

$$a_0=\frac{1}{c}\int_{-c}^{c} f(x)\,dx=\frac{1}{2}\int_0^2 x\,dx+\frac{1}{2}\int_{-2}^{0}(-x)\,dx$$

$$=\frac{1}{2}\left[\frac{x^2}{2}\right]_0^2+\frac{1}{2}\left[\frac{-x^2}{2}\right]_{-2}^{0}=\frac{1}{4}(4-0)+\frac{1}{4}(0+4)=2$$

$$a_n=\frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}\,dx=\frac{1}{2}\int_0^2 x\cos\frac{n\pi x}{2}\,dx+\frac{1}{2}\int_{-2}^{0}(-x)\cos\frac{n\pi x}{2}\,dx$$

$$=\frac{1}{2}\left[x\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right)-(1)\left(-\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)\right]_0^2+\frac{1}{2}\left[(-x)\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right)-(-1)\left(-\frac{4}{n^2\pi^2}\right)\cos\frac{n\pi x}{2}\right]_{-2}^{0}$$

$$=\frac{1}{2}\left[0+\frac{4}{n^2\pi^2}(-1)^n-\frac{4}{n^2\pi^2}\right]+\frac{1}{2}\left[0-\frac{4}{n^2\pi^2}+\frac{4}{n^2\pi^2}(-1)^n\right]$$

$$=\frac{1}{2}\frac{4}{n^2\pi^2}\left[(-1)^n-1-1+(-1)^n\right]=\frac{4}{n^2\pi^2}\left[(-1)^n-1\right]$$

$=-\dfrac{8}{n^2\pi^2}$ (If n is odd)

$=0$ (if n is even)

$b_n=0$ as $f(x)$ is even function

Therefore, the fourier series exapansion will be

$$f(x)=\frac{a_0}{2}+a_1\cos\frac{\pi x}{2}+a_2\cos\frac{2\pi x}{2}+\dots+b_1\sin\frac{\pi x}{2}+b_2\sin\frac{2\pi x}{2}+\dots$$

$$f(x)=1-\frac{8}{\pi^2}\left[\frac{\cos\frac{\pi x}{2}}{1^2}+\frac{\cos\frac{3\pi x}{2}}{3^2}+\frac{\cos\frac{5\pi x}{2}}{5^2}+\dots\right]$$

Example 10: The fourier series of the function $f(x)=1-|x|,\ -\pi<x<\pi$ is given as

(a) $\left(1-\dfrac{\pi}{2}\right)+\dfrac{4}{\pi}\left(\cos x+\dfrac{\cos 3x}{3^2}+\dfrac{\cos 5x}{5^2}+\cdots\right)$

(b) $\left(1-\dfrac{\pi}{2}\right)+\dfrac{4}{\pi}\left(\sin x+\dfrac{\sin 3x}{3^2}+\dfrac{\sin 5x}{5^2}+\cdots\right)$

(c) $\left(1-\dfrac{\pi}{2}\right)+\dfrac{4}{\pi}\left(\cos x+\dfrac{\sin 3x}{3^2}+\dfrac{\cos 5x}{5^2}+\dfrac{\sin 7x}{7^2}+\cdots\right)$

(d) None of the above

Soln: As we know, Fourier series of $f(x)$ in the interval (a, b) is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{2\pi nx}{b-a}\right)+\sum_{n=1}^{\infty}b_n\sin\left(\frac{2\pi nx}{b-a}\right)$$

Here, $f(x)=1-|x|,\ -\pi<x<\pi \Rightarrow f(x)=\begin{cases}1+x & ;\quad -\pi<x<0\\ 1-x & ;\quad 0<x<\pi\end{cases}$

Clearly, $f(-x)=f(x)$, Hence $f(x)$ is an even function.

Therefore, $b_n=0$.

$$\text{Now, } a_0=\frac{2}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx=\frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$=\frac{2}{\pi}\int_0^{\pi}(1-x)\,dx=\frac{2}{\pi}\left(x-\frac{x^2}{2}\right)_0^{\pi}=2\left(1-\frac{\pi}{2}\right)$$

$$a_n=\frac{2}{2\pi}\int_{-\pi}^{\pi} f(x)\cos\frac{2\pi nx}{2\pi}\,dx=\frac{1}{\pi}2\int_0^{\pi}(1-x)\cos(nx)\,dx$$

$$=\frac{2}{\pi}\int_0^{\pi}\cos nx\,dx-\frac{2}{\pi}\int_0^{\pi}x\cos nx\,dx=\frac{2}{\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi}-\frac{2}{\pi}\left\{x\frac{\sin nx}{n}+\frac{\cos nx}{n^2}\right\}_0^{\pi}$$

$$=0-\frac{2}{\pi}\left\{\frac{\cos n\pi}{n^2}-\frac{1}{n^2}\right\}=-\frac{2}{\pi}\left[\frac{(-1)^n-1}{n^2}\right]$$

$$\therefore\quad a_n=\begin{cases}\dfrac{4}{\pi n^2}, & \text{if} \quad n=\text{odd}\\ 0, & \text{if} \quad n=\text{even}\end{cases}$$

$\therefore$ Fourier series of $f(x)$ is

$$f(x)=\left(1-\frac{\pi}{2}\right)+\sum_{n=1,3,5}^{\infty}\frac{4}{\pi n^2}\cos nx=\left(1-\frac{\pi}{2}\right)+\frac{4}{\pi}\left(\cos x+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\cdots\right)$$

Correct option is (a)

Example 11: If $f(x)=\begin{cases}-k & ; -\pi<x<0\\ k & ; \ 0\le x\le\pi\end{cases}$, and a_0, a_n, b_n are the fourier co-efficients of fourier expansion of $f(x)$, then which of the following statement is CORRECT ?

(a) $a_0=0,\ a_n=0,\ b_n\neq 0$ (b) $a_0\neq 0,\ a_n=0,\ b_n=0$

(c) $a_0\neq 0,\ a_n\neq 0,\ b_n=0$ (d) $a_0=0,\ a_n\neq 0,\ b_n=0$

Soln: Since, $f(x)$ is an odd function.

Hence, $a_0=0,\ a_n=0,\ b_n\neq 0$.

Correct option is (a)

Example 12: If the fourier expansion of $f(x)=\begin{cases}\pi+x & ; \ -\pi<x<0\\ 0 & ; \ 0\le x\le\pi\end{cases}$ is given by

$$f(x)=\frac{\pi}{4}+\frac{2}{\pi}\left[\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\cdots\right]-\left[\frac{\sin x}{1}+\frac{\sin 2x}{2}+\cdots\right],$$

then the value of $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots$ is equal to

(a) $-\frac{\pi}{8}$ (b) $\frac{\pi}{8}$ (c) $\frac{\pi^2}{8}$ (d) $\frac{3\pi^2}{8}$

Soln: $f(x)=\frac{\pi}{4}+\frac{2}{\pi}\left[\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\cdots\right]-\left[\frac{\sin x}{1}+\frac{\sin 2x}{2}+\cdots\right],$

Putting $x=0$ on both sides, we get

$$\Rightarrow \frac{1}{2}\left(\lim_{x\to 0^+} f(x)+\lim_{x\to 0^-} f(x)\right)=\frac{\pi}{4}+\frac{2}{\pi}\left[\frac{1}{1^2}+\frac{1}{3^2}+\cdots\right]-0 \qquad \text{(since, } f(x) \text{ is discontinuous at } x=0\text{)}$$

$$\Rightarrow \frac{1}{2}(0+\pi)=\frac{\pi}{4}+\frac{\pi}{2}\left[\frac{1}{1^2}+\frac{1}{3^2}+\cdots\right]$$

Therefore, $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{8}$.

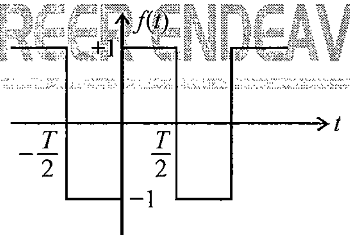
Correct option is (c)

Example 13: The fourier series of the square wave function, as shown in figure is

(a) $\frac{1}{\pi}(\sin\omega t+\sin 2\omega t+\cdots)$ (b) $\frac{1}{\pi}\left(\sin\omega t+\frac{\sin 2\omega t}{2}+\frac{\sin 3\omega t}{3}+\cdots\right)$

(c) $\frac{4}{\pi}\left(\sin\omega t+\frac{\sin 3\omega t}{3}+\frac{\sin 5\omega t}{5}+\cdots\right)$ (d) $\frac{4}{\pi}\left(1+\frac{\sin 2\omega t}{2}+\frac{\sin 4\omega t}{4}+\cdots\right)$

$\left(\text{Here, } \omega=\frac{2\pi}{T} \text{ is the angular frequency}\right)$



Soln: From the figure, $f(t)=\begin{cases}1 & ; \quad 0<t<\frac{T}{2}\\ -1 & ; \quad -\frac{T}{2}<t<0\end{cases}$

Clearly, $f(t)$ is an odd function. (anti-symmetric about $x=0$)

Hence, $a_0=a_n=0$

$$b_n=\frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\left(\frac{2\pi nt}{T}\right)dt$$

$$=\frac{2}{T}\times 2\int_0^{T/2}(+1)\sin\left(\frac{2\pi nt}{T}\right)dt=\frac{4}{T}\left(\frac{-\cos\left(\frac{2\pi nt}{T}\right)}{\frac{2\pi n}{T}}\right)_0^{T/2}=\frac{4}{T}\times\frac{T}{2\pi n}(-\cos n\pi+1)$$

Therefore, $b_n = \begin{cases} \dfrac{4}{2\pi n} \times 2 & \text{if} \quad n = \text{odd} \\ 0 & \text{if} \quad n = \text{even} \end{cases}$

Hence, fourier series $f(t) = \sum_{n=1}^{\infty} b_n \sin\left(\dfrac{2\pi nt}{T}\right)$

$$f(t) = \frac{4}{\pi}\left[\frac{\sin\left(\frac{2\pi t}{T}\right)}{2} + \frac{\sin\left(\frac{6\pi t}{T}\right)}{3} + \cdots\right]$$

$$f(t) = \frac{4}{\pi}\left[\frac{\sin \omega t}{2} + \frac{\sin 3\omega t}{3} + \cdots\right]$$

Correct option is (c)

Example 14: The fourier series for $|\sin\theta|$ in the range $-\pi \le \theta \le \pi$ is given by

(a) $\dfrac{1}{\pi} - \dfrac{2}{\pi}\sum_{m=1}^{\infty}\dfrac{\cos 2m\theta}{4m^2 - 1}$

(b) $\dfrac{2}{\pi} - \dfrac{4}{\pi}\sum_{m=1}^{\infty}\dfrac{\cos 2m\theta}{4m^2 - 1}$

(c) $-\dfrac{4}{\pi}\sum_{m=1}^{\infty}\dfrac{\cos 2m\theta}{4m^2 - 1}$

(d) None of these

Soln: We have $f(\theta) = |\sin\theta|, \ -\pi \le x \le \pi$

$$f(\theta) = \begin{cases} -\sin\theta & ; \ -\pi \le \theta < 0 \\ \sin\theta & ; \ 0 \le \theta \le \pi \end{cases}$$

Since, $f(\theta)$ is an even function, hence $b_n = 0$.

Now, $a_0 = \dfrac{2}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta = \dfrac{1}{\pi} \times 2\int_0^{\pi} \sin\theta\,d\theta = \dfrac{4}{\pi}$

and $a_n = \dfrac{2}{2\pi}\int_{-\pi}^{\pi} f(\theta)\cos\left(\dfrac{2\pi n\theta}{2\pi}\right)d\theta = \dfrac{1}{\pi} \times 2\int_0^{\pi} \sin\theta\cos n\theta\,d\theta$

$$a_n = \frac{2}{\pi}\left[\frac{\sin\theta\sin n\theta}{n} + \frac{\cos\theta\cos n\theta}{n^2}\right]_0^{\pi}\frac{n^2}{n^2 - 1}$$

$$= \frac{2}{\pi}\frac{n^2}{n^2 - 1}\left[\frac{-\cos\pi - 1}{n^2}\right] = -\frac{2}{\pi}\frac{1}{n^2 - 1}\left((-1)^n + 1\right)$$

$$a_n = \begin{cases} -\dfrac{4}{\pi} \times \dfrac{1}{n^2 - 1}, & \text{if} \quad n = \text{even} \\ 0, & \text{if} \quad n = \text{odd} \end{cases}$$

Hence, fourier series,

$$f(\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos\left(\frac{2\pi n\theta}{2\pi}\right)=\frac{2}{\pi}-\frac{4}{\pi}\sum_{n=2,4,6}^{\infty}\frac{\cos(n\theta)}{n^2-1}$$

Put $n=2m,\ m=1,2,3,\ldots$

$$\therefore\ f(\theta)=\frac{2}{\pi}-\frac{4}{\pi}\sum_{m=1,2,3,\ldots}^{\infty}\frac{\cos(2m\theta)}{4m^2-1}$$

Correct option is (b)

Example 15: If the fourier series of $f(x)$ in the interval $[-1,1]$, where

$$f(x)=\begin{cases} L/4 & ; \quad -L\le x\le -\frac{L}{2} \\ x^2/L & ; \quad -\frac{L}{2}\le x\le \frac{L}{2} \\ L/4 & ; \quad \frac{L}{2}\le x\le L \end{cases}$$

is given by $f(x)=\frac{L}{6}+\frac{2L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}A_n\cos\frac{n\pi x}{2}$, then A_n is equal to

(a) $\cos\left(\frac{n\pi}{2}\right)-\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)$ (b) $\sin\left(\frac{n\pi}{2}\right)-2\cos\left(\frac{n\pi}{2}\right)$

(c) $1-\frac{2}{n\pi}\cos\frac{n\pi}{2}$ (d) $1-\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)$

Soln: Since, $f(-x)=f(x)$. Hence, $f(x)$ is an even function. Hence, $b_n=0$

$$a_0=\frac{2}{2L}\int_{-L}^{L}f(x)\,dx=\frac{1}{L}2\int_{0}^{L}f(x)\,dx=\frac{2}{L}\int_{0}^{L/2}f(x)\,dx+\frac{2}{L}\int_{L/2}^{L}f(x)\,dx=\frac{2}{L}\int_{0}^{L/2}\frac{x^2}{L}dx+\int_{L/2}^{L}\frac{L}{4}dx=\frac{L}{3}$$

$$a_n=\frac{2}{2L}\int_{-L}^{L}f(x)\cos\left(\frac{2\pi nx}{2L}\right)dx=\frac{1}{L}2\int_{0}^{L}f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$=\frac{2}{L}\int_{0}^{L/2}\frac{x^2}{L}\cos\left(\frac{n\pi x}{L}\right)dx+\frac{2}{L}\int_{L/2}^{L}\frac{L}{4}\cos\left(\frac{n\pi x}{L}\right)dx$$

$$a_n=\frac{2}{L}\left[\frac{x^2}{L}\frac{\sin(n\pi x/L)}{n\pi/L}-\frac{2}{n\pi}\left\{x\times-\frac{\cos(n\pi x/L)}{n\pi/L}-\frac{\sin(n\pi x/L)}{(n\pi/L)^2}\right\}\right]_0^{L/2}+\frac{2}{L}\times\frac{L}{4}\left(\frac{\sin(n\pi x/L)}{n\pi/L}\right)_{L/2}^{L}$$

$$=\frac{2}{L}\left[\left\{\frac{L^2}{4}\times\frac{\sin(n\pi/2)}{n\pi}-\frac{2}{n\pi}\left(-\frac{L}{2}\times\frac{L}{n\pi}\cos\left(\frac{n\pi}{2}\right)-\frac{L^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right)\right\}-0\right]+\frac{1}{2}\times\frac{L}{n\pi}\left[-\sin\left(\frac{n\pi}{2}\right)\right]$$

$$=\frac{2L}{\pi^2n^2}\left[\cos\left(\frac{n\pi}{2}\right)-\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right]$$

Hence, on comparing with given fourier series, we have

$$A_n=\cos\left(\frac{n\pi}{2}\right)-\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right).$$

Correct option is (a)

Example 16: If $f(x)=\begin{cases}\cos x & ; \quad 0\le x\le\pi \\ -\cos x & ; \quad -\pi\le x\le 0\end{cases}$, then which of the following statements is/are CORRECT about the fourier expansion of $f(x)$ in the interval $[-\pi,\pi]$.

(a) Ratio of third harmonic to fourth harmonic is zero.

(b) Ratio of third harmonic to fourth harmonic is $32/15\pi$.

(c) $f(x)=\frac{4}{\pi}\left[\frac{2}{1\cdot 3}\sin 2x+\frac{4}{3\cdot 5}\sin 4x+\cdots\right]$, and $f(x)$ converges to zero at $x=0$.

(d) $f(x)=\frac{4}{\pi}\left[\frac{2}{1^2\cdot 3^2}\sin 2x+\frac{4}{3^2\cdot 5^2}\sin 4x+\cdots\right]$, and $f(x)$ converges to zero at $x=0$.

Soln: Since, $f(-x)=-f(x)$, hence $f(x)$ is an odd function. Therefore, $a_0=a_n=0$.

$$b_n=\frac{2}{2\pi}\int_{-\pi}^{\pi}f(x)\sin\left(\frac{2n\pi x}{2\pi}\right)dx=\frac{2}{\pi}\int_0^{\pi}\cos x\sin(n\pi)\,dx$$

$$=\frac{2}{\pi}\times\left(\frac{n^2}{n^2-1}\right)\left(-\frac{\cos x\cos nx}{n}-\frac{\sin x\sin nx}{n^2}\right)_0^{\pi}=\frac{2}{\pi}\left(\frac{n^2}{n^2-1}\right)\left(\frac{\cos n\pi}{n}+\frac{1}{n}\right)$$

$$b_n=\frac{2}{\pi}\left(\frac{n}{n^2-1}\right)\left((-1)^n+1\right)\Rightarrow b_n=\begin{cases}\frac{4n}{\pi(n^2-1)}, & \text{if} \quad n=\text{even}\\ 0, & \text{if} \quad n=\text{odd}\end{cases}$$

Hence, $b_1=b_3=b_5\cdots=0$

Therefore, ratio of third to fourth harmonic $=\frac{b_3}{b_4}=0$.

Also, fourier series, $f(x)=\sum_{n=1}^{\infty}b_n\sin(nx)=\frac{4}{\pi}\left(\frac{2}{3}\sin 2x+\frac{4}{15}\sin 4x+\frac{6}{35}\sin 6x+\cdots\right)$

$$=\frac{4}{\pi}\left(\frac{2}{1\cdot 3}\sin 2x+\frac{4}{3\cdot 5}\sin 4x+\frac{6}{5\cdot 7}\sin 6x+\cdots\right)$$

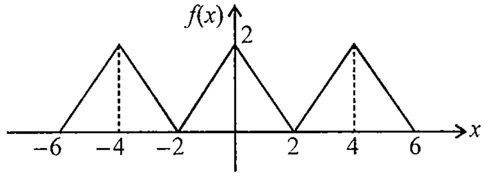
and at $x=0$, $f(x)$ converges to $\frac{1}{2}\left(\lim_{x\to 0^-}f(x)+\lim_{x\to 0^+}f(x)\right)=\frac{1}{2}(-1+1)=0.$

Correct options are (a) and (c)

Example 17: If the fourier expansion of function $f(x)$, $-a\le x\le a$, is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{a}\right)+\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi x}{a}\right),$$

then the value of b_2 for the function $f(x)$ which is shown in figure is __________.



Soln: Consider $f(x)$ in the interval $[-2, 2]$.

$$f(x)=\begin{cases}2+x & ; \quad -2\le x<0\\ 2-x & ; \quad 0\le x\le 2\end{cases}$$

Since, $f(-x)=f(x)$, $f(x)$ is an even function. Hence, $b_n=0$. Therefore, $b_2=0$.

Correct answer is (0)

Example 18: By considering the fourier expansion of the function $f(x)=\pi+x,\ -\pi<x<\pi$, value of $\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots\right)$ is equal to __________ $\times\,\pi$.

Soln: $f(x)=\pi+x,\ -\pi<x<\pi$

$$a_0=\frac{2}{2\pi}\int_{-\pi}^{\pi}(\pi+x)\,dx=\frac{1}{\pi}\left(\pi x+\frac{x^2}{2}\right)_{-\pi}^{\pi}=\frac{1}{\pi}\left(\pi^2+\frac{\pi^2}{2}+\pi^2-\frac{\pi^2}{2}\right)=2\pi$$

$$a_n=\frac{2}{2\pi}\int_{-\pi}^{\pi}(\pi+x)\cos(nx)\,dx=\frac{1}{\pi}\left[\int_{-\pi}^{\pi}\cos nx\,dx+\int_{-\pi}^{\pi}x\cos nx\,dx\right]=\frac{1}{\pi}\left[\pi\left(\frac{\sin nx}{n}\right)_{-\pi}^{\pi}+0\right]=0$$

$$b_n=\frac{2}{2\pi}\int_{-\pi}^{\pi}(\pi+x)\sin(nx)\,dx=\frac{1}{\pi}\left[\int_{-\pi}^{\pi}\pi\sin(nx)\,dx+\int_{-\pi}^{\pi}x\sin nx\,dx\right]$$

$$=\frac{1}{\pi}\left[0+x\times-\frac{\cos nx}{n}+\frac{\sin nx}{n^2}\right]_{-\pi}^{\pi}=\frac{1}{\pi}\left[-\frac{\pi\cos n\pi}{n}-\frac{\pi\cos n\pi}{n}\right]=-2\frac{(-1)^n}{n}$$

Hence, fourier series of $f(x)$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx=\pi+\sum_{n=1}^{\infty}-\frac{2(-1)^n}{n}\sin nx$$

$$\Rightarrow f(x)=\pi-2\left(-\sin x+\frac{\sin 2x}{2}-\frac{\sin 3x}{3}+\frac{\sin 4x}{4}-\frac{\sin 5x}{5}+\cdots\right)$$

At $x=\frac{\pi}{2}$, $\pi+\frac{\pi}{2}=\pi-2\left(-1+0+\frac{1}{3}+0-\frac{1}{5}+0+\frac{1}{2}\cdots\right)\quad\Rightarrow\frac{\pi}{4}=\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots\right)$

Correct answer is (0.25)

Example 19: $f(x)$ is symmetric periodic function of x i.e. $f(-x)=f(x)$. Then, in general, the Fourier series of the function $f(x)$ will be of the form

(a) $f(x)=\sum_{n=1}^{\infty}\left(a_n\cos(nkx)+b_n\sin(nkx)\right)$

(b) $f(x)=a_0+\left(a_n\cos(nkx)\right)$

(c) $f(x)=\sum_{n=1}^{\infty}\left(b_n\sin(nkx)\right)$

(d) $f(x)=a_0+\sum_{n=1}^{\infty}\left(b_n\sin(nkx)\right)$

Soln: For a symmetric periodic function i.e., $f(-x)=f(x)$.

The Fourier series contains constant term and cosine terms.

$$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos(nkx)$$

Correct option is (b)

Example 20: A periodic function $f(x)$ of period 2π is defined in the interval $(-\pi<x<\pi)$ as:

$$f(x)=\begin{cases}-1 & \text{for} \quad -\pi<x<0\\ 1 & \text{for} \quad 0<x<\pi\end{cases}$$

The approximate Fourier series expansion for $f(x)$ is

(a) $f(x)=\frac{4}{\pi}\left[\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+.........\right]$

(b) $f(x)=\frac{4}{\pi}\left[\sin x-\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x-.........\right]$

(c) $f(x)=\frac{4}{\pi}\left[\cos x+\frac{1}{3}\cos 3x+\frac{1}{5}\cos 5x+.........\right]$

(d) $f(x)=\frac{4}{\pi}\left[\cos x-\frac{1}{3}\cos 3x+\frac{1}{5}\cos 5x-.........\right]$

Soln: $f(x)=\begin{cases}-1 & -\pi<x<0\\ 1 & 0>x>\pi\end{cases}$

$$f(x)=\frac{a_0}{2}+\sum_n a_n\cos nx+\sum_n b_n\sin nx$$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx=\frac{1}{\pi}\left[\int_{-\pi}^{0}-dx+\int_{0}^{\pi}dx\right]=\frac{1}{\pi}[-\pi+\pi]=0$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx=\frac{1}{\pi}\left[\int_{-\pi}^{0}-\cos nxdx+\int_{0}^{\pi}\cos nxdx\right]=\frac{1}{\pi}\left\{\left[\frac{\sin nx}{n}\right]_{-\pi}^{0}+\left[\frac{\sin n\pi}{n}\right]_{0}^{\pi}\right\}=0$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx=\frac{1}{\pi}\left[\int_{-\pi}^{0}-\sin nxdx+\int_{0}^{\pi}\sin nxdx\right]$$

$$=\frac{1}{\pi}\left\{\left[\frac{\cos nx}{n}\right]_{-\pi}^{0}+\left[-\frac{\cos n\pi}{n}\right]_{0}^{\pi}\right\}=\frac{1}{\pi}\times\frac{1}{n}\left[\left(1-(-1)^n\right)+\left(1-(-1)^n\right)\right]$$

$$=\frac{2}{\pi}\left[\frac{1}{n}\left(1-(-1)^n\right)\right]$$

$$\Rightarrow \qquad b_n = \begin{cases} 0 & \text{for even n} \\ \dfrac{4}{\pi n} & \text{for odd n} \end{cases}$$

$$\therefore\ f(x) = \frac{4}{\pi}\sum_{n=1,3,5,\ldots\ldots\ldots} \frac{1}{n}\sin nx = \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots\ldots\right)$$

Correct option is (a)

4.3 Half Range Series

Let the function is defined in the interval $(0, L)$ and it is insignificant whatever be the form of function outside $(0, L)$.

(1) To get cosine half range cosine series, we take $f(x)$ is an even function in the interval $(-L, L)$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{2}{L}\int_{0}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = 0$$

(2) To get cosine half range sine series, we take $f(x)$ is an even function in the interval $(-L, L)$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = 0$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

4.4 Perseval's Formula

Full range Fourier series: $\displaystyle\int_{-l}^{l}[f(x)]^2\,dx = l\left\{\frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)\right\}$

Half range cosine series : $\displaystyle\int_{0}^{l}[f(x)]^2\,dx = \frac{l}{2}\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty}a_n^2\right]$

Half range sine series: $\int_0^l [f(x)]^2 dx = \frac{l}{2}\left[\sum_{n=1}^{\infty} b_n^2\right]$

Example 21: By using the half range sine series for $f(x)=1$ in $0<x<\pi$, show that

$$\frac{\pi^2}{8} = 1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+....$$

Soln: $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx$

$$=\frac{2}{\pi}\int_0^{\pi}(1)\sin nx dx = \frac{2}{\pi}\left(-\frac{\cos nx}{n}\right)_0^{\pi} = \frac{-2}{n\pi}[\cos n\pi - 1] = \frac{-2}{n\pi}\left[(-1)^n - 1\right]$$

$=\frac{4}{n\pi}$ if n is odd

$=0$ if n is even

Then the sine series is

$$1 = \frac{4}{\pi}\sin x + \frac{4}{3\pi}\sin 3x + \frac{4}{5\pi}\sin 5x + \frac{4}{7\pi}\sin 7x + ...$$

Using Perseval's formula, we get

$$\int_0^l [f(x)]^2 dx = \frac{l}{2}\left[b_1^2+b_2^2+b_3^2+b_4^2+b_5^2+.....\right]$$

$$\Rightarrow \int_0^{\pi}(1)^2 dx = \frac{\pi}{2}\left[\left(\frac{4}{\pi}\right)^2+\left(\frac{4}{3\pi}\right)^2+\left(\frac{4}{5\pi}\right)^2+\left(\frac{4}{7\pi}\right)^2+....\right]$$

$$\Rightarrow [x]_0^{\pi} = \left(\frac{\pi}{2}\right)\left(\frac{16}{\pi^2}\right)\left[1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+.....\right] \Rightarrow \pi = \frac{\pi}{2}\left(\frac{16}{\pi^2}\right)\left[1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+.....\right]$$

$$\Rightarrow \frac{\pi^2}{8} = 1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+.....$$

Example 22: If $f(x) = \begin{cases} \pi x & 0<x<1 \\ \pi(2-x), & 1<x<2 \end{cases}$

Using half range cosine series, show that

$$\frac{\pi^4}{96} = 1+\frac{1}{3^4}+\frac{1}{5^4}+......$$

Soln: $a_0 = \frac{2}{l}\int_0^l f(x)dx = \frac{2}{2}\left[\int_0^1 \pi x dx + \int_1^2 \pi(2-x)dx\right]$

$$=\pi\left(\frac{x^2}{2}\right)_0^1 + \pi\left(2x-\frac{x^2}{2}\right)_1^2 = \frac{\pi}{2}+\pi\left[(4-2)-\left(2-\frac{1}{2}\right)\right] = \pi$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx$$

$$= \frac{2}{2}\left[\int_0^1 \pi x\cos\frac{n\pi x}{2}dx + \int_1^2 \pi(2-x)\cos\frac{n\pi x}{2}dx\right]$$

$$= \pi\left[x\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}} - \left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right]_0^1 + \pi\left[(2-x)\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}} - (-1)\left[\frac{-\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right]\right]_1^2$$

$$= \pi\left[\frac{2}{n\pi}\sin\frac{n\pi}{2} + \frac{4}{n2}\pi^2\cos\frac{n\pi}{2} - \frac{4}{n^2\pi^2}\right] + \pi\left[0 - \frac{4}{n^2\pi^2}\cos n\pi - \frac{2}{n\pi}\sin\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi}{2}\right]$$

$$= \pi\left[\frac{8}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{4}{n^2\pi^2} - \frac{4}{n^2\pi^2}\cos n\pi\right] = \frac{4}{n^2\pi^2}\left[2\cos\frac{n\pi}{2} - 1 - \cos n\pi\right]$$

$a_1 = 0, a_2 = -\frac{4}{\pi},\ a_3 = 0,\ a_4 = 0, a_5 = 0,\ a_6 = -\frac{4}{9\pi}.....$

According to Perseval's formula:

$$\int_0^l [f(x)]^2 dx = \frac{l}{2}\left[\frac{a_0^2}{2} + a_1^2 + a_2^2 + a_3^2 + ...\right]$$

$$\Rightarrow \int_0^1 (\pi x)^2 dx + \int_1^2 \pi^2 (2-x)^2 dx = \frac{2}{2}\left[\frac{\pi^2}{2} + \frac{16}{\pi^2} + \frac{16}{81\pi^2} + ...\right]$$

$$\Rightarrow \pi^2\left[\frac{x^3}{3}\right]_0^1 - \pi^2\left[\frac{(2-x)^3}{3}\right]_1^2 = \frac{\pi^2}{2} + \frac{16}{\pi^2} + \frac{16}{81\pi^2} + ... \Rightarrow \frac{\pi^2}{3} - \pi^2\left(0 - \frac{1}{3}\right) = \frac{\pi^2}{2} + \frac{16}{\pi^2}\left[1 + \frac{1}{81} + ...\right]$$

$$\Rightarrow \frac{2\pi^2}{3} - \frac{\pi^2}{2} = \frac{16}{\pi^2}\left[1 + \frac{1}{3^4} + \frac{1}{5^4} + ...\right] \Rightarrow \frac{\pi^2}{6} = \frac{16}{\pi^2}\left[1 + \frac{1}{3^4} + \frac{1}{5^4} + ...\right]$$

$$\Rightarrow \frac{\pi^4}{96} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + ...$$

4.5 Fourier series in complex form

Suppose a function $f(x)$ is defined in the interval $(0, 2L)$

$$f(x) = c_0 + \sum_{n=1}^{\infty} c_n e^{i\frac{n\pi x}{L}} + \sum_{n=1}^{\infty} c_{-n} e^{-i\frac{n\pi x}{L}}$$

$$c_0 = \frac{1}{2L}\int_0^{2L} f(x)dx;\quad c_n = \frac{1}{2L}\int_0^{2L} f(x)e^{-i\frac{n\pi x}{L}}dx;\quad c_{-n} = \frac{1}{2L}\int_0^{2L} f(x)e^{i\frac{n\pi x}{L}}dx$$

Example 23: Obtain the complex form of the Fourier series of the function

$$f(x)=\begin{cases}0, & -\pi\le x\le 0\\ 1, & 0\le x\le \pi\end{cases}$$

Soln: $c_0=\frac{1}{2\pi}\int_0^{\pi}dx=\frac{1}{2}$

$$c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\,e^{-inx}dx=\frac{1}{2\pi}\left[\int_{-\pi}^{0}0.e^{-inx}dx+\int_0^{\pi}1.e^{-inx}dx\right]=\frac{1}{2\pi}\int_0^{\pi}e^{-inx}dx=\frac{1}{2\pi}\left[\frac{e^{-inx}}{-in}\right]_0^{\pi}$$

$$=-\frac{1}{2n\pi i}\left[e^{-in\pi}-1\right]=-\frac{1}{2n\pi i}\left[\cos n\pi-i\sin n\pi-1\right]=-\frac{1}{2n\pi i}\left[(-1)^n-0-1\right]$$

$$c_n=\begin{cases}\frac{1}{in\pi}, & n\text{ is odd}\\ 0, & \text{n is even}\end{cases}$$

Similarly, $c_{-n}=\begin{cases}-\frac{1}{in\pi}, & n\text{ is odd}\\ 0, & \text{n is even}\end{cases}$

$$f(x)=\frac{1}{2}+\frac{1}{i\pi}\left[\frac{e^{ix}}{1}+\frac{e^{3ix}}{3}+\frac{e^{5ix}}{5}+.....\right]+\frac{1}{i\pi}\left[\frac{e^{-ix}}{-1}+\frac{e^{-3ix}}{-3}+\frac{e^{-5ix}}{-5}+.......\right]$$

$$=\frac{1}{2}-\frac{1}{i\pi}\left[\left(e^{ix}-e^{-ix}\right)+\frac{1}{3}\left(e^{3ix}-e^{-3ix}\right)+\frac{1}{5}\left(e^{5ix}-e^{-5ix}\right)+......\right]$$

$$=\frac{1}{2}-\frac{2}{\pi}\left[\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+.......\right]$$

Example 24: The complex form of fourier series of $f(x)$, where $f(x)=\left\{e^{-|x|}, -2\le x\le 2\right.$ is

(a) $\sum_{n=-\infty}^{\infty}\frac{2}{4+n^2\pi^2}\left[1-(-1)^n e^{-2}\right]e^{i\frac{n\pi x}{2}}$ (b) $\sum_{n=-\infty}^{\infty}\frac{2}{4+n^2\pi^2}\left[1+(-1)^n e^{-2}\right]e^{in\pi x}$

(c) $\sum_{n=-\infty}^{\infty}\frac{2}{4+n^2\pi^2}\left[1-(-1)^n\right]e^{i\frac{n\pi x}{2}}$ (d) $\sum_{n=-\infty}^{\infty}\frac{2}{4+n^2\pi^2}\left[1+(-1)^n\right]e^{in\pi x}$

Soln: Complex form of fourier series is $f(x)=\sum_{-\infty}^{\infty}C_n e^{i\frac{2n\pi x}{4}}$

$$\text{Here, }C_n=\frac{1}{4}\int_{-2}^{2}f(x)e^{-i\frac{2n\pi x}{4}}dx=\frac{1}{4}\int_{-2}^{0}e^{x}e^{-i\frac{n\pi x}{2}}dx+\frac{1}{4}\int_0^2 e^{-x}e^{-i\frac{n\pi x}{2}}dx$$

$$=\frac{1}{4}\int_{-2}^{0}e^{\left(1-\frac{n\pi}{2}i\right)x}dx+\frac{1}{4}\int_0^2 e^{-\left(1+\frac{n\pi}{2}i\right)x}dx=\frac{1}{4}\left(\frac{e^{\left(1-\frac{n\pi i}{2}\right)x}}{1-\frac{n\pi}{2}i}\right)_{-2}^{0}+\frac{1}{4}\left(\frac{e^{-\left(1+\frac{n\pi i}{2}\right)x}}{-1+\frac{n\pi}{2}i}\right)_0^{2}$$

$$=\frac{1}{4}\left\{\frac{1}{1-\frac{n\pi}{2}i}\left(1-e^{-2\left(1-\frac{n\pi}{2}i\right)}\right)-\frac{1}{1+\frac{n\pi}{2}i}\left(e^{-2\left(1+\frac{n\pi}{2}i\right)}-1\right)\right\}$$

$$=\frac{1}{4}\left(\frac{1}{1-\frac{n\pi}{2}i}+\frac{1}{1+\frac{n\pi}{2}i}\right)+\frac{1}{4}\left(\frac{-1}{1-\frac{n\pi}{2}i}e^{-2}e^{n\pi i}-\frac{1}{1+\frac{n\pi}{2}i}e^{-2}e^{-n\pi i}\right)$$

$$=\frac{1}{4}\frac{2}{1+\frac{n^2\pi^2}{4}}+\frac{1}{4}\left(-\frac{1}{1-\frac{n\pi}{2}i}\cos n\pi-\frac{1}{1+\frac{n\pi}{2}i}\cos n\pi\right)e^{-2}$$

$$=\frac{2}{4+n^2\pi^2}-\left(\frac{2}{1+\frac{n^2\pi^2}{4}}\right)e^{-2}\cos n\pi\times\frac{1}{4}=\frac{2}{4+n^2\pi^2}\left[1-\cos n\pi\, e^{-2}\right]$$

$$=\frac{2}{4+n^2\pi^2}\left[1-(-1)^n e^{-2}\right]$$

$$\therefore\quad f(x)=\sum_{n=-\infty}^{\infty}\frac{2}{4+n^2\pi^2}\left[1-(-1)^n e^{-2}\right]e^{i\frac{n\pi x}{2}}.$$

Correct option is (a)

TIFR PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. A function $f(x)$ is defined in the range $-1\le x\le 1$ by

$$f(x)=\begin{cases}1-x & for\ \ x\ge 0\\ 1+x & for\ \ x<0\end{cases}$$

The first few terms in the Fourier series approximating this function are **[TIFR 2010]**

(a) $\frac{1}{2}+\frac{4}{\pi^2}\cos\pi x+\frac{4}{9\pi^2}\cos 3\pi x+...$

(b) $\frac{1}{2}+\frac{4}{\pi^2}\sin\pi x+\frac{4}{9\pi^2}\sin 3\pi x+...$

(c) $\frac{4}{\pi^2}\cos\pi x+\frac{4}{9\pi^2}\cos 3\pi x+...$

(d) $\frac{1}{2}-\frac{4}{\pi^2}\cos\pi x+\frac{4}{9\pi^2}\cos 3\pi x-...$

Soln. $f(x)=\begin{cases}1-x & x\ge 0\\ 1+x & x<0\end{cases}$ and $f(-x)=\begin{cases}1+x & x<0\\ 1-x & x\ge 0\end{cases}$

Hence, $f(x)$ is even $\quad(\because\ f(-x)=f(x))$

Therefore, $b_n=0$

$$a_0=\frac{2}{2}\int_{-1}^{1}f(x)\,dx=2\int_0^1(1-x)\,dx=2\left(x-\frac{x^2}{2}\right)_0^1=1$$

$$a_n=\frac{2}{2}\int_{-1}^{1}f(x)\cos\left(\frac{2\pi nx}{2}\right)dx=2\int_0^1(1-x)\cos(n\pi x)\,dx$$

$$= 2\left[\int_0^1 \cos n\pi x\,dx - \int_0^1 x\cos n\pi x\,dx\right] \qquad = 2\left[\frac{\sin n\pi x}{n\pi} - \frac{x\sin n\pi x}{n\pi} - \frac{\cos n\pi x}{(n\pi)^2}\right]_0^1$$

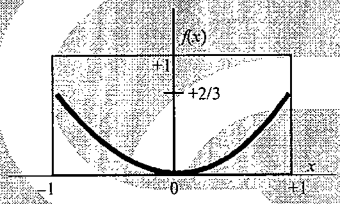
Therefore, $a_n = 2\left[-\frac{\cos n\pi}{n^2\pi^2} + \frac{1}{n^2\pi^2}\right] = \begin{cases} \frac{2}{n^2\pi^2}[2] & n = \text{odd} \\ 0 & n = \text{even} \end{cases}$

Therefore, fourier series of $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x$

$$\Rightarrow f(x) = \frac{1}{2} + \frac{4}{\pi^2}\left(\cos \pi x + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \cdots\right).$$

Correct option is (a)

Question 2. A student is asked to find a series approximation for the function $f(x)$ in the domain $-1 \le x \le +1$, as indicated by the thick line in the figure below.



The student represents the function by a sum of three terms

$$f(x) \approx a_0 + a_1 \cos\frac{\pi x}{2} + a_2 \sin\frac{\pi x}{2}$$ **[TIFR 2013]**

Which of the following would be the best choices for the coefficients a_0, a_1 and a_2?

(a) $a_0 = 1, a_1 = -\frac{1}{3}, a_2 = 0$ (b) $a_0 = \frac{2}{3}, a_1 = -\frac{2}{3}, a_2 = 0$

(c) $a_0 = \frac{2}{3}, a_1 = 0, a_2 = -\frac{2}{3}$ (d) $a_0 = -\frac{1}{3}; a_1 = 0, a_2 = -1$

Soln: Since, $f(x)$ is even (symmetric nature about x = 0), therefore $a_2 = 0$

At $x = 0\,; f(x) = 0 \Rightarrow 0 = a_0 + a_1 \Rightarrow a_0 = -a_1$

Correct option is (b)

JEST PREVIOUS YEAR QUESTIONS

Question 1. The function $f(x)=\cosh x$ which exists in the range $-\pi \le x \le \pi$, is periodically repeated between $x=(2m-1)\pi$ and $(2m+1)\pi$ where $m=-\infty$ to ∞. Using Fourier series, undicate the correct relation a $x=0$. **[JEST 2017]**

(a) $\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{1-n^2}=\frac{1}{2}\left(\frac{\pi}{\cosh\pi}-1\right)$

(b) $\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{1-n^2}=2\frac{\pi}{\cosh\pi}$

(c) $\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{1+n^2}=2\frac{\pi}{\sinh\pi}$

(d) $\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}=\frac{1}{2}\left(\frac{\pi}{\sinh\pi}-1\right)$

Soln: Since, $f(x)$ is even function, Hence $b_n=0$

$$a_0=\frac{2}{2\pi}\int_{-\pi}^{\pi}\cosh x\,dx=\frac{2}{\pi}\int_0^{\pi}\cosh x\,dx=\frac{2}{\pi}(\sinh x)_0^{\pi}=\frac{2}{\pi}\sinh\pi$$

$$a_n=\frac{2}{2\pi}\int_{-\pi}^{\pi}f(x)\cos\left(\frac{2\pi nx}{2\pi}\right)dx=\frac{2}{\pi}\int_0^{\pi}\cosh x\cos(nx)\,dx$$

Let $I=\int_0^{\pi}\cosh x\cos(nx)\,dx=\left(\cosh x\times\frac{\sin nx}{n}\right)_0^{\pi}-\int_0^{\pi}\sinh x\times\frac{\sin nx}{n}dx$

$$=0-\left\{\left(\frac{1}{n}\sinh x\times-\frac{\cos nx}{n}\right)_0^{\pi}-\int_0^{\pi}\frac{1}{n}\cosh x\times-\frac{\cos nx}{n}dx\right\}$$

$$I=\frac{1}{n^2}\sinh\pi\cos n\pi-\frac{1}{n^2}I\Rightarrow\left(1+\frac{1}{n^2}\right)I=\frac{\sinh\pi\cos n\pi}{n^2}$$

Therefore, $a_n=\frac{2}{\pi}\left(\frac{\sinh\pi\cos n\pi}{n^2+1}\right)=\frac{2}{\pi}(-1)^n\left(\frac{\sinh\pi}{n^2+1}\right)$

Hence, $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{2\pi nx}{2\pi}\right)\quad\Rightarrow\cosh x=\frac{1}{2}\times\frac{2}{\pi}\sinh x+\sum_{-\infty}^{\infty}\frac{2}{\pi}\frac{\sinh\pi(-1)^n}{n^2+1}\cos nx$

Putting $x=0$ on both sides, we get

$$\Rightarrow 1=\frac{1}{\pi}\sinh\pi+\frac{2}{\pi}\sum_{-\infty}^{\infty}\frac{(-1)^n\sinh\pi}{n^2+1}$$

$$\Rightarrow\frac{\pi}{2}=\sinh\pi\left(\frac{1}{2}+\sum_{-\infty}^{\infty}\frac{(-1)^n}{n^2+1}\right)\Rightarrow\sum_{-\infty}^{\infty}\frac{(-1)^n}{n^2+1}=\frac{1}{2}\left(\frac{\pi}{\sinh\pi}-1\right).$$

Correct option is (d)

DU PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. The Fourier series of a periodic function (with period 2π) is given by **[DU 2017]**

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

If $f(x)$ is defined by

$$f(x)=\begin{cases} x & \text{if } -\frac{\pi}{2}<x<\frac{\pi}{2} \\ \pi-x & \text{if } \frac{\pi}{2}<x<\frac{3\pi}{2} \end{cases}$$

then a_0 is equal to

(a) 0 (b) 2π (c) 1 (d) -2π

Soln: $a_0=\dfrac{2}{\dfrac{3\pi}{2}-\left(-\dfrac{\pi}{2}\right)}\displaystyle\int_{-\pi/2}^{3\pi/2}f(x)\,dx=\frac{2}{2\pi}\int_{-\pi/2}^{\pi/2}x\,dx+\frac{2}{2\pi}\int_{\pi/2}^{3\pi/2}(\pi-x)\,dx=0+\frac{1}{\pi}\left(\pi x-\frac{x^2}{2}\right)_{\pi/2}^{3\pi/2}$

$$=\frac{1}{\pi}\left[\pi\times\frac{3\pi}{2}-\frac{9\pi^2}{4\times2}-\pi\times\frac{\pi}{2}+\frac{\pi^2}{4\times2}\right]=\frac{\pi^2}{\pi}\left[\frac{3}{2}-\frac{9}{8}-\frac{1}{2}+\frac{1}{8}\right]=0$$

Correct option is (a)

HCU PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. The Fourier series of the functions **[HCU 2017]**

$$F(t)=\begin{cases} -1, & -\pi/\omega<t<0 \\ +1, & 0<t<\pi/\omega \end{cases}$$

in the interval $-\pi/\omega<t<\pi/\omega$ can be written as

(a) $\dfrac{4}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{\sin(n\omega t)}{n}$ (b) $\dfrac{4}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{\cos(n\omega t)}{n}$

(c) $\dfrac{4}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{\sin(n\omega t)}{n}+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(n\omega t)}{n}$ (d) $\dfrac{4}{\pi}\displaystyle\sum_{n=1,3,5,\ldots}^{\infty}\frac{\sin(n\omega t)}{n}$

Soln: $f(t)=\dfrac{a_0}{2}+\displaystyle\sum_{n=0}^{\infty}a_n\cos\frac{2n\pi}{b-a}+\sum_{n=0}^{\infty}b_n\sin\left(\frac{2n\pi}{b-a}\right)$

Since, $f(t)$ is an odd function

Therefore, $a_0=a_n=0$

$$b_n=\frac{2}{2\pi/\omega}\int_{-\pi/\omega}^{\pi/\omega}f(t)\sin\left(\frac{2\pi n}{2\pi/\omega}\right)dt \quad =\frac{\omega}{\pi}\left[\int_{-\pi/\omega}^{0}-\sin(n\omega t)\,dt+\int_{0}^{\pi/\omega}\sin(n\omega t)\,dt\right]$$

$$= \frac{\omega}{\pi}\times 2\int_0^{\pi/\omega}\sin(n\omega t)\,dt = \frac{2\omega}{\pi n\omega}(\cos(n\omega t))_0^{\pi/\omega} \quad = \frac{2}{\pi}[\cos(n\pi)-1]$$

$$\text{Therefore, } b_n = \begin{cases} \frac{2}{n\pi}[-1-1] & \text{if} \quad n = \text{odd} \\ \frac{2}{n\pi}\times 0 & \text{if} \quad n = \text{even} \end{cases}$$

$$\therefore \; f(t) = \sum_{n=1,3,5,\ldots}^{\infty} -\frac{4}{n\pi}\sin(n\omega t).$$

Correct option is (d)

BHU PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. The first term in the Fourier series of $f(x) = \frac{(\pi - x)^2}{4}$, in the range $(-\pi < x < \pi)$, is

[B.H.U-2016]

(a) $\frac{\pi}{12}$ (b) $\frac{\pi^2}{6}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi^2}{12}$

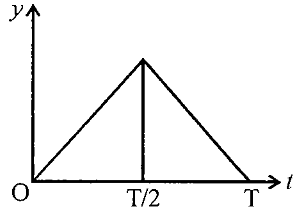
Soln: $a_0 = \frac{2}{2\pi}\int_{-\pi}^{\pi}\frac{(\pi - x)^2}{4}dx = \frac{1}{4\pi}\int_{-\pi}^{\pi}(\pi^2 + x^2 - 2\pi x)\,dx = \frac{1}{4\pi}\left[\pi^2 x + \frac{x^3}{3} - 2\pi\frac{x^2}{2}\right]_{-\pi}^{\pi}$

$$= \frac{1}{4\pi}\left[\left(\pi^3 + \frac{\pi^3}{3} - \pi^3\right) - \left(-\pi^3 - \frac{\pi^3}{3} - \pi^3\right)\right] = \frac{1}{4\pi}\left[\frac{8\pi^3}{3}\right] = \frac{2\pi^2}{3}.$$

Hence, 1st term $\frac{a_0}{2} = \frac{\pi^2}{3}$

Therefore, none of the options are correct.

Question 2. Consider the following triangular periodic wave of period T.



The first term in the Fourier series of this triangular periodic wave is **[B.H.U-2016]**

(a) a (b) $\frac{a}{2}$ (c) $-a$ (d) $-\frac{a}{2}$

Soln: $a_0 = \frac{2}{T}\int_0^T f(t)\,dt = \frac{2}{T}[\text{area under triangle}] \quad = \frac{2}{T}\times\frac{1}{2}\times T\times a = a$

Hence, 1st term $\frac{a_0}{2} = \frac{a}{2}$.

Correct option is (b)

Chapter 5

COMPLEX ANALYSIS

5.1 Basic Review of Complex Numbers

Various representation of Complex number:

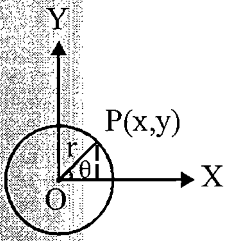
A complex number (z) is represented as $z = x + iy$, where x is the real part and y is imaginary part.

The conjugate of complex number z is represented by $\bar{z} = x - iy$

The modulus of complex number $z = |z| = r = \sqrt{x^2 + y^2}$ and argument of $z = Arg z = \theta = \tan^{-1}\left(\frac{y}{x}\right)$

The complex number (z) in polar form (r, θ) can also be written as

$$z = x + iy = r\, e^{i\theta} = r(\cos\theta + i\sin\theta)$$



where $x = r\cos\theta$

$y = r\sin\theta$

$\Rightarrow \quad x^2 + y^2 = r^2 = |z|^2$

This implies that $|z| = r$ represents a circle centred at the origin and having radius r.

Properties of modulus of z:

If z_1 and z_2 are two complex numbers, then

(i) $|z_1 + z_2| \le |z_1| + |z_2|$ (ii) $|z_1 - z_2| \ge |z_1| - |z_2|$

(iii) $|z_1 z_2| = |z_1||z_2|$ (iv) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

Properties of argument of z:

(i) $Arg(z_1.z_2.z_3.....z_n) = Arg.(z_1) + Arg.(z_2) + Arg.(z_3) + ... + Arg(z_n)$

(ii) $Arg\left(\frac{z_1}{z_2}\right) = Arg(z_1) - Arg(z_2)$

Some important Relations:

(i) $e^{i\theta} = \cos\theta + i\sin\theta$, (ii) $\cos\theta = (e^{i\theta} + e^{-i\theta})/2$, (iii) $\sin\theta = (e^{i\theta} - e^{-i\theta})/2i$

(iv) $\cos h\theta = \dfrac{(e^{\theta} + e^{-\theta})}{2}$, (v) $\sin h\theta = \dfrac{(e^{\theta} - e^{-\theta})}{2}$

De-Moiver's theorem:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Complex cube roots of unity:

$x^3 = 1 \Rightarrow x = 1, \omega = \dfrac{-1+i\sqrt{3}}{2}, \omega^2 = \dfrac{-1-i\sqrt{3}}{2}$ such that $1+\omega+\omega^2 = 0$ and $\omega^3 = 1$

Example 1: Multiplying a complex number z by $1 + i$ rotates the radius vector of z by an angle of

(a) 90° clockwise
(b) 45° anticlockwise
(c) 45° clockwise
(d) 90° anticlockwise

Soln: $z = r\,e^{i\theta} \Rightarrow (1+i)z = \left(\sqrt{2}\,e^{i\pi/4}\right) r\,e^{i\theta} = \sqrt{2}\,r\,e^{i(\theta+\pi/4)}$

Multiplying (1 + i) with z rotates the radius vector anticlockwise by 45° and increases the modulus by a factor of $\sqrt{2}$.

Correct option is (a)

Example 2: If $z = (\lambda+3) + i\sqrt{5-\lambda^2}$ (λ is a real parameter and $i = \sqrt{-1}$), then the locus of z will be

(a) circle
(b) ellipse
(c) parabola
(d) hyperbola

Soln: $z = (\lambda+3) + i\sqrt{5-\lambda^2} \Rightarrow$ real part $x = \lambda + 3$ and imaginary part $y = \sqrt{5-\lambda^2}$

$\Rightarrow y^2 = 5 - \lambda^2 = 5 - (x-3)^2 \Rightarrow (x-3)^2 + y^2 = 5$ (Equation of circle)

Correct option is (a)

Example 3: If $z = x + iy$, then for e^z to be real.

(a) $y = n\pi,\ n = 0, \pm 1, \pm 2, ...$
(b) $x = n\pi,\ n = 0, \pm 1, \pm 2, ...$
(c) $y = (2n+1)\dfrac{\pi}{2},\ n = 0, \pm 1, \pm 2, ...$
(d) $x = (2n+1)\dfrac{\pi}{2},\ n = 0, \pm 1, \pm 2, ...$

Soln: $e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y)$

Hence, for e^z to be real, $\sin y = 0 \Rightarrow y = n\pi,\ n = 0, \pm 1, \pm 2, ...$

Correct option is (a)

Example 4: If $z = x + iy$, then $\text{Im}\left(|z-4i|^2 + 3\bar{z}\right)$ is equal to

(a) $3y$
(b) $-3y$
(c) $(y-4)$
(d) $-(y-4)$

Soln: $|z-4i|^2+3\bar{z}=|x+iy-4i|^2+3(x-iy) \quad = x^2+(y-4)^2+3x-3iy$

$$=(x^2+(y-4)^2+3x)-i3y$$

Hence, imaginary part of $\left(|z-4i|^2+3\bar{z}\right)$ is $-3y$

Correct option is (b)

Example 5: $\left(1+\sqrt{3}\,i\right)^{1+i}$ can be expressed as

(a) $\left(1+\sqrt{3}\,i\right)e^{-\pi/3}\left[\cos(\log 2)+i\sin(\log 2)\right]$ (b) $\left(1+\sqrt{3}\,i\right)e^{\pi/3}\left[\cos(\log 2)-i\sin(\log 2)\right]$

(c) $(1+i)e^{\pi/3}\left[\cos\left(\log\sqrt{3}\right)+i\sin\left(\log\sqrt{3}\right)\right]$ (d) None of these

Soln: $\left(1+\sqrt{3}\,i\right)^{1+i}=e^{(1+i)\log\left(1+\sqrt{3}i\right)}=e^{(1+i)(\log 2+i(\pi/3))}$

$$=e^{\left(\log 2-\frac{\pi}{3}\right)+i\left(\log 2+\frac{\pi}{3}\right)} = e^{\log 2}e^{-\pi/3}\times e^{i\log 2}e^{i\frac{\pi}{3}}$$

$$=2e^{-\pi/3}\left[\cos(\log 2)+i\sin(\log 2)\right]\left[\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right]$$

$$=\left(1+\sqrt{3}\,i\right)e^{-\pi/3}\left[\cos(\log 2)+i\sin(\log 2)\right].$$

Correct option is (a)

Example 6: If z is a complex number such that $z=x+iy$, then z will be either purely real or purely imaginary if and only if z^2 is equal to

(a) $(\bar{z})^2$ (b) $(\bar{z})$ (c) $|\bar{z}|^2$ (d) None of these

Soln: If $z^2=(\bar{z})^2 \Rightarrow (x+iy)^2=(x-iy)^2 \Rightarrow (x+iy)^2-(x-iy)^2=0$

$\Rightarrow (x+iy+x-iy)(x+iy-x+iy)=0 \quad \Rightarrow (2x)(2iy)=0$

Hence, either $x=0$ or $y=0$.

Hence, z is purely imaginary or purely real accordingly.

Correct option is (a)

Example 7: Principle value of $\arg(z)$, where $z=-\frac{\sqrt{2}\,i}{1+i}$ is

(a) $\frac{3\pi}{4}$ (b) $\frac{\pi}{4}$ (c) $-\frac{3\pi}{4}$ (d) $-\frac{\pi}{4}$

Soln: $z=-\frac{\sqrt{2}\,i}{1+i}=-\frac{\sqrt{2}\,i}{1+i}\times\frac{1-i}{1-i} = -\frac{\sqrt{2}\,(1+i)}{2}=-\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}$

$$\arg(z)=\tan^{-1}\left(\frac{-1/\sqrt{2}}{-1/\sqrt{2}}\right)=\frac{\pi}{4}$$

Since, z lies in 3rd quadrant. Hence, principle value of arg (z) is $= \frac{\pi}{4} - \pi = -\frac{3\pi}{4}$.

Correct option is (c)

Example 8: If $|z| = 1$, then for all complex numbers 'a' and 'b' value of $\left|\frac{az+b}{\bar{b}z+\bar{a}}\right|$ is equal to

(a) $\sqrt{a^2+b^2}$ (b) 1 (c) $\frac{a}{b}$ (d) $\frac{b}{a}$

Soln: $\left|\frac{az+b}{\bar{b}z+\bar{a}}\right| = \frac{\left|z\left(a+\frac{b}{z}\right)\right|}{|\bar{b}z+\bar{a}|} = \frac{|z|\,|a+bz^{-1}|}{|\bar{b}z+\bar{a}|}$ $\left[\because |z| = 1,\ z\bar{z} = 1,\ \bar{z} = z^{-1}\right]$

$= \frac{|a+b\bar{z}|}{|\bar{b}z+\bar{a}|}$

Now, since $|\bar{z}| = |z|$

$|a+b\bar{z}| = |\overline{a+b\bar{z}}| = |\bar{a}+\bar{b}\bar{\bar{z}}| = |\bar{a}+\bar{b}z|$

$\therefore \quad \left|\frac{az+b}{\bar{b}z+\bar{a}}\right| = \frac{|a+b\bar{z}|}{|\bar{b}z+\bar{a}|} = \frac{|\bar{a}+\bar{b}z|}{|\bar{a}+\bar{b}z|} = 1.$

Correct option is (b)

Example 9: $\frac{1+i}{\sqrt{3}-i}$ is equal to

(a) $\sqrt{2}\,e^{i\frac{5\pi}{12}}$ (b) $\frac{1}{\sqrt{2}}e^{i\frac{5\pi}{12}}$ (c) $\sqrt{2}\,e^{i\frac{5\pi}{4}}$ (d) $\frac{1}{\sqrt{2}}e^{i\frac{5\pi}{4}}$

Soln: We have $1+i = \sqrt{2}\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right) = \sqrt{2}\,e^{i\frac{\pi}{4}}$ and $\sqrt{3}-i = 2\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right) = 2e^{i\frac{\pi}{6}}$

Hence, $\frac{1+i}{\sqrt{3}-i} = \frac{\sqrt{2}\,e^{i\frac{\pi}{4}}}{2e^{-i\frac{\pi}{6}}} = \frac{1}{\sqrt{2}}e^{i\frac{5\pi}{12}}$.

Correct option is (b)

Example 10: For the roots of unity $z = e^{2\pi\frac{i}{m}}$, $m > 1$. The value of $1+z+z^2+\cdots z^{m-1}$ is equal to __________.

Soln: $1+z+z^2+\cdots z^{m-1} = \frac{1-z^m}{1-z}$ (sum of 'm' terms of G.P.)

Here, $z^m = e^{2\pi\frac{i}{m}\times m} = e^{2\pi i} = 1$

Therefore, $\frac{1-z^m}{1-z}=0.$

Correct answer is (0)

Example 11: Imaginary part of $\sin^{-1}\left(\frac{3i}{4}\right)$ is ________.

Soln: Let $z=x+iy$

$$\sin^{-1}\left(\frac{3i}{4}\right)=z \Rightarrow \sin z=\frac{3i}{4} \Rightarrow \frac{e^{iz}-e^{-iz}}{2i}=\frac{3i}{4} \Rightarrow e^{iz}-e^{-iz}=-\frac{3}{2}$$

Let $e^{iz}=t$. Hence, $t-\frac{1}{t}=-\frac{3}{2} \Rightarrow t^2+\frac{3t}{2}-1=0$

Therefore, $t=\frac{-\frac{3}{4}\pm\sqrt{\frac{3}{4}-4\times-1}}{2} \Rightarrow t=-\frac{3}{4}\pm\frac{5}{4} \Rightarrow t=\frac{1}{2}$

$$\Rightarrow e^{iz}=\frac{1}{2} \Rightarrow iz=\log\left(\frac{1}{2}\right) \Rightarrow z=i\log 2$$

Hence, imaginary part $=\log 2=0.693$.

Correct answer is (0.693)

Example 12: If $|z|=1$, and $\left|z^2+2z+6+8i\right|\le n$, then value of n is ________. (Here, $z=x+iy$)

Soln: We have, $\left|z_1+z_2+z_3+\cdots z_n\right|\le\left|z_1\right|+\left|z_2\right|+\left|z_3\right|+\cdots\left|z_n\right|$

Hence, $\left|z^2+2z+6+8i\right|\le\left|z^2\right|+2|z|+|6+8i|\le|z|^2+2|z|+|6+8i|\le 1+2\times 1+\sqrt{36+64}\le 13$

Therefore, $n=13$.

Correct answer is (13)

5.2 Function of A complex Variable

Basic Representation:

$$f(z)=u(x,y)+iv(x,y)$$

Example: $f(z)=z^2=(x+iy)^2=\underbrace{(x^2-y^2)^2}_{u(x,y)\text{ is real part}}+\underbrace{i2xy}_{v(x,y)\text{ is imaginary part}}$

Existance of $\lim\limits_{z\to z_0} f(z)$:

The limit will exists only if the limiting value is independent of the path along which z approaches z_0

Example 13: Find whether the limit $\lim\limits_{z\to 0}\frac{z}{|z|}$ exist or not.

Soln: $z \to 0$ means $x \to 0 \;\&\; y \to 0$

For z = 0, we have to choose a path passing through a origin.

Therefore, we have choosen a straight line passing through the origin i.e. y = mx

$$\lim_{z\to 0}\frac{z}{|z|}=\lim_{\substack{z\to 0\\ y\to 0}}\frac{x+iy}{\sqrt{x^2+y^2}}=\lim_{x\to 0}\frac{x+imx}{\sqrt{x^2+m^2x^2}}=\frac{1+im}{\sqrt{1+m^2}}$$

Therefore, the limit depends on m i.e. slope of the straight line. Thus, the limiting values is dependent on the path and the limit does not exists.

Example 14: Calculate the value $\lim\limits_{z\to\infty}\frac{iz^3+iz-1}{(2z+3i)(z-i)^2}$

Soln: $$\lim_{z\to\infty}\frac{iz^3+iz-1}{(2z+3i)(z-i)^2}=\lim_{z\to\infty}\frac{z^3\left(i+\frac{i}{z^2}-\frac{1}{z^3}\right)}{z\left(2+\frac{3i}{z}\right)z^2\left(1-\frac{i}{z}\right)^2}=\frac{i}{2}$$

Differentiability of complex function:

$$f'(z)=\underset{\delta z\to 0}{Lt}\frac{[f(z+\delta z)-f(z)]}{\delta z}$$

The function will be differentiable if limit exists i.e the limiting value will be independent of path along with $\delta z \to 0$.

Example: $f(z) = (4x+y)+i(4y-x) \Rightarrow u = (4x+y)$ and $v = (4y-x)$

$\Rightarrow f(z+\delta z) = 4(x+\delta x)+(y+\delta y)+i\,[4(y+\delta y)-(x+\delta x)]$

$\Rightarrow f(z+\delta z) - f(z) = 4\delta x+\delta y+i(4\delta y-\delta x)$

$$\Rightarrow \frac{f(z+\delta z)-f(z)}{\delta z}=\frac{4\delta x+\delta y+i(4\delta y-\delta x)}{\delta z}\Rightarrow\frac{\delta f}{\delta z}=\frac{4\delta x+\delta y-i\delta x+4i\delta y}{\delta x+i\delta y}$$

Along real axis : $\delta x = \delta z$, $\delta y = 0$, $\Rightarrow \frac{\delta f}{\delta z}=4-i$

Along imaginary axis : $i\delta y = \delta z$, $\delta x = 0$, $\Rightarrow \frac{\delta f}{\delta z}=4-i$

Along a line : $y = x$, $\delta y = \delta x$, $\delta z = (1+i)\,\delta x$, $\Rightarrow \frac{\delta f}{\delta z}=\frac{5\delta x+3i\delta x}{(1+i)\delta x}=\frac{5+3i}{1+i}=4-i$

Therefore, $f(z) = (4x+y)+i(4y-x)$ will be differentiable.

5.3 Complex Analytic Function

A function $f(z)$ is said to be analytic at a point $z = z_0$ if it is defined and differentiable at every point in some neighhourhood of z_0. The function $f(z)$ is said to be analytic in a domain D if it is defined and is differentiable at every point of domain D.

Cauchy Reamann Equations in Cartesian Coordinates:

For a function $f(z) = u + iv$ to be analytic at all points in some region R,

Necessary conditions:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Sufficient Condition: $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous functions of x and y.

Derivative of $f(z)$**:** $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$

Example 15: Check whether $f(z) = \sin z$ is analytic or not.

Soln: $f(z) = \sin z = \sin(x+iy) = \sin x.\cos(iy) + \cos x.\sin(iy) = \sin x.\cosh y + i\cos x.\sinh y$

Therefore, u = sinx.coshy and v = cosx.sinhy

$\frac{\partial u}{\partial x} = \cos x.\cosh y$; $\frac{\partial u}{\partial y} = \sin x.\sinh y$; $\frac{\partial v}{\partial x} = -\sin x.\sinh y$; $\frac{\partial v}{\partial y} = \cos x.\cosh y$

So, C-R equation is satisfied, given f(z) is analytic.

Example 16: The harmonic conjugate function of $u(x,y) = 2x(1-y)$ corresponding to a complex analytic function $\omega = u(x,y) + iv(x,y)$ is given $v(x,y) = \alpha x^2 + \beta y + \gamma y^2$ (Taking the integration constant to be zero). Which of the following statement is true ?

(a) $\alpha - \gamma = \beta$ (b) $\alpha + \gamma + \beta = 0$ (c) $\alpha + \gamma = \beta$ (d) $\alpha\gamma\beta = 1$

Soln: $u(x,y) = 2x(1-y)$

$$\Rightarrow \frac{\partial u}{\partial x} = 2(1-y) = \frac{\partial v}{\partial y} \Rightarrow v = 2y - y^2 + f_1(x)$$

$$\Rightarrow \frac{\partial u}{\partial y} = -2x = -\frac{\partial v}{\partial x} \Rightarrow v = x^2 + f_2(y)$$

Therefore, the imaginary part of the complex function $v = x^2 - y^2 + 2y$

Comparing with the question, $\alpha = 1, \beta = 2, \gamma = -1 \quad \Rightarrow \alpha - \gamma = \beta$

Correct answer is (a)

Example 17: Let $u(x,y) = x + \frac{1}{2}(x^2 - y^2)$ be the real part of an analytic function *f*(*z*) of the complex variable $z = x + iy$. Then the imaginary part of *f*(*z*) is

(a) $y + xy$ (b) xy (c) y (d) $y^2 - x^2$

Soln: $u(x,y) = x + \frac{1}{2}(x^2 - y^2)$; $f(z) = u(x,y) + iv(x,y)$

Cauchy Reamann equation $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \frac{\partial u}{\partial x} = 1 + x \Rightarrow \frac{\partial v}{\partial y} = 1 + x$

$$\Rightarrow \quad v = y + xy + g(x) \qquad \ldots(1)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \Rightarrow \frac{\partial u}{\partial y} = -y \Rightarrow \frac{\partial v}{\partial x} = y \Rightarrow v = xy + h(y) \qquad \ldots(2)$$

Compare (1) & (2), taking $h(y) = y$; and $g(x) = 0$; $v = y + xy$

Correct answer is (a).

Cauchy Reamann equations in Polar co-ordiantes:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

Derivative of $f(z)$**:** $f'(z) = (\cos\theta - i\sin\theta)\frac{\partial f}{\partial r} = -\frac{i}{r}(\cos\theta - i\sin\theta)\frac{\partial f}{\partial \theta}$

5.4 Power Series Expansion of Complex Function

Every complex function which is analytic at $z = z_0$ can be expanded into power series about $z = z_0$. Power series expansion of $f(z)$ about $z = z_0$, is

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \ldots\ldots$$

where, z_0 is the centre of power series.

Radius of convergence of power series:

Consider a circle centered at $z = z_0$ and radius R i.e. $|z - z_0| = R$, such that the power series is convergent for all points of the region $|z - z_0| < R$ (i.e. within the circle) and divergent for some specific points of the region $|z - z_0| > R$ (outside the circle). Therefore, R is said to be the radius of convergence of power series and defined as

$$R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|$$

and $|z - z_0| < R$ is said to be the region of convergence of the power series.

.**Example 18:** Calculate the radius of convergence and region of convergence for the following power series

$$\sum_{n=0}^{\infty}\frac{(2n+3)}{(2n+5)(n+5)}z^n$$

Soln: For the given power series, $a_n=\dfrac{(2n+3)}{(2b+5)(n+5)},\ z_0=0$

$$R=\lim_{n\to\infty}\left|\frac{2n+3}{(2n+5)(n+5)}\,\frac{(2(n+1)+5)(n+1+5)}{2(n+1)+3}\right|=\lim_{n\to\infty}\left|\frac{\left(2+\frac{3}{n}\right)\left(2+\frac{7}{n}\right)\left(1+\frac{6}{n}\right)}{\left(2+\frac{5}{n}\right)\left(1+\frac{5}{n}\right)\left(2+\frac{5}{n}\right)}\right|=1$$

Region of convergence $|z|<1$

This series is convergent within a circle of radius 1 and centre (0, 0).

Example 19: Find the radius of convergence of the series

$$\frac{z}{2}+\frac{1.3}{2.5}z^2+\frac{1.3.5}{2.5.8}z^3+\ldots\ldots$$

Soln: The coefficient of z^n of the given power series is given by

$$a_n=\frac{1.3.5\ldots(2n-1)}{2.5.8\ldots(3n-1)}$$

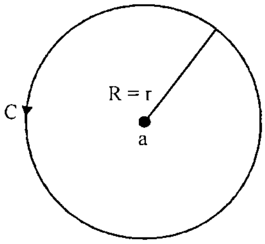
$$a_{n+1}=\frac{1.3.5\ldots(2n-1)(2n+1)}{25.8\ldots(3n-1)(3n+2)}$$

So, $$\frac{a_{n+1}}{a_n}=\frac{2n+1}{3n+2}=\frac{2}{3}\cdot\frac{\left(1+\frac{1}{2n}\right)}{\left(1+\frac{2}{3n}\right)}\Rightarrow R=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|=\lim_{n\to\infty}\left|\frac{3}{2}\frac{\left(1+\frac{1}{2n}\right)}{\left(1+\frac{2}{3n}\right)}\right|=\frac{3}{2}$$

Taylor Series Expansion

If a function $f(z)$ is analytic at all points inside the circle C, having center at $z=a$ and radius r, then at each point z inside C, the Taylor series expansion of $f(z)$ about $z=a$, is

$$f(z)=f(a)+\frac{f'(a)}{1!}(z-a)+\frac{f''(a)}{2!}(z-a)^2+\ldots.+\frac{f^n(a)}{n!}(z-a)^n+\ldots.$$



$$\Rightarrow f(z)=\sum_{n=0}^{\infty}\frac{f^n(z_0)}{n!}(z-z_0)^n$$

Example 20: Expand the function $f(x) = \sin x$ about the point $x = \frac{\pi}{6}$.

Soln: $f(x) = f\left(\frac{\pi}{6}\right) + \left(x - \frac{\pi}{6}\right) f'\left(\frac{\pi}{6}\right) + \frac{1}{2!}\left(x - \frac{\pi}{6}\right)^2 f''\left(\frac{\pi}{6}\right) + \ldots\ldots$

$$= \sin\frac{\pi}{6} + \left(x - \frac{\pi}{6}\right)\cos\frac{\pi}{6} + \frac{\left(x - \frac{\pi}{6}\right)^2}{2!}\left(-\sin\frac{\pi}{6}\right) + \ldots\ldots\ldots$$

$$= \frac{1}{2} + \left(x - \frac{\pi}{6}\right)\frac{\sqrt{3}}{2} - \frac{1}{2}\left(x - \frac{\pi}{6}\right)^2 \frac{1}{2} - \frac{1}{6}\left(x - \frac{\pi}{6}\right)^3 \frac{\sqrt{3}}{2} + \ldots\ldots$$

$$f(x) = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{4}\left(x - \frac{\pi}{6}\right)^2 - \frac{1}{4\sqrt{3}}\left(x - \frac{\pi}{6}\right)^3 + \ldots.$$

Example 21: Expand the following function $f(z) = \frac{1}{(z-1)(z-4)}$ about z = 2.

Soln: $\frac{1}{(z-1)(z-4)} = \frac{1}{3}\frac{1}{z-4} - \frac{1}{3}\frac{1}{z-1} = \frac{1}{3}\frac{1}{(z-2)-2} - \frac{1}{3}\frac{1}{(z-2)+1}$

$$= \frac{1}{3(-2)}\frac{1}{\left[1 - \frac{z-2}{2}\right]} - \frac{1}{3}\frac{1}{[1+(z-2)]} = -\frac{1}{6}\left[1 - \frac{z-2}{2}\right]^{-1} - \frac{1}{3}[1+(z-2)]^{-1}$$

$$= -\frac{1}{6}\left[1 + \frac{z-2}{2} + \frac{(z-2)^2}{4} + \frac{(z-2)^3}{8}\ldots\ldots\right] - \frac{1}{3}\left[1 - (z-2) + (z-2)^2 - (z-2)^3 + \ldots\ldots\right]$$

$$= -\frac{1}{2} + \frac{1}{4}(z-2) - \frac{3}{8}(z-2)^2 + \ldots\ldots\ldots$$

Example 22: The first few terms in the Taylor series expansion of the function f(x) = sinx around $x = \frac{\pi}{4}$ are

(a) $\frac{1}{\sqrt{2}}\left[1 + \left(x - \frac{\pi}{4}\right) + \frac{1}{2!}\left(x - \frac{\pi}{4}\right)^2 + \frac{1}{3!}\left(x - \frac{\pi}{4}\right)^3 + \ldots\ldots\ldots\right]$

(b) $\frac{1}{\sqrt{2}}\left[1 + \left(x - \frac{\pi}{4}\right) + \frac{1}{2!}\left(x - \frac{\pi}{4}\right)^2 - \frac{1}{3!}\left(x - \frac{\pi}{4}\right)^3 + \ldots\ldots\ldots\right]$

(c) $\left[\left(x - \frac{\pi}{4}\right) - \frac{1}{3!}\left(x - \frac{\pi}{4}\right)^3 + \ldots\ldots\ldots\right]$

(d) $\frac{1}{\sqrt{2}}\left[1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\ldots\right]$

Soln: $f(x)=f(a)+\dfrac{(x-a)}{1!}f'(a)+\dfrac{(x-a)^2}{2!}f''(a)+\dfrac{(x-a)^3}{3!}f'''(a)$

$f(x)=\sin x,\quad f'(x)=\cos x,\ f''(x)=-\sin x,\qquad f'''(x)=-\cos x$

$f\left(\dfrac{\pi}{4}\right)=\dfrac{1}{\sqrt{2}},\ f'\left(\dfrac{\pi}{4}\right)=\dfrac{1}{\sqrt{2}},\ f''\left(\dfrac{\pi}{4}\right)=-\dfrac{1}{\sqrt{2}}$

$\therefore\qquad f(x)=\dfrac{1}{\sqrt{2}}\left[1+\left(x-\dfrac{\pi}{4}\right)-\dfrac{1}{2!}\left(x-\dfrac{\pi}{4}\right)^2-\dfrac{1}{3!}\left(x-\dfrac{\pi}{4}\right)^3+......\right]$

Correct answer is (b)

Example 23: The taylor series expansion of the function $\ln(\cosh x)$, where x is real, about point x = 0 starts with the following terms:

(a) $-\dfrac{1}{2}x^2+\dfrac{1}{12}x^4+........$ (b) $\dfrac{1}{2}x^2-\dfrac{1}{12}x^4+........$

(c) $-\dfrac{1}{2}x^2+\dfrac{1}{6}x^4+........$ (d) $\dfrac{1}{2}x^2+\dfrac{1}{6}x^4+........$

Soln: Taylor series expansion of $f(x)$ about $x=x_0$ is:

$f(x)=f(x_0)+f'(x_0)(x-x_0)+\dfrac{f''(x_0)}{2!}(x-x_0)^2+\dfrac{f'''(x_0)}{3!}(x-x_0)+..............$

Now, in question $x_0=0$, $f(x)=\ell n(\cosh x)$

$f'(x)=\dfrac{1}{\cosh x}\sinh x=\tanh x$

$f''(x)=\sec h^2 x;$

$f'''(x)=2\text{sech}\,x(-\text{sech}\,x.\tanh x)=-2\text{sech}^2\,x.\tanh x$

$f''''(x)=-2(2\text{sech}\,x)(-\text{sech}\,x.\tanh x)\tanh x-2\text{sech}^2\,x.\text{sech}^2\,x\ =\ 4\text{sech}^2\,x.\tanh^2 x-2\text{sech}^4\,x$

Required Taylor series expansion: $\ell n(\cosh x)=0+0+\dfrac{1}{2}.1.x^2+0+\dfrac{1}{4!}(-2)x^4+................$

$\Rightarrow\qquad \ell n(\cosh x)=\dfrac{1}{2}x^2-\dfrac{1}{12}x^4+..............$

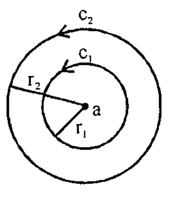
Correct answer is (b)

Laurent Series expansion:

Laurent's Theorem:

If $f(z)$ is analytic inside the ring-shaped region R bounded by two concentric circles C_1 and C_2 with centre at $z=a$ and respective radii r_1 and r_2 $(r_2>r_1)$, then for all points z in the region R, the Laurent series expansion of $f(z)$ about $z=a$ will be

$$f(z)=\sum_{n=0}^{\infty}a_n(z-a)^n+\sum_{n=1}^{\infty}\frac{b_n}{(z-a)^n}$$



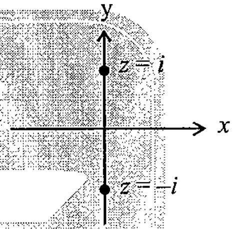
where, $a_n=\frac{1}{2\pi i}\oint_{C_1}\frac{f(z)}{(z-a)^{n+1}}dz$

$$b_n=\frac{1}{2\pi i}\oint_{C_2}\frac{f(z)}{(z-a)^{-(n+1)}}dz$$

Note: (i) The Laurrent series converges in the region R.

(ii) The point at which Laurrent series expansion has to be calculated, will be the centre of concentric circles.

Example 24: $f(z)=\frac{1}{1+z^2}$ about $z=i$



Soln: $f(z)=\frac{1}{1+z^2}=\frac{1}{(z+i)(z-i)}=\frac{1}{2i}\left[\frac{1}{z-i}-\frac{1}{z+i}\right]$

$$=\frac{1}{2i}\left[\frac{1}{z-i}-\frac{1}{(z-i)+2i}\right]$$

$$=\frac{1}{2i}(z-i)^{-1}-\frac{1}{2i}\left[\frac{1}{2i\left[1+\frac{(z-i)}{2i}\right]}\right]=\frac{1}{2i}(z-i)^{-1}-\frac{1}{4i^2}\left[1-\frac{(z-i)}{2i}+\frac{(z-i)^2}{4i^2}.....\right]$$

$$=\underbrace{\frac{1}{2i}(z-i)^{-1}}_{\text{Converges for }|z-i|\geq 0}+\underbrace{\frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\frac{(z-i)^n}{(2i)^n}}_{\text{Converges for }|z-i|<2}$$

So, the function will be convergent for $0\leq|z-i|<2$

Example 25: Exapand the function $f(z)=\frac{1}{(z-1)(z-2)}$ in the annulus $z=1$ and $z=2$.

Soln: $f(z)=\frac{1}{(z-1)(z-2)}=\frac{1}{(z-2)}-\frac{1}{(z-1)}$ [when $1<|z|<2$, then $\frac{1}{|z|}<1$ and $\frac{|z|}{2}<1$]

$$f(z)=\frac{1}{-2\left(1-\frac{z}{2}\right)}-\frac{1}{z\left(1-\frac{1}{z}\right)}=-\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}-\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$=-\frac{1}{2}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+....\right]-\frac{1}{z}\left[1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+.....\right]=-\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}-\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}$$

Example 26: Find the Laurent series of the function $f(z)=\dfrac{1}{z^2(1-z)}$ about z=0

Soln: $f(z)=\dfrac{1}{z^2(1-z)}=\dfrac{1}{z^2}(1-z)^{-1}=\dfrac{1}{z^2}\left(1+z+z^2+.....\right)=\dfrac{1}{z^2}+\dfrac{1}{z}+1+z+z^2+........$

Example 27: Expand $\dfrac{1}{z^2-3z+2}$ for (i) $0<|z|<1$ (ii) $1<|z|<2$ (iii) $|z|>2$.

Soln: $\dfrac{1}{z^2-3z+2}=\dfrac{1}{(z-1)(z-2)}=\dfrac{1}{(z-2)}-\dfrac{1}{(z-1)}$

(i) $0<|z|<1$: then $\dfrac{|z|}{2}<1$ and $|z|<1$

$$f(z)=\frac{1}{-2\left(1-\frac{z}{2}\right)^{-1}}-\frac{1}{-1(1-z)}=-\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}+(1-z)^{-1}$$

$$=-\frac{1}{2}\left[1+\frac{z}{2}+\frac{z^2}{4}+..........\right]+\left[1+z+z^2+..........\right]=\frac{-1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n+\sum_{n=0}^{\infty}z^n$$

(ii) $1<|z|<2$: then $\dfrac{1}{|z|}<1$ and $\dfrac{|z|}{2}<1$

$$\therefore\quad f(z)=-\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}-\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}=-\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+.....\right)-\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+.....\right)$$

$$=\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}-\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}$$

(iii) $|z|>2$: then $\dfrac{2}{|z|}<1$ and $\dfrac{1}{|z|}<1$

$$\therefore\quad f(z)=\frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}-\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}=\frac{1}{z}\left(1+\frac{2}{z}+\frac{4}{z^2}+.........\right)-\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+..........\right)$$

$$=\sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}}-\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}$$

Example 28: The Laurent series expansion of the function $f(z)=e^z+e^{1/z}$ about $z=0$ is given by

(a) $\displaystyle\sum_{n=-\infty}^{\infty}\frac{z^n}{n!}$ for all $|z|<\infty$

(b) $\displaystyle\sum_{n=0}^{\infty}\left(z^n+\frac{1}{z^n}\right)\frac{1}{n!}$ only if $0<|x|<1$

(c) $\displaystyle\sum_{n=0}^{\infty}\left(z^n+\frac{1}{z^n}\right)\frac{1}{n!}$ for all $0<|z|<\infty$

(d) $\displaystyle\sum_{n=-\infty}^{\infty}\frac{z^n}{n!}$, only if $|z|<1$

Soln: Given the complex function: $f(z) = e^z + e^{1/z}$

$$= \left[1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + ...\right] + \left[1 + \frac{1}{z} + \frac{1}{z^2.2!} + \frac{1}{z^3.3!} + ...\right]$$

$$= \sum_{n=0}^{\infty} \frac{z^n}{n!} + \sum_{n=0}^{\infty} \frac{1}{z^n} \frac{1}{n!} = \sum_{n=0}^{\infty} \left(z^n + \frac{1}{z^n}\right) \frac{1}{n!}$$

ince, e^z is not analytic at $z = \infty$ and $e^{1/z}$ is not analytic at $z = 0$,

Therefore, the laurrent series expansion will be convergent for $0 < |z| < \infty$

Correct Option is (c)

5.5 Singularity of Complex Function

Singular points of a function:

The points at which the function ceases to be analytic, are said to be the singular points of the function.

Example: $f(z) = \frac{1}{(z-2)}$ has a singularity at z = 2.

The singularity will be of the following types:

Isolated singularity:

A point $z = z_0$ is said to be isolated singularity of $f(z)$ if

(a) $f(z)$ is not analytic at $z = z_0$

(b) $f(z)$ is analytic in the neighbourhood of $z = z_0$ i.e. there exists a neighbourhood of $z = z_0$, containing no other singularity.

Example:

(i) Function $f(z) = \frac{1}{z}$ is analytic everywhere except at z = 0, therefore z = 0 is an isolated singularity.

(ii) The function $f(z) = \frac{z+2}{(z-1)(z-2)(z-3)}$ has three isolated singularities at $z = 1, 2$ and 3.

Non-isolated singularity:

A point $z = z_0$ is said to be non-isolated singularity of $f(z)$ if

(a) $f(z)$ is not analytic at $z = z_0$

(b) there exists a neighbourhood of $z = z_0$, containing other singularities of $f(z)$.

Example:

$$f(z)=\frac{1}{\left[\sin\frac{\pi}{z}\right]}$$

Condition of singularity: $\sin\frac{\pi}{z}=0 \Rightarrow \frac{\pi}{z}=n\pi \Rightarrow z=\frac{1}{n}\left(n=0,1,2,\ldots..\infty\right)$

The point $z = 0$ corresponding to $n\to\infty$, will be surrounded by infinite many singular points of $f(z)$. Thus, z = 0 is an non-isolated singularity of $f(z)$.

Types of isolated singularity

If $f(z)$ is an isolated singular point at z = a, then we can expand $f(z)$ about z = a into Laurent series as:

$$f(z)=\sum_{n=0}^{\infty}a_n(z-a)^n+\underbrace{\sum_{n=1}^{\infty}\frac{b_n}{(z-a)^n}}_{\substack{\text{principal part}\\ \text{of the expansion}}}=\left[a_0+a_1(z-a)+a_2(z-a)^2+\ldots..\right]+\left[\frac{b_1}{(z-a)}+\frac{b_2}{(z-a)^2}+\ldots..\right]$$

Therefore, three types of singulatiry are as follows:

(1) Removable singularity:

If the principal part of the Laurrent series expansion of $f(z)$ about z = a contains no term i.e. $b_n = 0$ for all 'n', then $f(z)$ has a removable singularity at z = a.

In this case, Laurent series expansion is $f(z)=\sum_{n=0}^{\infty}a_n(z-a)^n$

Example: $f(z)=\frac{\sin z}{z}=\frac{1}{z}\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\ldots\ldots..\right)=1-\frac{z^2}{3!}+\frac{z^4}{5!}-\ldots..$

Since, there is no negative term in the laurent series expansion of $f(z)$ about z = 0, hence z = 0 is a removable singularity of $f(z)$.

(2) Non-essential singularity or Pole:

If the principal part of the Laurrent series expansion of $f(z)$ about z = a contains finite number of terms, say m, i.e. $b_n = 0$ for all $n > m$, then $f(z)$ has a non-essential singularity or a pole of order m at z = a. A pole of order one is also known as simple pole.

Thus if z = a is pole of order m of function $f(z)$, then $f(z)$ will have the Laurent series expansion of the form

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\sum_{n=1}^{m}b_n(z-z_0)^{-n}$$

Example: $f(z)=\frac{z}{(z-1)(z+2)^2}$ has a simple pole at z = 1 and a pole of order 2 at z = -2.

(3) Essential singularity:

If the principal part of the Laurrent series expansion of $f(z)$ about z = a, contains infinite number of terms i.e. $b_n \neq 0$ for infinitely many values of n, then $f(z)$ has an essential singularity at z = a.

Example: $f(z)=e^{1/z}=1+\frac{1}{z}+\frac{1}{z^2\,2!}+\ldots\ldots$

Therefore, $f(z)$ has an essential singularity at z = 0.

Examples of Singular Points:

(1) $f(z)=\frac{1}{(z-1)(z+3)}=\frac{1}{4}\frac{1}{(z-1)}-\frac{1}{4}\frac{1}{(z+3)}$

For Laurent series expansion about $z = 1$

$$f(z)=\frac{1}{4}\frac{1}{(z-1)}-\frac{1}{4}\frac{1}{(z-1)+4}=\frac{1}{4}\frac{1}{(z-1)}-\frac{1}{4}\frac{1}{4\left[1+\frac{(z-1)}{4}\right]}$$

$$=\frac{1}{4}\frac{1}{(z-1)}-\frac{1}{16}\left[1+\frac{(z-1)}{4}\right]^{-1}=\frac{1}{4}\cdot\frac{1}{(z-1)}-\frac{1}{16}\left[1-\frac{(z-1)}{4}+\frac{(z-1)^2}{16}-\ldots\ldots\right]$$

$$=\frac{1}{4(z-1)}-\frac{1}{16}+\frac{(z-1)}{64}-\frac{(z-1)^2}{256}+\ldots\ldots$$

Therefore, $z = 1$ is a pole of order = 1
Similarly, we can check that $z = -3$ is also a pole of order 1

(2) $f(z)=\frac{1-e^{z^2}}{z^4}=\frac{1-\left[1+z^2+\frac{z^4}{2!}+\frac{z^6}{3!}+\ldots\ldots\right]}{z^4}=\frac{1}{z^2}-\frac{1}{2!}-\frac{z^2}{3!}-\ldots\ldots$

Therefore, $z = 0$ is a pole of order 2.

(3) $f(z)=\frac{z-\sin z}{z^3}=\frac{1}{z^3}\left[z-\left\{z-\frac{z^3}{3!}+\frac{z^5}{5!}-\ldots\ldots\right\}\right]$

$=+\frac{1}{3!}-\frac{z^2}{5!}+\ldots\ldots$

Therefore, $z = 0$ is a removable singular point.

Example 29: Examine the nature of singularity of the functions: (a) $\sin\left(\frac{1}{1-z}\right)$, (b) $(z-3)\sin\left(\frac{1}{z+2}\right)$.

Soln: (a) $\sin\left(\frac{1}{1-z}\right)=\frac{1}{1-z}-\frac{1}{(1-z)^3.3!}+\frac{1}{(1-z)^5.5!}-\ldots\ldots$

so, z = 1 is an isolated essential singular point.

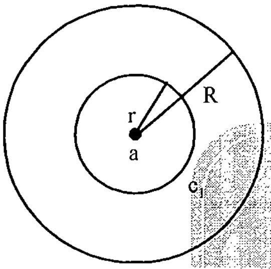
(b) $(z-3)\sin\left(\frac{1}{z+2}\right)=(z-3)\left[\frac{1}{z+2}-\frac{1}{(z+2)^3.3!}+\frac{1}{(z+2)^5.5!}-........\right]$

so, z = -2 is an isolated essential singular point.

5.6 Residue of a Complex Function

Residue at a pole:

Let, z = a be a pole of order 'm' of f (z) and C_1 is a circle of radius 'r' with center at z = a which does not contain singularities except z = a, then f(z) is analytic within the annular region r < | z – a | < R can be expanded into Laurrent series within the annulur region as:



$$f(z)=\sum_{n=0}^{\infty}a_n(z-a)^n+\sum_{n=1}^{\infty}b_n(z-a)^{-n}$$

Co-efficient b_1 is known as residue of f (z) at z = a i.e. Res. f(z = a) = $b_1 = \frac{1}{2\pi i}\oint_{C_1} f(z)dz$

Methods of finding residues:

CASE 1: Residue at simple pole:

(a) Method 1: Res. f(z = a) = $\underset{z\to a}{\text{Lt}}(z-a)f(z)$

(b) Method 2: If f(z) = $\frac{\phi(z)}{\psi(z)}$ where $\psi(a)=0$ but $\phi(a)\neq 0$, then Res. f(z = a) = $\frac{\phi(a)}{\psi'(a)}$

CASE 2: Residue at a pole of order 'n':

(a) Method 1: Res. f (z = a) = $\frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}\left[(z-a)^n f(z)\right]\right\}_{z=a}$

(b) Method 2: First put z - a = t and expand it into series, then Res. f (z = a) = co-efficient of 1/t.

CASE 3: Residue at z = ∞:

$$\text{Res.}f(z=\infty)=\underset{z\to\infty}{Lt}\left[-zf(z)\right]$$

Example 30: Find the singular points of the following function and the corresponding residues:

(a) $f(z)=\frac{1-2z}{z(z-1)(z-2)}$ (b) $f(z)=\frac{z^2}{z^2+a^2}$ (c) $f(z)=z^2e^{1/z}$

Soln: (a) $f(z)=\dfrac{1-2z}{z(z-1)(z-2)} \Rightarrow \text{Poles}: z=0, z=1, z=2$

$$\text{Res. } f(z=0)=\underset{z\to 0}{Lt}(z-0)f(z)=\underset{z\to 0}{Lt}\frac{1-2z}{(z-1)(z-2)}=\frac{1}{2}$$

$$\text{Res. } f(z=1)=\underset{z\to 1}{Lt}(z-1)f(z)=\underset{z\to 1}{Lt}\frac{1-2z}{z(z-2)}=1$$

$$\text{Res. } f(z=2)=\underset{z\to 2}{Lt}(z-2)f(z)=\underset{z\to 2}{Lt}\frac{1-2z}{z(z-1)}=-\frac{3}{2}$$

(b) $f(z)=\dfrac{z^2}{z^2+a^2} \Rightarrow \text{Poles}: z=ia, z=-ia$

$$\text{Res.} f(z=ia)=\left(\frac{z^2}{2z}\right)_{z=ia}=\frac{1}{2}ia; \text{Res.} f(z=-ia)=\left(\frac{z^2}{2z}\right)_{z=-ia}=-\frac{1}{2}ia$$

(c) $f(z)=z^2e^{1/z}=z^2\left[1+\dfrac{1}{z}+\dfrac{1}{z^2.2!}+\dfrac{1}{z^3.3!}+.......\right] \Rightarrow \text{Poles}: z=0$

$$\text{Res. } f(z=0)=\text{Coefficient of } \frac{1}{z}=\frac{1}{3!}=\frac{1}{6}$$

Example 31: The value of the residue of $\dfrac{\sin z}{z^6}$ is

(a) $-\dfrac{1}{5!}$ (b) $\dfrac{1}{5!}$ (c) $\dfrac{2\pi i}{5!}$ (d) $-\dfrac{2\pi i}{5!}$

Soln: $f(z)=\dfrac{\sin z}{z^6}=\dfrac{1}{z^6}\left[z-\dfrac{z^3}{3!}+\dfrac{z^5}{5!}-\dfrac{z^7}{7!}+.......\right]$

$$=\frac{1}{z^5}-\frac{1}{3!}\frac{1}{z^3}+\frac{1}{5!}\frac{1}{z}-......$$

Residue of $f(z)$ = coefficient of $\left(\dfrac{1}{z}\right)=\dfrac{1}{5!}$

Correct option is (b)

5.7 Cauchy Residue Theorem:

If f (z) in single-valued and analytic inside a simple closed curve 'C, except at a finite number of singular points within C, then

$$\oint_C f(z)dz=2\pi i(\text{sum of the residues at poles within 'C'})$$

Example 32: Evaluate the integral: $\oint\limits_C \frac{4-3z}{z(z-1)(z-3)}dz$ where $|z|=\frac{3}{2}$

Soln: $f(z)=\frac{4-3z}{z(z-1)(z-3)} \Rightarrow \text{Poles}: z=0, z=1, z=3$

But, the given contour is circle centered at the origin and radius 3/2 units.

Therefore, only z = 0 and z = 1 within the contour.

$$I=2\pi i\left[\operatorname{Res}.f(z=0)+\operatorname{Res}.f(z=1)\right]=2\pi i\left[\frac{4}{3}-\frac{1}{2}\right]=\frac{5\pi i}{3}$$

Example 33: Evaluate the integral: $\oint\limits_C \frac{e^{2z}+z^2}{(z-1)^5}dz \ where \ |z|=2$

Soln: $f(z)=\frac{e^{2z}+z^2}{(z-1)^5} \Rightarrow \text{Poles}: z=1\ (order\ 5)$

$$I=2\pi i\times\operatorname{Res}.f(z=1)=2\pi i\times\frac{1}{4!}\frac{d^4}{dz^4}\left[e^{2z}+z^2\right]_{z=1}=2\pi i\times\frac{2e^2}{3}=\frac{4\pi i e^2}{3}$$

Example 34: The value of the integral $\oint\limits_C \frac{z^3}{z^2-5z+6}dz$, where C is closed contour defined by the equation $2|z|-5=0$, traversed in the anti-clockwise direction, is

(a) $-16\pi i$ (b) $16\pi i$ (c) $8\pi i$ (d) $2\pi i$

Soln: $f(z)=\frac{z^3}{z^2-5z+6}$; Condition of singularity: $z^2-5z+6=0 \Rightarrow z=3,2$ (1st order pole)

Given closed contour: $c: 2|z|-5=0 \Rightarrow |z|=\frac{5}{2}$

Only z = 2 i.e. point (2, 0) is within 'c'

$$\operatorname{Res}.f(z)\big|_{z=2}=(z-2)\frac{z^3}{(z-3)(z-2)}\bigg|_{z=2}=-8$$

Given integral: $I=2\pi i\times(\text{sum of the residues})=-16\pi i$

Correct answer is (a)

Example 35: The value of the integral $\int\limits_C z^{10}dz$, C is the unit circle with the origin as the centre is

(a) 0 (b) $z^{11}/11$ (c) $2\pi i z^{11}/11$ (d) 1/1

Soln: $f(z)=z^{10}$ is analytic in the entire complex argand plane.

So, according to Cauchy integral theorem,

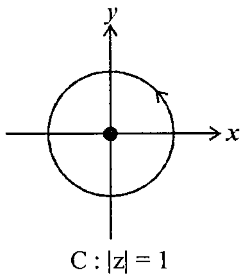
$$\int\limits_C z^{10}dz=0$$

Correct option is (a)

Example 36: The value of the integral $\oint_C dz/z^2$, where z is a complex variable and C is the unit circle with the origin as its centre, is:

(a) 0 (b) $2\pi i$ (c) $4\pi i$ (d) $-4\pi i$

Soln: $f(z)=\frac{1}{z^2}$ has a pole of order 2 at $z=0$



C : |z| = 1

Therefore, $\int_C \frac{dz}{z^2}=2\pi i$ [Residue at $z=0$]

$$=2\pi i\left[\frac{1}{(2-1)!}\frac{d}{dz}\left\{z^2\cdot\frac{1}{z^2}\right\}_{z=0}\right]=0$$

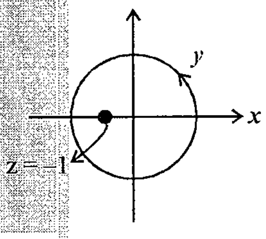
Correct option is (a)

Example 37: The value of $\int_C \frac{e^{2z}}{(z+1)^4}dz$, where C is a circle defined by $|Z|=3$, is

(a) $\frac{8\pi i}{3}e^2$ (b) $\frac{8\pi i}{3}e^{-1}$ (c) $\frac{8\pi i}{3}e$ (d) $\frac{8\pi i}{3}e^2$

Soln: $f(z)=\frac{e^{2z}}{(z+1)^4}$ has a pole of order 4 at $z=-1$.

C : |z| = 3



Residue of $f(z)$ at $z=-1$

$$=\frac{1}{(4-1)!}\frac{d^3}{dz^3}\left[(z+1)^4\frac{e^{2z}}{(z+1)^4}\right]_{z=-1}$$

$$=\frac{1}{3!}\left[8e^{2z}\right]_{z=-1}=\frac{4}{3}e^{-2}$$

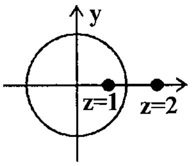
So, $I=2\pi i\left(\frac{4}{3}e^{-2}\right)=\frac{8\pi i}{3}e^{-2}$

Correct option is (a)

Example 38: The value of integral $\int_C \frac{e^z}{z^2-3z+2}dz$, where the contour C is the circle $|z|=\frac{3}{2}$ is

(a) $2\pi ie$ (b) πie (c) $-2\pi ie$ (d) $-\pi ie$

C:|Z| = 3/2



Soln: $f(z)=\frac{e^z}{z^2-3z+2}$ has simple poles at $z=1$ and $z=2$

Only $z=1$ lies within the circle.

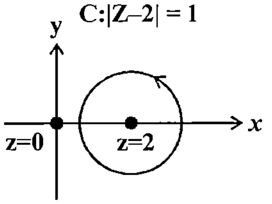
$I=2\pi i\left[\text{Residual of } f(z) \text{ at } z=1\right]$

$$=2\pi\left[\lim_{z\to 1}(z-1)\frac{e^z}{(z-1)(z-2)}\right]=2\pi i(-e)=-2\pi ie$$

Correct option is (c)

Example 39: The value of integral $\oint_C \frac{e^z \sin z}{z^2} dz$, where the contour C is the unit circle : $|z-2|=1$, is

(a) $2\pi i$ (b) $4\pi i$ (c) πi (d) 0



Soln: $f(z)=\frac{e^z \sin z}{z^2}$ has pole of order '1' at z = 0

Since, $z=0$ lies outside the circle

So, $\oint_C \frac{e^z \cdot \sin z}{z^2} dz = 0$

Correct option is (d)

Example 40: For the function $f(z)=\frac{16z}{(z+3)(z-1)^2}$, the residue at the pole $z=1$.........

(Your answer should be an integer)

Soln: $f(z)=\frac{16z}{(z+3)(z-1)^2}$ has a pole of order 2 at $z=1$

Residue of $f(z)$ at $z=1$

$$=\frac{1}{(2-1)!}\frac{d}{dz}\left[(z-1)^2\frac{16z}{(z+3)(z-1)^2}\right]_{z=1}=\left.\frac{(z+3)\cdot 16-16z\cdot 1}{(z+3)^2}\right|_{z=1}=\left.\frac{48}{(z+3)^2}\right|_{z=1}=3$$

Correct answer is (3)

Definite integrals of trigonometric functions of cosθ and sinθ:

Method:

(i) Consider the contour to be a circle centered at the origin and having radius one unit i.e. |z|=1

(ii) Assume, $z=e^{i\theta} \Rightarrow dz = ie^{i\theta}d\theta \Rightarrow d\theta = \frac{dz}{iz}$

(iii) Therefore, $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}=\frac{1}{2}\left(z+\frac{1}{z}\right)$ and $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}=\frac{1}{2i}\left(z-\frac{1}{z}\right)$.

Replacements regarding $\cos\theta$ and $\sin\theta$ is to be done only in the denominator of the given integral.

(iv) The limit will be changed from $0 \to 2\pi$ to $\oint_C$

(v) Find the singular points and find residues at only those singular points which lies inside the unit circle.

(vi) Finally use the Cauchy residue theorem.

Example 41: Evaluate the integral: $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}$; $a > b > 0$

Soln: $$\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \int_0^{2\pi} \frac{dz/iz}{a+b\left(\frac{z^2+1}{2z}\right)} = \int_0^{2\pi} \frac{2dz}{i(bz^2+2az+b)}$$

The singular points are at $\Rightarrow z = \alpha = \frac{-a+\sqrt{a^2-b^2}}{b}$ & $z = \beta = \frac{-a-\sqrt{a^2-b^2}}{b}$

The singular point $z = \beta$ will lie outside the unit circle as a>b>0 while the singular point $z = \alpha$ will lie inside the unit circle which is a simple pole.

$$\text{Res.} f(z=\alpha) = \lim_{z\to\alpha}(z-\alpha)f(z) = \lim_{z\to\alpha}\frac{2}{ib}\frac{(z-\alpha)}{(z-\alpha)(z-\beta)} = \frac{2}{ib(\alpha-\beta)} = \frac{2}{ib}\times\frac{b}{2\sqrt{a^2-b^2}} = \frac{1}{i\sqrt{a^2-b^2}}$$

Therefore, by cauchy Residue theorem.

$$I = 2\pi i \times \text{Residue} = 2\pi i \times \frac{1}{i\sqrt{a^2-b^2}} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

Example 42: Evaluate the integral $\int_0^{2\pi} \frac{d\theta}{5+4\cos\theta}$

Soln: $$\int_0^{2\pi} \frac{d\theta}{5+4\cos\theta} = \oint_C \frac{1}{5+4.\frac{1}{2}\left(z+\frac{1}{2}\right)}.\frac{dz}{iz} = \frac{1}{i}\oint_C \frac{dz}{(5z+2z^2+2)} = \oint_C \frac{dz}{(2z+1)(z+2)}$$

$f(z) = \frac{1}{(2z+1)(z+2)}$ $\Rightarrow$ poles are at $z = -\frac{1}{2}, -2$ (Only $z = -½$ is within the circle)

$$\text{Res. } f(z=-½) = \left(z+\frac{1}{2}\right).\frac{1}{(2z+1)(z+2)}\bigg|_{z=-\frac{1}{2}} = \frac{1}{2(z+2)}\bigg|_{z=-\frac{1}{2}} = \frac{1}{3}$$

Hence, the integral $= \frac{2\pi i}{3}$

Example 43: Evaluate the integral $\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta - n\theta)\,d\theta$

Soln: $$\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta-n\theta)\,d\theta = \text{Real part of } \int_0^{2\pi} e^{\cos\theta}e^{i(\sin\theta-n\theta)}d\theta$$

$$= \text{Real part of } \int_0^{2\pi} e^{(\cos\theta+i\sin\theta)}e^{-in\theta}d\theta = \text{Real part of } \oint_C e^z.z^{-n}\frac{dz}{iz}$$

$$= \text{Real part of } \frac{1}{i}\oint_C \frac{e^z}{z^{n+1}}dz$$

$f(z) = \frac{e^z}{z^{n+1}}$ $\Rightarrow$ poles are at $z = 0$ of order n $\Rightarrow$ $\text{Res. } f(z=0) = \frac{1}{n!}.\frac{d^n}{dz^n}\left(z^{n+1}.\frac{e^z}{z^{n+1}}\right)\bigg|_{z=0} = \frac{1}{n!}$

Hence the given integral = real part of $\left(\frac{1}{i}.2\pi i.\frac{1}{n!}\right)=\frac{2\pi}{n!}$

Evaluation of real improper integrals between the limit $-\infty$ to $+\infty$:

Case 1:

$$f(x)=\frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are alzebraic functions of x.

First assume $f(z)=\frac{p(z)}{q(z)}$ and find the singular points of $f(z)$.

$$\int_{-\infty}^{\infty} f(x)\,dx=\int_{-\infty}^{\infty} f(z)\,dz=2\pi i\left[\Sigma \text{ Res. at all singular points in the upper half of the complex plane}\right]$$
$$+\pi i\left[\Sigma \text{ Res. at all singular points on the real axis}\right]$$

Example 44: Calculate the following integral $\int_{-\infty}^{\infty}\frac{dx}{1+x^2}$

Soln: $f(x)=\frac{1}{1+x^2}$; Assume $f(z)=\frac{1}{1+z^2}$

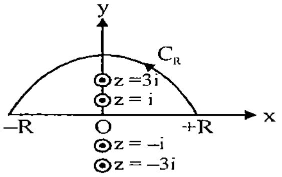
$f(z)=\frac{1}{1+z^2}$ has simple poles at $z=\pm i$.

Here, $z=i$ lies in the upper half of the complex plane.

So, $\int_{-\infty}^{\infty}\frac{dx}{1+x^2}=\int_{-\infty}^{\infty}\frac{dz}{1+z^2}=2\pi i\left[\text{Res. } f(z=i)\right]=2\pi i\left[\frac{1}{2i}\right]=\pi$

Example 45: Apply calculus of residues to show that $\int_0^{\infty}\frac{dx}{(x^2+1)(x^2+9)}=\frac{\pi}{24}$

Soln: $I=\int_0^{\infty}\frac{dx}{(x^2+1)(x^2+9)}=\frac{1}{2}\int_{-\infty}^{\infty}\frac{dx}{(x^2+1)(x^2+9)}$



Let $f(z)=\frac{1}{(z^2+1)(z^2+9)}$

The singular points of f(z) are

$(z^2+1)(z^2+9)=0 \quad\Rightarrow\quad z=i,-i,3i,-3i$ (Simple poles)

The only poles lying in the upper half plane are at $z=+i$ and $+3i$

$$\int_{-\infty}^{\infty} \mathrm{f}(\mathrm{z})\,\mathrm{dz} = \int_{-\infty}^{\infty} \mathrm{f}(\mathrm{x})\,\mathrm{dx} = 2\pi \mathrm{i}\left(\text{Res. at } (\mathrm{z}=\mathrm{i}) + \text{Res. at } (\mathrm{z}=3\mathrm{i})\right)$$

Therefore, $\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+9)} = 2\pi i\left[\frac{1}{16i} - \frac{1}{48i}\right] = \frac{\pi}{12} \Rightarrow \int_{0}^{\infty} \frac{dx}{(x^2+1)(x^2+9)} = \frac{\pi}{24}$

Example 46: The value of the integral $\int\limits_{-\infty}^{\infty} \frac{1}{1+x^4}\,dx$ is

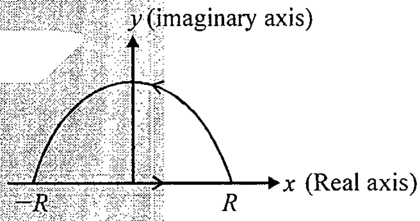
(a) $\pi/\sqrt{2}$ (b) $\pi/2$ (c) $\sqrt{2}\pi$ (d) 2π

Soln: Assume, $f(z) = \frac{1}{1+z^4}$

Condition of singularity : $1+z^4 = 0 \Rightarrow z^4 = -1 = e^{i(2\pi+1)\pi} \Rightarrow z = e^{i(2n+1)\pi/4}$ $(n = 0, 1, 2, 3)$

Poles of $f(z)$ will be $z = e^{i\pi/4}, e^{i3\pi/4}, e^{i5\pi/4}, e^{i7\pi/4}$ and will be order 1.

Out of them, $z = e^{i\pi/4}, e^{3i\pi/4}$ lies in the upper half of complex plane



$$\text{Res}\cdot f\left(z = e^{i\pi/4}\right) = \frac{1}{4z^3} = \frac{1}{4e^{i3\pi/4}}$$

$$\text{Res}\cdot f\left(z = e^{i3\pi/4}\right) = \frac{1}{4z^3} = \frac{1}{4e^{i9\pi/4}}$$

$$\therefore \qquad \int\limits_{-\infty}^{\infty} \frac{1}{1+x^4}\,dx = 2\pi i\frac{1}{4}\left[\frac{1}{e^{3i\pi/4}} + \frac{1}{e^{i9\pi/4}}\right]$$

$$= \frac{2\pi i}{4}\left[e^{-3i\pi/4} + e^{-i9\pi/4}\right] = \frac{2\pi i}{4}\left[\frac{1}{\sqrt{2}}(-1-i) + \frac{1}{\sqrt{2}}(1-i)\right] = \frac{4\pi}{4\sqrt{2}} = \frac{\pi}{\sqrt{2}}$$

Correct option is (a)

Case 2:

$$f(x) = \frac{p(x)}{q(x)}\cos mx \ \ or \ \ \frac{p(x)}{q(x)}\sin mx$$

where $p(x)$ and $q(x)$ are alzebraic functions of x.

First assume $f(z) = \frac{p(z)}{q(z)}e^{imz}$ and find the singular points of $f(z)$.

$$\int\limits_{-\infty}^{\infty} f(x)\,dx = \text{R.P. or I.P. of} \int\limits_{-\infty}^{\infty} f(z)\,dz$$

$$= \text{R.P. or I.P. of } \{2\pi i[\Sigma \text{ Res. at all singular points in the upper half of the complex plane}]$$
$$+ \pi i[\Sigma \text{ Res. at all singular points on the real axis}]\}$$

Example 47: Prove $\int_{-\infty}^{\infty}\frac{\cos x dx}{\left(x^2+a^2\right)}=\frac{\pi}{a}e^{-a} \quad ; a>0$

Soln: Let $f(z)=\frac{e^{iz}}{\left(z^2+a^2\right)}$; Condition of singularity: $z^2+a^2=0 \Rightarrow z=\pm ia$.

The only pole which lies in the upper half of the circle is at z = ia.

Therefore, Residue at (z = ia) = $\lim_{z\to ia}\frac{(z-ia)e^{iz}}{(z-ia)(z+ai)}=\frac{e^{-a}}{2ia}$

So, $\int_{-\infty}^{\infty}\frac{e^{iz}}{z^2+a^2}dz=2\pi i[\text{Residue at } z=ia\ ]=2\pi i\left[\frac{e^{-a}}{2ia}\right]=\frac{\pi}{a}e^{-a}$

Hence, $\int_{-\infty}^{\infty}\frac{\cos x}{\left(x^2+a^2\right)}dx$ = R.P. of $\int_{-\infty}^{\infty}\frac{e^{ix}}{\left(x^2+a^2\right)}dx=\frac{\pi}{a}e^{-a} \Rightarrow \int_0^{\infty}\frac{\cos x}{\left(x^2+a^2\right)}dx=\frac{\pi}{2a}e^{-a}$

Example 48: If $a>0$, $m>0$, then

$$\int_0^{\infty}\frac{\cos mx dx}{\left(x^2+a^2\right)^2}=\frac{\pi}{4a^3}(1+ma)e^{-ma}$$

Soln: $f(z)=\frac{e^{imz}}{\left(a^2+z^2\right)^2}$ has poles of order 2 at $z=\pm ia$

Only $z=ia$ is in the upper half of the complex plane.

$$\int_{-\infty}^{\infty}\frac{e^{imz}}{\left(z^2+a^2\right)^2}dx=2\pi i[\text{Res}(z=ia)]=2\pi i e^{-ma}\frac{(1+ma)}{4a^3 i}=\frac{\pi}{2a^3}e^{-ma}(1+ma)$$

Equating real parts from both sides,

$$\int_{-\infty}^{\infty}\frac{\cos mx dx}{\left(x^2+a^2\right)^2}=\frac{\pi}{2a^3}(1+ma)e^{-ma}$$

But $\int_{-\infty}^{\infty}\frac{\cos mx dx}{\left(x^2+a^2\right)^2}dx=2\int_0^{\infty}\frac{\cos mx}{\left(x^2+a^2\right)^2}dx$

Hence, the last gives $\int_0^{\infty}\frac{\cos mx}{\left(x^2+a^2\right)^2}dx=\frac{\pi}{4a^3}(1+ma)e^{-ma}$

Example 49: The principal value of the integral $\int_{-\infty}^{\infty}\frac{\sin(2x)}{x^3}dx$ is

(a) −2π (b) −π (c) π (d) 2π

Soln: Given integral: $\int_{-\infty}^{\infty}\frac{\sin(2x)}{x^3}dx = \text{Imaginary part of } \int_{-\infty}^{\infty}\frac{e^{i2x}}{x^3}dx$

Assume, the complex function to be $f(z)=\frac{e^{i2z}}{z^3}$

Singular points of $f(z)$: $z = 0$ is pole of order 3
Since, the pole lie on the real axis, therefore,

$$\int_{-\infty}^{\infty}\frac{e^{i2z}}{z^3}dz = \pi i\left[\text{Residue of } f(z) \text{ at } z=0\right]$$

$$=\pi i\left[\frac{1}{(3-1)!}\frac{d^2}{dz^2}\left\{(z-0)^3\frac{e^{i2z}}{z^3}\right\}\right]_{z=0}$$

$$=\pi i\left[\frac{1}{2}\left(4i^2\right)e^{i2z}\right]_{z=0} = -2\pi i$$

So, the required integral $\int_{-\infty}^{\infty}\frac{\sin(2x)}{x^3}dx = \text{Imaginary part of } (-2\pi i) = -2\pi$

Correct Option is (a)

TIFR PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. If $z = x + iy$ then the function

$$f(x,y) = (1+x+y)(1+x-y) + a(x^2-y^2) - 1 + 2iy(1-x-ax)$$

where a is a real parameter, is analytic in the complex z plane if a is equal to

(a) -1 (b) $+1$ (c) 0 (d) i

Soln: $f(x,y) = (1+x+y)(1+x-y) + a(x^2-y^2) - 1 + 2iy(1-x-ax)$

Comparing this with, $f(x,y) = u(x,y) + iv(x,y)$

We have, $u = (1+x+y)(1+x-y) + a(x^2-y^2) - 1$ and $v = 2y(1-x-ax)$

For $f(x,y)$ to be analytic, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\Rightarrow (1+x+y) + (1+x-y) + 2ax = 2(1-x-ax)$

$\Rightarrow 2 + 2x + 2ax = 2 - 2x - 2ax$

$\Rightarrow 4x + 4ax = 0 \Rightarrow 4x(1+a) = 0$

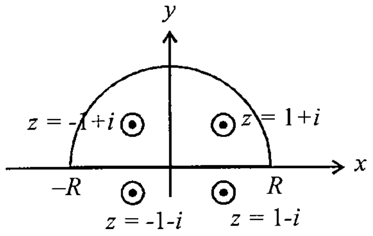
Since, $x \neq 0 \Rightarrow 1 + a = 0$ or $a = -1$.

Correct option is (a)

Question 2. The integral $\int_0^{\infty} \frac{dx}{4+x^4}$ evalautes to **[TIFR 2014, 2017]**

(a) π (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{8}$

Soln: We have, $\int_0^{\infty} \frac{dx}{4+x^4}$



Let $f(z) = \frac{1}{z^4+4}$

The singular points of $f(z)$ are

$$z^4 = -4 = 4(\cos\pi + i\sin\pi) = 4(\cos(2n+1)\pi + i\sin(2n+1)\pi)$$

$$\Rightarrow \quad z = \sqrt{2}\left[\cos(2n+1)\frac{\pi}{4} + i\sin(2n+1)\frac{\pi}{4}\right] \qquad n = 0,1,2,3$$

$z_1 = (1+i)$, $z_2 = (-1+i)$, $z_3 = (-1-i)$, $z_4 = (1-i)$ out of this four poles only $(1+i)$ and $(-1+i)$ lie with in the contour.

Now, $\int_0^{\infty} \frac{dz}{z^4+4} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dz}{z^4+4} = \pi i$ (sum of the residues with in contour)

$$= \pi i\left[\text{Residue at } z = \sqrt{2}e^{\frac{i\pi}{4}} + \text{Residue at } z = \sqrt{2}\,e^{\frac{3i\pi}{4}}\right]$$

$$= \pi i\left[\frac{1}{8\sqrt{2}}\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) + \frac{1}{8\sqrt{2}}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)\right]$$

$$= \frac{\pi i}{16}[-1-i+1-i] = \frac{\pi}{8}$$

Correct option is (d)

Question 3. The integral $\int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta + a^2}$ where $0 < a < 1$, evaluates to **[TIFR 2015]**

(a) $\frac{2\pi}{1-a^2}$ (b) $\frac{2\pi}{1+a^2}$ (c) 2π (d) $\frac{4\pi}{1-a^2}$

Soln: $$\int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta + a^2} = \int_0^{2\pi} \frac{d\theta}{1-2a\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)+a^2} = \int_0^{2\pi} \frac{d\theta}{1-a\left(e^{i\theta}+e^{-i\theta}\right)+a^2}$$

Let $e^{i\theta} = z$ and C is the unit circle $|z| = 1$

Therefore, $ie^{i\theta}\,d\theta = dz \Rightarrow d\theta = \frac{dz}{iz}$

Therefore, $\int_0^{2\pi} \frac{d\theta}{1-a\left(e^{i\theta}+e^{-i\theta}\right)+a^2} = \int_C \frac{dz}{\left[1-a\left(z+\frac{1}{z}\right)+a^2\right]iz}$

$$= \frac{1}{i}\int_C \frac{dz}{z-az^2-a+za^2} = \frac{1}{i}\int_C \frac{dz}{(z-a)\,(1-az)}$$

Poles are at $z = a, \frac{1}{a}$, since $a<1$, only $z=a$ lies within the contour.

Residue at $z = a$, $\lim\limits_{z\to a}\left((z-a)\frac{1}{(z-a)\,(1-az)}\right) = \frac{1}{1-a^2}$

Therefore, value of integral $= \frac{2\pi i}{i}$ (Sum of residue)

$$= 2\pi\left(\frac{1}{1-a^2}\right)$$

$$= \frac{2\pi}{1-a^2}.$$

Correct option is (a)

Question 4. The value of the integral $\oint_C \frac{\operatorname{Sin} z}{z^6}\,dz$, where C is the circle with centre $z = 0$ and radius 1 unit **[TIFR 2016]**

(a) $i\pi$ (b) $\frac{i\pi}{120}$ (c) $\frac{i\pi}{60}$ (d) $-\frac{i\pi}{6}$

Soln: $f(z) = \frac{\sin z}{z^6}$ has a pole at z = 0.

$$\frac{\sin z}{z^6} = \frac{1}{z^6}\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+\ldots\right)$$

Residue at ($z=0$) = co-efficient of $1/z = \frac{1}{5!} = \frac{1}{120}$

Therefore, $\oint_C \frac{\sin z}{z^6}\,dz = 2\pi i \times \frac{1}{120} = \frac{\pi i}{60}$.

Correct option is (c)

Question 5. The value of the integral $\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}\,dx$ is

(a) $\frac{1}{2a}$ (b) $\frac{1}{2\pi a}$ (c) $\pi a\exp(-a)$ (d) $\exp(-a)/a$

Soln: $\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}\,dx$ = Real part of $\frac{1}{\pi}\int_C \frac{e^{iz}}{z^2+a^2}\,dz,$

Taken round the closed contour C, consisting of the upper half of a large circle $|z| = R$ and the real axis from $-R$ to R.

Poles are at $z^2 + a^2 = 0$, $\boxed{z = \pm i\,a}$ and Only $z = i\,a$ lies inside the contour

$$\text{Residue at } (z = ia) = \lim_{z \to ia}\left(\frac{e^{iz}(z - i\,a)}{(z + i\,a)(z - ia)}\right) = \frac{e^{i \times i\,a}}{i\,a + i\,a} = \frac{e^{-a}}{2ia}$$

$$\text{Therefore, value of integral} = \frac{1}{\pi} 2\pi\, i \times \frac{e^{-a}}{2i\,a} = \frac{e^{-a}}{a}.$$

Correct option is (d)

JEST PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. The value of integral $\int_0^\infty \frac{\ln x}{\left(x^2 + 1\right)^2} dx$ is [JEST 2012]

(a) 0 (b) $-\frac{\pi}{4}$ (c) $-\frac{\pi}{2}$ (d) $\frac{\pi}{2}$

Soln: $I = \int_0^\infty \frac{\ln x}{(x^2 + 1)^2} dx$

Let $f(z) = \frac{\ln z}{(z^2 + 1)^2}$

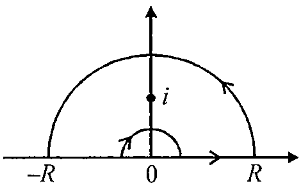
Singular points are $z = 0$ (pole of order 1), and $z = \pm i$ (poles of order 2).

Hence, residue at $z = 0$

$$\lim_{z \to 0} (z - 0)\frac{\ln(z)}{(z^2 + 1)^2} \text{ replace z by } \frac{1}{t}$$

$$\text{Therefore, } \lim_{t \to 0} \frac{1}{t}\frac{\ln\left(\frac{1}{t}\right)}{\left(\frac{1}{t^2} + 1\right)^2} \Rightarrow \lim_{t \to 0} -\frac{\ln(t)}{t} \times \frac{1}{\left(\frac{1}{t^2} + 1\right)^2} = 0 \quad \text{(Using L-Hospital's Rule)}$$

Residue at $z = i$ (only $z = i$ lies within upper half)



$$\lim_{z \to i} \frac{d}{dz}\left[\frac{(z - i)^2 \ln(z)}{(z + i)^2 (z - 1)^2}\right]$$

$$\lim_{z \to i} \frac{d}{dz}\left[\frac{\ln(z)}{(z + i)^2}\right] \Rightarrow \lim_{z \to i}\left[\frac{(z + i)^2 \times \frac{1}{z} - (\ln z)\,2(z + i)}{(z + i)^4}\right]$$

$$\Rightarrow \quad \lim_{z \to i}\left[\frac{(z + i)\frac{1}{z} - 2\ln z}{(z + i)^3}\right] = \frac{2i \times \frac{1}{i} - 2\ln(i)}{(2i)^3} = \frac{2 - 2\ln(e^{i\pi/2})}{-8i} = \frac{2 - i\pi}{-8i}$$

Hence, according to Cauchy residue theorem,

$$\int_{-\infty}^{\infty} \frac{\ln x}{(x^2+1)^2}\,dx = \pi i \text{ (residue at } z=0) + 2\pi i \text{ (residue at } z=i)$$

$$= \pi i \times 0 + 2\pi i \times \left(\frac{2-i\pi}{-8i}\right) = \frac{2\pi}{-8}\times 2 + \frac{2\pi}{8} i\pi = -\frac{\pi}{2} + i\frac{\pi^2}{4}.$$

or $$\int_{-\infty}^{0} \frac{\ln x}{(x^2+1)^2}\,dx + \int_{0}^{\infty} \frac{\ln x}{(x^2+1)^2}\,dx = -\frac{\pi}{2} + i\frac{\pi^2}{4} \qquad \ldots (1)$$

Now, $\displaystyle\int_{-\infty}^{0} \frac{\ln x}{(x^2+1)^2}\,dx$ (Replacing x by $-x$)

$$= \int_{\infty}^{0} \frac{\ln(-x)}{(x^2+1)^2}(dx) = -\int_{\infty}^{0} \frac{\ln(e^{i\pi}x)}{(x^2+1)^2}\,dx$$

$$= i\pi \int_{0}^{\infty} \frac{dx}{(x^2+1)^2}\,dx + \int_{0}^{\infty} \frac{\ln x}{(x^2+1)^2}\,dx$$

$$= i\pi \times \frac{\pi}{4} + \int_{0}^{\infty} \frac{\ln x}{(x^2+1)^2}\,dx$$

Therefore, from Eq. (1) ;

$$i\frac{\pi^2}{4} + \int_{0}^{\infty} \frac{\ln x}{(x^2+1)^2}\,dx = -\frac{\pi}{2} + i\frac{\pi^2}{4} \text{ or } \int_{0}^{\infty} \frac{\ln x}{(x^2+1)^2}\,dx = -\frac{\pi}{2}.$$

Correct option is (c)

Question 2. Compute $\displaystyle\lim_{z\to 0} \frac{\mathrm{Re}(z^2)+\mathrm{Im}(z^2)}{z^2}$ **[JEST 2013]**

(a) the limit does not exist
(b) 1
(c) –i
(d) –1

Soln: Let $z = x + iy$, then $z^2 = x^2 - y^2 + 2ixy$

Therefore, $\mathrm{Re}(z^2) = x^2 - y^2,\ \mathrm{Im}(z^2) = 2xy$

Now, $$\lim_{z\to 0} \frac{\mathrm{Re}(z^2)+\mathrm{Im}(z^2)}{z^2} = \lim_{\substack{x\to 0\\ y\to 0}} \frac{(x^2-y^2)+2xy}{(x^2-y^2)+i2xy}$$

Let $y = mx$, then $$\lim_{\substack{x\to 0\\ y\to 0}} \frac{(x^2-m^2x^2)+2mx^2}{(x^2-m^2x^2)+2imx^2} = \frac{1-m^2+2m}{1-m^2+2im}$$

Since, this depends on m, and hence on the path along which $x \to 0,\ y \to 0$.
Hence, limit does not exists.
Correct option is (a)

Question 3. The value of integral $I = \oint_c \frac{\sin z}{2z-\pi}dz$

with c a is circle $|z| = 2$, is **[JEST 2014]**

(a) 0 (b) $2\pi i$ (c) πi (d) $-\pi i$

Soln: $\frac{\sin z}{2z-\pi} = \frac{\sin z}{2\left(z-\frac{\pi}{2}\right)} = \frac{\cos\left(z-\frac{\pi}{2}\right)}{2\left(z-\frac{\pi}{2}\right)} = \frac{\cos t}{2t}$ (Let $z-\frac{\pi}{2}=t$)

$$= \frac{1}{2t}\left[1-\frac{t^2}{2!}+\frac{t^4}{4!}-\frac{t^6}{6!}+........\right] = \frac{1}{2}\frac{1}{t}-\frac{t}{2\times 2!}+\frac{t^3}{2\times 4!}-\frac{t^5}{2\times 6!}$$

The residue at $t=0$ is the coefficient of $\frac{1}{z}$ i.e. $\frac{1}{2}$

Therefore, the pole $z=\frac{\pi}{2}$ lies within the contour C and the residue at $z=\frac{\pi}{2}$ is $\frac{1}{2}$

$$\therefore \quad \oint_C \frac{\sin z}{2z-\pi}dz = 2\pi i\left[\text{sum of residue}\right] = 2\pi i\times\frac{1}{2} = \pi i$$

Correct option is (c)

Question 4. The value of limit $\lim_{z\to i}\frac{z^{10}+1}{z^6+1}$ is equal to **[JEST 2014]**

(a) 1 (b) 0 (c) −10/3 (d) 5/3

Soln: $\lim_{z\to i}\frac{z^{10}+1}{z^6+1}$, Here $(i)^{10}=(i^2)^5=-1$; $i^6=(i^2)^3=-1$

Hence, given that limit is of 0/0 form.

Therefore, using L-Hospital's rule,

$$\lim_{z\to i}\frac{10z^9}{6z^5} \Rightarrow \frac{10(i)^9}{6(i)^5} = \frac{5}{3}\frac{i}{i} = \frac{5}{3}.$$

Correct option is (d)

Question 5. Given an analytic function $f(x,y)=\phi(x,y)+i\,\psi(x,y)$ where **[JEST 2015]**

$\phi(x,y)=x^2+4x-y^2+2y$. If C is a constant, then which of the following relations is true?

(a) $\psi(x,y)=x^2y+4y+C$ (b) $\psi(x,y)=2xy-2x+C$

(c) $\psi(x,y)=2xy+4y-2x+C$ (d) $\psi(x,y)=x^2y-2x+C$

Soln: The condition of analytic function are

$$\frac{\partial \phi}{\partial y}=\frac{\partial \psi}{\partial x} \qquad \frac{\partial \phi}{\partial y}=-\frac{\partial \psi}{\partial x}$$

We have, $\phi(x,y)=x^2+4x-y^2+2y$

Therefore, $\frac{\partial \phi}{\partial x}=2x+4$ And $\frac{\partial \phi}{\partial y}=-2y+2$

$\Rightarrow \quad \frac{\partial \psi}{\partial y}=2x+4$ $\Rightarrow \quad -\frac{\partial \psi}{\partial x}=-2y+2$

$\Rightarrow \quad \psi=2xy+4y+c_1(x)$ $\Rightarrow \quad \psi=2xy-2x+c_1(y)$

Therefore, $\psi(x,y)=2xy+4y-2x+C$ (Numerical constant)

Correct option is (c)

Question 6. The value of the integral $\int_0^{\infty}\frac{\ln x}{\left(x^2+1\right)}dx$ is **[JEST 2016]**

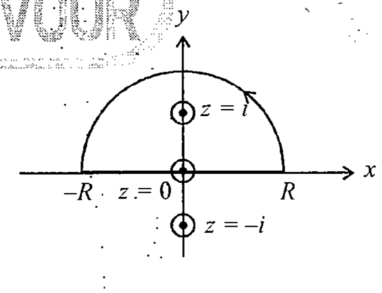
(a) $\frac{\pi^2}{4}$ (b) $\frac{\pi^2}{2}$ (c) π^2 (d) 0

Soln: $\int_0^{\infty}\frac{\ell nx}{\left(x^2+1\right)}dx$

Say, $f(z)=\frac{\ell nz}{\left(z^2+1\right)}$ Singular points : $z=0,\pm i \Rightarrow$ pole of order 1

$$\text{Res}\cdot f(z=0)=\underset{z\to 0}{Lt}(z-0)\frac{\ell nz}{\left(z^2+1\right)}$$

$$=\underset{z\to\infty}{Lt}\left(\frac{1}{t}\right)\frac{\ell n\left(\frac{1}{t}\right)}{\left(1/t^2+1\right)} \quad \left[\text{Replacing } z \text{ by } \frac{1}{t}\right]$$



$$=\underset{t\to\infty}{Lt}\left(-\frac{\ell nt}{t}\right)\frac{1}{\left(1/t^2+1\right)}=0 \quad \text{(Using L'Hospital's rule)}$$

$$\text{Res}\cdot f(z=i)=\underset{z\to i}{Lt}(z-i)\frac{\ell nz}{\left(z^2+1\right)}=\underset{z\to i}{Lt}\frac{nz}{(z+i)}=\frac{\ell n(i)}{2i}=\frac{\ell n\left(e^{i\pi/2}\right)}{2i}=\frac{i\pi/2}{2i}=\frac{\pi}{4}$$

According to Cauchy's Residue Theorem,

$$\int_{-\infty}^{\infty}\frac{\ell nx}{\left(x^2+1\right)}dx=\pi i\left[\text{Res}\cdot f(z=0)\right]+2\pi i\left[\text{Res}\cdot f(z=i)\right]=i\pi^2/2$$

$$\Rightarrow \quad \int_{-\infty}^{0} \frac{\ell nx}{\left(x^2+1\right)} dx + \int_{0}^{\infty} \frac{\ell nx}{\left(x^2+1\right)} dx = \frac{i\pi^2}{2} \qquad \text{... (i)}$$

Now, $\int_{-\infty}^{0} \frac{\ell nx}{\left(x^2+1\right)} dx$ (Replacing x by $-x$)

$$= \int_{\infty}^{0} \frac{\ell n(-x)}{\left(x^2+1\right)}(-dx) = -\int_{\infty}^{0} \frac{\ell n\left(e^{i\pi}x\right)}{\left(x^2+1\right)} dx$$

$$= i\pi \int_{0}^{\infty} \frac{dx}{\left(x^2+1\right)} + \int_{0}^{\infty} \frac{\ell nx}{\left(x^2+1\right)} dx \qquad \text{... (ii)}$$

Using (ii) in equation (i), we get

$$i\pi \int_{0}^{\infty} \frac{dx}{\left(x^2+1\right)} + 2\int \frac{\ell nx}{\left(x^2+1\right)} dx = \frac{i\pi^2}{2}$$

Comparing real parts, $\int_{0}^{\infty} \frac{\ell nx}{\left(x^2+1\right)} dx = 0$

Correct option is (d)

Question 7. The integral $I = \int_{1}^{\infty} \frac{\sqrt{x-1}}{(1+x)^2} dx$ is **[JEST 2017]**

(a) $\frac{\pi}{\sqrt{2}}$ (b) $\frac{\pi}{2\sqrt{2}}$ (c) $\frac{\sqrt{\pi}}{2}$ (d) $\sqrt{\frac{\pi}{2}}$

Soln: $I = \int_{1}^{\infty} \frac{\sqrt{x-1}}{(1+x)^2} dx$

Let $x-1 = e^{2i\theta}$ $\Rightarrow$ $dx = e^{2i\theta} \times 2i d\theta$

and $x+1 = e^{2i\theta}+2$, as $x \to 1$ $\theta \to -\infty$

and as $x \to \infty$ $\theta \to \infty$

$$\int_{-\infty}^{\infty} \frac{e^{i\theta}}{(e^{2i\theta}+2)^2} 2ie^{2i\theta} d\theta$$

Let $z = e^{i\theta} i d\theta \Rightarrow dz = e^{i\theta} i d\theta$

Therefore, $\int_{-\infty}^{\infty} \frac{2z^2 dz}{(z^2+2)^2}$

Singular points are $z^2+2=0 \Rightarrow z = \pm\sqrt{2}i$.

Only $z = \pm\sqrt{2}i$ lies in the upper half.

Hence Residue at $z = \sqrt{2}i$

$$= \lim_{z\to\sqrt{2}i} \frac{d}{dz}\left(\left(z-\sqrt{2}i\right)^2 \frac{2z^2}{(z+\sqrt{2}i)^2(z-2i)^2}\right) = \lim_{z\to\sqrt{2}i} \frac{d}{dz}\left(\frac{2z^2}{(z+\sqrt{2}i)^2}\right)$$

$$= \lim_{z\to\sqrt{2}i}\left(\frac{(z+\sqrt{2}i)^2\times 2z-2z^2 2(z+\sqrt{2}i)}{(z+\sqrt{2}i)^4}\right) = \lim_{z\to\sqrt{2}i}\left(\frac{(z+\sqrt{2}i)4z-4z^2}{(z+\sqrt{2}i)^3}\right)$$

$$= \lim_{z\to\sqrt{2}i}\frac{4\sqrt{2}iz}{(z+\sqrt{2}i)^3} = \frac{i4\sqrt{2}\times\sqrt{2}i}{(2\sqrt{2}i)^3} = \frac{-8}{-8\times 2\sqrt{2}i} = \frac{-i}{2\sqrt{2}}$$

Hence, $$I = 2\pi i\times\frac{-i}{2\sqrt{2}} = \frac{\pi}{\sqrt{2}}$$

Correct option is (a)

JNU PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. The function $\frac{1}{\cosh x}$ may be expressed around the point $x = 0$ as a power series as **[JNU 2012]**

(a) $1-\frac{1}{2}x^2+\frac{1}{24}x^4-\frac{1}{720}x^6+....$

(b) $1-\frac{1}{2}x^2+\frac{5}{24}x^4-\frac{61}{720}x^6+....$

(c) $1-\frac{1}{2}x^2+\frac{11}{24}x^4-\frac{331}{720}x^6+....$

(d) $1-\frac{1}{2x^2}+\frac{1}{24x^4}-\frac{1}{720x^6}+....$

Soln: $$S = \frac{1}{\cosh x} = (\cosh x)^{-1} = \left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\cdots\right)^{-1}$$

$$= 1+\left(\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\cdots\right)+\left(\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\cdots\right)^2$$

$$= 1-\frac{x^2}{2}+\left(-\frac{x^4}{4!}+\frac{x^4}{(2)^2}\right)+\left(-\frac{x^6}{6!}+\frac{x^2 x^4}{2!\,4!}\right)+\cdots$$

$$= 1-\frac{x^2}{2}+\frac{5}{24}x^4-\frac{61}{720}x^6+\cdots$$

Correct option is (b)

Question 2. The contour integral

$$\oint_C \frac{zdz}{(z-1)^2}$$

where the contour C is a circle of radius 2, is obtained as **[JNU 2013]**

(a) $2\pi i$ (b) 0 (c) πi (d) $4\pi i$

Soln: $$I = \oint\frac{z\,dz}{(z-1)^2}$$

Poles are at $z=1$ of second order.

Hence residue at ($z=1$) $= \lim\limits_{z\to 1}\dfrac{d}{dz}\left((z-1)^2\dfrac{z}{(z-1)^2}\right) = \lim\limits_{z\to 1}(1) = 1.$

Therefore, $I = 2\pi i$ (residue at $z=1$)

$= 2\pi i$.

Correct option is (a)

Question 3. What is the residue of the complex function $f(z)=\exp(1/z)$ at $z=0$? **[JNU 2017]**

(a) 1 (b) 0 (c) Undefined (d) e (e) ∞

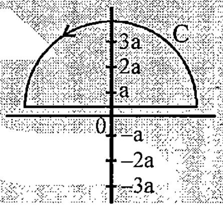
Soln: $f(z) = \exp\left(\dfrac{1}{z}\right) = 1+\dfrac{1}{z}+\dfrac{1}{2!z^2}+\cdots$

Therefore, residue at $z=0$, co-efficient of $\dfrac{1}{z}=1$.

Correct option is (a)

DU PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. Consider the integral of the function f(z) in the complex plane along the contour C as shown in the figure. For $f=[z^2+a^2]^{-1}$, the value of the integral is: **[DU 2014]**



(a) $\dfrac{\pi}{a}$

(b) $\dfrac{2\pi}{a}$

(c) $\dfrac{i\pi}{2a}$

(d) $\dfrac{i\pi}{a}$

Soln: $I = \oint\limits_C f(z)\,dz = \oint\limits_C \dfrac{1}{z^2+a^2}dz$

Poles are at $z^2+a^2=0 \Rightarrow z=\pm ia$

Only $z=ia$ lies within the given contour

Therefore, residue at $z=ia$

$R = \lim\limits_{z\to ia}\left((z-ia)\times\dfrac{1}{(z+ia)(z-ia)}\right) \Rightarrow R=\dfrac{1}{2ia}$

Therefore, $I = 2\pi i\times R = 2\pi i\times\dfrac{1}{2ia}=\dfrac{\pi}{a}.$

Correct option is (a)

Question 2. The function $f(z)=\dfrac{(z-2)}{(z^2-4)(z+3)}$ **[DU 2014]**

(a) Has a simple pole at z = 2 and a simple pole at z = 3

(b) Has a simple pole at z = –2 and a simple pole at z = –3

(c) Has a pole of order 2 at z = 2 and a simple pole at z = –3

(d) Has a pole of order 2 at z = –2 and a simple pole at z = 3

Soln: $f(z) = \frac{z-2}{(z^2-4)(z+3)} = \frac{z-2}{(z-2)(z+2)(z+3)}$

Hence, $f(z)$ has poles at $z = 2, -2, -3$, which are simple.

Correct option is (b)

Question 3. The complex function f(z) = $z^{1/2}$ **[DU 2014]**

(a) Has a simple pole at z = 0

(b) Has a pole of order 2 at z = 0

(c) Has a branch point

(d) Is analytic in the entire complex plane

Soln: Correct option is (c)

Question 4. The imaginary part of an analytic function in Cartesian coordinate is $2xy$. The real part is given by

(a) $x^2 + y^2 = a^2$

(b) $x^2 - y^2 = a^2$ **[DU 2015]**

(c) $-x^2 + y^2 = a^2$

(d) $-x^2 - y^2 = a^2$

Soln: Let $f(z) = u(x,y) + v(x,y)$ and $z = x + iy$

Given : $v(x,y) = 2xy$

$v_x = \frac{\partial v}{\partial x} = 2y \Rightarrow v_x(z,0) = 0$ and $v_y = \frac{\partial v}{\partial y} = 2x \Rightarrow v_y(z,0) = 2z$

Therefore, $f(z) = \int \left(v_y(z,0) + i v_x(z,0)\right) dz + c$

$= \int \left(2z\,dz + 0\right) dz + c$

$= 2\frac{z^2}{2} + c$

$f(z) = z^2 = (x^2 - y^2 + 2ixy)$

Hence, real part is $\left(x^2 - y^2\right)$.

Correct option is (b)

Question 5. The Laurent series of a function $f(z)$ about $z = 2$ is given by **[DU 2015]**

$$f(z) = \frac{1}{2(z-2)^3} - \frac{1}{4(z-2)^2} + \frac{1}{8(z-2)} - \frac{1}{16} + \frac{z-2}{32} - \ldots..$$

The order of the pole and the value of the residue are

(a) Order = 3 and Residue = $\frac{1}{2}$

(b) Order = 2 and Residue = $-\frac{1}{4}$

(c) Order = 1 and Residue = $\frac{1}{8}$

(d) Order = 3 and Residue = $\frac{1}{8}$

Soln: Laurent series of $f(z)$ about $z = 2$

$$f(z)=\frac{1}{2(z-2)^3}-\frac{1}{4(z-2)^2}+\frac{1}{8(z-2)}-\frac{1}{16}+\frac{z-2}{3^2}\cdots$$

From this expansion, order = 3 (Highest negative power) and residue at $z=2$ is 1/8 (Co-efficient of $1/z-2$).

Correct option is (d)

Question 6. In which quadrants of the complex plane, is the function $f(z)=|x|-i|y|$ analytic? **[DU 2015]**

(a) Only in the fourth quadrant

(b) Only in the first and third quadrants

(c) Only in the second and fourth quadrants

(d) In all the quadrants

Soln: In 4th quadrant $f(z)=x+iy$.

Hence, $u=x,\ v=y\,;\ \frac{\partial u}{\partial x}=1,\ \frac{\partial v}{\partial y}=1,$ and $\frac{\partial u}{\partial y}=0=\frac{\partial v}{\partial y}$

In 2nd quadrant $f(z)=-x-iy$

Hence, $u=-x,\ v=-y\,;\ \frac{\partial u}{\partial x}=-1,\ \frac{\partial v}{\partial y}=-1,$ and $\frac{\partial u}{\partial y}-\frac{\partial v}{\partial y}=0$

Correct option is (c)

Question 7. If $z=x+iy$, then the real part of cos (z) is : **[DU 2017]**

(a) $\cos(x)\cosh(y)$ (b) $\cos(x)\cos(y)$ (c) $\sin(x)\sinh(y)$ (d) $-\sin(x)\sinh(y)$

Soln: $\cos z=\cos(x+iy)\ =\cos x\cos iy-\sin x\sin iy\ =\cos x\cosh y-i\sin x\sinh y$

Therefore, real part is $\cos x\cosh y$.

Correct option is (a)

Question 8. If C is the unit circle traversed clockwise, then the integral $\int_C \frac{dz}{z^2-3z}$ is **[DU 2017]**

(a) 0 (b) $\frac{\pi}{3}i$ (c) $\frac{2\pi}{3}i$ (d) $-\frac{2\pi}{3}i$

Soln: $f(z)=\frac{1}{z^2-3z}$ has simple poles are at z =0, 3 and only $z=0$ will lie within the unit circle.

$$\text{Res. at }(z=0)=\lim_{z\to 0}\left[z\times\frac{1}{z(z-3)}\right]=\frac{1}{-3}$$

Therefore, $I=-2\pi i\times-\frac{1}{3}=\frac{2\pi i}{3}$. (Since, the contour is in clockwise direction)

Correct option is (c)

HCU PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. The sqaure roots of $(-15-8i)$ are **[H.C.U.-2010]**

(a) $1-4i, -1+4i$ (b) $1+4i, 1-4i$ (c) $\sqrt{15}-\sqrt{8}i$ (d) $2\sqrt{2}-\sqrt{15}i$

Soln: Let $(-15-8i)^{1/2}=a+ib \Rightarrow -15-8i=(a+ib)^2=a^2-b^2+2iab$

$\therefore\ a^2-b^2=-15$ ... (i)

$2ab=-8$ ... (ii)

Solving equation (i) and (ii), we get $b=\pm 4,\ a=\mp 1$

$\therefore$ Square roots are $-1+4i$ and $1-4i$

Correct option is (a)

Question 2. $|z|=2$, where $z=x+iy$, represents a

(a) straight line (b) circle (c) ellipse (d) parabola **[H.C.U.-2011]**

Soln: $|z|=2 \Rightarrow |x+iy|=2$; $\sqrt{x^2+y^2}=2 \Rightarrow x^2+y^2=4$

Hence, circle.

Correct option is (b)

Question 3. Given $\left(\frac{1-i}{1+i}\right)^{100}=a+ib$, the values of a and b are **[H.C.U.-2012]**

(a) $a=2, b=-1$ (b) $a=0, b=0$ (c) $a=0, b=1$ (d) $a=-1, b=2$

Soln: $\left(\frac{1-i}{1+i}\right)^{100}=a+ib \Rightarrow \left\{\left(\frac{1-i}{1+i}\right)\times\frac{1-i}{1-i}\right\}^{100}=\left(\frac{1-1-2i}{1+1}\right)^{100}=a+ib$

$\Rightarrow (-i)^{100}=a+ib \Rightarrow (i^2)^{50}=a+ib \Rightarrow 1=a+ib$

$\therefore\ a=1,\ b=0$

None of the options is correct.

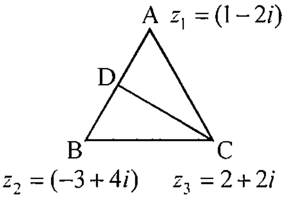
Question 4. Let $(1-2i), (-3+4i), (2+2i)$ represent the three vertices *A, B, C* of a triangle *ABC*. the length of the meridian from *C* to the side *AB* is

(a) $\sqrt{10}$ (b) $\sqrt{12}$ (c) 6 (d) 8 **[H.C.U.-2013]**

Soln: From figure, $D=\frac{z_1+z_2}{2} \Rightarrow -\frac{2+2i}{2}=-1+i$

and, $CD=\sqrt{(2+1)^2+(2-1)^2}=\sqrt{10}$

Correct option is (a)



Question 5. Which of the following numbers does not lie on the unit circle $|z|=1$?

(a) $\left(\frac{1+i}{\sqrt{2}}\right)^{201}$ (b) $\left(\frac{1+i}{1-i}\right)^{57}$ (c) $\left(\frac{3+4i}{2-\sqrt{21}i}\right)$ (d) $\sqrt{\frac{1+2i}{3+2i}}$ **[H.C.U.-2013]**

Soln: Lets check option (c)

$$\frac{3+4i}{2-\sqrt{21}\,i}\times\frac{2-1\sqrt{21}\,i}{2+1\sqrt{21}\,i}=\frac{6+3\sqrt{21}\,i+8i-4\sqrt{21}}{4+21}=\frac{\left(6-4\sqrt{21}\right)+\left(8+3\sqrt{21}\right)i}{25}$$

$$\text{Now, }\left|\frac{\left(6-4\sqrt{21}\right)+\left(8+3\sqrt{21}\,i\right)}{25}\right|>1$$

Hence, this will not lie on the unit circle $|z|=1$.

Correct option is (c)

Question 6. The coefficient of x^4 in the expansion of $\sqrt{1+2x^2}$ is

(a) $-1/2$ (b) $-15/32$ (c) $-65/28$ (d) $-21/128$ **[H.C.U.-2013]**

Soln: $$\left(1+2x^2\right)^{1/2}\Rightarrow 1+\frac{1}{2}2x^2+\frac{1}{2}\left(\frac{1}{2}-1\right)\times\frac{(2x^2)^2}{2!}+\cdots$$

Therefore, co-efficient of x^4 is $\frac{1}{2}\times-\frac{1}{2}\times\frac{4x^4}{2}=-\frac{1}{2}$.

Correct option is (a)

Question 7. The real and imaginary parts of the function $f(z)=z\,e^z$ where $z\equiv x+iy$ respectively are

(a) $xe^x;\ ye^y$ (b) $xe^y;\ ye^x$ **[H.C.U.-2014]**

(c) $x\cos y;\ y\sin y$ (d) $[x\cos y-y\sin y]e^x;\ [y\cos y+x\sin y]e^x$

Soln:
$$\begin{aligned}f(z)&=z\,e^z\\&=(x+iy)\,e^{x+iy}\Rightarrow(x+iy)\,e^x\,e^{iy}\\&=e^x(x+iy)(\cos y+i\sin y)\\&=e^x(x\cos y-y\sin y)+i\,e^x(y\cos y+x\sin y)\end{aligned}$$

Correct option is (d)

Question 8. The first three terms in the Taylor series expansion of $\sin x$ around $x=\frac{\pi}{2}$ are

(a) $\left(x-\frac{\pi}{2}\right)-\frac{\left(x-\frac{\pi}{2}\right)^3}{3!}+\frac{\left(x-\frac{\pi}{2}\right)^5}{5!}$ (b) $\left(x-\frac{\pi}{2}\right)+\frac{\left(x-\frac{\pi}{2}\right)^3}{3!}+\frac{\left(x-\frac{\pi}{4}\right)^5}{5!}$

(c) $1-\frac{\left(x-\frac{\pi}{2}\right)^2}{2!}+\frac{\left(x-\frac{\pi}{2}\right)^4}{4!}$ (d) $1+\left(x-\frac{\pi}{2}\right)+\frac{1}{2!}\left(x-\frac{\pi}{2}\right)^2$ **[H.C.U.-2014]**

Soln: $f(x)=\sin x$ $f(\pi/2)=1$

$f'(x)=\cos x$ $f'(\pi/2)=0$

$f''(x)=-\sin x$ $f''(\pi/2)=-1$

$f'''(x)=-\cos x$ $f'''(\pi/2)=0$ and so on......

Therefore, Taylor expansion of $f(x)$ about $x=\frac{\pi}{2}$

$$=f\left(\frac{\pi}{2}\right)+\left(x-\frac{\pi}{2}\right)f'\left(\frac{\pi}{2}\right)+\left(x-\frac{\pi}{2}\right)^2\frac{f''(\pi/2)}{2!}+\left(x-\frac{\pi}{2}\right)^3\frac{f'''(\pi/2)}{3!}+\cdots$$

$$=1-\left(\frac{x-\frac{\pi}{2}}{2!}\right)^2+\left(\frac{x-\frac{\pi}{4}}{4!}\right)^4-\cdots$$

Correct option is (c)

Question 9. In the complex z-plane, the equation $|z|=2|z-1|$ represents

(a) a circle of radius 2/3 (b) a circle of radius 1/2 **[H.C.U.-2015]**

(c) a circle of radius 3/2 (d) a straight line through (0, 0) and (1, 1)

Soln: $|z|=2|z-1| \Rightarrow |x+iy|=2|x+iy-1|$

$\Rightarrow \sqrt{x^2+y^2}=2\sqrt{(x-1)^2+y^2} \Rightarrow x^2+y^2=4(x^2+1-2x+y^2)$

$\Rightarrow 3x^2+3y^2-8x+4=0 \Rightarrow \left(x-\frac{4}{3}\right)^2+y^2=\left(\frac{2}{3}\right)^2$

Correct option is (a)

Question 10. Given, $z=x+iy, e^{iz}$ reduces to

(a) $(\cos x+i\sin x)e^{-y}$ (b) $(\cos x+i\sin x)e^{y}$

(c) $(\cos x-i\sin x)(\cos y-i\sin y)$ (d) $(\cos x+i\sin x)(\cos y-i\sin y)$ **[H.C.U.-2016]**

Soln: $e^{iz}=e^{i(x+iy)}=e^{-y+ix}=e^{-y}e^{ix}=e^{-y}(\cos x+i\sin x)$

Correct option is (a)

Chapter 6

FOURIER TRANSFORM

6.1 Basic Formula

The fourier transform of $f(x)$ $[-\infty < x < \infty]$:

$$g(s) = F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

Inverse fourier transform of $g(s)$ $[-\infty < s < \infty]$:

$$f(x) = F^{-1}\{g(s)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(s)\, e^{-isx}\, ds$$

Important Formula:

(i) $\int_0^{\infty} \frac{\sin ax}{x} dx = \frac{\pi}{2}$ (ii) $\int_0^{\infty} e^{-x^2} dx = \sqrt{\frac{\pi}{2}}$ (iii) $\int_0^{\infty} e^{-ax} \sin bx\, dx = \frac{b}{a^2+b^2}$

(iv) $\int_0^{\infty} e^{-ax} \cos bx\, dx = \frac{a}{a^2+b^2}$ (v) $\int_0^{\infty} e^{-ax^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$ (vi) $\int_0^{\infty} e^{-ax^2} \cos bx\, dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\, e^{-b^2/4a}$

(vii) $\int_{-\infty}^{\infty} e^{-ax^2+bx} dx = \sqrt{\frac{\pi}{a}}\, e^{b^2/4a}$

Example 1: Find the fourier transform of Gaussian distribution function $f(x) = Ne^{-ax^2}$ where N and 'a' are constant.

Soln: $F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} Ne^{-ax^2} e^{isx} dx = \frac{N}{\sqrt{2\pi}} \sqrt{\frac{\pi}{a}} e^{-s^2/4a} = \frac{N}{\sqrt{2a}} e^{-s^2/4a}$

Example 2: Find the fourier transform of the function $f(x) = \delta(x)$

Soln: $F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x) e^{isx} dx = \frac{1}{\sqrt{2\pi}} \left[e^{isx}\right]_{x=0} = \frac{1}{\sqrt{2\pi}}$

Example 3: Find the fourier transform of $f(x) = e^{-a|x|}$ $(-\infty < x < \infty)$ $(a > 0)$

Soln: $$F\left[e^{-a|x|}\right] = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} e^{ax} e^{isx} dx + \int_{0}^{\infty} e^{-ax} e^{isx} dx\right] = \frac{1}{\sqrt{2\pi}}\left[\left(\frac{e^{(a+is)x}}{(a+is)}\right)_{-\infty}^{0} + \left(\frac{e^{-(a-is)x}}{(is-a)}\right)_{0}^{\infty}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{a+is} + \frac{1}{a-is}\right] = \frac{2a}{\sqrt{2\pi}\left(s^2+a^2\right)}$$

Example 4: If $F\left\{e^{-i2\pi vat}\right\} = f(\omega + a)$, then $F(\cos 2\pi vat) = ?$

Soln: $$\mathrm{F}[\cos 2\pi \nu a t] = \mathrm{F}\frac{\left[e^{i2\pi\nu at} + e^{-i2\pi\nu at}\right]}{2} = \frac{1}{2}\left\{F\left[e^{i2\pi vat}\right] + F\left[e^{-i2\pi vat}\right]\right\} = \frac{1}{2}\left[f(\omega - a) + f(\omega + a)\right]$$

Example 5: Find the fourier cosine transform of the function $f(x) = e^{-a^2x^2}$

Soln: $$f_c(s) = F_c\left[f(x)\right] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-a^2x^2} \cos sx dx = \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} e^{-a^2x^2} \cos sx dx$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}} \text{ Real part of } \int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx} dx = \frac{1}{2}\sqrt{\frac{2}{\pi}} \text{ Real part of } \left(\sqrt{\frac{\pi}{a^2}} e^{\frac{i^2s^2}{4a^2}}\right) = \frac{1}{\sqrt{2a^2}} e^{-s^2/4a^2}$$

6.2 Properties of Fourier Transform

(1) Linearity theorem:

If $F(x) = a_1 f_1(x) + a_2 f_2(x) + \ldots$ then, $g(s) = F\left[f(x)\right] = a_1 g_1(s) + a_2 g_2(s) + \ldots$

(2) Change of scale property:

If $F\left[f(x)\right] = g(s)$, then $F\left\{f(ax)\right\} = \frac{1}{a} g(\mathrm{s}/a)$

(3) Shifting property:

If $F\left[f(x)\right] = g(s)$, then $F\left\{f(x \pm a)\right\} = e^{\mp isa} g(s)$

(4) Modulation theorem:

If $F\left\{f(x)\right\} = g(s)$ then $F\left\{f(x)\cos ax\right\} = \frac{1}{2} g(s-a) + \frac{1}{2} g(s+a)$

(5) If $F\{f(x)\}=g(s)$, then $F\{f^*(x)\}=g^*(-s)$

(6) Derivative theorem:

If $F\{f(x)\}=g(s)$, then $F\left[\dfrac{df}{dx}\right]=-isg(s)$

If $F\{f(x)\}=g(s)$, then $F\left[\dfrac{d^n f}{dx^n}\right]=(-is)^n g(s)$

Example 6: Find Fourier transform of $f(x)=\begin{cases} x^2, & |x|<a \\ 0, & |x|>a \end{cases}$

Soln: $$F\{f(x)\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx} f(x)\,dx=\frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{isx}x^2 dx$$

$$=\frac{1}{\sqrt{2\pi}}\left[\left(\frac{e^{isx}}{is}.x^2\right)_{x=-a}^{a}-\frac{2}{is}\int_{-a}^{a} xe^{isx}dx\right]$$

$$=\frac{1}{\sqrt{2\pi}}\left[\frac{a^2}{is}\left(e^{isa}-e^{-isa}\right)-\frac{2}{is}\left[\left(\frac{xe^{isx}}{is}\right)_{x=-a}^{a}-\frac{1}{is}\int_{-a}^{a} e^{isx}dx\right]\right]$$

$$=\frac{1}{\sqrt{2\pi}}\left[\frac{2a^2}{s}\sin(sa)+\frac{2a}{s^2}\cos(sa)-\frac{4}{s^3}\sin(sa)\right]$$

Example 7: Find the complex fourier transform $e^{-|x|}$

Soln: $$F\left\{e^{-|x|}\right\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-|x|}\,e^{isx}dx=\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} e^{-|x|}\,e^{isx}dx+\int_{0}^{\infty} e^{-|x|}.e^{isx}dx\right]$$

$$=\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} e^{x}.e^{isx}dx+\int_{0}^{\infty} e^{-x(1-is)}dx\right]=\frac{1}{\sqrt{2\pi}}\left[\left[\frac{e^{x(1+is)}}{1+is}\right]_{x=-\infty}^{0}+\left[\frac{e^{-x(1-is)}}{-(1-is)}\right]_{x=0}^{\infty}\right]$$

$$=\frac{1}{\sqrt{2\pi}}\left[\frac{1}{1-is}+\frac{1}{1+is}\right]=\frac{1}{\sqrt{2\pi}}\left(\frac{2}{1+s^2}\right)$$

Example 8: Fourier transform of the derivative of the Dirac δ-function, namely $\delta'(x)$, is proportional to

(a) 0 (b) 1 (c) $\sin k$ (d) ik

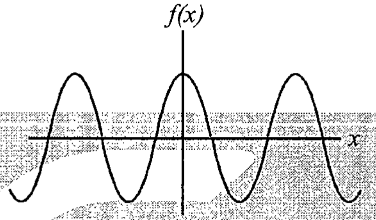
Soln: Fourier transform of $\delta'(x) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\delta'(x)\,e^{ikx}dx$

Using the property of Dirac delta function $\int\limits_{-\infty}^{\infty} f(x)\,\delta'(x-a)\,dx = -f'(a)$

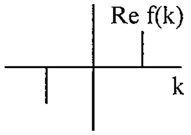
So, $F[\delta'(x)] = -\frac{1}{\sqrt{2\pi}}(ik) \qquad \Rightarrow F[\delta'(x)] \propto ik$

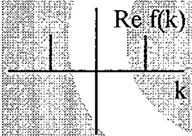
Correct answer is (d)

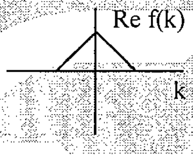
Example 9: The graph of a real periodic function $f(x)$ for the range $[-\infty, \infty]$ is shown below

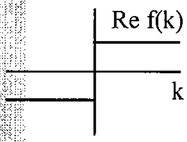


Which of the following graphs represents the real part of its Fourier transform?

(a)


(b)


(c)


(d)


Soln: The given graph in the question is of $f(x) = \cos\omega x$

$$F(\cos\omega x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\cos\omega x\, e^{ikx}dx = \frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{+\infty}\frac{e^{i(k+\omega)x}}{2}dx + \int_{-\infty}^{+\infty}\frac{e^{i(k-\omega)x}}{2}dx\right)$$

$$\Rightarrow F(\cos\omega x)\frac{1}{\sqrt{2\pi}}\left(\frac{1}{2}\delta(k+\omega)+\frac{1}{2}\delta(k-\omega)\right)$$

Therefore, the fourier transform of $f(x) = \cos x\omega x$ consist of two delta functions of same height at $k = \omega$ and $k = -\omega$.

Correct answer is (b)

Example 10: The Fourier transform of $f(x)$ is $\tilde{f}(k) = \int\limits_{-\infty}^{\infty} dx\, e^{ikx} f(x)$. If $f(x) = \alpha\,\delta(x) + \beta\,\delta'(x) + \gamma\,\delta''(x)$, where $\delta(x)$ is the Dirac Delta function (and prime denotes derivative), what is $\tilde{f}(k)$?

(a) $\alpha + i\beta k + i\gamma k^2$ (b) $\alpha + \beta k - \gamma k^2$ (c) $\alpha - i\beta k - \gamma k^2$ (d) $i\alpha + \beta k - i\gamma k^2$

Soln: Given: Fourier transform of $f(x)$ is $\tilde{f}(x) = \int\limits_{-\infty}^{\infty} dx\, e^{ikx} f(x)$

Fourier transform of $f''(x)$ is $-ik\,\tilde{f}(k)$

Fourier transform of $f''(x)$ is $(-ik)^2\,\tilde{f}(x) = -k^2\tilde{f}(x)$

So, $F[\delta(x)] = \int\limits_{-\infty}^{\infty} dx\, e^{ikx}\delta(x) = 1$

$F[\delta'(x)] = -ik(1) = -ik$ and $F[\delta''(x)] = -k^2(1) = -k^2$

Therefore, $F[\alpha\delta(x) + \beta\delta'(x) + \gamma\delta''(x)] = \alpha - ik\beta - \gamma k^2$

Correct option is (c)

Example 11: The Heaviside function is defined as $H(t) = \begin{cases} +1 & \text{for } t > 0 \\ -1 & \text{for } t < 0 \end{cases}$ and its Fourier transform is given by $-2i/\omega$. The Fourier transform of $\frac{1}{2}\left[H\left(t+\frac{1}{2}\right) - H\left(t-\frac{1}{2}\right)\right]$ is

(a) $\dfrac{\sin\left(\frac{\omega}{2}\right)}{\omega/2}$ (b) $\dfrac{\cos\left(\frac{\omega}{2}\right)}{\omega/2}$ (c) $\sin\left(\frac{\omega}{2}\right)$ (d) 0

Soln: Shifting property:

If $F[f(t)] = f(\omega)$, then $F[f(t \pm a)] = e^{\mp i\omega a} f(\omega)$

$$F[H(t)] = -\frac{2i}{\omega}$$

$$F\left\{\frac{1}{2}H\left(t+\frac{1}{2}\right) - \frac{1}{2}H\left(t-\frac{1}{2}\right)\right\}$$

$$= \frac{1}{2}e^{-i\omega/2}\left(-\frac{2i}{\omega}\right) - \frac{1}{2}e^{i\omega/2}\left(-\frac{2i}{\omega}\right)$$

$$= \frac{i}{\omega}\left[e^{i\omega/2} - e^{-i\omega/2}\right] = \frac{i}{\omega}\left(2i\sin\frac{\omega}{2}\right) = -\frac{\sin\frac{\omega}{2}}{(\omega/2)}$$

Correct option is (a)

Chapter 7

LAPLACE TRANSFORM

7.1 Basic Formula

Laplace transform of function $f(x)$ defined in the interval $(0 < x < \infty)$:

$$f(s) = L[f(x)] = \int_0^{\infty} e^{-sx} f(x)\,dx$$

Important Laplace Transforms:

(a) $L(1) = \frac{1}{s}$

(b) $L(x^n) = \frac{n!}{s^{n+1}} \; (n = 0, 1, 2, \dots\dots\dots)$

(c) $L(e^{ax}) = \frac{1}{s-a} \; (s > a)$

(d) $L(\cos ax) = \frac{s}{s^2 + a^2} \quad (s > 0)$

(e) $L(\sin ax) = \frac{a}{s^2 + a^2} \quad (s > 0)$

(f) $L(\cosh ax) = \frac{s}{s^2 - a^2} \quad (s^2 > a^2)$

(g) $L(\sinh ax) = \frac{a}{s^2 - a^2} \quad (s^2 > a^2)$

7.2 Properties of Laplace Transform:

(1) Linear Property:

$$L[a_1 f_1(x) + a_2 f_2(x)] = a_1 L[f_1(x)] + a_2 L[f_2(x)]$$

(2) Shifting Property:

If $L[f(x)] = g(s)$, then $L[e^{mx} f(x)] = g(s-m)$

Example: $L[\cos ax] = \frac{s}{s^2 + a^2}$, then $L\left[e^{mx} \cos ax\right] = \frac{(s-m)}{(s-m)^2 + a^2}$

(3) Scaling Property:

If $L[f(x)] = g(s)$, then $L[f(mx)] = \frac{1}{m} g\left(\frac{s}{m}\right)$

Example: $L\left[\cos^2 x\right]=\dfrac{s^2+2}{s\left(s^2+4\right)}$,

then $L\left[\cos^2(mx)\right]=\dfrac{1}{m}\dfrac{(s/m)^2+2}{(s/m)\left[(s/m)^2+4\right]}=\dfrac{s^2+2m^2}{s\left(s^2+4m^2\right)}$

(4) If $L\left[f(x)\right]=g(s)$, then $L\left[x^n f(x)\right]=(-1)^n\dfrac{d^n}{ds^n}\left[g(s)\right]$

Example: $L[\sin 3x]=\dfrac{3}{s^2+9}$,

then $L[x\sin 3x]=(-1)^1\dfrac{d}{ds}\left[\dfrac{3}{s^2+9}\right]=(-1)\left[-\dfrac{3(2s)}{\left(s^2+9\right)^2}\right]=\dfrac{6s}{\left(s^2+9\right)^2}$

(5) If $L\left[f(x)\right]=g(s)$, then $L\left[\dfrac{f(x)}{x}\right]=\int_s^\infty g(s)\,ds$

Example: $L[\sin x]=\dfrac{1}{s^2+1}$.

then $L\left[\dfrac{\sin x}{x}\right]=\int_s^\infty\dfrac{1}{s^2+1}ds=\left[\tan^{-1}s\right]_s^\infty=\dfrac{\pi}{2}-\tan^{-1}s=\cot^{-1}s$

(6) If $L\left[f(x)\right]=g(s)$, then $L\left[\int_0^t f(x)\,dx\right]=\dfrac{1}{s}g(s)$

Example: $L\left[\dfrac{\sin x}{x}\right]=\cot^{-1}s$

then $L\left[\int_0^t\dfrac{\sin x}{x}dx\right]=\dfrac{1}{s}L\left[\dfrac{\sin x}{x}\right]=\dfrac{1}{s}\cot^{-1}s$

(7) Laplace transforms of derivatives of a function:

If $L\left[f(x)\right]=g(s)$, then

(a) $L\left[f'(x)\right]=s.g(s)-f(x=0)$

(b) $L\left[f''(x)\right]=s^2.g(s)-f'(x=0)-s.f(x=0)$

(c) $L\left[f^{(n)}(x)\right]=s^n.g(s)-s^{n-1}.f(x=0)-s^{n-2}.f'(x=0)......s.f^{(n-2)}(x=0)-f^{(n-1)}.(x=0)$

(9) Laplace transform of periodic functions:

Suppose $f(x)$ is a periodic function having period 'T' such that

$$f(x+nT)=f(x) \quad \text{for } n=1,2,3.....$$

then $$L\left[f(x)\right]=\frac{1}{1-e^{-sT}}\int_0^T e^{-sx}f(x)\,dx$$

Example: Given $f(x)=x^2$ for $0<x<2$ and $f(x+2)=f(x)$

Then, $$L\left[f(x)\right]=\frac{1}{1-e^{-2s}}\int_0^2 e^{-sx}x^2dx$$

$$=\frac{1}{1-e^{-2s}}\left[-e^{-sx}\frac{x^2}{s}+2e^{-sx}\left(-\frac{x}{s^2}-\frac{1}{s^3}\right)\right]_0^2$$

$$=\frac{1}{1-e^{-2s}}\frac{2}{s^3}\left[1-e^{-2s}\left(1+2s+2s^2\right)\right]$$

(10) Laplace transform of unit step function:

$$u(x-a)=u_a(x)=\begin{cases}0 & \text{for } x<a\\ 1 & \text{for } x\geq a\end{cases}$$

then $$L\left[u(x-a)\right]=\int_a^\infty e^{-sx}.1.dx=\frac{e^{-sa}}{s}$$

(11) Laplace transform of Dirac delta function:

$$L\left[\delta(x-a)\right]=\int_0^\infty e^{-sx}\delta(x-a)\,dx=e^{-sa}$$

Example 1: Find the laplace transform of $f(x)=(1+\cos 2x)$

Soln. $L\left[f(x)\right]=\int_0^\infty e^{-sx}(1+\cos 2x)\,dx=\left(\frac{e^{-sx}}{-s}\right)_0^\infty+\int_0^\infty e^{-sx}\cos 2x\,dx$

$$=\frac{1}{s}+\frac{s}{4+s^2}=\frac{2s^2+4}{s(s^2+4)}$$

Example 2: Find the laplace transform of $f\left(x\right)=2\sin 2x\cos 4x$

Soln. $L\left[f\left(x\right)\right]=L\left[2\sin 2x.\cos 4x\right]=L\left[\sin 6x-\sin 2x\right]=\left(\dfrac{6}{s^2+36}-\dfrac{2}{s^2+4}\right)$

Example 3: Find the laplace transform of $f\left(x\right)=e^{-x}\left(3\sinh 2x-5\cosh 2x\right)$

Soln. $L\left[f\left(x\right)\right]=L\left(3e^{-x}\sinh 2x\right)-L\left(5e^{-x}\cosh 2x\right)$

$$=3\frac{2}{\left(s+1\right)^2-4}-5.\frac{\left(s+1\right)}{\left(s+1\right)^2-4}=\frac{\left(1-5\text{s}\right)}{\left(\text{s}^2+2\text{s}-3\right)}\quad \text{(Using 2nd property)}$$

Example 4: Find the laplace transform of $f\left(x\right)=\left(1+xe^{-x}\right)^3$

Soln. $f\left(x\right)=\left(1+xe^{-x}\right)^3=1+x^3e^{-3x}+3xe^{-x}+3x^2e^{-2x}$

$$L\left[f\left(x\right)\right]=L\left(1\right)+L\left(x^3e^{-3x}\right)+L\left(3xe^{-x}\right)+L\left(3x^2e^{-2x}\right)$$

$$=\frac{1}{s}+\left(-1\right)^3\frac{d^3}{ds^3}\left(\frac{1}{s+3}\right)+3\left(-1\right)^1\frac{d}{ds}\left(\frac{1}{s+1}\right)+3\left(-1\right)^2\frac{d^2}{ds^2}\left(\frac{1}{s+2}\right)\quad \text{(Using 4th property)}$$

$$=\frac{1}{s}+\frac{6}{\left(s+3\right)^4}+\frac{3}{\left(s+1\right)^2}+\frac{6}{\left(s+2\right)^3}$$

Example 5: Find the laplace transform of $f\left(x\right)=\dfrac{1-e^x}{x}$

Soln. $L\left(\dfrac{1-e^x}{x}\right)=\displaystyle\int_s^{\infty}\left(\frac{1}{s}-\frac{1}{s-1}\right)ds=\left[\ell n\, s-\ell n\left(s-1\right)\right]_s^{\infty}=\left[\ell n\frac{s}{s-1}\right]_s^{\infty}$ (Using 5th property)

$$=\left[\ell n\, s-\ell n\left(s-1\right)\right]_s^{\infty}=\left[\ell n\frac{s}{s-1}\right]_s^{\infty}=\ell n\left(\frac{1}{1-\frac{1}{s}}\right)_s^{\infty}$$

$$=0-\ell n\frac{s}{s-1}=\ell n\frac{\left(s-1\right)}{s}$$

Example 6: Find the Laplace transform of the Bessel function of zeroth order $J_0\left(x\right)$.

Soln. $J_0\left(x\right)=\left[1-\dfrac{x^2}{2^2}+\dfrac{x^4}{2^2.4^2}-\dfrac{x^6}{2^2.4^2.6^2}+..............\right]$

$$\Rightarrow L\left[J_0\left(x\right)\right]=\left[\frac{1}{s}-\frac{1}{2^2}\frac{2!}{s^3}+\frac{1}{2^2.4^2}\frac{4!}{s^5}-\frac{1}{2^2.4^2.6^2}\frac{6!}{s^7}+..............\right]$$

$$=\frac{1}{s}\left[1-\frac{1}{2}\left(\frac{1}{s^2}\right)+\frac{1.3}{2.4}\left(\frac{1}{s^2}\right)^2-\frac{1.3.5}{2.4.6}\left(\frac{1}{s^2}\right)^3+..............\right]=\frac{1}{s}\left[1+\frac{1}{s^2}\right]^{-1/2}=\frac{1}{\sqrt{s^2+1}}$$

Example 7: Find the inverse laplace transform of

$$f(s)=\frac{s-2}{(s-2)^2+25}+\frac{s+4}{(s+4)^2+81}+\frac{1}{(s+2)^2+9}$$

Soln. $$L^{-1}[f(s)]=L^{-1}\left[\frac{s-2}{(s-2)^2+25}\right]+L^{-1}\left[\frac{s+4}{(s+4)^2+81}\right]+L^{-1}\left[\frac{1}{(s+2)^2+9}\right]$$

$$=e^{2x}\cos 5x+e^{-4x}\cos 9x+\frac{e^{-2x}}{3}\sin 3x$$

Example 8: Find the inverse Laplace transform of $f(s)=\frac{1}{s^2}\left(\frac{s+1}{s^2+1}\right)$

Soln. $$f(s)=\frac{1}{s^2}\left(\frac{s+1}{s^2+1}\right)=\frac{s+1}{s^2}-\frac{s+1}{s^2+1}=\frac{1}{s}+\frac{1}{s^2}-\frac{s}{s^2+1}-\frac{1}{s^2+1}$$

$$\Rightarrow L^{-1}\left[\frac{1}{s^2}\left(\frac{s+1}{s^2+1}\right)\right]=1+x-\cos x-\sin x$$

Example 9: The inverse Laplace transform of $\frac{1}{s^2(s+1)}$ is

(a) $\frac{1}{2}t^2e^{-t}$ (b) $\frac{1}{2}t^2+1-e^{-t}$ (c) $t-1+e^{-t}$ (d) $\frac{1}{2}t^2(1-e^{-t})$

Soln. $$f(s)=\frac{1}{s^2(s+1)}=\frac{(s+1)-s(s+1)+s^2}{s^2(s+1)}=\frac{1}{s^2}-\frac{1}{s}+\frac{1}{s+1}$$

$$\Rightarrow L^{-1}[f(s)]=t-1+e^{-t}$$

Correct answer is (c).

Example 10: The Laplace transform of $6t^3+3\sin 4t$ is

(a) $\frac{36}{s^4}+\frac{12}{s^2+16}$ (b) $\frac{36}{s^4}+\frac{12}{s^2-16}$ (c) $\frac{18}{s^4}+\frac{12}{s^2-16}$ (d) $\frac{36}{s^3}+\frac{12}{s^2+16}$

Soln. $$L[f(t)]=L[6t^3]+L[3\sin 4t]=6L[t^3]+3[L[\sin 4t]]$$

Using $L[t^n]=\frac{n!}{s^{n+1}}, L[\sin at]=\frac{a}{s^2+a^2}$

We get, $L[f(t)]=6\cdot\frac{3!}{s^4}+3\cdot\frac{4}{s^2+16}=\frac{36}{s^4}+\frac{12}{s^2+16}$

Correct option is (a)

Example 11: The laplace transform of $f(t)=\sin\pi t$ is $F(s)=\frac{\pi}{\left(s^2+\pi^2\right)}, s>0$. Therefore, the laplace transform of $t\sin\pi t$ is:

(a) $\frac{\pi}{s^2\left(s^2+\pi^2\right)}$ (b) $\frac{2\pi}{s^2\left(s^2+\pi^2\right)}$ (c) $\frac{2\pi s}{\left(s^2+\pi^2\right)^2}$ (d) $\frac{2\pi}{\left(s^2+\pi^2\right)^2}$

Soln. **Property :** $L\left[f(x)\right]=f(s)$ then $L\left[x^n f(x)\right]=(-1)^n\frac{d^n}{ds^n}\left[f(s)\right]$

$$L[t\sin\pi t]=(-1)^1\frac{d}{ds}\left[L(\sin\pi t)\right]=-\frac{d}{ds}\left[\frac{\pi}{s^2+\pi^2}\right]=\frac{2\pi s}{\left(s^2+\pi^2\right)^2}$$

Correct option is (c)

Example 12: If $f(x)=\begin{cases}0 & \text{for } x<3\\ x-3 & \text{for } x\geq 3\end{cases}$ then, the laplace transform of $f(x)$ is

(a) $s^{-2}e^{3s}$ (b) s^2e^{-3s} (c) s^{-2} (d) $s^{-2}e^{-3s}$

Soln. $L\left[f(x)\right]=\int_3^\infty e^{-sx}(x-3)\,dx=(x-3)\left.\frac{e^{-sx}}{-s}\right|_3^\infty-\int_3^\infty 1\cdot\frac{e^{-sx}}{-s}dx=-\left.\frac{e^{-sx}}{s^2}\right|_3^\infty=\frac{e^{-3s}}{s^2}$

Correct option is (d)

Example 13: Which of the following pairs of the given function $F(t)$ and its Laplace transform $f(s)$ is not correct?

(a) $F(t)=\delta(t), f(s)=1$ (b) $F(t)=1, f(s)=\frac{1}{s}(s>0)$

(c) $F(t)=\sin kt, f(s)=\frac{s}{s^2+k^2}(s>0)$ (d) $F(t)=te^{kt}, f(s)=\frac{1}{(s-k)^2}(s>k, s>0)$

Soln. $L[\delta(t)]=\int\limits_0^\infty\delta(t)e^{-sx}dx=1$

$$L[1]=\int\limits_0^\infty 1\cdot e^{-sx}dx=\left[\frac{e^{-sx}}{-s}\right]_0^\infty=\frac{1}{s}$$

$$L[\sin kt]=\int\limits_0^\infty\sin kt\,e^{-st}dt=\frac{k}{s^2+k^2}$$

$$L[te^{kt}]=L[t]_{s\to s-k}=\left.\frac{1}{s^2}\right|_{s\to s-k}=\frac{1}{(s-k)^2}$$

Correct option is (c)

Chapter 8

PROBABILITY

8.1 Basic Review of Probability

Definition:

Probability is the numerical measurement of degree of certainity of the occurrence of an event. If an event A can happen in m ways and fail in n ways then the probability of happening of event A will be equal to

$$P(A) = \frac{m}{m+n}$$

and probability of failing of event A = $P'(A) = \frac{n}{m+n}$

Therefore, $P(A) + P'(A) = 1$

Probability of any event A = $\dfrac{\text{number of events or ways favourable to A}}{\text{total number of events or ways}}$

(i) Sample Space: The set at all possible outcomes in a single performance of an random experiment is known as sample space. Each outcome is called sample point.

Example: Tossing a coin, we can get head or tail. Therefore, sample space S = (H, T)

(ii) Random experiment : These are experiments, in which result may be altogether different, even though they are performed under identical conditions.

Example: Tossing a coin, Throwing a die

(iii) Trial and Event: Performing a random experiment is called 'trial' and outcome is called 'event'.

Example: Tossing a coin is a trial and turning up head /tail are events.

(iv) Equally like events: Two events are said to be equally like events, if one of them cannot be expected in preference to the other.

Example: Drawing a card from a well shuffled pack,we may get any card, then all 52 different cases are equally like.

(v) Independent Events: If happening of one event does not influence occurrence of other, then they are independent events.

Example: If two coins are simultanously, then the event of getting head on first coin and the event of getting tail on second coin are independent.

(vi) Mutually Exclusive Events: Two events are known as mutually exclusive events, when the occurrence of one of them excludes the occurance of the other.

Example: Tossing a coin, we can get either head or tail but not both. Therefore, getting a head and getting a tail are mutually exclusive events.

(vii) Conditional Probability: The probability of happening of event A, such that B has already occurred, is called conditional probability of happening of A. It is usually denoted as P(A | B)

Example 1: What is the probability of throwing an even number in case of a dice?

Soln: $$P=\frac{3\,[\text{getting 2, 4, 6 will be favourable cases}]}{6\,[\text{total no. of outcomes possibles are 1, 2, 3, 4, 5, 6}]}$$

Example 2: A bag contains 7 red balls and 8 black balls. Find the probability of drawing a red ball.

Soln: $$P=\frac{{}^{7}c_1[\text{No. of ways in which 1 red ball is chosen from 7 red balls}]}{{}^{15}c_1\ [\text{No. of ways in which 1 ball is chosen from 15 balls}]}$$

Example 3: A unbiased coin is tossed two times. What is the probability of at least one head?

Soln: $$P=\frac{3\left[\text{Outcomes having at least one head are: HT, TH,HH}\right]}{4\left[\text{Different outcomes possible: HH,HT,TH,TT}\right]}$$

Theorem 1: Probability that either event A or event B or both will occur (when they are mutually exclusive) is given by

$$P(A \cup B) = P(A) + P(B)$$

Theorem 2: Probability that either event A or event B or both will occur(when they are not mutually exclusive) is given by,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Theorem 3: Probability that both A, B events occur (when they are independent events) is given by,

$$P(A \cap B) = P(A)\,P(B)$$

Theorem 4: Probability that both A, B events occur (when they ar not independent events) is given by,

$$\begin{aligned} P(A\cap B) &= P(B\mid A).P(A) \quad [\text{Here A occurs first, then } B] \\ &= P(A\mid B).P(B) \quad [\text{Here B occurs first, then } A] \end{aligned}$$

Theorem 5: For every event A, $0 \le P(A) \le 1$ i.e. a probability is between 0 and 1.

Theorem 6: $P(\varnothing) = 0$ i.e. the impossible event has probability zero.

Theorem 7: If A' is the complement of A, then $P(A') = 1 - P(A)$

Example 4: Find the probability of throwing 9 with two dice.

Soln: Total number of possible ways of throwing two dice $= 6 \times 6 = 36$
Number of ways getting 9 i.e. (3+6), (4+5), (5+4), (6+3) = 4
Therefore, the required probability = 4/36 =1/9

Example 5: What is the probability that a leap year, selected at random, will contain 53 sundays?

Soln: Every year has minimum of 52 weeks = 364 days i.e. there will be 52 sundays, 52 mondays, ... etc. Since a leap year contains 366 days, there are 2 extra days. If the year starts on either saturday or sunday, we will get the 53rd sunday. Therefore, probability of getting 53 sundays in a leap year is $\frac{2}{7}$.

Example 6: A unbiased coin is tossed 3 times. What is the probability of obtaining two heads?

Soln: The possible outcomes are HHH, HHT, HTH, THH, HTT, THT, TTH, TTT.

The favorable outcomes are HHT, HTH, THH.

The required probability of getting 2 heads is $\frac{3}{8}$.

Example 7: A problem is given to three students A, B and C whose chances of solving it are $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$ respectively. What is the probability that this problem will be solved?

Soln: Since $P(A) = \frac{1}{2}, P(B) = \frac{1}{3}, P(C) = \frac{1}{4}$, then $P'(A) = \frac{1}{2}, P'(B) = \frac{2}{3}, P'(C) = \frac{3}{4}$

The problem will not be solved if all three of them will not be able to solve the problem.

Then, probability that the problem will not be solved $= P'(A)\,P'(B)\,P'(C) = \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} = \frac{1}{4}$

Therefore, the probability that the problem will be solved $= 1 - \frac{1}{4} = \frac{3}{4}$

Example 8: From a pack of 52 cards one are drawn at random. What is the probability that the card is either a king or a spade?

Soln: Probability that one spade will be choosen from a pack of 52 cards $= \frac{13}{52} = \frac{1}{4}$

Probability that one king will be choosen from a pack of 52 cards $= \frac{4}{52} = \frac{1}{13}$

Since, there is one card which is both king and spade. Then, the probability that the choosen card is either a king or a spade $= \frac{1}{4} + \frac{1}{13} - \frac{1}{52} = \frac{4}{13}$

Example 9: Let $P\begin{bmatrix} 1 & a \\ b & c \end{bmatrix}$, where a, b, c are chosen randomly from the set {1, 2, 3, 4, 5}. What is the probability that P is singular?

Soln: $P=\begin{bmatrix} 1 & a \\ b & c \end{bmatrix}$; for the matrix to be singular $|P| = c - ab = 0$

$$\text{Required probability} = \frac{\text{number of singular matrices}}{\text{total number of matrices formed}}$$

Each a, b, c can be chosen in five ways, therefore, total number of matrices formed = 5×5×5=125

For the following combinations of a, b, c , $|P| = c - ab = 0$:

c	1	2 2	3 3	4 4 4	5 5
a	1	1 2	3 1	1 4 2	1 5
b	1	2 1	1 3	4 1 2	5 1

So, (Probability) $P = \frac{10}{125} = \frac{2}{25}$

Example 10: A six faced die is a biased one. it is thrice more likely to show an odd number than to show an even number. It is thrown twice. What is the probability that the sum of the numbers in the two throws is even?

Soln: $\text{Prob.}(\text{odd}) = \text{Prob.}(\text{even})$

$\text{Prob.}(\text{odd}) + \text{Prob.}(\text{even}) = 1$

$\Rightarrow\ 3\text{Prob.}(\text{even}) + \text{Prob.}(\text{even}) = 1\ \Rightarrow \text{Prob.}(\text{even}) = \frac{1}{4}$ and $\text{Prob.}(\text{odd}) = \frac{3}{4}$

Sum of the numbers will be even if both numbers will be either even or odd.

Required probability = $\text{Prob.}(\text{even-even}) + \text{Prob.}(\text{odd-odd}) = \left(\frac{1}{4}\times\frac{1}{4}\right) + \left(\frac{3}{4}\times\frac{3}{4}\right) = \frac{5}{8}$

8.2 Baye's Theorem

If B_1, B_2, B_3...B_n are mutually exclusive events (with non-zero probability) of an random experiment, then for any arbitrary event A of the sample space of above random experiment, the probability of event B_k such that A has already occured is

$$P(B_k \mid A) = \frac{P(B_k)\,P(A \mid B_k)}{\sum_{i=1}^{n} P(B_i) P(A \mid B_i)}$$

Example 11: Suppose bag 1 contains 2 white, 3 red balls and bag 2 contains 3 white and 4 red balls. If a bag is selected at random and a ball is drawn from it. If the drawn ball is white, what is probability that the ball is from bag 1?

Soln: Event A: The drawn ball is white, Event B_1: Choosing bag 1, Event B_2: Choosing bag 2

Probability that the ball is chosen from bag 1 given that the ball is white =

$$P(B_1 \mid A)=\frac{P(B_1)\,P(A\mid B_1)}{\sum_{i=1}^{2} P(B_i)P(A\mid B_i)}=\frac{P(B_1)\,P(A\mid B_1)}{P(B_1)\,P(A\mid B_1)+P(B_2)\,P(A\mid B_2)}=\frac{\frac{1}{2}\times\frac{{}^2C_1}{{}^5C_1}}{\frac{1}{2}\times\frac{{}^2C_1}{{}^5C_1}+\frac{1}{2}\times\frac{{}^3C_1}{{}^7C_1}}=\frac{14}{29}$$

8.3 Random Variable

It is defined as a real number 'x' connected with the outcomes of a random experiment. There are of two types as follows:

(i) Discrete random variables

(ii) Continuous random variables

Discrete Random Variable:

In this case the random variable x will take discrete values.

Example:

Two coins are tossed together. The sample place will be

S = {HH, HT, TH, TT}

Let X be a ranodm variable which denote the number of heads that we have obtained. Then, X can take values 0, 1, 2

Continuous Random Variable:

In this case random variable x can take contiuous values.

Example : Height and weight of a students in a class is a continuous random variable

8.4 Probability Distribution

This gives the all possible outcomes of a random experiment with their respective probabilities. These are of two types

(i) Discrete probability distribution

(ii) Continuous probability distribution

Discrete probability distribution:

Let X is a discrete random variable which can take the following discrete values $X_i\ (i=1,2,3,......n)$ with corresponding probabilities $P(X_i)(i=1,2,3......n)$ such that

$$\sum_{i=1}^{n} P(x_i)=1$$

then it will be discrete probability distribution with $P(x_i)$ as a probability distribution function.

(i) Mean of the distribution:

It can be defined as the average or expected value of the random variable 'x' and given by

$$E(x)=\mu=\langle x\rangle=\sum_{i=1}^{n}x_iP(x_i)$$

(ii) Variance and standard deviation of the distribution:

It can be defined as the measurement of deviation of the random variable from its mean value and given by

$$\text{variance } \sigma^2(x)=\sum_{i=1}^{n}(x_i-\mu)^2P(x_i)=\langle x^2\rangle-\langle x\rangle^2$$

$$\text{Standard Deviation, } \sigma(x)=\sqrt{\langle x^2\rangle-\langle x\rangle^2}$$

Note:

(i) $E(Kx)=K\,E(x)$

(ii) $E(x+y)=E(x)+E(y)$

(iii) $E(xy)=E(x)\cdot E(y)$

(iv) $\sigma^2(x+y)=\sigma^2(x)+\sigma^2(y)$

Example 12: A dice thrown and the outcomes are observed. Find the mean, variance, standard deviation of the random experiment.

Soln: The probability distirbution of the random experiment is as follows

x_i	1	2	3	4	5	6
$P(x_i)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$\text{Mean } \mu=\sum_{i=1}^{6}x_iP(x_i)=1\times\frac{1}{6}+2\times\frac{1}{6}+3\times\frac{1}{6}+4\times\frac{1}{6}+5\times\frac{1}{6}+6\times\frac{1}{6}=3.5$$

Variance

$$\sigma^2=\sum_{i=1}^{6}(x_i-\mu)^2P(x_i)$$

$$=(1-3.5)^2\times\frac{1}{6}+(2-3.5)^2\times\frac{1}{6}+(3-3.5)^2\times\frac{1}{6}+(4-3.5)^2\times\frac{1}{6}+(5-3.5)^2\times\frac{1}{6}+(6-3.5)^2\times\frac{1}{6}=2.9$$

Standard deviation $\sigma=\sqrt{2.9}\simeq 1.7$

Continuous probability distribution:

Suppose x is a continuous random variable and the continuous probability distribution of x is defined by a probability density function or probability distribution function $f(x)$ such that

$$\int_{-\infty}^{\infty}f(x)\,dx=1$$

i.e. probability of finding the continuous random variable x from $-\infty$ to $+\infty$ is equal to 1.

Mean of the distribution (μ) = $\langle x\rangle = \int_{-\infty}^{\infty} xf(x)\,dx$

Variance $\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$

Standard deviation $\sigma = \left[\int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx\right]^{1/2}$

Example 13: Suppose, 'x' is a continuos random varible with probability density function given by,

$$
\begin{aligned}
f(x) &= kx\,;(0 \le x < 2)\\
&= 2k\,;(2 \le x < 4)\\
&= -kx + 6k\,;(4 \le x < 6)
\end{aligned}
$$

What is the value of k and mean of the distribution?

Soln: $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\Rightarrow \int_0^2 kxdx + \int_2^4 2kdx + \int_4^6 (-kx+6k)\,dx = 1 \qquad \Rightarrow k = \frac{1}{8}$$

Mean $\mu = \int_{-\infty}^{\infty} xf(x).dx = \int_0^2 k.x^2 dx + \int_2^4 2kxdx + \int_4^6 (-kx+6x)\,xdx = 3$

8.5 Binomial Distribution

Binomial distribution can be used under the following conditions

(a) A random experiment is repeated many times but the number of trials will be finite

(b) There are only two possible outcomes i.e. success and failure, for each trial.

(c) All trials are independent of each other

If a random experiment is repeated 'n' times (i.e. there are 'n' trials of a random experiment), the probability that event A will occur 'r' times is

$$P(r) = {}^nC_r\, p^r q^{n-r}$$

where, n = total number of independent trials
r = number of success
p = probability of success in a single trial
q = probability of failure in a single trial.

Mean of distribution $= \mu = \langle x\rangle = np$

Standard Deviation of the ditribution $\sigma = \sqrt{npq}$

Variance of the distribution $\sigma^2 = npq$

Note : (i) $\langle x\rangle = np$

(ii) $\langle x^2\rangle = np(1-p+np)$

(iii) $\langle x^3\rangle = np\left(1-3p+3np+2p^2-3np^2+n^2p^2\right)$

Example 14: Find the proability of getting at least 4 head in 5 tosses of a fair coin.

Soln: Number of trials $n = 5$, number of success $r = 4$ or 5.

$$\text{Required Probability} = P(4)+P(5) = {}^5C_4\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^{5-4} + {}^5C_5\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^{5-5} = 5.\left(\frac{1}{2}\right)^5+\left(\frac{1}{2}\right)^5 = \frac{3}{16}$$

Example 15: Assuming that each face of six faced dice is equally likely to land uppermost. Consider a game which involves the tossing of five such dices. Calculate the probability that the number 4 appears uppermost.

(i) In exactly one dice. (ii) In exactly two dice.

(iii) In exactly five dice (iv) in atleast one dice.

Soln: (i) $P = {}^5c_1\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^4 = 0.4$ (ii) $P = {}^5c_2\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3 = 0.16$ (iii) $P = {}^5c_5\left(\frac{1}{6}\right)^5\left(\frac{5}{6}\right)^0 = \left(\frac{1}{6}\right)^5$

(iv) At least one means it will 1,2,3,4,5 . Therefore, $P = 1-{}^5c_0\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^5 = 0.6$

8.6 Poisson Distribution

Poisson distribution is a limiting form of binomial distribution. It can be used under the following conditions:

(a) the number of trials are infinitely large i.e $n \to \infty$

(b) the constant probability of success for each trial is infinitely small i.e. $p \to 0$

(c) np is a finite quantity

If a random experiment is repeated 'n' times, then the probability that event A will occur 'r' times

$$P(r) = \frac{e^{-m}\cdot m^r}{r!}$$

where 'm' is the mean of the distribution.

Mean of the distribution, $\mu = \langle x\rangle = np$

Variance of the ditribution $\sigma^2 = np$

Standard Deviation of the distribution $\sigma = \sqrt{np}$

Note: (i) $\langle x\rangle = np$

(ii) $\langle x^2\rangle = np(1+np)$

(iii) $\langle x^3\rangle = np\left(1+3np+n^2p^2\right)$

Example 16: Ten percent of the tools produced in a certain manufacturing process turn out to be defective. Find the probability that in a sample of 10 tools chosen at random, exactly 2 will be defective, by using (a) the binomial distribution, (b) the poisson distribution.

Soln: (a) The probability of a defective tool is p = 0.1. Let X denote the number of defective tools out of 10 chosen. Then, according to the binomial distribution,

$$P(r=2)=\left({}^{10}c_2\right)(0.1)^2(0.9)^8=0.1937 \text{ or } 0.19$$

(b) We have $m = np = (10)(0.1) = 1$. Then, according to the Poisson distribution,

$$P(r=2)=\frac{m^r e^{-m}}{r!}=\frac{(1)^2 e^{-1}}{2!}=0.1839 \text{ or } 0.18$$

Example 17: A car hire firm has two cars which it hires out day by day. The number of demands for a car on each day is distributed as a poisson's distribution with mean 1.5. Calculate the number of days in a year on which (i) car is not used (ii) The number of days in a year on which some demand is refused.

Soln: Mean of the distribution 'm' = 1.5

(i) If the car is not used, then demand r=0

$$\mathrm{P}(\mathrm{r})=\frac{\mathrm{e}^{-\mathrm{m}}\mathrm{m}^{\mathrm{r}}}{\mathrm{r}!} \Rightarrow P(0)=\frac{e^{-1.5}(1.5)^0}{0!}=e^{-1.5}=0.2231$$

Therefore, number of days in a year when there is no demand of car. $= 365\times0.2231 = 81.4315 = 81$ days

(ii) Some demand is refused if the number of demands is more than 2 i.e. r > 2.

$$P(r>2)=p(3)+p(4)+\ldots=1-\left[P(0)+P(1)+P(2)\right]$$

$$=1-\left[\frac{e^{-1.5}(1.5)^0}{0!}+\frac{e^{-1.5}(1.5)^1}{1!}+\frac{e^{-1.5}(1.5)^2}{2!}\right]=0.1912625$$

Therefore, number of days in a year when some demand is refused $= 365\times0.1912625 = 69.81 = 70$ days

Example 18: If the probability of unfavorable reaction from a certain injection is 0.001, what is the probability that out of 2000 individuals, more than 2 will suffer from unfavorable reaction?

Soln: Number of trials n = 2000

Probability of success in single trial p = 0.001

Probability that more than two persons will suffer from the unfavourable reaction =

$$P(r>2)=1-\left[P(0)+P(1)+P(2)\right]=1-\left[\frac{e^{-2}2^0}{0!}+\frac{e^{-2}2^1}{1!}+\frac{e^{-2}2^2}{2!}\right]=1-\frac{5}{e^2}$$

8.7 Gaussian Distribution

The Gaussian probability distribution function for a continuous random variable x is given by

$$f(x)=\frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

where, μ = mean of distribution $(-\infty<\mu<\infty)$

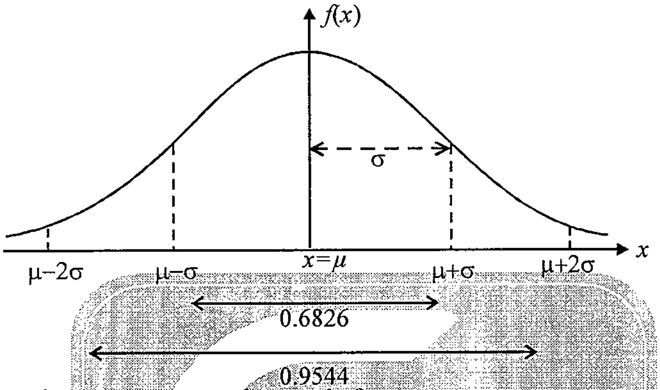
σ = S.D. of the distribution $(\sigma>0)$

Replacing $\frac{x-\mu}{\sigma}$ by z, we get the probability distribution function for the normal distribution.

i.e. $$f(z)=\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{z^2}{2}\right]$$

where, mean of the distribution $\mu = 0$ and standard deviation $\sigma = 1$

Normal curve and its properties:



(a) The curve is symmetric about $x = \mu$ and extends from $x = -\infty$ to $x = \infty$

(b) The curve attains a peak about $x = \mu$ and the peak value $f(x)\big|_{\max} = \frac{1}{\sqrt{2\pi}\sigma}$

(c) As we change x on both sides from μ, y decreases rapidly.

(d) Area under the curve is the measurement of probability for the ranodm variable 'x'.

$$P(\mu-\sigma \le x \le \mu+\sigma) = 0.6826$$
$$P(\mu-2\sigma \le x \le \mu+2\sigma) = 0.9544$$
$$P(\mu-2\sigma \le x \le \mu+3\sigma) = 0.9974$$

Note:
(i) $\langle x \rangle = \mu$
(ii) $\langle x^2 \rangle = \mu^2 + \sigma^2$
(iii) $\langle x^3 \rangle = (\mu^2 + 3\sigma^2)\mu$
(iv) $\langle x^4 \rangle = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4$

Example 19: A manufacturer produces airmail envelopes whose weight is normally distributed with a mean 1.95 gm and standard deviation 0.05 gm. The envelopes are sold in lots of 1000. How many envelopes in a lot will heavier than 2 gm?

Soln: Maximum number of envelops have weight $\mu = 1.95$ gram.

Since the standard deviation in the weight of envelops $= \sigma = 0.05$g $\Rightarrow \mu + \sigma = 2$ g, $\mu - \sigma = 1.90$ g

Percentage of envelops having weight between 1.90 gram and 2 gram = 68.7 %

Percentage of envelops having weight less than 1.90 g and more 2 gram = 100% − 68.7 % = 31.3%

Percentage of envelops having weight more than 2 grams = 15.65 % (since the normal curve is symmetric.)

Number of envelops having weight more than 2 grams = 1000 × 15.65 $\approx$ 156.

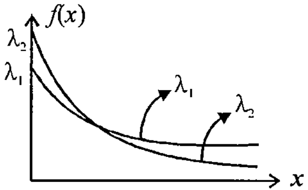
8.8 Exponential Distribution

The probability distribution function is given as follows

$$f(x) = \lambda e^{-\lambda x} \quad \text{for } x \geq 0$$
$$= 0 \quad \text{for } x < 0$$

where λ is known as rate parameter (+ve) and its denotes the decay rate of probability.

If λ_1, λ_2 are two rate parameter and $\lambda_2 > \lambda_1$, then



Mean : $\mu = \langle x \rangle = \int_{-\infty}^{\infty} x\, f(x)\, dx$

$$\langle x \rangle = \int_{-\infty}^{\infty} x.\lambda e^{-\lambda x} dx = \frac{1}{\lambda}\int_{-\infty}^{\infty} t\, e^{-t} dt = \frac{1}{\lambda}$$

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx = \int_{-\infty}^{\infty} \left(x - \frac{1}{\lambda}\right)^2 \lambda\, e^{-\lambda x} dx = \frac{1}{\lambda^2}$$

TIFR PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. A 100 page book is known to have 200 printing errors distributed randomly through the pages. The probability that one of the pages will be found to be completely free of errors is closest to

[TIFR 2011]

(a) 67% (b) 50% (c) 25% (d) 13%

Soln: Since, 100 page contains 200 error hence, on an average, error per page = 2.
Here, R.E. = checking 1 page for error.
Success = getting error in a page
From poission distribution, Probability of not getting an error in a page ($r = 0$)

$$P(r) = \frac{e^{-nP}(nP)^r}{r!}$$

Here, r = number of times of success, nP = average error per page.

Therefore, $r = 0,\ nP = 2$

Therefore, $P(r=0) = \frac{e^{-2} \times (2)^0}{0!} = \frac{1}{e^2} \simeq 13\,\%.$

Correct option is (d)

Question 2. The probability function for a variable x which assumes only positive values is

$$f(x) = x \exp\left(-\frac{x}{\lambda}\right)$$

where $\lambda > 0$. The ratio $\langle x \rangle / \hat{x}$, where $\hat{x}$ is the most probable value and $\langle x \rangle$ is the mean value of the variable x, is **[TIFR 2014]**

(a) 2 (b) $\frac{1+\lambda}{1-\lambda}$ (c) $\frac{1}{\lambda}$ (d) 1

Soln: $f(x) = x\exp\left(\frac{-x}{\lambda}\right) \rightarrow$ Probability function

$$\text{Mean value } \langle x\rangle = \frac{\int\limits_0^\infty xf(x)\,dx}{\int\limits_0^\infty f(x)\,dx} = \frac{\int\limits_0^\infty x^2\cdot e^{-x/\lambda}dx}{\int\limits_0^\infty xe^{-x/\lambda}dx} = \frac{\int\limits_0^\infty \lambda^2p^2\cdot e^{-p}\cdot\lambda dp}{\int\limits_0^\infty e^{-p}\cdot\lambda^2 p dp} = \frac{\lambda\int\limits_0^\infty e^{-p}\cdot p^2dp}{\int\limits_0^\infty e^{-p}\cdot pdp}$$

Assuming, $\frac{x}{\lambda} = p, \Rightarrow x = \lambda p \Rightarrow dx = \lambda dp$

[Using gamma integral, $\int\limits_0^\infty e^{-p}\cdot p^{n-1}dp = \Gamma n$] $= \frac{\lambda\,\Gamma 3}{\Gamma 2} = 2\lambda$

At $x = \hat{x}$ (most probable value), $\frac{df}{dx} = 0$

$$\Rightarrow \quad e^{-x/\lambda} + x\cdot e^{-x/\lambda}\cdot\left(-\frac{1}{\lambda}\right) = 0 \quad \Rightarrow \quad e^{-x/\lambda}\left(1-\frac{x}{\lambda}\right) = 0 \Rightarrow x = \hat{x} = \lambda$$

Therefore, $\frac{\langle x\rangle}{\hat{x}} = 2$

Correct option is (a)

Question 3. A random number generator outputs +1 or –1 with equal probability every time it is run. After it is run 6 times, what is the probability that the sum of the answers generated is zero? Assume that the individual runs are independent of each other. **[TIFR 2015]**

(a) 1/2 (b) 5/6 (c) 5/16 (d) 15/32

Soln: R.E. = running a random number generator, Number of trials $n = 6$

p = getting '+1' in single run and q = getting '–1' in single run.

Hence, $p = \frac{1}{2}$, $q = \frac{1}{2}$

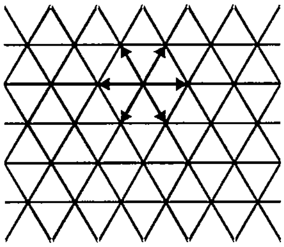
Sum will be zero, if (+1) occurs 3 times and (–1) occurs 3 times.

Hence, required probability $P(r) = {}^nC_3\, p^r\, q^{n-r}$

Here, $P(r=3) = {}^6C_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^3 = \frac{5}{16}$.

Correct option is (c)

Question 4. In a triangular lattice a particle moves from a lattice point to any of its 6 neighbouring points with equal probability, as shown in the figure on the right. **[TIFR 2016]**

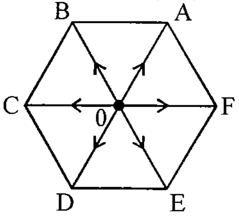


The probability that the particle is back at its starting point after 3 moves is

(a) 5/18 (b) 1/6 (c) 1/18 (d) 1/36

Soln: Total number of ways along which particle can move in 3 steps is $6\times6\times6=216$.

To return to the starting point after 3 steps,
Number of ways to take 1st step 6. (Particle may go either to A, B, C, D, E or F).
Number of ways to take 2nd step is 2.
Number of ways to take 3rd step 1.



Therefore, required probability $=\dfrac{6\times2\times1}{216}=\dfrac{1}{18}$

Correct option is (c).

JEST PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. An unbiased die is cast twice. The probability that the positive difference (bigger-smaller) between the two numbers is 2 is **[JEST 2012]**

(a) 1/9 (b) 2/9 (c) 1/6 (d) 1/3

Soln: Sample space $=S=36$.

$E=$ event of getting the difference 2.

$\{(3,1),(4,2),(5,3),(6,4),(1,3),(2,4),(3,5),(4,6)\}$

$n(E)=8$

Therefore, required probability $P=\dfrac{8}{36}=\dfrac{2}{9}$.

Correct option is (b)

Question 2. A box contains 100 coins out of which 99 fair coins and 1 is a double-headed coin. Suppose you choose a coin at random and toss it 3 times. It turns out that the results of all 3 tosses are heads. What is the probability that the coin you have drawn is the double-headed one? **[JEST 2013]**

(a) 0.99 (b) 0.925 (c) 0.075 (d) 0.01

Soln: Total 100 coins = 99 fair coins + 1 double headed coin

Suppose Event A : getting 3 heads in 3 tosses

Event B_1 : selecting a fair coin

Event B_2 : selecting a double headed coin.

$$P(B_2\mid A)=\frac{P(B_2)P(A\mid B_2)}{P(B_1)P(A\mid B_1)+P(B_2)P(A\mid B_2)}=\frac{\dfrac{{}^{1}C_1}{{}^{100}C_1}\times1}{\dfrac{{}^{99}C_1}{{}^{100}C_1}\times\dfrac{1}{8}+\dfrac{{}^{1}C_1}{{}^{100}C_1}\times1}=0.075$$

Correct option is (c)

Question 3. There are on average 20 buses per hour at a point, but at random times. The probability that there are no buses in five minutes is closest to **[JEST 2013]**

(a) 0.07 (b) 0.60 (c) 0.36 (d) 0.19

Soln: On an average number of buses in one min $= \frac{20}{60} = \frac{1}{3}$.

Hence, number of buses (in average) in 5 min = 5/3.
R.E. = checking for bus for five minutes.
Success = getting a bus.

Therefore, required probability $P(r) = \frac{e^{-nP}(nP)^r}{r!}$.

Here r = number of times of success = 0.
nP = average number of buses in 5 min = 5/3.

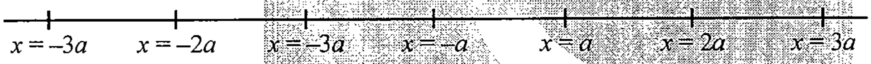
Therefore, $P(r=0) = \frac{e^{-5/3}(5/3)^0}{0!} = e^{-5/3} \simeq 0.18$.

Correct option is (d)

Question 4. Two drunks start out together at the origin, each having equal probability of making a step simultaneously to the left or right along the x-axis. The probability that they meet after n steps is **[JEST 2014]**

(a) $\frac{1}{4^n}\frac{2n!}{n!^2}$ (b) $\frac{1}{2^n}\frac{2n!}{n!^2}$ (c) $\frac{1}{2^n}2n!$ (d) $\frac{1}{4^n}n!$

Soln:



Let us keep random walker 1 at rest and let random walker 2 takes '$2n$' steps. Then random walker 2 will return back to origin or at random walker 1, if he takes n steps right and 'n' steps left.
Total steps = $2n$
p = probability of taking a step right = 1/2.
q = probability of taking a step left = 1/2.

Therefore, required probability $p(r=n) = {}^{2n}C_n\left(\frac{1}{2}\right)^n\left(\frac{1}{2}\right)^{2n-n} = \frac{1}{4^n}\frac{(2n)!}{(n!)^2}$.

Correct option is (a)

Question 5. If the distribution function of x is $f(x) = xe^{-x/\lambda}$ over the interval $0 < x < \infty$, then mean value of x is **[JEST 2014]**

(a) λ (b) 2λ (c) $\lambda/2$ (d) 0

Soln: $f(x) = x\,e^{-x/\lambda}$

$$\langle x \rangle = \text{mean value of } x = \frac{\int_{-\infty}^{\infty} x\,f(x)\,dx}{\int_{-\infty}^{\infty} f(x)\,dx}$$

Here, $\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{\infty} x\,e^{-x/\lambda}dx = \left(x\left(-\lambda\,e^{-x/\lambda}\right) - \lambda^2\left(e^{-x/\lambda}\right)\right)_0^{\infty} = \lambda^2$

and $\int_{-\infty}^{\infty} x\,f(x)\,dx = \int_0^{\infty} x^2\,e^{-x/\lambda}\,dx = \left(x^2\left(-\lambda\,e^{-x/\lambda}\right)\right)_0^{\infty} + 2\lambda\int_0^{\infty} x\,e^{-x/\lambda}\,dx$

$\Rightarrow \quad 0 + 2\lambda \times \lambda^2 = 2\lambda^3.$

Therefore, $\langle x\rangle = \dfrac{2\lambda^3}{\lambda^2} = 2\lambda.$

Question 6. If two ideal dice are rolled once, what is the probability of getting atleast one '6'?

[JEST 2015]

(a) 11/36 (b) 1/36 (c) 10/36 (d) 5/36

Soln: $n(s) = 36$

E = event of getting at least one six.

$= \{(1,6), (2,6), (3,6), (4,6), (5,6), (6,6), (6,1), (6,2), (6,3), (6,4), (6,5)\}$

$n(E) = 11$

Therefore, required probability = 11/36.

Correct option is (a)

Question 7. You receive on average 5 emairs per day during a 365-days year. The number of days on average on which you do not receive any emails in that year are **[JEST 2016]**

(a) More than 5 (b) More than 2 (c) 1 (d) None of the above

Soln: Average emails per day = 5

R.E. = Checking that email is received or not

Success = Getting an email in a day

Therefore, $P(r) = \dfrac{e^{-nP}(nP)^r}{r!}$

Here, $r = 0,\ nP = 5$

$P(r = 0) = \dfrac{e^{-5}(5)^0}{0!} = e^{-5}.$

Therefore, number of days on average on which we do not receive any emails.

$= e^{-5} \times 365 =$ more than 2 days.

Correct option is (b)

Question 8. The mean value of random variable x with probability density $p(x) = \dfrac{1}{\sigma\sqrt{2\pi}}\exp\left[-\left(x^2 + \mu x\right)/2\sigma^2\right]$, is

[JEST 2016]

(a) 0 (b) $\mu/2$ (c) $-\mu/2$ (d) σ

Soln: We have probability function, $P(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{\left(x^2+\mu x\right)}{2\sigma^2}}$

Now, $\frac{dP}{dx}=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{\left(x^2+\mu x\right)}{2\sigma^2}}\left(-2x-\mu\right)$

For $x=x_0$, $\left.\frac{dP}{dx}\right|_{x=x_0}=0\Rightarrow\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{\left(x_0^2+\mu x_0\right)}{2\sigma^2}(-2x_0-\mu)}=0 \qquad \Rightarrow x_0=-\frac{\mu}{2}$

For gaussian distribution most probable and mean value are equal

Therefore, mean value of x is $-\frac{\mu}{2}$

Correct option is (c)

Question 9. Suppose that we toss two fair coins hundred times each. The probability that the same number of *heads* occur for both coins at the end of the experiment is **[JEST 2017]**

(a) $\left(\frac{1}{4}\right)^{100}\sum_{n=0}^{100}\binom{100}{n}$

(b) $2\left(\frac{1}{4}\right)^{100}\sum_{n=0}^{100}\binom{100}{n}^2$

(c) $\frac{1}{2}\left(\frac{1}{4}\right)^{100}\sum_{n=0}^{100}\binom{100}{n}^2$

(d) $\left(\frac{1}{4}\right)^{100}\sum_{n=0}^{100}\binom{100}{n}^2$

Soln: Required probability

$$=\sum_{n_1}{}^{100}C_{n_1}\left(\frac{1}{2}\right)^{n_1}\left(\frac{1}{2}\right)^{100-n_1}\sum_{n_2}{}^{100}C_{n_2}\left(\frac{1}{2}\right)^{n_2}\left(\frac{1}{2}\right)^{100-n_2}$$

$$=\left(\frac{1}{4}\right)^{100}\sum_{n_1}{}^{100}C_{n_1}\sum_{n_2}{}^{100}C_{n_2}$$

For $n_1=n_2=n$

$$=\left(\frac{1}{4}\right)^{100}\left(\sum_{n}{}^{100}C_n\right)^2.$$

Correct option is (b)

JNU PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. Three dics with faces marked 1, 2, ..., 6 are thrown together. Assuming that they are unbiased, what is the probability that the sum of the numbers that turn up is 15? **[JNU 2010]**

(a) 1/108 (b) 3/108 (c) 5/108 (d) 7/108

Soln: $x(s) = 6 \times 6 \times 6$

$E =$ sum is 15

$= \{(3,6,6), (6,3,6), (6,6,3), (4,6,5), (4,5,6), (5,6,4), (6,5,4), (5,4,6), (6,4,5), (5,5,5)\}$

Hence, required probability, $\dfrac{10}{6\times 6\times 6} = \dfrac{5}{108}$.

Correct option is (c)

HCU PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. If one person speaks the truth in 70% of the cases and another in 80% of the cases, then the probability that they will contradict each other in describing a single event is

(a) 0.36 (b) 0.38 (c) 0.40 (d) 0.42 **[H.C.U.-2012]**

Soln: $P(A) =$ probability that 1st person speaks truth = 70/100.

$P(B) =$ probability that 2nd person speaks truth = 80/100.

Hence, required probability = (1st speaks truth and 2nd lies) + (2nd speaks truth and 1st lies).

$$= \frac{70}{100} \times \frac{20}{100} + \frac{80}{100} \times \frac{30}{100} = \frac{14+24}{100} = 0.38.$$

Correct option is (b)

Question 2. A number is choosen at random from the set of integers {1, 2, ..., 1000}. What is the probability that it is divisible by 3 or 5? [i.e., either 3 or 5 or both]

(a) 0.333 (b) 0.2 (c) 0.1 (d) 0.467 **[H.C.U.-2013]**

Soln: $A =$ event that choosen number is divisible by 3.

$= \{3, 6, 9, 12, 15, ..., 999\}$

$n(A) = 333$

$B =$ even that choosen number is divisible by 5.

$= \{5, 10, 15, 20, ..., 1000\}$

$n(B) = 200$

Numbers divisible by both 3 and 5.

$\{15, 30, 45, ..., 990\}$

$n(A \cap B) = 66.$

Therefore, probability that choosen number is divisible by either 3 or 5.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{333}{1000} + \frac{200}{1000} - \frac{66}{1000} = 0.467.$$

Question 3. If a bag contains blue, green and red marbles, all but 15 are blue, all but 13 are red and all but 12 are green. How many are red?

(a) 13 (b) 7 (c) 8 (d) 20 **[H.C.U.-2014]**

Soln: Let number of blue marbles = X
number of green marbles = Y
number of red marbles = Z

Since, all but 15 are blue, Y + Z = 15 ... (1)

All but 13 are red, X + Y = 13 ... (2)

All but 12 are green, X + Z = 12 ... (3)

Solving Eqs. (1), (2) and (3) for Z,

Therefore, Z = 7.

Correct option is (b)

BHU PREVIOUS YEAR QUESTIONS & SOLUTIONS

Question 1. The probability of drawing two aces in succession from a pack of 52 cards is

(a) $\frac{2}{26}$ (b) $\frac{2}{13}$ (c) $\frac{1}{221}$ (d) $\frac{1}{169}$ **[B.H.U-2010]**

Soln: Required probability $= \frac{{}^{4}C_1}{52} \times \frac{{}^{3}C_1}{52} = \frac{4}{52} \times \frac{3}{51} = \frac{1}{221}$.

Correct option is (c)

Question 2. Two non-interacting particles are distributed in three distinct states. Let P_c be the probability for both of them in the same state in case particles are distinguishable and P_b the probability for them to be in the same state in case they are indistinguishable bosons. The ratio P_c/P_b is

(a) $\frac{3}{2}$ (b) 1 (c) $\frac{2}{3}$ (d) $\frac{1}{3}$ **[B.H.U-2010]**

Soln: If particles are distinguishable (a, b)

ε_1	ε_2	ε_3
ab		
	ab	
		ab
a	b	
	a	b
b	a	
	b	a
a		b
b		a

Hence, probability for both of them to be in the same state $= P_c = \frac{3}{9} = \frac{1}{3}$.

If particles are indistinguishable (a, a)

ε_1	ε_2	ε_3
aa		
	aa	
		aa
a	a	
	a	a
a		a

Hence, probability for both them to be in the same state $= P_b = \frac{3}{6} = \frac{1}{2}$.

Therefore, $\frac{P_c}{P_b} = \frac{1}{3} \times \frac{2}{1} = \frac{2}{3}$.

Correct option is (c)

Question 3. A card is drawn a pack containing 52 cards with 4 aces and another card is drawn from a pack of 48 cards with 8 aces. What is the probability that both are aces? **[B.H.U-2012]**

(a) $\frac{4}{52}$ (b) $\frac{8}{48}$ (c) $\frac{32}{52\times 48}$ (d) $\frac{1}{78}$.

Soln: Probability of drawing an ace in 1st case, $P_1 = \frac{4}{52}$, and probability of drawing an ace in 2nd case, $P_2 = \frac{8}{48}$.

Then probability that both cards are ace is $= \frac{4}{52} \times \frac{8}{46} = \frac{1}{78}$.

Correct option is (d)

Question 4. In a coded telegram, the letters are arranged in groups of 5 letters called 'words'. How many different `such words are there which contains each letter at most once? [**B.H.U-2012]**

(a) $\frac{26!}{\{26-5\}!}$ (b) $\frac{26!}{5!}$ (c) $\frac{26!}{5!\{26-5\}!}$ (d) $\{26-5\}!$

Soln: 5 letters can be chosen from 26 letters in ${}^{26}C_5$ way.

Also, all five letters can be arranged among themselves in 5! way.

Hence, total number of words

$${}^{26}C_5 \times 5! = \frac{26!}{(26-5)!\,5!} \times 5! = \frac{26!}{(26-5)!}.$$

Correct option is (a)

Question 5. In a random walk problem, if the probability that a particle is found between x to $x+dx$ is given as $P(x)=e^{-\pi x^2}$, the mean $x(\bar{x})$ is **[B.H.U-2013]**

(a) 0 (b) 1 (c) π (d) none of these

$$\langle x\rangle=\frac{\int_{-\infty}^{\infty} x\,p(x)\,dx}{\int_{-\infty}^{\infty} p(x)\,dx}$$

Soln: Here, $\int\limits_{-\infty}^{+\infty} p(x)\,dx=\int\limits_{-\infty}^{\infty} e^{-\pi x^2}\,dx=\sqrt{\frac{\pi}{\pi}}=1$

$$\int\limits_{-\infty}^{\infty} x\,p(x)\,dx=\int\limits_{-\infty}^{\infty} x e^{-\pi x^2}\,dx=0 \qquad (\because \text{ odd function})$$

Therefore, $\langle x\rangle=0$.

Correct option is (a)

Question 6. If the probability in a total number of N steps making n_1 steps to the right and n_2 steps to the left is described by a binomial distribution, the mean number (n_1) is proportional to **[B.H.U-2015]**

(a) N (b) N^{-1} (c) $N^{1/2}$ (d) none of these

Soln: $n_1+n_2=N$

$p(n_1)={}^{N}C_{n_1}(p)^{n_1}(q)^{n_2}$, where $p+q=1$

$$\langle n_1\rangle=\sum n_1\,p(n_1)=\sum_{n_1,n_2} n_1\frac{N!}{n_1!n_2!}(p)^{n_1}(q)^{n_2}$$

$$=p\frac{\partial}{\partial p}\sum_{n_1,n_2}\frac{N!}{n_1\,n_2}(p)^{n_1}(q)^{n_2}=p\frac{\partial}{\partial p}(p+q)^{N}$$

$$=Np(p+q)^{N-1}=Np.$$

Correct option is (a)

Chapter 9

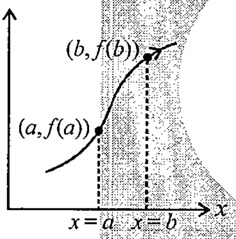
CALCULUS OF SINGLE & MULTIPLE VARIABLES

9.1 Monotonic Function

A monotonic function is a function which is either entirely non-increasing or non-decreasing. The first derivative of the monotonic function (which need not be continuous) does not change sign.

(1) A function is called monotonically increasing (also increasing or non-decreasing), if for all $x = a$ and b such that $a \le b$, one has $f(a) \le f(b)$, so 'f' preservers the order.

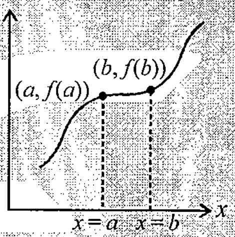
Example: (i)



$f(a) < f(b)$

(Strictly increasing)

(ii)

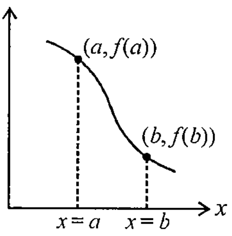


$f(a) \le f(b)$

(Only increasing)

(2) A function is called monotonically decreasing (also decreasing or non-increasing), if for all $x = a$ and b, such that $a \le b$, one has $f(a) \ge f(b)$, so, 'f' reverse the order.

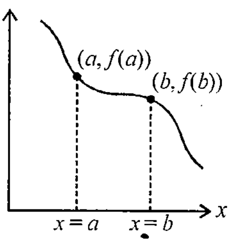
Example: (i)



$f(a) > f(b)$

(Strictly decreasing)

(ii)



$f(a) \ge f(b)$

(Only decreasing)

9.2 Test for increasing and decreasing nature of a function at any point:

(i) If $f'(a) > 0$, then $f(x)$ is increasing at $x = a$.

(ii) If $f'(a) < 0$, then $f(x)$ is decreasing at $x = a$.

(iii) If $f'(a) = 0$, then check the sign of $f'(a^+)$ and $f'(a^-)$.

(a) If $f'(a^+) > 0$ and $f'(a^-) > 0$, then increasing.

(b) If $f'(a^+) < 0$ and $f'(a^-) < 0$, then decreasing.

(c) Otherwise neither increasing nor decreasing.

Example 1. Let $f(x) = x^3 - 3x^2 + 3x + y$. Examine the nature of function at $x = 0$ and 1.

Soln: $f(x) = x^3 - 3x^2 + 3x + 4 \quad \Rightarrow f'(x) = 3x^2 - 6x + 3$

At $x = 0$, $f'(0) = 3 > 0$. Hence $f(x)$ is increasing at $x = 0$.

At $x = 1$, $f'(1) = 3 - 6 + 3 = 0$. Hence, let's check $f'(1^+)$ and $f'(1^-)$.

Clearly, $f'(1^+) = f(x = 1.001) > 0$, and $f'(1^-) = f(x = 0.999) > 0$.

Hence $f(x)$ is increasing at $x = 1$.

9.3 Steps to find the intervals for increasing and decreasing behaviour of a function:

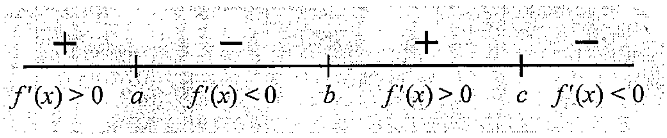
Let given function is $y = f(x)$.

(i) Find $f'(x)$ and equate it to zero. i.e., $f'(x) = 0$

(ii) Find the roots of $f'(x) = 0$ and Let it is a, b, c etc.

(iii) Now use sign scheme for $f'(x)$.

Suppose it is like this



Then, intervals for increasing $(f'(x) > 0)$ is $x \in (-\infty, a) \cup (b, c)$

and, intervals for decreasing $(f'(x) < 0)$ is $x \in (a, b) \cup (c, \infty)$

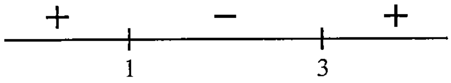
Example 2. Find the intervals of monotonicity for the following function

$$f(x) = -x^3 + 6x^2 - 9x - 2$$

Soln: (i) $f(x) = -x^3 + 6x^2 - 9x - 2 \quad \Rightarrow f'(x) = -3x^2 + 12x - 9$

Now, $f'(x) = 0 \Rightarrow x^2 - 4x + 3 = 0 \Rightarrow (x-3)(x-1) = 0 \Rightarrow x = 1, 3$

Now, using sign scheme for $f'(x)$ (quadratic).



Hence, intervals for increasing is $x \in (-\infty, 1) \cup (3, \infty)$ and intervals for decreasing is $x \in (1, 3)$.

9.4 Steps to find points of maxima and minima of a function containing single variable:

Method 1:

Let given function is $y = f(x)$

(i) Find roots for $f'(x) = 0$ and Let roots are $x = a, b$ etc.

(ii) Use sign scheme. The point(s) at which $f'(x)$ changes sign from (+) to (–) is/are the points of maxima (also called local maxima) and the point(s) at which $f'(x)$ changes sign from (–) to (+) is/are the points of minima (also called local minima).

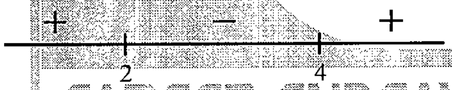
Example 3. Find the maxima and minima of the function $f(x) = x^3 - 9x^2 + 24x$.

Soln: Here, $f'(x) = 3x^3 - 18x^2 + 24$

Applying $\quad f'(x) = 0 \Rightarrow 3x^2 - 18x + 24 = 0$

$$\Rightarrow x^2 - 6x + 8 = 0 \Rightarrow (x-4)(x-2) = 0 \Rightarrow x = 4, 2$$

Now, applying sign scheme



At $x = 2$, sign changes from (+) to (–), hence $x = 2$ is a point of local maxima.

At $x = 4$, sign changes from (–) to (+), hence $x = 4$ is a point of local minima.

Method 2:

Let given function is $y = f(x)$

(i) Find roots of $f'(x) = 0$ and Let roots are $x = a, b$ etc.

(ii) Now find $f''(x)$, if $f''(a) > 0$, $x = a$ is a point of local minima, and if $f''(a) < 0$, $x = a$ is a point of local maxima.

Example 4. Find the maxima and minima of the function $f(x) = \dfrac{x+1}{\sqrt{5x^2+35}}$

Soln: Here, $f'(x) = \dfrac{(5x^2+35)^{1/2}(1) - (x+1) \times \dfrac{10x}{2(5x^2+35)^{1/2}}}{(5x^2+35)}$

Applying $f'(x)=0 \Rightarrow 2(5x^2+35)-10x(x+1)=0 \Rightarrow x=7$

Now, $f''(x=7)=-10<0$

Hence, $x=7$ is a point of local maxima.

Example 5. Let $F(x)=\int_0^x \left(t^2-3t+2\right)dt$, then F has

(a) a local maximum at $x=1$ and a local minimum at $x=2$

(b) a local minimum at $x=1$ and a local maximum at $x=2$

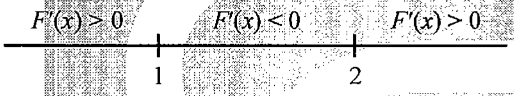
(c) local maxima at $x=1$ and $x=2$

(d) local minima at $x=1$ and $x=2$

Soln: $F(x)=\int_0^x \left(t^2-3t+2\right)dt \Rightarrow F'(x)=x^2-3x+2$

Hence for maxima and minima, $F'(x)=0 \Rightarrow x^2-3x+2=0 \Rightarrow x=1, 2$

Now, use sign scheme for $F'(x)$



Hence, $x=1$ is a point of local maxima and $x=2$ is a point of local minima.

Correct option is (a)

Example 6. Suppose that f is a continuous and differentiable on $[a,b]$. If $f'(x)\le 0$ on $[a,c)$ and $f'(x)\ge 0$ on $(c,b]$, $a<c<b$, then on $[a,b]$.

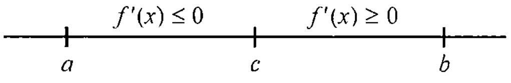
(a) $f(x)$ is never less than $f(c)$ (b) $f(x)$ is always less than $f(c)$

(c) $f(x)$ is always less than $f(a)$ (d) $f(x)$ is always greater than $f(b)$

Soln: Since, $f'(x)\le 0$ on $[a,c]$ and $f'(x)\ge 0$ on $(c,b]$

Hence, $f'(c)=0$

Now, use sign scheme



Hence, point $x=c$ is a point of local minima and $f(c)$ is minimum value of $f(x)$ on $[a,b]$. Hence, $f(x)$ is always greater than $f(c)$.

Correct option is (a)

Example 7. Let $f(x)=2x^3-x^2+2x-5$. Consider the following statements about the roots of $f(x)=0$.

A: At least one root is positive.

B: At least one root is negative.

C: There is a root between $x=1$ and .

Which one of the following is TRUE ?

(a) A, B and C are valid statements.
(b) A and B are valid statements, but C is not a valid statement.
(c) A and C are valid statements, but B is not a valid statement.
(d) A is valid statement, but B and C are not valid statements.

Soln: $f(x) = 2x^3 - x^2 + 2x - 5$

Since, there are three sign change in the expression of the function, thus there will be at least three positive roots of $f(x)$ will exist, i.e., it will have no negative roots.

Also, $f(1) = -2$ and $f(2) = 11$

Thus, $f(x)$ will have at least one root between $x = 1$ and $x = 2$.

So, A and C are valid statements.

Correct option is (c)

9.5 Steps to find points of maxima and minima of a function containing multiple variables:

Let given function is $z = f(x, y)$

(i) Find roots for $f_x(x, y) = \dfrac{\partial f}{\partial x} = 0$ and $f_y(x, y) = \dfrac{\partial f}{\partial y} = 0$. Let roots are $x = x_0$, $y = y_0$ and this points are known as critical points.

(ii) Find $D(x_0\, y_0) = f_{xx}(x_0\, y_0)\, f_{yy}(x_0\, y_0) - f_{xy}(x_0\, y_0)^2$,

$$\left(\text{where } f_{xx} = \frac{\partial^2 f(x,y)}{\partial x^2},\ f_{yy} = \frac{\partial^2 f(x,y)}{\partial y^2} \text{ and } f_{xy} = \frac{\partial^2 f(x,y)}{\partial x \partial y}\right)$$

(iii) (a) If $D > 0$ and $f_{xx}(x_0\, y_0) < 0$, there is a local maximum at $(x_0\, y_0)$

(b) If $D > 0$ and $f_{xx}(x_0\, y_0) > 0$, there is a local minimum at $(x_0\, y_0)$

(c) If $D < 0$, there is neither a maximum nor a minimum at $(x_0\, y_0)$

(d) If $D = 0$, the test fails.

Example 8. Find all local maxima and minima for $f(x, y) = x^2 - y^2$.

Soln: Here, $f_x = 2x,\ f_y = -2y,\ f_{xx} = 2,\ f_{yy} = -2,\ f_{xy} = 0$

Applying $f_x = 0,\ f_y = 0$, we get $x = 0$ and $y = 0$. So, the critical ponit of the function is (0, 0).

Here, $D(0, 0) = f_{xx}(0, 0)\, f_{yy}(0, 0) - f_{xy}(0, 0)^2 = (2 \times -2) - 0 = -4 < 0$.

So, there is neither a maximum nor minimum at (0, 0), and so there are no local maxima or minima of the function.

Example 9. Find all local maximum and minimum points of $f_x(x, y) = x^2 + 4y^2 - 2x + 8y - 1$.

Soln: Here, $f_x = 2x - 2,\ f_y = 8y + 8,\ f_{xx} = 2,\ f_{yy} = 8\ ;\ f_{xy} = 0$

Applying $f_x = 0,\ f_y = 0$, we get $x = 1$ and $y = -1$. So, the critical ponit of the function is (1, -1).

Here, $D(1, -1) = f_{xx}(1, -1)\, f_{yy}(1, -1) - f_{xy}(1, -1)^2 = 2 \times 8 - 0 = 16 > 0$ and $f_{xx}(1, -1) > 0$

So, there is a local minima at $(1, -1)$.

Example 10. The function $f(x, y) = x^3 + 3xy^2 - 4y^2 - 4y^3 - 15x$ has a local

(a) minima at $\left(-\sqrt{5}, 0\right)$ (b) minima at $\left(\sqrt{5}, \sqrt{5}\right)$

(c) maxima at $\left(\sqrt{5}, 0\right)$ (d) maxima at $\left(-\sqrt{5}, 0\right)$

Soln: $f(x, y) = x^3 + 3xy^2 - 4y^2 - 4y^3 - 15x$

$$f_x = \frac{\partial f}{\partial x} = 3x^2 + 3y^2 - 15 = 0 \Rightarrow x^2 + y^2 = 5 \quad ...(1)$$

$$f_y = \frac{\partial f}{\partial y} = 6xy - 12y^2 = 0 \quad \Rightarrow 6y(x - 2y) = 0 \quad \Rightarrow y = 0 \quad \text{and } y = x/2$$

If $y = 0$, from equation (1), $x = \pm\sqrt{5}$ and if $y = \frac{x}{2}$, from equation (1), $x^2 + \frac{x^2}{4} = 5 \Rightarrow x = \pm 2$

Hence, critical points of the function are $\left(\pm\sqrt{5}, 0\right)$ and $(\pm 2, \pm 1)$

We have to check only at $\left(\pm\sqrt{5}, 0\right)$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = 9x \quad ; f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = 6y \quad ; f_{yy} = \frac{\partial^2 f}{\partial y^2} = 6x - 24y$$

Here, $D\left(\sqrt{5}, 0\right) = \left(f_{xx} f_{yy} - f_{xy}^2\right)\Big|_{\left(\sqrt{5}, 0\right)} > 0$ and $f_{xx}\left(\sqrt{5}, 0\right) < 0$. Therefore, the function has a local minima at $\left(\sqrt{5}, 0\right)$.

Here, $D\left(-\sqrt{5}, 0\right) = \left(f_{xx} f_{yy} - f_{xy}^2\right)\Big|_{\left(-\sqrt{5}, 0\right)} > 0$ and $f_{xx}\left(-\sqrt{5}, 0\right) < 0$. Therefore, the function has a local maxima at $\left(-\sqrt{5}, 0\right)$.

Correct option is (d)

Example 11. Let $f(x, y) = x^2 - 2xy + y^2 + x^3 - y^3 + x^5$. Choose the correct statement about $f(x, y)$.

(a) $f(x, y)$ has neither a maximum nor a minimum at (0, 0)

(b) $f(x, y)$ has a local maxima at (0, 0)

(c) $f(x, y)$ has a local minima at (0, 0)

(d) None of the above are correct

Soln: $f(x, y) = x^2 - 2xy + y^2 + x^3 - y^3 + x^5$

$$f_x = \frac{\partial f}{\partial x} = 2x - 2y + 3x^2 + 5x^4 \text{ and } f_y = \frac{\partial f}{\partial y} = -2x + 2y - 3y^2$$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = 2 + 6x + 20x^3, \; f_{yy} = \frac{\partial^2 f}{\partial y^2} = 2 - 6y \text{ and } \frac{\partial^2 f}{\partial x \partial y} = -2$$

Applying $f_x = 0, f_y = 0$, we get $x = 0$ and $y = 0$. So, the critical ponit of the function is (0, 0).

Now, $D(0,0)=\left(f_{xx}f_{yy}-f_{xy}^2\right)\Big|_{(0,0)}=0$

Hence, the point (0, 0) is a doubtful point, hence needs further investigation.

We can write $f(x,y)=(x-y)^2+(x-y)(x^2+xy+y^2)+x^5$
We have $f(0,0)=0$

In the neighbourhood of the point (0, 0), where $x=y$, we have $f(x,y)=x^5$, which is positive when $x>0$ and is negative when $x<0$.
Thus, in the neighbourhood of the point (0, 0), there are points at which $f(x,y)$ takes values less than its value at the point (0, 0) and there are points at which $f(x,y)$ takes values greater than its value at (0, 0).
Hence, $f(x,y)$ can not have a maximum or a minimum at the point (0, 0).

Correct option is (a)

9.5 Multiple Integral

A multiple integral is a generalization of the usual integral in one dimension to functions of multiple variables in higher-dimensional spaces. The most common multiple integrals are double and triple, involving two or three variables, respectively.

Double Integral

The double integral is written as $\iint f(x,y)\,dx\,dy$.
Here, $f(x,y)$ can be function of x and y both or function of x or y or it can be constant.

To integrate,

(i) One should treat y as a constant and perform the integration with respect to x,

(ii) After that, perform the integration with respect to y.

The limit are of the form

$$\int_{y=c}^{y=d}\int_{x=f_1(y)}^{x=f_2(y)} f(x,y)\,dx\,dy$$

Similarly, for $\iint f(x,y)\,dy\,dx$, $f(x,y)$ is first integrated with respect to y and then it is integrated with respect to x.
The limit are the form

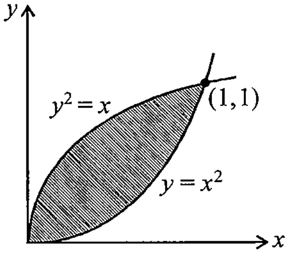
$$\int_{x=c}^{x=d}\int_{y=f_1(x)}^{y=f_2(x)} f(x,y)\,dy\,dx.$$

Example 12. Compute the double integral $I=\int_0^1\int_0^1\frac{x^2+y^2}{2}\,dx\,dy$.

Soln: $I=\frac{1}{2}\int_0^1\left(\frac{x^3}{3}+y^2x\right)_0^1 dy=\frac{1}{2}\int_0^1\left(\frac{1}{3}+y^2\right)_0^1 dy=\frac{1}{2}\left(\frac{1}{3}y+\frac{y^3}{3}\right)_0^1=\frac{1}{2}\left(\frac{1}{3}+\frac{1}{3}\right)=\frac{1}{3}.$

Example 13. Evaluate $\iint (x^2+y^2)\,dx\,dy$ over the region bounded by $y=x^2$, $y^2=x$.

Soln: The region of integration is bounded by $C_1: y=x^2$ and $C_2: y^2=x$



$$I=\iint (x^2+y^2)\,dx\,dy=\int_{y=0}^{1}\int_{x=y^2}^{\sqrt{y}}(x^2+y^2)\,dx\,dy.$$

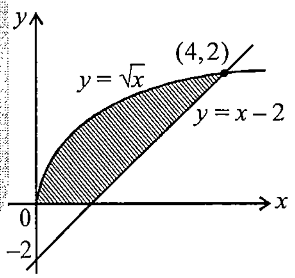
$$I=\int_{y=0}^{1}\left(\frac{x^3}{3}+xy^2\right)_{y^2}^{\sqrt{y}}dy=\int_{y=0}^{1}\left\{\left(\frac{1}{3}y^{3/2}+\sqrt{y}\,y^2\right)-\left(\frac{y^6}{3}+y^4\right)\right\}dy$$

$$=\int_{y=0}^{1}\left(\frac{1}{3}y^{3/2}+y^{5/2}-\frac{y^6}{3}-y^4\right)dy=\left(\frac{1}{3}\frac{y^{5/2}}{5/2}+\frac{y^{7/2}}{7/2}-\frac{y^7}{3\times7}-\frac{y^5}{5}\right)_0^1$$

$$=\frac{2}{15}+\frac{2}{7}-\frac{1}{21}-\frac{1}{5}=\frac{6}{35}.$$

Example 14. Find the area of the region in the 1st quadrant that is bounded by $y=\sqrt{x}$, $y=x-2$ and the x-axis.

Soln: Region is bounded by the curves, $C_1: y=\sqrt{x}$ and $C_2: y=x-2$.



$$\text{Required area}=\int_{y=0}^{2}\int_{x=y^2}^{x=y+2}dx\,dy=\int_{y=0}^{2}(x)_{y^2}^{y+2}dy=\int_0^2(y+2-y^2)\,dy$$

$$=\left(\frac{y^2}{2}+2y-\frac{y^3}{3}\right)_0^2=2+4-\frac{8}{3}=\frac{10}{3}.$$

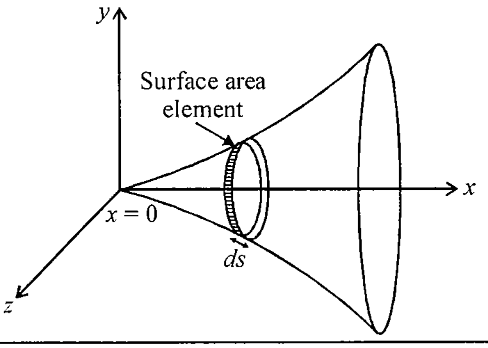
Example 15. Consider the curve $y=x^2$ from $x=0$ to $x=1$. Now, rotate the area about the x-axis to form a volume and surface of revolution. Then, calculate the area of the curved surface from $x=0$ to $x=1$.

Soln: We find the area of the surface of revolution by using as element the curved surface of a thin slab as shown in figure. This is a strip of circumference $2\pi y$ and width ds.

Here, $ds=\text{arc length}=\left(1+\left(\frac{dy}{dx}\right)^2\right)^{1/2}dx$

Hence, total area of revolution,

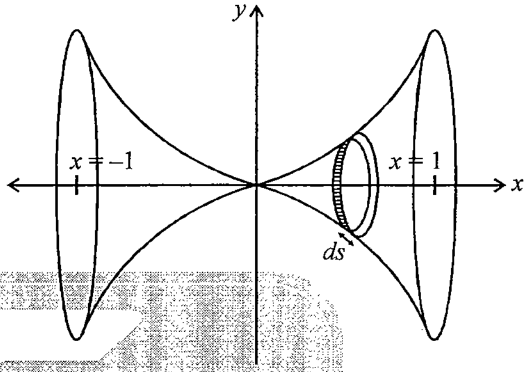
$$A=\int_{x=0}^{1}2\pi y\,ds=\int_0^1 2\pi x^2\sqrt{1+4x^2}\,dx$$



$$=\frac{2\pi}{64}\left[18\sqrt{5}-\sinh^{-1}(2)\right].$$

Example 16. Find the area of the surface obtained by rotating about the x-axis the arc of the curve $y=\frac{x^3}{3}$, from $x=-1$ to $x=1$.

Soln: $A=2\int 2\pi y\,ds$



$$A=4\pi\int_0^1\frac{x^3}{3}\sqrt{1+x^4}\,dx$$

$$1+x^4=t^2,\ 4x^3dx=2t\,dt$$

$$A=\frac{4\pi}{3}\int_1^{\sqrt{2}}t\times\frac{t\,dt}{2}$$

$$=\frac{4\pi}{3}\times\frac{1}{2}\left(\frac{t^3}{3}\right)_1^{\sqrt{2}}\qquad=\frac{2\pi}{9}\left(2\sqrt{2}-1\right)$$

Triple Integral

The triple integral is written as $\iiint f(x,y,z)\,dx\,dy\,dz$.

Here, $f(x,y,z)$ can be function of x, y and z both or function of x or y or z or it can be a constant. To integrate,

(i) One should treat y, z as constants and perform the integration with respect to x,

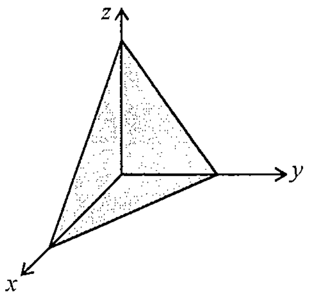
(ii) After that, perform the integration with respect to y, treating z as a constant.

(iii) At last, perform the integration with respect to z.

The limit are of the form

$$\int_{z=c}^{z=d}\int_{y=f_3(z)}^{y=f_4(z)}\int_{x=f_1(y,z)}^{x=f_2(y,z)}f(x,y,z)\,dx\,dy\,dz$$

Example 17. Evaluate $\iiint_E 2x\,dV$, where E is the region under the plane $2x+3y+z=6$ that lies in the first octant.



Soln: $V = \iiint 2x\, dV = \iint 2x\, dx\, dy\, dz$

Plane is $2x + 3y + z = 6$

Hence, z will vary from 0 to $(6-3y-2x)$

x will vary from 0 to $\left(-\frac{3}{2}y+3\right)$

y will vary from 0 to 2

Therefore, $V = \iint\int_{z=0}^{6-3y-2x} 2x\, dz\, dy\, dx$

$$= \int_{y=0}^{2}\int_{x=0}^{-\frac{3}{2}y+3} 2x(6-2x-3y)\, dx\, dy = 9$$

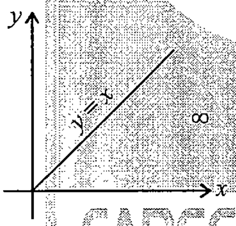
Example 18. The value of the integral $\int_0^{\infty}\int_{y=0}^{x} x e^{-x^2/y}\, dy\, dx$ is

(a) 0 (b) 1/2 (c) 4 (d) 1

Soln: $I = \int_0^{\infty}\int_{y=0}^{x} x e^{-x^2/y}\, dy\, dx$

Domain of integration is bounded by $y = 0$, $y = x$, $x = 0$; $x = \infty$

Changing the order



Therefore, $I = \int_0^{\infty}\int_y^{\infty} x e^{-x^2/y}\, dx\, dy = -\int_0^{\infty}\left(\frac{y}{2}e^{-x^2/y}\right)_y^{\infty} dy$

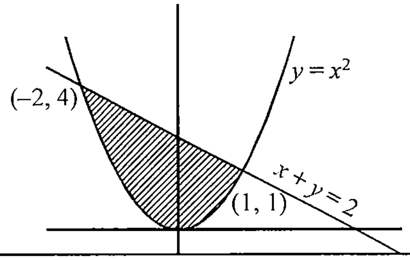
$$= \frac{1}{2}\int_0^{\infty} y e^{-y}\, dy = \frac{1}{2}\left[-ye^{-y} - e^{-y}\right]_0^{\infty} = \frac{1}{2}$$

Correct option is (b)

Example 19. The area of the region enclosed by the curve $y = x^2$ and the straight line $x + y = 2$ is

(a) 3 (b) 27/2 (c) 9/2 (d) 9

Soln: Required region is shown in figure :



Solving the two curves $y = x^2$ and $x + y = 2$, we get the points of intersection as $(-2, 4)$ and $(1, 1)$.

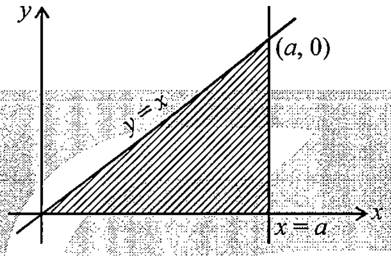
Hence, required area = $\displaystyle\int_{x=-2}^{1}\int_{y=x^2}^{y=2-x} dy\,dx = \int_{-2}^{1}(2-x-x^2)\,dx = \frac{9}{2}$.

Correct option is (c)

Example 20. Let $F : R \to R$ be a continuous function and $a > 0$, then the integral $\displaystyle\int_0^a\left[\int_0^x F(y)\,dy\right]dx$ equals

(a) $\displaystyle\int_0^a yF(y)\,dy$ (b) $\displaystyle\int_0^a (a-y)\,F(y)\,dy$ (c) $\displaystyle\int_0^a (y-a)\,F(y)\,dy$ (d) $\displaystyle\int_0^a yF(y)\,dy$

Soln: Domain of integration is bounded by $y = 0,\ y = x,\ x = 0,\ x = a$



Hence, changing the order

$$I = \int_0^a\left[\int_0^x F(y)\,dy\right]dx = \int_{y=0}^{a}\left[\int_{x=y}^{a} F(y)\,dx\right]dy = \int_{y=0}^{a}(a-y)\,F(y)\,dy.$$

Correct option is (b)

Chapter 10

PARTIAL DIFFERENTIATION

Introduction:

Let z denote the function $f(x,y)$ which posseses one definite value for every pair of values of x and y, and moreover if y is kept constant, $z = f(x,y)$ varies because of x, and similarly if x is kept constant, z varies because of the variation in a single variable y. A similar definition can be given for functions of more than two variables, i.e., $u = f(x,y,z)$ denotes a function of three variables x, y, z.

Partial derivatives of $f(x,y)$:

The ordinary derivative of $f(x,y)$ with respect to x keeping the other variable constant is called the partial derivtive of $f(x,y)$ with respect to x. Similarly ordinary derivative of $f(x,y)$ with respect to y keeping the other variable x constant is called the partial derivative of $f(x,y)$ with respect to y.

The partial derivative of $f(x,y)$ with respect to x and y are generally denoted by $f_x = \frac{\partial f}{\partial x}$ and $f_y = \frac{\partial f}{\partial y}$. Thus by definition we write

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h} \quad \text{and} \quad \frac{\partial f}{\partial y} = \lim_{k \to 0} \frac{f(x,k+y) - f(x,y)}{k}$$

when these limits exist.

At a particular point (a,b), these partial derivatives are often denoted by $f_x(a,b)$ and $f_y(a,b)$. Thus,

$$f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}\,;\ f_y(a,b) = \lim_{k \to 0} \frac{f(a,b+k) - f(a,b)}{k}$$

Again $f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h}$; $f_y(0,0) = \lim_{k \to 0} \frac{f(0,k) - f(0,0)}{k}$

provided the limit exist.

Example 1. Find the first and second order partial derivatives of $x^3 + y^3 + 3axy$

Soln: Let $f(x,y) = x^3 + y^3 + 3axy$, then $\frac{\partial f}{\partial x} = 3x^2 + 3ay$, $\frac{\partial f}{\partial y} = 3y^2 + 3ax$

$$\frac{\partial^2 f}{\partial y \partial x} = 3a, \quad \frac{\partial^2 f}{\partial x \partial y} = 3a\,; \ \frac{\partial^2 f}{\partial x^2} = 6x, \quad \frac{\partial^2 f}{\partial y^2} = 6y\,;$$

Example 2. If $u = e^{xyz}$, show that $\dfrac{\partial^3 u}{\partial x \partial y \partial z} = \left(1+3xyz+x^2y^2z^2\right)e^{xyz}$

Soln: We have $u = e^{xyz}$, $\Rightarrow \dfrac{\partial u}{\partial x} = e^{xyz}.yz \Rightarrow \dfrac{\partial^2 u}{\partial y \partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial x}\right) = \dfrac{\partial}{\partial y}\left(yze^{xyz}\right) = z\left[ye^{xyz}.xz + e^{xyz}\right]$

$$\Rightarrow \frac{\partial}{\partial z}\left(\frac{\partial^2 u}{\partial x \partial y}\right) = \frac{\partial}{\partial z}\left[xyz^2e^{xyz} + ze^{xyz}\right] = xy\left[z^2e^{xyz}.xy + e^{xyz}.2z\right] + \left[ze^{xyz}.xy + e^{xyz}\right]$$

$$= e^{xyz}\left(1+3xyz+x^2y^2z^2\right)$$

Example 3: If $u = \log\left(x^3+y^3+z^3-3xyz\right)$ show that

(a) $\left(\dfrac{\partial}{\partial x}+\dfrac{\partial}{\partial y}+\dfrac{\partial}{\partial z}\right)u = -\dfrac{9}{(x+y+z)^2}$ and (b) $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2} = \dfrac{z}{(x+y+z)^2}$

Soln: (a) $u = \log\left(x^3+y^3+z^3-3xyz\right) \Rightarrow \dfrac{\partial u}{\partial x} = \dfrac{3x^2-3yz}{x^3+y^3+z^3-3xyz}$

$$\Rightarrow \frac{\partial u}{\partial y} = \frac{3y^2-3xz}{x^3+y^3+z^3-3xyz} \text{ and } \frac{\partial u}{\partial z} = \frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}$$

$$\Rightarrow \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{3\left(x^2+y^2+z^2-xy-yz-zx\right)}{x^3+y^3+z^3-3xyz} = \frac{3}{x+y+z}$$

Now, $\left(\dfrac{\partial}{\partial x}+\dfrac{\partial}{\partial y}+\dfrac{\partial}{\partial z}\right)^2 u = \left(\dfrac{\partial}{\partial x}+\dfrac{\partial}{\partial y}+\dfrac{\partial}{\partial z}\right)\left(\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}+\dfrac{\partial u}{\partial z}\right) = \left(\dfrac{\partial}{\partial x}+\dfrac{\partial}{\partial y}+\dfrac{\partial}{\partial z}\right)\left(\dfrac{3}{x+y+z}\right)$

$$= \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$= -\frac{3}{(x+y+z)^2}-\frac{3}{(x+y+z)^2}-\frac{3}{(x+y+z)^2} = -\frac{9}{(x+y+z)^2}$$

(b) $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{6x\left(x^3+y^3+z^3-3xyz\right)-\left(3x^2-3yz\right)\left(3x^2-3yz\right)}{\left(x^3+y^3+z^3-3xyz\right)^2} = -\dfrac{3\left(x^4-2xy^3-2xz^3+3y^2z^2\right)}{\left(x^3+y^3+z^3-3xyz\right)^2}$

Similarly, $\dfrac{\partial^2 u}{\partial y^2} = -\dfrac{3\left(y^4-2yz^3-2yx^3+3z^2x^2\right)}{\left(x^3+y^3+z^3-3xyz\right)^2}$ and $\dfrac{\partial^2 u}{\partial z^2} = -\dfrac{3\left(z^4-2zx^3-2zy^3+3x^2y^2\right)}{\left(x^3+y^3+z^3-3xyz\right)^2}$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = -\frac{3\left(x^2+y^2+z^2-xy-yz-zx\right)^2}{\left(x^3+y^3+z^3-3xyz\right)^2} = -\frac{3}{(x+y+z)^2}$$

Example 4. If $x^x.y^y.z^z = c$ show that at $x = y = z$, $\frac{\partial^2 z}{\partial x \partial y} = -(x \log ex)^{-1}$

Soln: Given that, $x^x y^y z^z = c$

Taking log of both sides, we have, $x \log x + y \log y + z \log z = \log c$

Differentiating partially w.r.t. x, we get,

$$x.\frac{1}{x} + \log x + \left(z.\frac{1}{z} + \log z\right)\frac{\partial z}{\partial x} = 0 \quad \Rightarrow \quad \frac{\partial z}{\partial x} = -\frac{(1+\log x)}{(1+\log z)}$$

Similarly, we have, $\frac{\partial z}{\partial y} = -\frac{(1+\log y)}{(1+\log z)}$

$$\Rightarrow \frac{\partial^2 z}{\partial x \partial y} = \frac{(1+\log x)}{(1+\log z)^2}.\frac{1}{z}\frac{\partial z}{\partial y} = -\frac{(1+\log x)}{(1+\log z)^2}.\frac{1}{z}.\frac{(1+\log y)}{(1+\log z)} = -\frac{(1+\log x)^2}{(1+\log x)^3}.\frac{1}{x} \text{ (since } x = y = z)$$

$$= -\frac{1}{x(1+\log x)} = -\frac{1}{x \log ex} = -(x \log ex)^{-1}$$

Example 5: If $z = x^2 \tan^{-1}(y/x) - y^2 \tan^{-1}(x/y)$, prove that $\frac{\partial^2 z}{\partial y \partial x} = \frac{x^2 - y^2}{x^2 + y^2}$

Soln: We have, $\frac{\partial z}{\partial x} = 2x \tan^{-1}\frac{y}{x} + \frac{x^2}{1+(y/x)^2}\left(-\frac{y}{x^2}\right) - \frac{y^2}{1+(x/y)^2}\left(\frac{1}{y}\right)$

$$= 2x \tan^{-1}(y/x) - \frac{x^2 y}{x^2 + y^2} - \frac{y^3}{x^2 + y^2} = 2x \tan^{-1}(y/x) - \frac{y(x^2 + y^2)}{(x^2 + y^2)} = 2x \tan^{-1}(y/x) - y$$

Now, $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left\{2x \tan^{-1}\frac{y}{x} - y\right\} = \frac{2x}{1+(y/x)^2}\left(\frac{1}{x}\right) - 1 = \frac{2x^2}{x^2 + y^2} - 1 = \frac{x^2 - y^2}{x^2 + y^2}$

Example 6. If $u = f(r)$, where $r^2 = x^2 + y^2$, show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f'(r) + \frac{1}{r}f'(r)$$

Soln: We have, $2r\frac{\partial r}{\partial x} = 2x$ or $\frac{\partial r}{\partial x} = \frac{x}{r}$ and $2\text{r}\frac{\partial r}{\partial y} = 2y$ or $\frac{\partial r}{\partial y} = \frac{y}{r}$

Now, $u = f(r) \Rightarrow \frac{\partial u}{\partial x} = f'(r).\frac{\partial r}{\partial x} = f'(r)\cdot\frac{x}{r}$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{f'(r)x}{r}\right] = \frac{r\left\{f'(r).1 + x f''(r)\left(\frac{\partial r}{\partial x}\right)\right\} - x f'\left(\frac{\partial r}{\partial x}\right)}{r} = \frac{1}{r^2}\left[r f'(r) + x^2 f''(r) - \frac{x^2}{r}f'(r)\right]$$

Similary, $\frac{\partial^2 u}{\partial y^2}=\frac{1}{r^2}\left[r\,f'(r)+y^2 f''(r)-\frac{y^2}{r}f'(r)\right]$

Hence, $\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial y^2}=\frac{1}{r^2}\left[2rf'(r)+(x^2+y^2)f''(r)-\frac{(x^2+y^2)}{r}f'(r)\right]$

$$=\frac{1}{r^2}\left[2rf'(r)+r^2 f''(r)-rf'(r)\right]=\frac{1}{r}f'(r)+f''(r)$$

Example 7. If $\theta=t^n e^{-r^2/4t}$, find what value of n makes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2.\frac{\partial\theta}{\partial r}\right)=\frac{\partial\theta}{\partial t}$$

oln: We have, $\theta=t^n e^{-r^2/4t}\Rightarrow\frac{\partial\theta}{\partial r}=t^n.e^{-r^2/4t}\left(-\frac{2r}{4t}\right)=-\frac{1}{2}rt^{n-1}e^{-r^2/4t}$

$$\Rightarrow\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=\frac{\partial}{\partial r}\left[-\frac{1}{2}r^3t^{n-1}e^{-r^2/4t}\right]=-\frac{1}{2}t^{n-1}\left[r^3.e^{-r^2/4t}\left(-\frac{2r}{4t}\right)+3r^2.e^{-r^2/4t}\right]$$

$$\Rightarrow\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=\frac{1}{4}t^{n-2}\left[r^2-6t\right]e^{-r^2/4t}$$

$$\frac{\partial\theta}{\partial t}=t^n.e^{-r^2/4t}\frac{r^2}{4t^2}+e^{-r^2/4t}.nt^{n-1}=\frac{1}{4}t^{n-2}e^{-r^2/4t}\left(r^2+4nt\right)$$

Now to satisfy the condition given in the question, we have

$$\frac{1}{4}t^{n-2}\left[r^2-6t\right]e^{-r^2/4t}=\frac{1}{4}t^{n-2}\left(r^2+4nt\right)\Rightarrow r^2-6t=r^2+4nt\Rightarrow n=-3/2$$

Homogeneous Functions:

An expression of the type $a_0x^n+a_1x^{n-1}y+a_2x^{n-2}y^2+a_3x^{m-3}y^3+.....a_ny^n$ in which every term is of degree n, is called a homogenous function of degree n in x and y.

Euler's theorem : If u is a homogenous function of degree 'n' in x and y then

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$$

For any number of variables,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}+..........+w\frac{\partial u}{\partial w}=nu$$

Example 8. If u be a homogenous function of degree n, then show that

(i) $x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=(n-1)\frac{\partial u}{\partial x}$ (ii) $x\frac{\partial^2 u}{\partial x\partial y}+y\frac{\partial^2 u}{\partial y^2}=(n-1)\frac{\partial u}{\partial y}$

(iii) $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u$

Soln: Since u is a homogeneous function of degree n, hence by Euler's theorem, we have

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu \qquad \text{... (i)}$$

(i) Differentiating both sides partialy with respect to x, we get

$$x\frac{\partial^2 u}{\partial x^2}+1.\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=n\frac{\partial u}{\partial x} \Rightarrow x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=(n-1)\frac{\partial u}{\partial x} \qquad \text{... (ii)}$$

(ii) Again differentiating (i) partially w.r.t. y

$$x\frac{\partial u}{\partial y\partial x}+y\frac{\partial u}{\partial y^2}+1.\frac{\partial u}{\partial y}=n\frac{\partial u}{\partial y} \Rightarrow x\frac{\partial^2 u}{\partial x\partial y}+y\frac{\partial^2 u}{\partial y^2}=(n-1)\frac{\partial u}{\partial y} \qquad \text{... (iii)}$$

(iii) Mutiplying (ii) by x and (iii) by y and adding,

$$x\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=(n-1)\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)=n(n-1)u,$$

Example 9. If $u=\tan^{-1}\frac{x^3+y^3}{x-y}$, prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u$

Soln: We have $u=\tan^{-1}\frac{x^3+y^3}{x-y}$, *i.e*, $\tan u=\frac{x^3+y^3}{x-y}$

Let $z=\tan u=\frac{x^3+y^3}{x-y}$

Hence z is a homogeneous function of degree (3-1)=2; $\Rightarrow x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2z$

$$\Rightarrow x\frac{\partial}{\partial x}(\tan u)+y\frac{\partial}{\partial y}(\tan u)=2\tan u \Rightarrow x\sec^2 u\frac{\partial u}{\partial x}+y\sec^2 u\frac{\partial u}{\partial y}=2\tan u$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{2\tan u}{\sec^2 u}=2\sin u\cos u=\sin 2u$$

Note:

(1) If $u=f(x_1,x_2,x_3,.....x_n)$ and $x_1,x_2,x_3,........x_n$ are all functions of t, then

$$\frac{du}{dt}=\frac{\partial u}{\partial x_1}.\frac{dx_1}{dt}+\frac{\partial u}{\partial x_2}.\frac{dx_2}{dt}+\frac{\partial u}{\partial x^3}.\frac{dx_3}{dt}+....+\frac{\partial u}{\partial x_n}.\frac{dx_n}{dt}$$

(2) If $u = f(x_1, x_2, x_3, ...x_n)$ and $x_2, x_3, ...x_n$ are all functions of x_1, then

$$\frac{du}{dx_1} = \frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2}.\frac{dx_2}{dx_1} + ... + \frac{\partial u}{\partial x_n}.\frac{dx_n}{dx_1}$$

(3) If $u = f(x_1, x_2, x_3, ...x_n)$, then

$$du = \frac{\partial u}{\partial x_1}dx_1 + \frac{\partial u}{\partial x_2}dx_2 + \frac{\partial u}{\partial x_3}dx_3 + ...\frac{\partial u}{\partial x_n}dx_n$$

(4) If x and y are related by the equation $f(x, y) = \text{constant}$, then

$$\frac{dy}{dx} = -\frac{(\partial f / \partial x)}{(\partial f / \partial y)}$$

Example 10. Find $\frac{dz}{dt}$ when $z = xy^2 + x^2y, x = at^2, y = 2at$

Soln: We have, $z = xy^2 + x^2y \Rightarrow \frac{\partial z}{\partial x} = y^2 + 2xy$ and $\frac{\partial z}{\partial y} = 2xy + x^2$

Also, $x = at^2, y = 2at \Rightarrow \frac{dx}{dt} = 2at, \frac{dy}{dt} = 2a$

Hence, $\frac{dz}{dt} = \frac{\partial z}{\partial x}.\frac{dx}{dt} + \frac{\partial z}{\partial y}.\frac{dy}{dt} = (y^2 + 2xy)(2at) + (2xy + x^2)2a$

$= (4a^2t^2 + 2at^2.2at)2at + (2at^2.2at + a^2t^4)2a$

$= 8a^3t^3 + 8a^3t^4 + 8a^3t^3 + 2a^4t^3 = 16a^3t^3 + 8a^3t^4 + 2a^4t^3$

Example 11. If $u = 4x^2 - y^2, x = e^t \cos t, y = 2e^t \sin t$, find $\frac{du}{dt}$.

Soln: We have, $u = 4x^2 - y^2 \Rightarrow \frac{\partial u}{\partial x} = 8x$ and $\frac{\partial u}{\partial y} = -2y$

Also $\quad x = e^t \cot t, \qquad y = 2e^t \sin t$

Hence, $\frac{dx}{dt} = e^t \cos t - e^t \cdot \sin t, \frac{dy}{dt} = 2e^t \sin t + 2e^t \cos t$

Therefore, $\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt} = 8x(e^t \cos t - \sin t\, e^t) + (-2y).2e^t(\sin t + \cos t)$

$= 8e^t \cos t(e^t \cos t - \sin t\, e^t) - 4e^t(2e^t \sin t)(\sin t + \cos t) = 8e^{2t}(\cos 2t - \sin 2t)$

Example 12. Find $\frac{du}{dx}$ if $u = \sin(x^2 + y^2)$, where $a^2x^2 + b^2y^2 = c^2$

Soln: We have, $u = \sin(x^2 + y^2) \Rightarrow \frac{\partial u}{\partial x} = \cos(x^2 + y^2).2x$ and $\frac{\partial u}{\partial y} = \cos(x^2 + y^2).2y$

Given: $a^2x^2+b^2y^2=c^2 \Rightarrow 2a^2x+2b^2y\dfrac{dy}{dx}=0 \;\Rightarrow\; \dfrac{dy}{dx}=-\dfrac{a^2x}{b^2y}$

$$\Rightarrow \frac{du}{dx}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}.\frac{dy}{dx}=2x\cos\left(x^2+y^2\right)+2y\cos\left(x^2+y^2\right)\left(-\frac{a^2x}{b^2y}\right)=2x\cos\left(x^2+y^2\right)\left(1-\frac{a^2x}{b^2y}\right)$$

Example 13. If $x^y+y^x=c$, find the value of $\dfrac{dy}{dx}$

Soln: Let $f(x,y)=x^y+y^x$

Then, $\dfrac{\partial f}{\partial x}=yx^{y-1}+y^x\log_e y$; Similarly, $\dfrac{\partial f}{\partial y}=x^y\log_e x+xy^{x-1}$

$$\therefore \qquad \frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial y}=-\frac{y^x\log_e y+y^{y-1}}{x^y\log_e x+x\,y^{x-1}}$$

Chapter 11

JACOBIAN

Definition: If $u_1, u_2, u_3, \ldots\ldots, u_n$ be the functions of n variables $x_1, x_2, x_3, \ldots\ldots\ldots, x_n$ then the Jacobian of $u_1, u_2, \ldots\ldots u_n$ with respect to $x_1, x_2, x_3, \ldots\ldots x_n$ is

$$\frac{\partial(u_1, u_2, u_3, \ldots\ldots\ldots, u_n)}{\partial(x_1, x_2, x_3, \ldots\ldots\ldots, x_n)} = \mathrm{J}(u_1, u_2, u_3, \ldots\ldots u_n) = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_4}{\partial x_3} & \cdots & \frac{\partial u_3}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \frac{\partial u_n}{\partial x_3} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

Example 1. If $x = r\cos\theta$, $y = r\sin\theta$, then find $\frac{\partial(x, y)}{\partial(r, \theta)}$ and $\frac{\partial(r, \theta)}{\partial(x, y)}$

Soln: $$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\left(\cos^2\theta + \sin^2\theta\right) = r$$

Again we have, $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\frac{y}{x}$

$$\frac{\partial r}{\partial x} = \frac{1}{2}\left(x^2 + y^2\right)^{-1/2} .2x = \cos\theta \ ; \qquad \frac{\partial r}{\partial y} = \frac{1}{2}\left(x^2 + y^2\right)^{-1/2} .2y = \sin\theta \ ;$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1 + \frac{y^2}{x^2}} . \left(-\frac{y}{x^2}\right) = -\frac{\sin\theta}{r} \ ; \qquad \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}$$

$$\therefore \quad \frac{\partial(r, \theta)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{vmatrix} = 1/r\left(\cos^2\theta + \sin^2\theta\right) = 1/r$$

Example 2. If $u_1 = \dfrac{x_2x_3}{x_1}$, $u_2 = \dfrac{x_3x_1}{x_2}$ and $u_3 = \dfrac{x_1x_2}{x_3}$, then prove that, $J(u_1,u_2,u_3) = 4$

Soln: We have $J(u_1,u_2,u_3) = \begin{vmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1}{\partial x_2} & \dfrac{\partial u_1}{\partial x_3} \\ \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_2}{\partial x_2} & \dfrac{\partial u_2}{\partial x_3} \\ \dfrac{\partial u_3}{\partial x_1} & \dfrac{\partial u_3}{\partial x_2} & \dfrac{\partial u_3}{\partial x_3} \end{vmatrix} = \begin{vmatrix} -\dfrac{x_2x_3}{x_1^2} & \dfrac{x_3}{x_1} & \dfrac{x_2}{x_1} \\ \dfrac{x_3}{x_2} & -\dfrac{x_3x_1}{x_2^2} & \dfrac{x_1}{x_2} \\ \dfrac{x_2}{x_3} & \dfrac{x_1}{x_3} & -\dfrac{x_1x_2}{x_3^2} \end{vmatrix}$

$$= \frac{1}{x_1^2x_2^2x_3^2}\begin{vmatrix} -x_2x_3 & x_3x_1 & x_1x_2 \\ x_2x_3 & -x_3x_1 & x_1x_2 \\ x_2x_3 & x_3x_1 & x_1x_2 \end{vmatrix} = \frac{(x_2x_3)(x_3x_1)(x_1x_2)}{x_1^2x_2^2x_3^2}\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = 4$$

Example 3. *If* $y_1 = (1-x_1)$, $y_2 = x_1(1-x_2)$, $y_3 = x_1x_2(1-x_3)$,....

$y_n = x_1x_2...x_{n-1}(1-x_n)$, then prove that $\dfrac{\partial(y_1,y_2,....,y_n)}{\partial(x_1,x_2,.....x_n)} = (-1)^n x_1^{n-1}x_2^{n-2}.....x_{n-1}$

Soln: $$\frac{\partial(y_1,y_2,....,y_n)}{\partial(x_1,x_2,.....,x_n)} = \frac{\partial y_1}{\partial x_1}.\frac{\partial y_2}{\partial x_2}.....\frac{\partial y_n}{\partial x_n} = (-1).(-x_1).(-x_1x_2).....(-x_1x_2.....x_{n-1})$$

$$= (-1)^n x_1^{n-1}.x_2^{n-2}......x_{n-1}$$

Jacobian of Implicit functions:

If $u_1,u_2,u_3,......u_n$ and $x_1,x_2,.......x_n$ are implicitly connected by n equations as:

$f_1(u_1,u_2,........u_n,x_1,x_2,......,x_n) = 0$

$f_2(u_1,u_2,.....,u_n,x_1,x_2,.....x_n) = 0$

....

....

$f_n(u_1,u_2,.....,u_n,x_1,x_2,......,x_n) = 0$

then $$\frac{\partial(f_1,f_2,......f_n)}{\partial(u_1,u_2,.....u_n)}.\frac{\partial(u_1,u_2,......,u_n)}{\partial(x_1,x_2,....,x_n)} = (-1)^n\frac{\partial(f_1,f_2,....,f_n)}{\partial(x_1,x_2,....x_n)}$$

Note : The above result is a generalised result of the elementary theorem $\dfrac{dy}{dx} = -\dfrac{\partial f/\partial x}{\partial f/\partial y}$

Example 4. If $x+y+z=u$, $y+z=uv$, $z=uvw$, find the value of the Jacobian of x, y, z with respect to u, v, w.

Soln: Let, $f_1 = x+y+z-u = 0$

$$f_2 = y+z-uv = 0$$

$$f_3 = z-uvw = 0$$

Then, $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = (-1)^3 \dfrac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)} \div \dfrac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}$

$$= -\begin{vmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial v} & \dfrac{\partial f_1}{\partial w} \\ \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial v} & \dfrac{\partial f_2}{\partial w} \\ \dfrac{\partial f_3}{\partial u} & \dfrac{\partial f_3}{\partial v} & \dfrac{\partial f_3}{\partial w} \end{vmatrix} \div \begin{vmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} & \dfrac{\partial f_1}{\partial z} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} & \dfrac{\partial f_2}{\partial z} \\ \dfrac{\partial f_3}{\partial x} & \dfrac{\partial f_3}{\partial y} & \dfrac{\partial f_3}{\partial z} \end{vmatrix} = -\begin{vmatrix} -1 & 0 & 0 \\ -v & -u & 0 \\ -vw & -uw & -uv \end{vmatrix} \div \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= -(-1)(-u)(-uv) \div (1)(1)(1) = u^2v$$

Jacobian of function of function:

Theorem : If $u_1, u_2, u_3, \ldots\ldots u_n$ are functions of $y_1, y_2, y_3, \ldots, y_n$ and $y_1, y_2, y_3, \ldots, y_n$ are functions of $x_1, x_2, x_3, \ldots\ldots x_n$ then,

$$\frac{\partial(u_1,u_2,\ldots,u_n)}{\partial(x_1,x_2,\ldots,x_n)} = \frac{\partial(u_1,u_2,\ldots,u_n)}{\partial(y_1,y_2,\ldots,y_n)} \cdot \frac{\partial(y_1,y_2,\ldots,y_n)}{\partial(x_1,x_2,\ldots,x_n)}$$

Example 5. If $u^3+v^3+w^3 = x+y+z$, $u^2+v^2+w^2 = x^3+y^3+z^3$, $u+v+w = x^2+y^2+z^2$, then show that

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{(y-z)(z-x)(x-y)}{(u-v)(v-w)(w-u)}$$

Soln: Let, $f_1 \equiv u^3+v^3+w^3-x-y-z=0$, $f_2 \equiv u^2+v^2+w^2-x^3-y^3-z^3=0$

and $f_3 \equiv u+v+w-x^2-y^2-z^2=0$

We have, $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \cdot \dfrac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)} \div \dfrac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}$

$$= -\begin{vmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} & \dfrac{\partial f_1}{\partial z} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} & \dfrac{\partial f_2}{\partial z} \\ \dfrac{\partial f_3}{\partial x} & \dfrac{\partial f_3}{\partial y} & \dfrac{\partial f_3}{\partial z} \end{vmatrix} \div \begin{vmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial v} & \dfrac{\partial f_1}{\partial w} \\ \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial v} & \dfrac{\partial f_2}{\partial w} \\ \dfrac{\partial f_3}{\partial u} & \dfrac{\partial f_3}{\partial v} & \dfrac{\partial f_3}{\partial w} \end{vmatrix} = -\begin{vmatrix} -1 & -1 & -1 \\ -3x^2 & -3y^2 & -3z^2 \\ -2x & -2y & -2z \end{vmatrix} \div \begin{vmatrix} 3u^2 & 3v^2 & 3w^2 \\ 2u & 2b & 2w \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 6\begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x & y & z \end{vmatrix} \div 6\begin{vmatrix} u^2 & v^2 & w^2 \\ u & v & w \\ 1 & 1 & 1 \end{vmatrix} = \frac{(y-z)(z-x)(x-y)}{(u-v)(u-w)(w-u)}$$

Theorem: Let $u_1, u_2, \ldots\ldots, u_n$ be functions of n independent variables $x_1, x_2, \ldots\ldots x_n$. The necessary and sufficient condition that the function be connected by a relation $f(u_1, u_2, \ldots\ldots\ldots, u_n) = 0$ is that the Jacobian $\frac{\partial(u_1, u_2, \ldots\ldots\ldots, u_n)}{\partial(x_1, x_2, \ldots, x_n)}$ vanishes identically.

Example 6. Show that the functions: $u = 3x + 2y - z$, $v = x - 2y + z$ and $w = x(x + 2y - z)$ are not independent, and find the relation between them.

Soln: If the given functions u, v, w are not independent, then we must have $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ equal to zero, or

$$J = \begin{vmatrix} 3 & 2 & 1 \\ 1 & -2 & -1 \\ (2x+2y-z) & 2x & -x \end{vmatrix} = 0$$

Adding R_2 to R_1, we get

Since, $J = 0$, hence the functions are not independent. Now we have to find the relation between u, v, w.

Clearly, $u + v = 4x$ and $u - v = 2(x + 2y - z)$

$$\therefore \quad (u+v).(u-v) = 8x(x+2y-z) = 8w \Rightarrow u^2 - v^2 = 8w$$

Example 7. If $u = \frac{x+y}{z}$, $v = \frac{y+z}{x}$, $w = \frac{y(x+y+z)}{xz}$, show that u, v, w are not independent and find the relation between them.

Soln: We have

$$J = \begin{vmatrix} \frac{1}{z} & \frac{1}{z} & -\frac{(x+y)}{z^2} \\ -\frac{(y+z)}{x^2} & \frac{1}{x} & \frac{1}{x} \\ -\frac{y^2-yz}{x^2z} & \frac{x+2y+z}{xz} & \frac{-xy-y^2}{xz^2} \end{vmatrix}$$

taking, $\frac{1}{z^2}, \frac{1}{x^2}$ and $\frac{1}{x^2z^2}$ common from R_1, R_2 and R_3 respectively, we get

$$J = \frac{1}{x^4z^4}\begin{vmatrix} z & z & -(x+y) \\ -(y+z) & x & x \\ -(y^2z+yz^2) & \frac{x+2y+z}{xz} & \frac{-xy-y^2}{xz^2} \end{vmatrix}$$

taking $\frac{1}{z^2}, \frac{1}{x^2}$ and $\frac{1}{x^2z^2}$ common from R_1, R_2 and R_3 respectively, we get

$$J = \frac{1}{x^4z^4}\begin{vmatrix} z & z & -(x+y) & \\ -(y+z) & x & x & z \\ -(y^2z+yz^2) & x^2z+2xyz+xz^2 & -(x^2y+xy^2) & \end{vmatrix}$$

Applying $C_2 - C_1$ and $C_3 - C_1$,

$$= \frac{1}{x^4 z^4} \begin{vmatrix} z & 0 & -(x+y+z) \\ -(y+z) & (x+y+z) & (x+y+z) \\ -yz(y+z) & z(x^2+y^2+2xy)+z^2(x+y) & -xy(x+y)+yz(y+z) \end{vmatrix}$$

$$= \frac{1}{x^4 z^4} \begin{vmatrix} z & 0 & -(x+y+z) \\ -(y+z) & (x+y+z) & (x+y+z) \\ -yz(y+z) & z(x+y)(z+y+z) & (x+y+z)(yz-xy) \end{vmatrix}$$

$$= \frac{(x+y+z)^2}{x^4 y^4} \begin{vmatrix} z & 0 & -1 \\ -(y+z) & 1 & 1 \\ -yz(y+z) & z(x+y) & (yz-xy) \end{vmatrix}$$

Applying $C_1 + zC_3$, we get

$$= \frac{(x+y+z)^2}{x^4 z^4}(-1) \begin{vmatrix} -y & 1 \\ -yz(y+x) & z(x+y) \end{vmatrix} = \frac{(x+y+z)^2}{x^4 z^4}\left[-yz(x+y)+yz(x+y)\right] = 0$$

Since $J = 0$, hence the given functions are not independent.

Again. $uv = \frac{xy+y^2+yz+zx}{zx} = \frac{y(x+y+z)}{xz} + 1 = w+1$

$\therefore$ $uv = w+1$ is the required relation between them.

Example 8. Consider the integral $\iiint\limits_V f(x,y,z)\,dV_1 = \iiint\limits_V f(r,\theta,\phi)\,dV$, if $dV_1 = dx\,dy\,dz$ and $dV_2 = J\,dr\,d\theta\,d\phi$, then the value of J is

(a) $r\sin\theta$ (b) $r^2\sin\theta$ (c) $r\sin\phi$ (d) $r^2\sin\phi$

Soln: Here, J is the Jacobian from Cartesian (x, y, z) to spherical polar (r, θ, ϕ)

We have, $x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta & \partial x/\partial\phi \\ \partial y/\partial r & \partial y/\partial\theta & \partial y/\partial\phi \\ \partial z/\partial r & \partial r/\partial\theta & \partial r/\partial\phi \end{vmatrix} = \begin{vmatrix} \cos\theta\sin\phi & -r\sin\theta\sin\phi & r\cos\theta\cos\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\phi & 0 & -r\sin\phi \end{vmatrix}$$

On solving, this gives $= r^2\sin\theta$. Therefore, $J = r^2\sin\theta$.

Correct option is (b)

Example 9. If $z_1 = \frac{\omega_2\,\omega_3}{\omega_1}$, $z_2 = \frac{\omega_1\,\omega_3}{\omega_2}$, $z_3 = \frac{\omega_1\,\omega_2}{\omega_3}$, then the Jacobian ($J$) from $(z_1\ z_2\ z_3)$ to $(\omega_1\ \omega_2\ \omega_3)$ is equal to

(a) 3 (b) 2 (c) 4 (d) 5

Soln: $$J=\frac{\partial(z_1\, z_2\, z_3)}{\partial(\omega_1\, \omega_2\, \omega_3)}=\begin{vmatrix}\partial z_1/\partial\omega_1 & \partial z_1/\partial\omega_2 & \partial z_1/\partial\omega_3\\ \partial z_2/\partial\omega_1 & \partial z_2/\partial\omega_2 & \partial z_2/\partial\omega_3\\ \partial z_3/\partial\omega_1 & \partial z_3/\partial\omega_2 & \partial z_3/\partial\omega_3\end{vmatrix}$$

$$=\begin{vmatrix}-\omega_2\,\omega_3/\omega_1^2 & \omega_3/\omega_1 & \omega_2/\omega_1\\ \omega_3/\omega_2 & -\omega_1\,\omega_3/\omega_2^2 & \omega_1/\omega_2\\ \omega_2/\omega_3 & \omega_1/\omega_3 & -\omega_1\,\omega_2/\omega_3^2\end{vmatrix}$$

$$=\frac{1}{\omega_1^2\,\omega_2^2\,\omega_3^2}\begin{vmatrix}-\omega_2\,\omega_3 & \omega_1\,\omega_3 & \omega_1\,\omega_2\\ \omega_2\,\omega_3 & -\omega_1\,\omega_3 & \omega_1\,\omega_2\\ \omega_2\,\omega_3 & \omega_1\,\omega_3 & -\omega_1\,\omega_2\end{vmatrix}$$

$$=\frac{(\omega_2\,\omega_3)\,(\omega_1\,\omega_3)\,(\omega_1\,\omega_2)}{\omega_1^2\,\omega_2^2\,\omega_3^2}\begin{vmatrix}-1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & -1\end{vmatrix}=4$$

Therefore, $J=4$.

Correct option is (c)

Example 10. If $u=x^2-y^2,\ v=2xy,\ x=r\cos\theta,\ y=r\sin\theta$, then the value of Jacobian $J=\dfrac{\partial(u,v)}{\partial(r,\theta)}$ is

(a) $\frac{1}{2}\left(x^2+y^2\right)^{3/2}$ (b) $2\left(x^2+y^2\right)^{1/2}$ (c) $4\left(x^2+y^2\right)^{3/2}$ (d) $4\left(x^2+y^2\right)^{1/2}$

Soln: We have $J=\dfrac{\partial(u,v)}{\partial(r,\theta)}=\dfrac{\partial(u,v)}{\partial(x,y)}\times\dfrac{\partial(x,y)}{\partial(r,\theta)}$

Here, $\dfrac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\partial u/\partial x & \partial u/\partial y\\ \partial v/\partial x & \partial v/\partial y\end{vmatrix}=\begin{vmatrix}2x & -2y\\ 2y & 2x\end{vmatrix}=4\left(x^2+y^2\right)$

and $\dfrac{\partial(x,y)}{\partial(r,\theta)}=\begin{vmatrix}\partial x/\partial r & \partial x/\partial\theta\\ \partial y/\partial r & \partial y/\partial\theta\end{vmatrix}=\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}=r$

Therefore, $J=4\left(x^2+y^2\right)r=4\left(x^2+y^2\right)^{3/2}$.

Correct option is (c)

Example 11. If $u=xyz,\ v=x^2+y^2+z^2,\ w=x+y+z$, then the value of Jacobian, $J=\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$ is

(a) $-2\,(x-y)\,(y-z)\,(z-x)$

(b) $-\dfrac{1}{2\,(x-y)\,(y-z)\,(z-x)}$

(c) $-\dfrac{(x-y)\,(y-z)\,(z-x)}{2}$

(d) None of these

Soln: $J=\dfrac{\partial(u,v,w)}{\partial(x,y,z)}=\begin{vmatrix}\partial u/\partial x & \partial u/\partial y & \partial u/\partial z\\ \partial v/\partial x & \partial v/\partial y & \partial v/\partial z\\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z\end{vmatrix}=\begin{vmatrix}yz & xz & xy\\ 2x & 2y & 2z\\ 1 & 1 & 1\end{vmatrix}$

$$=\begin{vmatrix}z(y-x) & x(z-y) & xy\\ -2(y-x) & -2(z-y) & 2z\\ 0 & 0 & 1\end{vmatrix}\quad [C_1\to C_1-C_2 \text{ and } C_2\to C_2-C_3]$$

$$=(y-x)(z-y)\begin{vmatrix}z & x & xy\\ -2 & -2 & 2z\\ 0 & 0 & 1\end{vmatrix}$$

$$=(x-y)(y-z)\{-2z+2x\}$$

$$=-2(x-y)(y-z)(z-x)$$

Therefore, $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}=-\dfrac{1}{2(x-y)(y-z)(z-x)}$.

Correct option is (b)

Example 12. Let x and y are Cartesian variable which transformed to another variable u and v such that $x=2u+3v$ and $y=2u-3v$, then the value of $|J|$ is __________ . ($|J|$ denotes the modulus of J).

Soln: $J=\dfrac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\partial x/\partial u & \partial x/\partial v\\ \partial y/\partial u & \partial y/\partial v\end{vmatrix}=\begin{vmatrix}2 & 3\\ 2 & -3\end{vmatrix}=-6-6=-12$

Hence, $|J|=12$.

Correct answer is (12)

Chapter 12

DIRAC DELTA , BETA & GAMMA FUNCTION

Dirac Delta function:

$$\delta(x-a)=0 \quad x\neq a$$
$$=\infty \quad x=a \quad (\text{very high})$$

Such that $\int_{-\infty}^{\infty}\delta(x-a)\,dx=1$ i.e. area under the curve should be equal to unity.

Note: If x has a dimension of length, then Dirac delta function will have a dimension of $(\text{length})^{-1}$.

Some representations of Dirac Delta function:

(1) $\lim\limits_{\sigma\to 0} R_\sigma(x)=\delta(x-a)$ where rectangular function $R_\sigma(x)=\dfrac{1}{2\sigma} \quad a-\sigma<x<a+\sigma$

$=0$ otherwise

(2) $\lim\limits_{\sigma\to 0} G_\sigma(x)=\delta(x-\mu)$ where Gaussian function $G_\sigma(x)=\dfrac{1}{\sqrt{2\pi}\,\sigma}\exp\left[\dfrac{-(x-\mu)^2}{2\sigma^2}\right]$

(3) $\underset{\varepsilon\to 0}{Lim}\, L_\sigma(x)=\delta(x-x_0)$ where Lorentzian function $L_\sigma(x)=\dfrac{1}{\pi}\dfrac{\varepsilon}{(x-x_0)^2+\varepsilon^2}$

(4) Integral representation: $\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} e^{\pm ik(x-x_0)}dk=\delta(x-x_0)$

Some properties of Dirac Delta function:

(1) $\int_{-\infty}^{\infty} f(x)\delta(x-a)dx=f(a)$

(2) $\int_{-\infty}^{\infty} f'(x)\delta(x-a)dx=-f'(a)$

(3) $\delta(x-a)=\delta(x-a)$

(4) $\delta\left[c(x-a)\right]=\dfrac{1}{|c|}\delta(x-a) \quad (c \text{ is constant})$

(5) $\delta(x^2-a^2)=\dfrac{1}{|2a|}\left[\delta(x-a)+\delta(x+a)\right]$

(6) $\int_{-\infty}^{\infty}\delta(x-a)\delta(x-b)dx=\delta(a-b)$

(7) $F[\delta(x-a)]=\frac{1}{\sqrt{2\pi}}e^{isa}$

(8) $L[\delta(x-a)]=e^{-sa}$

Example 1: Evaluate the following integrals

Soln. (i) $\int_0^{\pi}\cos x\,\delta(x-\pi)\,dx=\cos\pi=-1$

(ii) $\int_{-2}^{2}x^3\delta(x+1)\,dx=(-1)^3=-1$

(iii) $\int_{-2}^{2}(2x+3)\delta(3x)\,dx=\frac{1}{3}\int_{-2}^{2}(2x+3)\delta(x)\,dx=1$

(iv) $\int_0^{2}(x^3+3x+2)\delta(x-1)\,dx=2$

(v) $\int_{-\infty}^{\infty}\cos x\,\delta(x^2-\pi^2)\,dx=\int_{-\infty}^{\infty}\cos x\frac{1}{|2\pi|}[\delta(x-\pi)+\delta(x+\pi)]\,dx=\frac{1}{2\pi}[\cos\pi+\cos(-\pi)]=-\frac{1}{\pi}$

Representation of Dirac delta function in 3-D:

(1) Cartesian Co-ordinates: $\delta^3(\vec{r}-\vec{r}_0)=\delta(x-x_0)\delta(y-y_0)\delta(z-z_0)$

(2) Spherical polar Co-ordinates: $\delta^3(\vec{r}-\vec{r}_0)=\frac{1}{r^2\sin\theta}\delta(r-r_0)\delta(\theta-\theta_0)\delta(\varphi-\varphi_0)$

(3) Cylindrical Co-ordinates: $\delta^3(\vec{r}-\vec{r}_0)=\frac{1}{\rho}\delta(\rho-\rho_0)\delta(\varphi-\varphi_0)\delta(z-z_0)$

Gamma Function: $\int_0^{\infty}e^{-x}x^{n-1}dx=\Gamma n$

Some important relations:

(1) $\Gamma(n+1)=n\Gamma n=n!$ (2) $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$

(3) Gamma of zero and negative integers is not defined.

Example 2: Evaluate the following integral $\int_0^{\infty}x^2\,e^{-2x^2}dx$

Soln. $\int_0^{\infty}x^2\,e^{-2x^2}dx$, assuming $2x^2=t\Rightarrow 4x\,dx=dt\quad\Rightarrow dx=\frac{dt}{4x}=\frac{1}{2\sqrt{2t}}dt$,\

we get $I=\frac{1}{4\sqrt{2}}\int_0^{\infty}e^{-t}.t^{1/2}dt=\frac{1}{4\sqrt{2}}\Gamma\left(\frac{3}{2}\right)=\frac{1}{8\sqrt{2}}\sqrt{\pi}$

Beta function:

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \beta(m,n)$$

Some important relations:

(1) $\beta(m,n) = \dfrac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$ (2) $\int_0^{\pi/2} \sin^p\theta\cos^q\theta\,d\theta = \dfrac{1}{2}\beta\left(\dfrac{p+1}{2},\dfrac{q+1}{2}\right) = \dfrac{1}{2}\dfrac{\Gamma\left(\dfrac{p+1}{2}\right).\Gamma\left(\dfrac{q+1}{2}\right)}{\Gamma\left(\dfrac{p+q+2}{2}\right)}$

(where p and q are positive integers of fractions)

(3) $\Gamma x.\Gamma(1-x) = \dfrac{\pi}{\sin\pi x}$

Example 3: Evaluate the integral $\int_0^{\pi/2}\sqrt{\tan\theta}\,d\theta$

Soln. $\int_0^{\pi/2}\sqrt{\tan\theta}\,d\theta = \int_0^{\pi/2}\sin^{1/2}\theta\cos^{-1/2}\theta\,d\theta = \dfrac{1}{2}\dfrac{\Gamma\left(\dfrac{3}{4}\right).\Gamma\left(\dfrac{1}{4}\right)}{\Gamma(1)} = \dfrac{1}{2}\dfrac{\pi}{\sin\dfrac{\pi}{4}} = \dfrac{\pi}{\sqrt{2}}$

Example 4: The value of $\int_0^3 t^2\delta(3t-6)\,dt$ is______ (upto one decimal place)

Soln. $\int\limits_0^3 t^2\delta(3t-6)\,dt\int\limits_0^3 t^2\delta\left[3(t-2)\right]dt = \dfrac{1}{3}\int\limits_0^3 t^2\delta(t-2)\,dt = \dfrac{4}{3} = 1.3$

Example 5: The value of the integral $\int\limits_0^\infty dx\,x^9\exp(-x^2)$ is **[TIFR 2013]**

(a) 20160 (b) 12 (c) 18 (d) 24

Soln. $\int\limits_0^\infty dx\,x^9 e^{-x^2}$ [put $x^2 = z \Rightarrow 2xdx = dz$]

$$= \int\limits_0^\infty e^{-z}\cdot z^4\cdot\frac{dz}{2}$$

$$= \frac{1}{2}\int\limits_0^\infty e^{-z}\cdot z^4\cdot dz$$ [Using Gamma integral]

$$= \frac{1}{2}\Gamma 5 = 12$$ $$\int\limits_0^\infty e^{-p}\cdot p^{n-1}dp = \Gamma n$$

Correct option is (b)

Example 6: The integral $\int_{-\infty}^{\infty} dx\,\delta(x^2-\pi^2)\cos x$, evaluates to **[TIFR 2013]**

(a) -1 (b) 0 (c) $\frac{1}{\pi}$ (d) $-\frac{1}{\pi}$

Soln. $\int_{-\infty}^{\infty} dx\delta\left(x^2-\pi^2\right)\cos x dx$

Applying the property, $\delta\left(x^2-a^2\right)=\frac{1}{|2a|}\left[\delta\left(x+a\right)+\delta\left(x-a\right)\right]$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty} dx\left[\delta\left(x+\pi\right)+\delta\left(x-\pi\right)\right]\cos x dx$$

Applying the property, $\int_{-\infty}^{\infty} f\left(x\right)\delta\left(x-a\right)dx=f\left(a\right)$

$$=\frac{1}{2\pi}\left[\cos\pi+\cos\left(-\pi\right)\right]=-\frac{1}{\pi}$$

Correct option is (d)

Example 7: The Dirac delta function $\delta\left(x\right)$ satisfies the relation $\int_{-\infty}^{\infty} f\left(x\right)\delta\left(x\right)dx=f\left(0\right)$ for a well behaved function $f\left(x\right)$. If x has the dimension of momentum then **[JEST 2014]**

(a) $\delta\left(x\right)$ has the dimension of momentum (b) $\delta\left(x\right)$ has the dimension of (mometum)^2

(c) $\delta\left(x\right)$ is dimensionless (d) $\delta\left(x\right)$ has the dimension of (momentum)$^{-1}$.

Soln. $\int_{-\infty}^{\infty} f\left(x\right)\delta\left(x\right)dx=f\left(0\right)$

Since, $f\left(x\right)$ and $f\left(0\right)$ have same dimension, then $\delta\left(x\right)dx$ should be dimensionless.

If x has dimension of momentum, then $\delta\left(x\right)$ has a dimension of (momentum)$^{-1}$

Correct option is (d)

MECHANICS

for

M.Sc. Entrance

IIT-JAM | JNU | TIFR | BHU | ISM | JNCASR | DU

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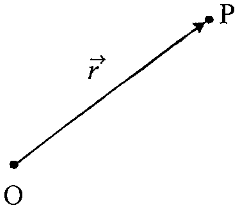
Chapter 1

Velocity and Accceleration in Different Coordinate system

In physics basic laws are first introduced for a point partile and then laws are extended to system of particles or continuous bodies. Therefore, we also begin the discussion with point particle and later on we will study collection of particles or rigid body.

To write equations governing the dynamics of a aprticle we need its position vectors, velocity, acceleration etc. Therefore we first introduce these elementaty concepts.

Position vector:
It is a vector directed from some point to location of particle. In the figure shown $\vec{r}$ is position vector of particle 'P' with respect to point 'O'. If we specify the coordinate of particle then position vector can be expressed in terms of coordinates and unit vectors used in that coordinate system.



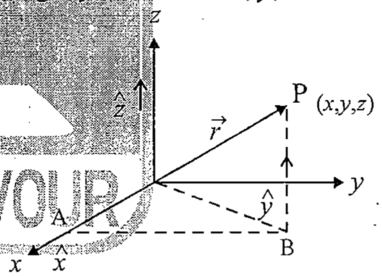
In cartesian coordinate system:
Coordinates of particle are written as (x, y, z) and unit vectors along x, y, z axes are $\hat{x}, \hat{y}$, and $\hat{z}$ respectively. Therefoer, from figure,

$$OA = x,\ AB = y,\ BP = z$$

and $$\vec{OP} = \vec{OB} + \vec{BP}$$

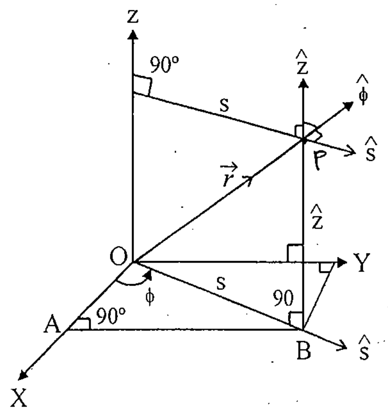
$$\vec{r} = \vec{OA} + \vec{AB} + \vec{BP} = x\hat{x} + y\hat{y} + z\hat{z}$$

$$\therefore \quad \boxed{\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}}$$



Unit vectors are taken in the directions in which coordinates increase.

In cylindrical system coordinates of particle are written as (s, ϕ, z) and unit vectors along the increasing direction of coordinates are $(\hat{s}, \hat{\phi}, \hat{z})$. s is perpendicular distance of particle from z-axis, ϕ is its angular position with respect to x-axis and z is its distance above x-y plane.



Therefore, from figure $OB = s$, $\angle AOB =$, $BP = z$ and $\vec{OP} = \vec{OB} + \vec{BP}$

$$\boxed{\vec{r} = s\hat{s} + z\hat{z}}$$

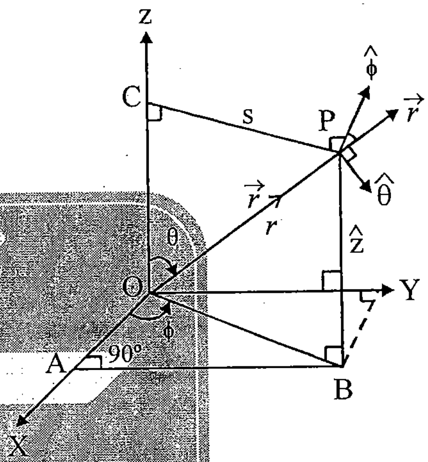
relation with cartesian coordinates

$$\boxed{x = s\cos\phi,\ y = s\sin\phi,\ z = z} \text{ and } \boxed{\hat{s} = \cos\phi\hat{x} + \sin\phi\hat{y},\ \hat{\phi} = -\sin\phi\hat{x} + \cos\phi\hat{y},\ z = \hat{z}}$$

In spherical polar coordinates system, coordinates of particle are written as (r,θ,ϕ) and unit vector in increasing direction of coordinates are $(\hat{r},\hat{\theta} \text{ and } \hat{\phi})$. r is the distance of particle from origin, θ and ϕ are angular position with respect to z and x axes respectively.

From figure $OP = r$, $\angle COP = \theta$, $\angle AOB = \phi$ and $\vec{OP} = r\hat{r}$

$\therefore \quad \boxed{\vec{r} = r\hat{r}}$



$OB = CP = r\sin\theta, \qquad BP = r\cos\theta$

$\therefore \qquad AB = OB\sin\phi = r\sin\theta.\sin\phi$ and

$OA = OB\cos\phi = r\sin\theta\cos\phi$

Relation with cartesian coordinate,

$$\boxed{x = r\sin\theta.\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta}$$

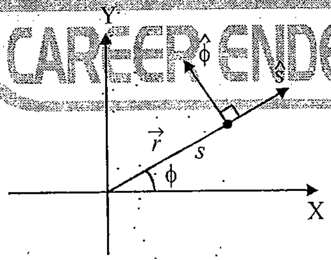
And $\quad \hat{r} = \sin\theta(\cos\phi\hat{x} + \sin\phi\hat{y}) + \cos\theta\hat{z}$

$\hat{\theta} = \cos\theta(\cos\phi\hat{x} + \sin\phi\hat{y}) - \sin\theta\hat{z}$

$\hat{\phi} = -\sin\phi\hat{x} + \cos\phi\hat{y}$

If motion of a particle is confined in one plane then only two coordinates are required to describe its position. We can either use cartesian coordinates (x, y) or plane polar coordinates (s,ϕ).

Thus if a particle is moving on a plane then its position vector can be written as



$\vec{r} = x\hat{x} + y\hat{y}$

Or, $\quad \vec{r} = s\hat{s}$ in (plane polar coordinate)

Plane polar coordinates (s,ϕ) are the same coordinates which are used in cylindrical coordinates system.

Notice that, $\hat{x}, \hat{y}$ and $\hat{z}$ have a fixed direction as they are along the x, y and z axes, whereas $\hat{r}, \hat{s}, \hat{\theta}, \hat{\phi}$ etc do not have fixed directions. Therefore, $\hat{x}, \hat{y}, \hat{z}$ are constant unit vectors but $\hat{r}, \hat{s}, \hat{\theta}, \hat{\phi}$ are not constant unit vectors.

Thus, $\dfrac{d\hat{x}}{dt} = 0, \dfrac{d\hat{y}}{dt} = 0, \dfrac{d\hat{z}}{dt} = 0$ and $\dfrac{d\hat{r}}{dt} \neq 0, \dfrac{d\hat{s}}{dt} \neq 0, \dfrac{d\hat{\theta}}{dt} \neq 0, \dfrac{d\hat{\phi}}{dt} \neq 0$

Derivative of unit vectors $(\hat{r},\hat{\theta},\hat{s},\hat{\phi})$ can easily be found by using their relation with $(\hat{x},\hat{y},\hat{z})$.

For example: In plane polar or cylindrical coordinates, $\hat{s}=\cos\phi\hat{x}+\sin\phi\hat{y}$ and $\hat{\phi}=-\sin\phi\hat{x}+\cos\phi\hat{y}$

$$\therefore \qquad \frac{d\hat{s}}{dt}=-\sin\phi\frac{d\phi}{dt}\hat{x}+\cos\phi\frac{d\phi}{dt}\hat{y}$$

$$=\left(-\sin\phi\hat{x}+\cos\phi\hat{y}\right)\frac{d\phi}{dt}=\hat{\phi}\dot{\phi}$$

and $$\frac{d\hat{\phi}}{dt}=-\cos\phi\frac{d\phi}{dt}\hat{x}-\sin\phi\frac{d\phi}{dt}\hat{y}$$

$$=-\left(\cos\phi\hat{x}+\sin\phi\hat{y}\right)\frac{d\phi}{dt}=-\hat{s}\dot{\phi}$$

$$\therefore \qquad \boxed{\frac{d\hat{s}}{dt}=\hat{\phi}\dot{\phi}, \qquad \frac{d\hat{\phi}}{dt}=-\hat{s}\dot{\phi}}$$

Velocity: Average velocity is defined as,

$$\vec{u}_{av}=\frac{\text{total displacement}}{\text{total time taken}}=\frac{\text{change in position vector}}{\text{total time taken}}$$

$$=\frac{\vec{r}_2-\vec{r}_1}{t_2-t_1}$$

Instantaneous velocity (velocity at any instant of time) is defined as time derivative of position vector.

Instantaneous velocity, $\vec{v}=\frac{d\vec{r}}{dt}$.

By expressing $\vec{r}$ in different coordinate system.

In cartesian coordinate system:

$$\vec{r}=x\hat{x}+y\hat{y}+z\hat{z}$$

Therefore, $$\vec{v}=\frac{d}{dt}\left(x\hat{x}+y\hat{y}+z\hat{z}\right)=\frac{dx}{dt}\hat{x}+\frac{dy}{dt}\hat{y}+\frac{dz}{dt}\hat{z}$$

$$\boxed{\vec{v}=\dot{x}\hat{x}+\dot{y}\hat{y}+\dot{z}\hat{z}}$$

Or, $\quad v_x\hat{x}+v_y\hat{y}+v_z\hat{z}$

$v_x=\dot{x}=\frac{dx}{dt}$ is component of velocity in x direction.

$v_y=\dot{y}=\frac{dy}{dt}$ is component of velocity in y direction

$v_z=\dot{z}=\frac{dz}{dt}$ is component of velocity in z direction

In plane coordinate system:

$$\vec{r}=s\hat{s}$$

Therefore, $$\vec{v}=\frac{d}{dt}\left(s\hat{s}\right)=\frac{ds}{dt}\hat{s}+s\frac{d\hat{s}}{dt}$$

$$\boxed{\vec{v} = \dot{s}\hat{s} + s\dot{\phi}\hat{\phi}} \qquad \left\{\because \frac{d\hat{s}}{dt} = \dot{\phi}\hat{\phi}\right\}$$

Or, $\vec{v} = v_s\hat{s} + v_\phi\hat{\theta}$

$v_s = \dot{s} = \dfrac{ds}{dt}$ is component velocity in $\hat{s}$ direction and it is called radial velocity

$v_\phi = s\dot{\phi} = s\dfrac{d\phi}{dt}$ is component velocity in $\hat{\phi}$ direction and it is called transverse velocity.

In cylindrical coordinate system:

$$\vec{r} = s\hat{s} + z\hat{z}$$

Therefore, $\vec{v} = \dfrac{d\vec{r}}{dt} = \dfrac{d}{dt}(s\hat{s} + z\hat{z})$

$$= \frac{ds}{dt}\hat{s} + s\frac{d\hat{s}}{dt} + \frac{dz}{dt}\hat{z}$$

$$\boxed{\vec{v} = \dot{s}\hat{s} + s\dot{\phi}\hat{\phi} + \dot{z}\hat{z}} \qquad \left\{\because \frac{d\hat{s}}{dt} = \dot{\phi}\hat{\phi}\right\}$$

Acceleration : Average acceleration is defined as

$$\vec{a}_{av} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1}$$

Instantaneous acceleration is defined as time derivative of velocity vector.

$$\vec{a} = \frac{d\vec{v}}{dt}$$

By expressing $\vec{v}$ indifferent coordinatge system. We can writge acceleration of a particle in different coordinate system.

In cartesian coordinate:

$$\vec{v} = \dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z}$$

Therefore, $\vec{a} = \dfrac{d\vec{v}}{dt} = \dfrac{d}{dt}(\dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z})$

$$\boxed{\vec{a} = \ddot{x}\hat{x} + \ddot{y}\hat{y} + \ddot{z}\hat{z}}$$

or, $\vec{a} = (a_x\hat{x} + a_y\hat{y} + a_z\hat{z})$

$a_x = \ddot{x} = \dfrac{d}{dt}(\dot{x}) = \dfrac{dv_x}{dt}$ is component of acceleration along x-direction.

$a_y = \ddot{y} = \dfrac{d}{dt}(\dot{y}) = \dfrac{dv_y}{dt}$ is component of acceleration along y-direction.

$a_z = \ddot{z} = \dfrac{d}{dt}(\dot{z}) = \dfrac{dv_z}{dt}$ is component of acceleration along z-direction.

Since, $a_x = \dfrac{dv_x}{dt}$, therefore, if velocity along x-direction is constant then acceleration along x-direction must be zero.

In Plane polar coordinate :

$$\vec{v} = \dot{s}\hat{s} + s\dot{\phi}\hat{\phi}$$

Therefore, $$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\dot{s}\hat{s} + s\dot{\phi}\hat{\phi}\right)$$

$$= \ddot{s}\hat{s} + \dot{s}\frac{d\hat{s}}{dt} + \dot{s}\dot{\phi}\hat{\phi} + s\ddot{\phi}\hat{\phi} + s\dot{\phi}\frac{d\hat{\phi}}{dt}$$

$$= \ddot{s}\hat{s} + \dot{s}\left(\dot{\phi}\hat{\phi}\right) + \dot{s}\dot{\phi}\hat{\phi} + s\ddot{\phi}\hat{\phi} + s\dot{\phi}\left(-\dot{\phi}\hat{s}\right)$$

$$\boxed{\vec{a} = \left(\ddot{s} - s\dot{\phi}^2\right)\hat{s} + \left(2\dot{s}\dot{\phi} + s\ddot{\phi}\right)\hat{\phi}}$$

Or, $$\vec{a} = a_s\hat{s} + a_\phi\hat{\phi}$$

$a_s = \ddot{s} - s\dot{\phi}^2$ is component of acceleration along $\hat{s}$ and its called radial acceleration. Clearly, it is not equal to time derivative of radial component of velocity $\left(v_s = \dot{s}\right)$.

Therefore, if v_s = constant , then a_s may not be zero.

$a_\phi = 2\dot{s}\dot{\phi} + s\ddot{\phi}$ is component of acceleration in $\hat{\phi}$ direction. It is called transverse acceleration. Clearly it is also not equal to time derivative of radial component of velocity $\left(v_\phi = s\dot{\phi}\right)$

i.e. $$a_\phi \neq \frac{d}{dt}\left(v_\phi\right)$$

Therefore, if v_ϕ = constant then a_ϕ may not be zero.

In cylindrical coordinate system:

$$\vec{v} = \dot{s}\hat{s} + s\dot{\phi}\hat{\phi} + \dot{z}\hat{z}$$

First two terms are same as in plane polar coordinate.

Therefore, $$\boxed{\vec{a} = \frac{d\vec{v}}{dt} = \left(\ddot{s} - s\dot{\phi}^2\right)\hat{s} + \left(2\dot{s}\dot{\phi} + s\ddot{\phi}\right)\hat{\phi} + \ddot{z}\hat{z}}$$

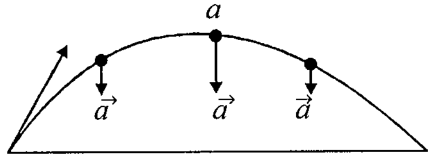
How to choose a coordinate system:

Generally speaking, dynamics of a particle can be studied using any coordinate system. However judicious selection of coordinate system makes task easy.

One dimensional cases is simple because we just need one coordinate which we can take either x or y or z.

However in two dimensional cases one has to be contious. In most of the two dimensional problems if acceleration of the particle is directed towards a point then plane polar coordinate is conveninet to use. However, if accleration is not directed towards a point then cartesian coordinate is right choice.

For example : A projectile motion in which particle is projected with a small speed at some angle with horizontal. It moves with constant acceleration which is directed downward. In this case acceleration is not directed towards a point. Therefore, we can conveniently use cartesian coordinates.



In planetary motion, acceleration of planet is always directed towards centre of the sun. Therefore in this case plane polar coordinate makes calculation simple.

If a particle is attached to a point with a string or spring then also generally polar coordinate is helpfull. For example, in case of simple pendulum.

Formulae of kinematics (uniformly accelerated motions): When a constant force acts on a particle its acceleration is constant. If u_x, v_x be initial and final velocity of a particle along x-direction and a_x be its acceleration along x direction then.

$$\left.\begin{aligned} v_x &= u_x + a_x t \\ v_x^2 &= u_x^2 + 2a_x x \\ x &= u_x t + \tfrac{1}{2} a_x t^2 \end{aligned}\right\} \text{if } a_x = \text{constant}$$

where x is displacement along x-direction and t is time taken. We can write similar relations for y and z direction also.

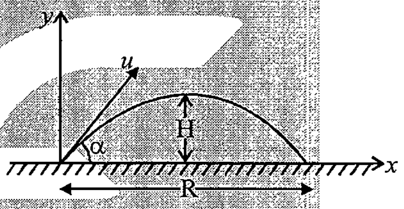
Note: If acceleration of particle is not constant then we cannot use formulae of kinematics. In that case we start with either definition of velocity or definition of acceleration i.e.

$$v_x = \frac{dx}{dt} \quad \text{or} \quad a_x = \frac{dv_x}{dt}$$

we may also have to use $a_x = \dfrac{F_x}{m}$ and $\dfrac{dv_x}{dt} = \dfrac{dv_x}{dx} \cdot \dfrac{dx}{dt} = \dfrac{v_x dv_x}{dx}$ as the case may be

Projectile motion: If u be the projection speed and α be angle of projection then equation of path of projectile is,

$$y = x \tan\alpha - \frac{gx^2}{2u^2 \cos^2\alpha}$$



Range, $R = \dfrac{u^2 \sin 2\alpha}{g}$

Maximum height, $H = \dfrac{u^2 \sin^2 \alpha}{2g}$

Total time of flight, $T = \dfrac{2u \sin\alpha}{g}$

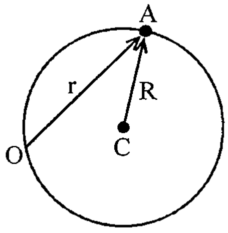
Coordinate of projectile at any time $t < T$,

$$x = u\cos\alpha\, t, \; y = u \sin\alpha\, t - \frac{1}{2} g t^2$$

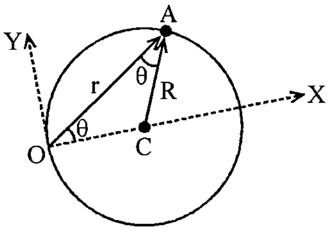
Note: Above results for projectile are applicable only if projectile is thrown from ground and it finally lands on ground. And projection speed is small so that projectile does not go too high. If projection speed is not small then height will be large and in that case we need to consider variation of acceleration due to gravity.

SOLVED PROBLEMS

1. A particle A moves along a circle of radius $R = 50$ cm so that its radius vector r relative to point O (figure (a)) rotates with the constant angular velocity $\omega = 0.40$ rad/sec. Find the modulus of the velocity of the particle and modulus and direction of its total acceleration.



(a)



(b)

Soln. Consider X and Y axes as shown in (figure (b)). Using sine law in triangle CAO, we get $\dfrac{r}{\sin(\pi - 2\theta)} = \dfrac{R}{\sin\theta}$

or $\dfrac{r}{2\sin\theta\cos\theta} = \dfrac{R}{\sin\theta}$

$\therefore \quad r = 2R\cos\theta$

Now $\mathbf{r} = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j} = 2R\cos^2\theta\,\mathbf{i} + 2R\cos\theta\sin\theta\,\mathbf{j}$

Now $\mathbf{v} = \dfrac{d\mathbf{r}}{dt} = -4R\cos\theta\sin\theta\dfrac{d\theta}{dt}\mathbf{i} + 2R\cos 2\theta\dfrac{d\theta}{dt}\mathbf{j} = -2R\sin 2\theta\,\omega\,\mathbf{i} + 2\theta\,\omega\,\mathbf{j}$

$\therefore \quad |\mathbf{v}| = 2\omega R$

Further

$$\mathbf{a} = \frac{dv}{dt} = 4R\cos 2\theta\,\omega\frac{d\theta}{dt}\mathbf{i} - 4R\omega\sin 2\theta\frac{d\theta}{dt}\mathbf{j} = -4R\omega^2\cos 2\theta\,\mathbf{i} - 4R\omega^2\sin 2\theta\,\mathbf{j}$$

$$|\mathbf{a}| = 4R\omega^2.$$

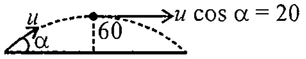
2. After being hit, a golf ball reaches a maximum height of 60m with speed of 20m/s. Right after being hit the speed of the ball is (take $g = 10$ m/s^2) **[GP-2008]**

(a) 100m/s (b) 80m/s (c) 60m/s (d) 40m/s

Soln. (d) If u be the speed just after being hit at angle α then

$u\cos\alpha = 20$ ... (i)

$\dfrac{u^2\sin^2\alpha}{2g} = 60$



$\therefore \quad u^2\sin^2\alpha = 1200$ ... (ii)

on squaring (i) and adding to (ii) we get

$u^2 = 1200 + 400$

$\therefore \quad u = 40$

3. A particle starts moving with acceleration $a = \alpha - \beta v$ where α and β are constants and v is its instantaneous speed. Find velocity of particle as a function of time and also find its terminal velocity.

Soln: Acceleration is variable therefore we start with definition of acceleration

$$a = \frac{dv}{dt} \text{ or } \alpha - \beta v = \frac{dv}{dt} \text{ or } \int_0^v \frac{dv}{\alpha - \beta v} = \int_0^t dt$$

$$-\frac{1}{\beta}\ln\left(\frac{\alpha - \beta v}{\alpha}\right) = t \text{ or, } 1 - \frac{\beta v}{\alpha} = e^{-\beta t} \qquad \therefore v = \frac{\alpha}{\beta}(1 - e^{-\beta t})$$

as $t \to \infty$, $v = \dfrac{\alpha}{\beta}$ = constant. This is terminal velocity.

4. A projectile is thrown at an angle α with horizontal with initial speed u after what time the projectile's velocity will be perpendicular to its initial direction.

Soln. Suppose velocity after time t becomes perpendicular to initial velocity. Velocity after time t is

$$\vec{v}(t) = \dot{x}\hat{i} + \dot{y}\hat{j} \quad = u\cos\alpha\hat{i} + (u\sin\alpha - gt)\hat{j}$$

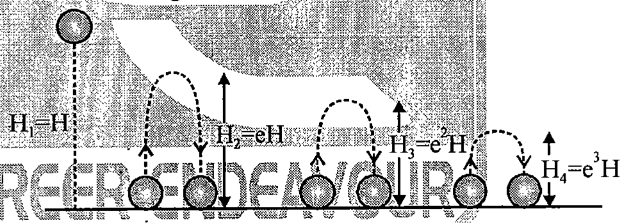
Initial velocity $\vec{u} = u(\cos\alpha\hat{i} + \sin\alpha\hat{j})$

$\vec{v}$ is $\perp$ to $\vec{u}$, therefore, $\vec{v}.\vec{u} = 0 \quad \therefore u\cos^2\alpha + u\sin^2\alpha - gt\sin\alpha = 0 \quad (\because u \neq 0)$

$$\therefore t = \frac{u}{g\sin\alpha}$$

5. A ball is dropped from a height H. It bounces back up to a height e times after hitting the ground. If $e<1$, after what time the ball will finally come to rest.

Soln. Ball moves under the effect of gravity. Therefore magnitude of acceleration of ball during upward or downward movement remains constant.



For first bounce

$$y = H,\ u_y = 0,\ a_y = g$$

$$y = u_y t + \frac{1}{2}a_y t^2$$

$$\therefore H = 0 + \frac{1}{2}gt^2 \qquad \therefore t = \sqrt{\frac{2H}{g}}$$

Therefore time of fall before first bounce $t_1 = \sqrt{\dfrac{2H}{g}}$

Under gravity, time for going up a certain distance is equal to time for going down. Therefore time interval between first bounce and second bounce is

$$t_2 = 2\sqrt{\frac{2H_2}{g}} = 2\sqrt{\frac{2eH}{g}}$$

Similarly $t_3 = 2\sqrt{\dfrac{2e^2H}{g}},\ t_4 = 2\sqrt{\dfrac{2e^3H}{g}}, \ldots\ldots$

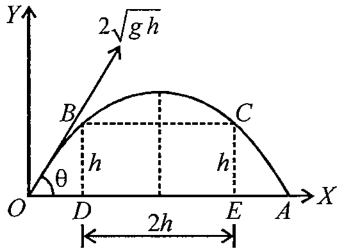
Therefore, total time elapsed before ball stops is

$$T = t_1 + t_2 + t_3 + \ldots\ldots$$

$$= \sqrt{\frac{2H}{g}}\left[1 + 2\left(e^{1/2} + e^{2/2} + e^{3/2} + \ldots\right)\right] = \sqrt{\frac{2H}{g}}\left[1 + 2\frac{e^{1/2}}{1 - e^{1/2}}\right] = \sqrt{\frac{2H}{g}}\left(\frac{1+\sqrt{e}}{1-\sqrt{e}}\right)$$

6. A particle is projected with velocity $2\sqrt{(h\,g)}$ so that it just clears two walls of equal height h which are at a distance $2h$ from each other. Show that the time of passing between the walls is $2\sqrt{(h/g)}$.

Soln. The situation is shown in figure.



Suppose the particle is at a height h after time t. Then,

$$h = 2\sqrt{g\,h}\,\sin\theta\cdot t - \frac{1}{2}g\,t^2$$

or $$g\,t^2 - 4\sqrt{g\,h}\,\sin\theta\cdot t + 2h = 0 \qquad \text{... (i)}$$

This equation gives the two values of time at which the height of the particle is h. Let these times be t_1 and t_2 respectively. Then,

$$(t_1 - t_2)^2 = (t_1 + t_2) - 4t_1 t_2$$

From equation (i)

$$t_1 + t_2 = \frac{4\sqrt{(g\,h)}\,\sin\theta}{g} \text{ and } t_1\,t_2 = \frac{2h}{g}$$

$$\therefore \quad (t_1 - t_2)^2 = \frac{16gh\sin^2\theta}{g^2} - \frac{8h}{g} = \frac{8h}{g}[2\sin^2\theta - 1]$$

or $$(t_1 - t_2) = \sqrt{\left(\frac{8h}{g}\right)}\,[2\sin^2\theta - 1]^{1/2} \qquad \text{... (ii)}$$

During this time, the horizontal distance moved is $2h$. The horizontal velocity being $2\sqrt{(g\,h)}\,\cos\theta$.

Using $s = ut$, we have

$$2h = 2\sqrt{(g\,h)}\,\cos\theta\cdot t$$

or $$2h = 2\sqrt{(g\,h)}\,\cos\theta \times \sqrt{(8h/g}\,[2\sin^2\theta - 1]^{1/2}$$

Squaring $1 = 8\cos^2\theta\,(1 - 2\cos^2\theta)$

or $$16\cos^4\theta - 8\cos^2\theta + 1 = 0,$$

or $$(4\cos^2\theta - 1)^2 = 0.$$

$$\therefore \quad \cos^2\theta = \frac{1}{4} \text{ and } \sin^2\theta = \frac{3}{4}. \qquad \text{... (iii)}$$

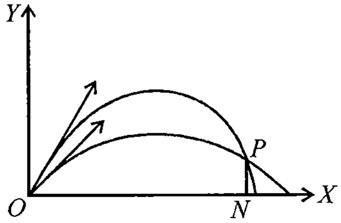
Substituting these values in equation (ii) we have

$$(t_1 - t_2) = \sqrt{[8\,h/g]}\left[2\cdot\frac{3}{4} - 1\right]^{1/2}$$

or $$t_1 - t_2 = 2\sqrt{(h/g)}.$$

7. Two particles are projected from a point at the same instant with velocities whose horizontal and vertical components are u_1, v_1 and u_2, v_2 respectively. Prove that the interval between their passing through the other common point of their path is $\frac{2(v_1 u_2 - v_2 u_1)}{g(u_1 + u_2)}$.

Soln. The situation is shown in figure.



Let the two particles pass through the common point P. Again let, t_1 and t_2 be the time taken by the particles to go to P from O.

Along X-axis, we have

$$u_1 t_1 = u_2 t_2 \quad \text{...(i)}$$

In the vertical direction

$$v_1 t_1 - \frac{1}{2} g t_1^2 = PN = v_2 t_2 - \frac{1}{2} g t_2^2 \quad \text{...(ii)}$$

From equation (ii), we have

$$t_1^2 - t_2^2 = 2/g \cdot (v_1 t_1 - v_2 t_2) \quad \text{...(iii)}$$

But from equation (i), $\frac{u_1}{u_2} = \frac{t_2}{t_1}$.

Addition one to both sides, we have $\frac{u_1 + u_2}{u_2} = \frac{t_2 + t_1}{t_1}$.

$$\therefore \quad (t_1 + t_2) = \frac{(u_1 + u_2)}{u_2} \times t_1 \quad \text{...(iv)}$$

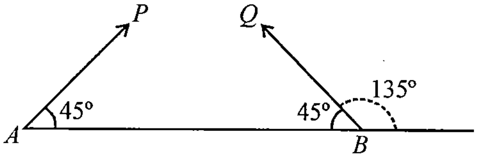
From equation (iii), we get

$$(t_1 + t_2)(t_1 - t_2) = \frac{2}{g} \cdot (v_1 t_1 - v_2 t_2)(t_1 - t_2) \times t_1 = \frac{2}{8}\left(v_1 t_1 - v_2 \frac{u_1}{u_2} \cdot t_1 \right)$$

$$(t_1 + t_2)\frac{(u_1 + u_2)}{u_2} = \frac{2}{g}\left(v_1 - v_2 \frac{u_1}{u_2} \right)(t_1 + t_2) = \frac{2}{g} \cdot \frac{u_2}{(u_1 + u_2)} \times \frac{(v_1 u_2 - v_2 u_1)}{u_2}$$

or $\quad (t_1 - t_2) = \frac{2}{g} \frac{(v_1 u_2 - v_2 u_1)}{(u_1 + u_2)}$.

8. Particles P and Q of mass 20 gm and 40 gm respectively are simultaneously projected from points A and B on the ground. The initial velocities of P and Q make 45° and 135° angles respectively with the horizontal as shown in the figure. Each particle has an initial speed of 49 m/sec. The separation AB is 245 m. Both particles travel in the same vertical plane and undergo a collision. After the collision P retraces its path. Determine the position of Q when it hits the ground. How much time after the collision does the particle Q take to reach the ground. Take $g = 9.8$ m/sec^2.



Soln. The horizontal velocity of either particle is $49 \cos 45° = 49/\sqrt{2}$ m/sec. Initially, the vertical component of velocity of either particle $= 49 \sin 45° = 49/\sqrt{2}$ m/sec. The horizontal distance travelled by each particle at the time of collision = 245/2 metre. The vertical component of the velocity of each particle at the time of collision is

$$\frac{49}{\sqrt{2}} - 9.8 \times \frac{245}{2} \times \frac{\sqrt{2}}{49} = 0.$$

At the time of collision, the velocity of each particle is horizontal and is equal to $49/\sqrt{2}$ m/sec.

Let v_p and v_q be the velocities after collision, then

$$0.02 \times \frac{49}{\sqrt{2}} - 0.04 \times \frac{49}{\sqrt{2}} = 0.02\, v_p + 0.04\, v_q.$$

As P retraces its path, $v_p = -\frac{49}{\sqrt{2}}$.

$$\therefore \quad \frac{49}{\sqrt{2}}(0.02 - 0.04) = -0.02 \times \frac{49}{\sqrt{2}} + 0.04\, v_q$$

The particle falls vertically downward from the point of collision, i.e., at a distance of 122.5 metre from A.

Let t be the time taken, then $\frac{49}{\sqrt{2}} = 9.8 \times t$.

Since collision does not after the vertical component of velocities.

$$\therefore \quad t = \frac{49}{\sqrt{2} \times 9.8} = \frac{5}{\sqrt{2}} = 3.53 \text{ sec.}$$

9. Two shots are projected from a gun at the top of a hill with the same velocity u at angles of projection α and β respectively. If the shots strike the horizontal ground through the foot of the hill at the same point, show that the height h of the hill above the plane is given by

$$h = \frac{2u^2(1 - \tan\alpha \tan\beta)}{g(\tan\alpha + \tan\beta)^2}.$$

Soln. Let R be the horizontal range for both. Taking the point of projection as origin, the point struck is (R, h). This point satisfies the equations of both the trajectories.

$$-h = R\tan\alpha - \frac{1}{2} g \frac{R^2}{u^2\cos^2\alpha} \qquad \text{...(i)}$$

and $$-h = R\tan\beta - \frac{1}{2} g \frac{R^2}{u^2\cos^2\beta} \qquad \text{...(ii)}$$

From these equations, we have

$$R\tan\alpha - \frac{1}{2} g \frac{R^2}{u^2\cos^2\alpha} = R\tan\beta - \frac{1}{2} g \frac{R^2}{u^2\cos^2\beta}$$

or $R(\tan\alpha - \tan\beta) = \frac{1}{2}g\frac{R^2}{u^2\cos^2\alpha} - \frac{1}{2}g\frac{R^2}{u^2\cos^2\beta}$

$$2u^2/g\,(\tan\alpha - \tan\beta) = R(\sec^2\alpha - \sec^2\beta)$$

or $2u^2/g\,(\tan\alpha - \tan\beta) = R[(1+\tan^2\alpha) - (1+\tan^2\beta)]$

or $2u^2/g = R(\tan\alpha + \tan\beta)$. ... (iii)

$$R = \frac{2u^2}{g(\tan\alpha + \tan\beta)}$$

Putting the value of R in equation (i), we have

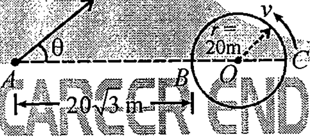
$$h = R\left[\frac{gR}{2u^2\cos^2\alpha} - \tan\alpha\right] = \frac{2u^2}{g(\tan\alpha+\tan\beta)}\left[\frac{\sec^2\alpha}{\tan\alpha+\tan\beta} - \tan\alpha\right]$$

$$= \frac{2u^2}{g(\tan\alpha+\tan\beta)^2}\left[\sec^2\alpha - \tan^2\alpha - \tan\alpha\tan\beta\right] = \frac{2u^2}{g(\tan\alpha+\tan\beta)^2}(1-\tan\alpha\tan\beta).$$

10. A particle is moving along a vertical circle of radius $r = 20$ m with a constant speed $v = 31.4$ m/s as shown in figure below. Straight line ABC is horizontal and passes through the center of the circle. A shell is fired from point A at the instant when the particle is at C. If distance AB is $20\sqrt{3}$ m and the shell collide with the particle at B, then prove

$$\tan\theta = \frac{(2n-1)^2}{\sqrt{3}},$$

where n is an integer. Further show that smallest value of θ is 30°.



Soln. As at the time of firing of the shell, the particle was at C and the shell collides with it at B, therefore the number of revolutions completed by the particle is odd multiple of half i.e., $(2n-1)/2$, where n is an integer.

Let T be the time period of the particle, then

$$T = \frac{2\pi r}{v} = \frac{2\times 3.14\times 20}{31.4} = 4 \text{ second}$$

If t be the time of flight of the shell, then t = time of $[(2n-1)/2]$ revolutions of the particle

$$= \frac{(2n-1)}{2}\times 4 = 2(2n-1) \text{ second}$$

for a projectile, the time of flight is given by $t = \frac{2u\sin\theta}{g}$.

Hence, $\frac{2u\sin\theta}{g} = 2(2n-1)$ ... (i)

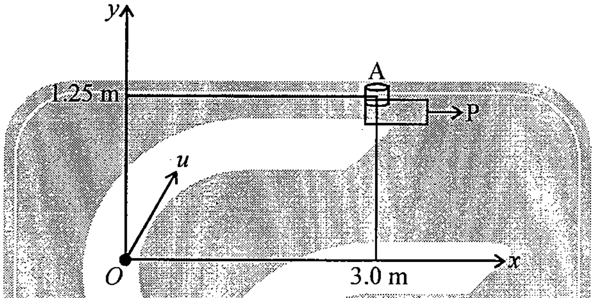
The range of the projectile is given by $R = \frac{u^2\sin 2\theta}{g}$.

Hence, $\dfrac{u^2 \sin 2\theta}{g} = 20\sqrt{3}$ ... (ii)

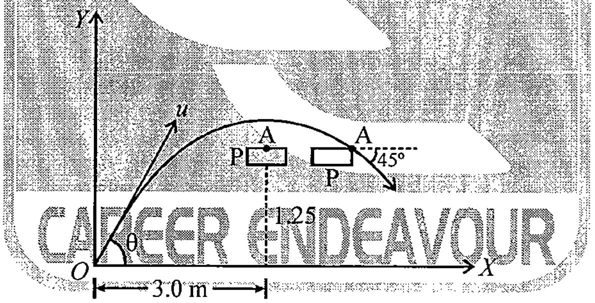
From equations (i) and (ii), we get $\tan\theta = \dfrac{(2n-1)^2}{\sqrt{3}}$.

For θ to be smallest, $n = 1$, so $\theta_{\min} = 30°$.

11. An object A is kept fixed at the point $x = 3\,\text{m}$ and $y = 1.25\,\text{m}$ on a plank P raised above the ground. At time $t = 0$ the plank starts moving along the $+x$ direction with an acceleration $1.5\ \text{m/s}^2$. At the same instant a stone is projected from the origin with a velocity u as shown. A stationary person on the ground observes the stone hitting the object during its downward motion at an angle of 45° to the horizontal. All the motions are in the xy-plane. Find u and the time after which the stone hits the object. Take $g = 10\ \text{m/s}^2$.



Soln. See figure.



Let θ be the angle of projectile. Further, suppose the stone hits the object at time t, then

$$(u\cos\theta)t = 3.0 + \frac{1}{2}\times 1.5 \times t^2 \quad \text{... (i)}$$

and $$(u\sin\theta)t - \frac{1}{2}gt^2 = 1.25 \quad \text{... (ii)}$$

As the stone hits the object A, moving along 45° with the horizontal during downward motion, we have

$$\tan(-45°) = \left(\frac{v_y}{v_x}\right) = \frac{u\sin\theta - gt}{u\cos\theta}$$

or $$\frac{u\sin\theta - gt}{u\cos\theta} = -1$$

or $$u\sin\theta - gt = -u\cos\theta \quad \text{... (iii)}$$

From equation (i), $u\cos\theta = \frac{3.0}{t} + 0.75t$

From equation (ii), $u\sin\theta = \frac{1.25}{t} + \frac{g}{2}t$

Substituting these values in equation (iii), we get

$$\frac{1.25}{t} + 5t - 10t = -\frac{3.0}{t} - 0.75t$$

$$\frac{4.25}{t} = 4.25t \text{ or } t^2 = 1 \text{ or } t = 1 \text{ sec.}$$

From equation (i) and (ii), we get

$$u\cos\theta = 3.0 + 0.75 = 3.75 \qquad \text{... (iv)}$$

and $u\sin\theta = 6.25$ ... (v)

Dividing equation (v) equation (iv), we get

$$\tan\theta = \frac{6.25}{3.75} = \frac{5}{3} \text{ or } \theta = \tan^{-1}\left(\frac{5}{3}\right)$$

Squaring and adding equation (iv) and (v), we get

$$u^2 = (3.75)^2 + (6.25)^2 = (14.06) + (39.06)$$

$$u^2 = 53.12 \text{ or } u = 7.29 \text{ m/s.}$$

12. A rotation disc (figure below) moves in the positive direction of the x-axis. Find the $y(x)$ describing the positive of the instantaneous axis of rotation, if at the initial moment the axis C of the disc was located at the point O after which it moved

(a) with a constant velocity v, while the disc started rotating counter clockwise with a constant angular acceleration α (the initial angular velocity is equal to zero);

(b) with a constant acceleration a (and the zero initial velocity), the disc rotates counter clockwise with a constant angular velocity ω.

Soln. (a) When the disc rotates with constant angular velocity v and counter clockwise angular acceleration α :

We have the relation $\omega = \omega_0 + \alpha t = \alpha t, t = \frac{x}{v}$.

Now consider any point P of the disc at a distance y from the x-axis; i.e. $\mathbf{r} = -y\,\mathbf{j}$. An axis passing through P and parallel to the axis of rotation. Now the linear velocity of point P.

$$\mathbf{v}_p = \vec{\omega}\times\mathbf{r} \text{ and } \vec{\omega} = \alpha\frac{x}{v}\mathbf{k} = \left(\frac{\alpha x}{v}\right)\mathbf{k}\times y(-\mathbf{j}) = \frac{\alpha x}{v}y\,\mathbf{i};$$

that is, the point P has a velocity along the x-axis which is the same as the velocity of any point of the disc $= v$.

Hence, $y = \frac{v^2}{\alpha x}$: (equation of a hyperbola)

(b) When the disc rotates with constant angular velocity ω and constant acceleration a : In this case

$$x = ut + \frac{1}{2}at^2 = 0 + \frac{1}{2}at^2 \Rightarrow t = \sqrt{\left(\frac{2x}{a}\right)}$$

Hence, $v = at = \sqrt{2xa}$

Again $\mathbf{v}_p = \vec{\omega} \times \mathbf{r} = \vec{\omega}\mathbf{k} \times y(-\mathbf{j})$,

Therefore, $|\mathbf{v}| = \omega y$ or $\sqrt{2xa} = \omega y$ or $y = \dfrac{\sqrt{2xa}}{\omega}$, a parabola].

13. A balloon rises from rest on the ground with constant acceleration $g/8$. A stone is dropped when the balloon has risen to a height H meter. Show that the time taken by the stone to reach the ground is given by $2\sqrt{(H/g)}$.

Soln. The velocity v of the balloon when it has risen to a height H is given by $v^2 = 0 + 2 \times \frac{g}{8} \times H$ taking upward direction positive.

$$\therefore \quad v = \sqrt{\left\{\frac{(gH)}{4}\right\}} = \frac{\sqrt{(gH)}}{2} \text{ m/sec.}$$

This will also be the velocity of the stone in upward direction when it is dropped.

Taking upward direction position for the stone, $-H = \frac{\sqrt{(gH)}}{2}t - \frac{1}{2}t - \frac{1}{2}gt^2$ $(\because g = -g \text{ and } h = -H)$

or $\quad gt^2 - \sqrt{(gH)}\,t - 2H = 0$

$$t = \frac{\sqrt{(gH)} \pm \sqrt{\{(gH) + (8gH)\}}}{2g} = \frac{\sqrt{(gH)} \pm 3\sqrt{(gH)}}{2g} = 2\sqrt{\left(\frac{H}{8}\right)} \text{ taking positive value.}$$

14. A particle is projected vertically upwards. Prove that it will be at 3/4 of its greatest height at time which are in the ratio 1 : 3.

Soln. We know that the greatest height attained $= u^2/2g$ (where u is the initial velocity). Let t be the time when the particle is at a height $\frac{3}{4}\left(\frac{u^2}{2g}\right)$.

Using the formula $s = ut + \frac{1}{2}gt^2$, we have $\frac{3}{4}\left(\frac{u^2}{2g}\right) = ut - \frac{1}{2}gt^2 \Rightarrow t^2 - \frac{2u}{g}t + \frac{6u^2}{8g^2} = 0.$

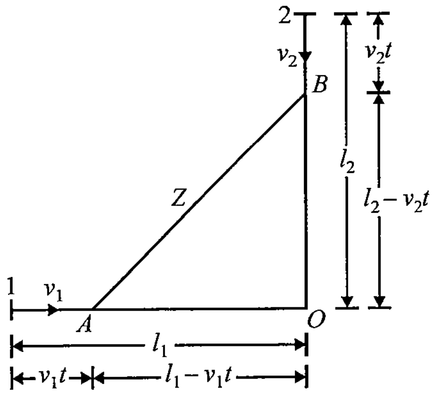
Solving for t, we have $t = \dfrac{\frac{2u}{g} \pm \sqrt{\left(\frac{4u^2}{g^2} - \frac{3u^2}{g^2}\right)}}{2} = \frac{u}{g} \pm \frac{u}{2g}.$

Taking negative sign, $t_1 = \frac{u}{2g}$.

Taking only positive sign, $t_2 = \frac{3u}{2g}$.

$$\therefore \quad \frac{t_1}{t_2} = \frac{(u/2g)}{(3u/2g)} = \mathbf{1:3.}$$

15. Two particles, 1 and 2, move with constant velocities v_1 and v_2 along two mutually perpendicular straight lines towards the intersection point O. At the moment $t = 0$ the particles were located at the distances l_1 and l_2 from the point O. How soon will the distance the particles become the smallest ? What is it equal to ?



Soln. At a time t, the positions of the particle 1 and 2 are separately, $(l_1 - v_1 t, 0)$ and $(0, l_2 - v_2 t)$. Hence the distance between them at this instant.

$$Z = \sqrt{(l_1 - v_1t)^2 + (l_2 - v_2t)^2}$$

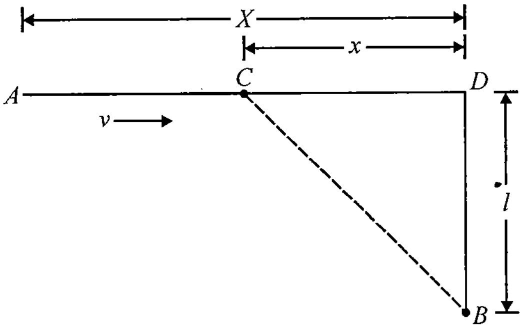
To be minimum at some instant $t = t'$, we use the condition

$$\frac{dZ}{dt} = 0 \Rightarrow \frac{1}{2}\frac{2(l_1 - v_1t)(-v_1) + 2(l_2 - v_2t)(-v_2)}{\sqrt{(l_1 - v_1t)^2 + (l_2 - v_2t)^2}} = 0$$

or $\quad (v_1^2 + v_2^2)t = l_1 v_1 + l_2 v_2 \quad$ or $\quad t = t' = \dfrac{l_1 v_1 + l_2 v_2}{v_1^2 + v_2^2}$

Hence $Z_{\min} = \sqrt{\left(l_1 - v_1\dfrac{l_1 v_1 + l_2 v_2}{v_1^2 + v_2^2}\right)^2 + \left(l_2 - v_2\dfrac{l_1 v_1 + l_2 v_2}{v_1^2 + v_2^2}\right)^2} = \dfrac{|l_1 v_2 - l_2 v_1|}{\sqrt{(v_1^2 + v_2^2)}}.$

16. From point A located on a highway, one has to get by car as soon as possible to point B located in the field at a distance l from the highway. It is known that the car moves in the field η times slower than on the highway. At what distance from the point D one must turn off the highway ?



Soln. Let the car turn off the road at a distance x from D. Hence the total time of travel by car from A to B.

$$T = \frac{X - x}{v} + \frac{\sqrt{l^2 + x^2}}{v/\eta}$$

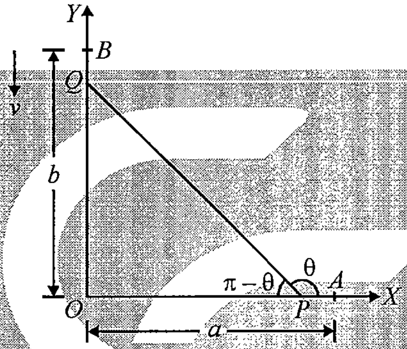
For this to be minimum :

$$\frac{dT}{dx}=0 \Rightarrow -\frac{1}{v}+\frac{\eta}{v}\frac{1}{2}\left(\frac{2x}{\sqrt{l^2+x^2}}\right)=0 \Rightarrow \frac{\eta x}{\sqrt{l^2+x^2}}=1$$

or $$x=\frac{\sqrt{l^2+x^2}}{\eta} \Rightarrow x^2\left(1-\frac{1}{\eta^2}\right)=\frac{l^2}{\eta^2} \Rightarrow x=\frac{l}{\sqrt{\eta^2-1}}.$$

17. Two points are moving with uniform velocities u and v along the perpendicular axes, OX and OY. The motion is directed towards O, the origin. When $t=0$, they are at a distance a and b respectively from O. Calculate the angular velocity of the line joining them at time t. Show that it is greatest, when $t=\dfrac{au+bv}{u^2+v^2}$.

Soln. See figure. At $t=0$, the particles are at A and B respectively.



Let at time t, the particles are at P and Q respectively.

From figure, $OP = a - ut$ ... (i)

and $OQ = b - vt$ ... (ii)

Let θ be the angle which the line PQ makes with the direction OX at time t. From ΔQOP,

$$\tan(\pi-\theta)=\frac{OQ}{OP}=\frac{b-vt}{a-ut} \text{ or } -\tan\theta=\frac{b-vt}{a-ut}$$

or $$\tan\theta=\left(\frac{b-vt}{ut-a}\right) \text{ or } \theta=\tan^{-1}\left(\frac{b-vt}{ut-a}\right) \quad \text{... (iii)}$$

Differentiating equation (iii), we get $$\frac{d\theta}{dt}=\frac{1}{1+\left(\frac{b-vt}{ut-a}\right)^2}\times\frac{(ut-a)(-v)-(b-vt)u}{(ut-a)^2}$$

Simiplifying it, we get $\omega=\dfrac{d\theta}{dt}=\dfrac{av-bu}{(a-ut)^2+(b-vt)^2}$, ω will be maximum when denominator is minimum because numerator is constant.

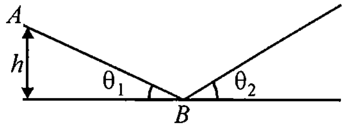
Let $Z=(a-ut)^2+(b-vt)^2$

$$\frac{dZ}{dt}=2(a-ut)(-u)+2(b-vt)(-v)=0$$

Solving it for t, we get $t=\dfrac{au+bv}{u^2+v^2}$.

It can be seen that $\frac{d^2Z}{dt^2}$ is positive. Hence ω is maximum, when $t=\frac{au+bv}{u^2+v^2}$.

18. A particle of mass m starts from rest moving down the inclined plane AB and rises up to point C on the inclined plane BC. Assuming no losses of energy, the time period of the to and for motion of the particle is............



Soln. The time taken t_1 in moving from A to B, is given by $\frac{h}{\sin\theta_1}=0+\frac{1}{2}(g\sin\theta_1)$ $\left(\because s=AB=\frac{h}{\sin\theta_1}\right)$.

Therefore, $t_1=\sqrt{\left(\frac{2h}{g}\right)\cdot\frac{1}{\sin\theta_1}}$

Similarly, the time taken, t_2 from C to B is given by $t_2=\sqrt{\left(\frac{2h}{g}\right)\cdot\frac{1}{\sin\theta_2}}$.

Time period, $T=2(t_1+t_2)=2\sqrt{\left(\frac{2h}{g}\right)}\left[\frac{1}{\sin\theta_1}+\frac{1}{\sin\theta_2}\right]$.

19. A particle moves in the plane xy with constant acceleration a directed along the negative y-axis. The equation of motion of the particle has the form $y=px-qx^2$ where p and q are positive constants. Find the velocity of the particle at the origin of coordinates.

Soln. Given that $y=px-qx^2$

$$\therefore \quad \frac{dy}{dt}=p\frac{dx}{dt}-q\cdot 2x\frac{dx}{dt} \text{ and } \frac{d^2y}{dt^2}=p\frac{d^2x}{dt^2}-2qx\frac{d^2x}{dt^2}-2q\left(\frac{dx}{dt}\right)^2 \text{ or } -a=-2q\left(\frac{dx}{dt}\right)^2=-2qv_x^2$$

$$\because \quad \frac{d^2x}{dt^2}=0 \text{ (no acceleration along } x\text{-axis), and } \frac{d^2y}{dt^2}=-a.$$

$$\therefore \quad v_x^2=\frac{a}{2q} \text{ or } v_x=\sqrt{\frac{a}{2q}}$$

Further, $\left(\frac{dy}{dt}\right)_{x=0}=p\frac{dx}{dt}$ or $v_y=pv_x$

$$\therefore \quad v_y=p\sqrt{\left(\frac{a}{2q}\right)}$$

Now, $v=\sqrt{(v_x^2+v_y^2)}=\sqrt{\left(\frac{a}{2q}+\frac{ap^2}{2q}\right)}$ or $v=\sqrt{\left[\frac{a(p^2+1)}{2q}\right]}$.

Chapter 2

Newton's Laws of Motion

" *What we know is a drop, what we don't know is an ocean.*" **Isaac Newton**

First Law: It states that a particle which is not acted upon by any external force, maintains its state of rest or state of uniform motion.

When Newton's law is applied to an object or system which consists of more than one particles, it is centre of mass of the object or system which obeys Newton's law. For example if two particles are placed on a smooth table at some separation, because of their natural gravitational attraction the two will move towards each other but centre of the system of two particles remains at rest.

Second Law: It states that rate of change of momentum of a system is equal to resultant external force on the system.

$$\vec{F} = \frac{d\vec{p}}{dt} \qquad \text{... (2.1)}$$

This form of Newton's law is valid in all cases.

If m be mass of system and $\vec{v}$ be its velocity then $\vec{p} = m\vec{v}$

Therefore, $\vec{F} = \frac{d}{dt}(m\vec{v})$

If mass of system is constant then, $\vec{F} = m\frac{d\vec{v}}{dt}$

or $$\vec{F} = m\vec{a} \qquad \text{... (2.2)}$$

where $\vec{a} = \frac{d\vec{v}}{dt}$ is acceleration, evidently, equation (2.2) can be used only if mass is constant. If mass is variable, we use equation (2.1).

If F_x, F_y, F_z be Cartesian components of $\vec{F}$ and (x, y, z) be coordinate of the particle on which force is acting then equation (2.2) can be written in component form as

$$\left.\begin{aligned} F_x &= m\frac{d^2x}{dt^2} \\ F_y &= m\frac{d^2y}{dt^2} \\ F_z &= m\frac{d^2z}{dt^2} \end{aligned}\right\} \qquad \text{... (2.3)}$$

However, instead of Cartesian coordinates if we use some curvilinear coordinate then we do not get similar relations. For example if F_r, F_θ, F_ϕ be components of force $\vec{F}$ in spherical polar coordinate and r, θ, ϕ be spherical polar coordinates of particle then,

$$\left.\begin{aligned} F_r &\neq m\frac{d^2r}{dt^2} \\ F_\theta &\neq m\frac{d^2\theta}{dt^2} \\ F_\phi &\neq m\frac{d^2\phi}{dt^2} \end{aligned}\right\} \quad \ldots (2.4)$$

Thus, we cannot write Newton's law in other coordinate system by merely replacing x, y, z by corresponding coordinates.

Due to this reason we say that Newton's law is not covariant under coordinate transformation.

Third Law: It states that to every action there is equal and opposite reaction. If $\vec{F}_{12}$ be force on an object 1 due to an object 2, and F_{21} be force on 2 due to 1 then

$$\vec{F}_{12} = -\vec{F}_{21}$$

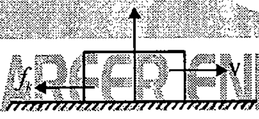
Newton's third law assumes that to an action there is prompt reaction. This implies that in Newton's law it is assumed that information travels with infinite speed.

Violation of Newton's Laws: Newton's 1st and 2nd laws are strictly valid in inertial frames. In non-inertial frames Newton's laws do not hold good. In non-inertial frames we can apply Newton's law if we include pseudo forces.

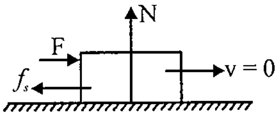
There are some cases where Newton's third law also does not hold. For example magnetic force exerted by two non-parallel current carrying wires on each other are not in opposite in direction.

Friction force: Its arise because of interaction (electromagnetic) among the particles of surfaces in contact. It is of two types

(i) Kinetic friction: It acts on objects which are sliding on a surface $f_k = \mu_k N$



(ii) Static friction: It acts on object which are subjected to a force but not sliding on the contact surface. It is an adjustable force $f_s \leq \mu_s N$



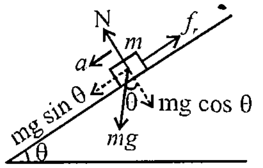
Example : A block slides down a rough inclined plane of inclination θ. Coefficient of friction between the block and the plane is μ. Calculate acceleration of block.

Soln. The block slides on the inclined plane.

Therefore, to get acceleration of block we resolve all forces parallel and perpendicular to the plane.

Therefore, for equilibrium perpendicular to the plane

$$N = mg\cos\theta \quad \ldots (i)$$



equation of motion parallel to plane, $mg\sin\theta - f_r = ma \quad \ldots$ (ii)

and, $f_r = \mu N = \mu\, mg\cos\theta$

from (ii), $mg\sin\theta - \mu mg\cos\theta = ma \quad \therefore\ a = g(\sin\theta - \mu\cos\theta)$

Example: A block of mass '*m*' is pulled with a force F at an angle θ with horizontal. The block does not move on the surface. Calculate force of friction on the block and value of coefficient of friction.

Soln. The block is stationary, therefore horizontal components of F must be balanced by friction force.

$\therefore f_r = F\cos\theta$

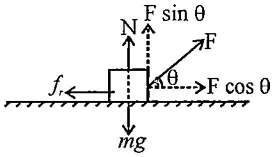
for vertical equilibrium, $N + F\sin\theta = mg$

or, $N = mg - F\sin\theta$

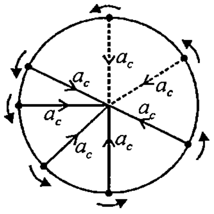
if μ be coefficient of static friction then,

$f_r \le \mu N$

or $F\cos\theta \le \mu(mg - F\sin\theta)$ $\quad\therefore \mu \ge \dfrac{F\cos\theta}{mg - F\sin\theta}$

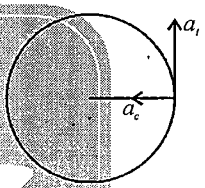


Centripetal acceleration: When a particle moves in a circular path, due to continuous change in direction of velocity of the particle, particle always have an acceleration towards centre of circle.



This is called centripetal acceleration. Its value is $a_c = \dfrac{v^2}{r}$

If the particle's velocity is non-uniform then in addition to centripetal acceleration, the particle also has a tangential acceleration which is equal to $a_t = \dfrac{dv}{dt}$



Centripetal force: It is not a new kind of force. When an object moves in a circular path the net components of known forces towards centre of circle is called centripetal force.
It turns out that if a particle of mass *m* is moving in a circle of radius *r*. Then at any instant when its speed is v, net force towards centre $= ma_c = \dfrac{mv^2}{r}$.

Example : A point mass *m* is tied to a light string of length '*l*'. It is whirled in a vertical circle. Its speeds at lowest point, highest point and at a point where string makes 60° with upward vertical are $\sqrt{6gl}, \sqrt{2gl}$ and $\sqrt{3gl}$ respectively. Calculate tension in string at these points.

Soln. The point mass is moving in a circle, therefore at each point of circle net force towards centre must be equal to $\dfrac{mv^2}{r}$.

At lowest point: Net force towards centre $= T - mg$

$$\therefore\ T - mg = \frac{mv^2}{l} \text{ or } T = mg + \frac{m6gl}{l} = 7mg$$

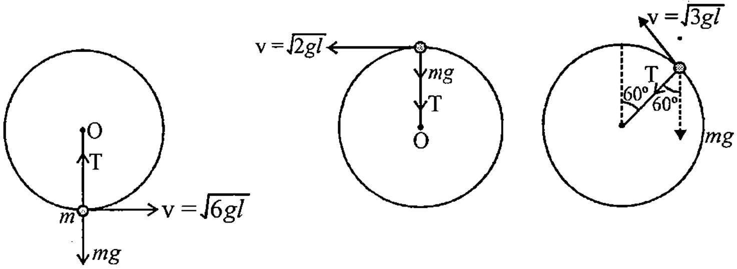
At highest point: Net force towards centre $= T + mg$

$$\therefore T + mg = \frac{mv^2}{l} \text{ or } T = \frac{m2gl}{l} - mg = mg$$

At 60° with upward vertical: Net force towards centre $= T + mg\cos 60°$

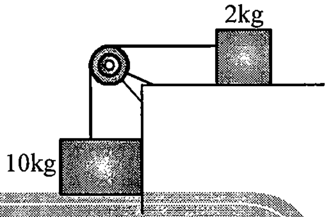
$$\therefore T + mg\cos 60° = \frac{mv^2}{l}$$

$$T = \frac{m.3gl}{l} - \frac{mg}{2} = \frac{5mg}{2}$$


O
T
m
v = $\sqrt{6gl}$
mg
v = $\sqrt{2gl}$
mg
T
O
v = $\sqrt{3gl}$
T
60°
60°
mg

SOLVED PROBLEMS

1. In the figure, the tension in the inelastic string is T when all the surfaces are frictionless. If 2kg block is glued to the surface, the tension in the string will be **[GP-2007]**



(a) Zero (b) Greater than T
(c) Less than T (d) Equal to T

Soln. (b)

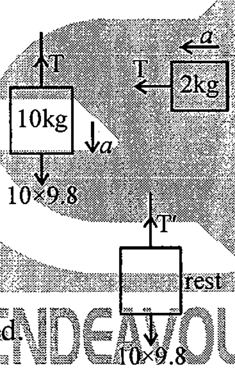
When all surfaces are smooth the two blocks will accelerate. If 'a' be the acceleration

then, $10\times 9.8 - T = 10a$

and $T = 2a$

$$\therefore \quad \frac{98-T}{T} = \frac{5}{1}$$

$$T = \frac{98}{6} = 16.33N$$


T
a
T
2kg
10kg
a
10×9.8
T'
rest
10×9.8

when 2kg is glued,

$T' = 10\times 9.8 = 98\text{N}$

$\therefore$ Tension in string increases when 2kg block is glued.

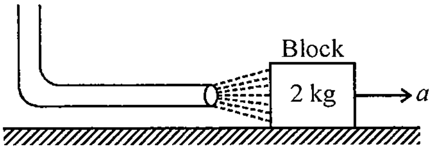
2. A block of metal weighting 2 kg is resting on a frictionless plane. It is struck by a jet releasing water at at rate of 1 kg/s and at a speed of 5 m/s. Calculate the initial acceleration of the block.

Soln. $F = \frac{d}{dt}(mv) = m\frac{dv}{dt} + v\frac{dm}{dt}$

$$\therefore \quad F = v\frac{dm}{dt} \qquad (\because\ v = \text{constant})$$

$$= 5\times 1 = 5\text{ N}$$

$$F = ma \text{ or } a = \frac{F}{m} = \frac{5}{2} = 2.5\text{ m/s}^2.$$

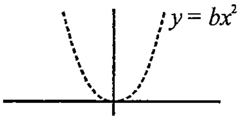


3. A particle of mass moves along a curve on a horizontal surface. The equation of curve is $y = bx^2$. Without using curvature formula find radius of curvature at $x = 0$.

Soln. Radial acceleration of particle at $x = 0$ is

$$a_y = \left.\frac{d^2y}{dt^2}\right|_{x=0}$$

$$= \left[\frac{d^2}{dt^2}(bx^2)\right]_{x=0} = \left[2b(\dot{x}^2 + x\ddot{x})\right]_{x=0} = 2b\dot{x}^2$$



Since at $x = 0$, $\dot{x} = v_0$

$$\therefore \quad a_y = 2bv_0^2$$

Since radial acceleration is given as

$$a_y = \frac{v_0^2}{r}$$

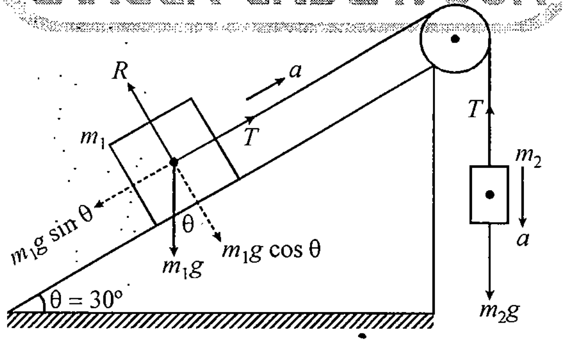
where 'r' is radius of curvature

$$\therefore \quad 2bv_0^2 = \frac{v_0^2}{r}$$

$$\therefore \quad r = \frac{1}{2b}$$

4. A block of mass $m_1 = 4$ kg on a smooth inclined plane of 30° is connected by a cord over a small, frictionless pulley to a second block of mass $m_2 = 5$ kg hanging vertically. Calculate the acceleration with which the block moves and also the tension in the cord. Take $g = 10$ m/sec^2.

Soln. The different forces acting on the masses are shown in figure.



We have $T - m_1g\sin\theta = m_1a$ ... (i)

and $m_2g - T = m_2a$ ... (ii)

Solving equations (i) and (ii), we get

$$a = \frac{(m_2 - m_1\sin\theta)g}{(m_1 + m_2)} \quad \text{... (iii)}$$

and $$T = m_2 g\left[1 - \frac{(m_2 - m_1 \sin\theta)}{(m_1 + m_2)}\right]$$

or $$T = \frac{m_1 m_2 (1 + \sin\theta) g}{(m_1 + m_2)} \qquad \text{... (iv)}$$

Here $m_1 = 4\,\text{kg},\ m_2 = 5\,\text{kg},\ \theta = 30^\circ$ and $g = 10\ \text{m/s}$

Substituting these values in equation (iii), we get

$$T = \frac{5 \times 4\left(1 + \frac{1}{2}\right)10}{9} = \frac{300}{9} = 33.33\ \text{N}.$$

5. A uniform rope of length L is pulled by a constant force F. What is the tension in the rope at a distance l from the end, where it is applied ?

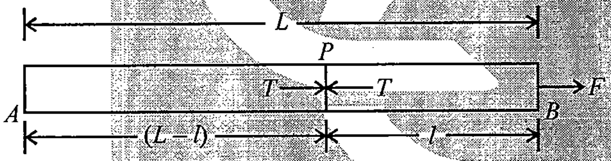
Soln. See figure below,

For part PB,

$$F - T = (\text{Mass of } PB)\, a, \text{ where } a = \text{acceleration.}$$

or $$F - T = \left(\frac{M}{L}\right) \times l \times a, \qquad \text{... (i)}$$

where $\frac{M}{L}$ is mass per unit length.



For part AP,

$$T = (\text{Mass of } AP)\, a = \left(\frac{M}{L}\right)(L - l)\, a. \qquad \text{... (ii)}$$

Solving the two, we get $a = T\dfrac{L}{M}(L - l)$.

Substituting a in equation (i) and simplifying, we get

$$T = \left(1 - \frac{l}{L}\right)F$$

6. At the moment $t = 0$ a stationary particle of mass m experiences a time-dependent force $F = kt(t' - t)$, where k is a constant vector, t' is the time during which the given force acts. Find :

(a) the momentum of the particle when the action of the force discontinued ;

(b) the distance covered by particle while the force acted.

Soln. (a) Since $F = kt(t' - t)$, the direction of the force is fixed, as k is a constant vector and the particle starts from rest, the motion will be in the direction of the force.

$$m\frac{dv}{dt} = kt(t' - t), \text{ from Newton's II law}$$

Integrating, $\int_0^{v} dv = \frac{k}{m}\int_0^{t'} t\,(t'-t)\,dt = \frac{k}{m}\left[t'\frac{t^2}{2}-\frac{t^3}{3}\right]_0^{t'}$ $\qquad v = \frac{k}{m}\cdot\frac{t'^3}{6}$

Therefore, momentum of the particle $p = mv = kt\frac{t'^3}{6}$.

(b) $ds = v\,dt \Rightarrow s = \int_0^{s} ds = \int_0^{t'} v\,dt = \frac{kt'^4}{12m}$.

7. A motor-boat of mass m moves along a lake with velocity v_0. At the moment $t = 0$ the engine of the boat is shut down. Assuming the resistance of the particle to be proportional to the velocity of the boat $F = -rv$, find:

(a) how long the motorboat moved with the shutdown engine;

(b) the velocity of the motorboat as a function of the distance covered with the shutdown engine, as well as total distance covered till the complete stop.

(c) the mean velocity of the motorboat over the time interval (beginning with the moment $t = 0$), during which its velocity decreases η times.

Soln. (a) $F = rv \Rightarrow m\frac{dv}{dt} = -rv \Rightarrow \frac{dv}{v} = -\frac{r}{m}dt$

Integrating, $\log_e v = -\frac{r}{m}t + C$

At $t = 0,\ v = v_0,\ \therefore\ C = \log_e v_0$

Therefore, $v = v_0\,e^{-rt/m}$; $v \to 0$ as $t \to \infty$

Hence, the velocity becomes zero after in infinite time.

(b) From above, $m\frac{dv}{dt} = m\frac{dv}{ds}\cdot\frac{ds}{dt} = mv\frac{dv}{ds} = -rv$ or $dv = -\frac{r}{m}ds$

Integrating, we get $v = -\frac{r}{m}s + C_1$

But $v = v_0$ when $s = 0,\ \therefore C_1 = v_0$

Therefore, $v = v_0 - \frac{rs}{m}$

Total distance covered : $s_0 = \frac{mv_0}{r}$, by putting $v = 0$ at $s = s_0$

(c) Let the velocity becomes $\frac{v_0}{\eta}$ at time t',

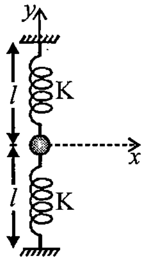
$\therefore\quad \frac{v_0}{\eta} = v_0\,e^{-rt'/m} \Rightarrow t' = -\frac{m}{r}\log_e\frac{1}{\eta} = \frac{m}{r}\log_e \eta$

$\therefore\quad \bar{v}$ = average velocity during time interval t'

$$= \frac{\int_0^{t'} v(t)\,dt}{t'} = \frac{\overset{\frac{m}{}}{-\frac{r}{r}}\,v_0\,e^{-rt/m}}{t'} \text{ or } \bar{v} = \frac{-\frac{m}{r}v_0\left(e^{-rt'/m}-1\right)}{\frac{m}{r}\log_e \eta} = \left.\frac{v_0(\eta-1)}{\eta\log_e \eta}\right].$$

8. A particle of mass m is connected to two springs of unstretched length 'l' and spring constant k as shown in figure. Calculate acceleration of particle if it is slightly displaced along X-direction.

Soln. Let the particle is displaced by a distance x along $+x$ direction as shown in figure.

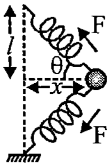


Therefore, stretched length of spring $= \sqrt{l^2+x^2}$

elongation in springs $= \sqrt{l^2+x^2} - l$

Restoring force on the particle due to each spring $F = k(\sqrt{l^2+x^2} - l)$

Because of restoring force the particle moves back towards its initial position.



Equation of motion of particle $F_x = m\dfrac{d^2x}{dt^2}$

or $-2F\cos\theta = m\dfrac{d^2x}{dt^2}$ or $-2k(\sqrt{l^2+x^2} - l).\dfrac{x}{\sqrt{l^2+x^2}} = m\dfrac{d^2x}{dt^2}$

Therefore, $\dfrac{d^2x}{dt^2} = -\dfrac{2kx}{m}\left(1-\dfrac{l}{\sqrt{l^2+x^2}}\right)$ or $\dfrac{d^2x}{dt^2} = -\dfrac{2kx}{m}\left[1-\left(1+\dfrac{x^2}{l^2}\right)^{-1/2}\right]$

Since, $x << l$, $\left(1+\dfrac{x^2}{l^2}\right)^{-1/2} \approx 1-\dfrac{x^2}{2l^2}$

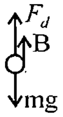
Therefore, $\dfrac{d^2x}{dt^2} = -\dfrac{2kx}{m}\left[1-\left(1-\dfrac{x^2}{2l^2}\right)\right]$ or $\dfrac{d^2x}{dt^2} = -\dfrac{kx^3}{ml^2}$

9. Spherical particles of a given material of density ρ are released from rest inside a liquid medium of lower density. The viscous drag force may be approximated by the Stoke's law, i.e, $F_d = 6\pi\eta R\mathrm{v}$, where η is the viscosity of the medium, R the radius of a particle and v its instantaneous velocity. If $\tau(m)$ is the time taken by a particle of mass m to reach half its terminal velocity, then the ratio $\tau(8m)/\tau(m)$ is

Soln. Each particle has same density but different radii and masses. We have to calculate time of fall in terms of mass of particle therefore we will write our equations explicitly in terms of mass and we will remove radius from our equations wherever it appears.

Drag force on particles

$F_d = 6\pi\eta R\mathrm{v}$, Mass of a particle $m = \rho.\dfrac{4}{3}\pi R^3$



$\therefore R = \left(\dfrac{3m}{4\pi\rho}\right)^{1/3}$ $\therefore F_d = 6\pi\eta\left(\dfrac{3m}{4\pi\rho}\right)^{1/3} v = Km^{1/3}v$, where $K = 6\pi\eta\left(\dfrac{3}{4\pi\rho}\right)^{1/3}$

If B is buoyancy force, then equation of motion of a particle is

$m\dfrac{dv}{dt} = mg - F_d - B$ where B $= \dfrac{4}{3}\pi R^3\sigma g = \dfrac{4}{3}\pi R^3\rho g.\dfrac{\sigma}{\rho} = mg\dfrac{\sigma}{\rho}$, σ = density of medium

$$\therefore\ m\frac{dv}{dt} = mg - Km^{1/3}v - mg\frac{\sigma}{\rho},\quad m\frac{dv}{dt} = mg\left(1-\frac{\sigma}{\rho}\right) - Km^{1/3}v$$

$$\text{or}\quad \frac{dv}{dt} = g\left(1-\frac{\sigma}{\rho}\right) - Km^{-2/3}v \qquad \text{... (a)}$$

when terminal velocity is reached $\frac{dv}{dt} = 0$

$$\therefore\ 0 = g\left(1-\frac{\sigma}{\rho}\right) - Km^{-2/3}v_t \qquad \therefore\ v_t = \frac{g\left(1-\frac{\sigma}{\rho}\right)}{Km^{-2/3}} \qquad \text{...(b)}$$

Now, if τ be the time to reach half the terminal velocity then from (a)

$$\int_0^{v_t/2} \frac{dv}{g\left(1-\frac{\sigma}{\rho}\right) - Km^{-2/3}v} = \int_0^{\tau} dt,\quad \therefore\ -\frac{1}{Km^{-2/3}}\ln\left[\frac{g\left(1-\frac{\sigma}{\rho}\right) - \frac{Km^{-2/3}v_t}{2}}{g\left(1-\frac{\sigma}{\rho}\right)}\right] = \tau$$

using value of v_t from (b) we get, $\tau = \frac{m^{2/3}}{K}\ln 2$ or $\tau(m) = \frac{m^{2/3}\ln 2}{K}$

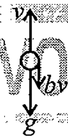
$$\therefore\ \tau(8m)/\tau(m) = (8m)^{2/3}/m^{2/3} = 4$$

10. A particle of unit mass is thrown vertically upward with initial speed v_0. It is acted upon by a drag force bv^2 in addition to gravity where b is constant and v is instantaneous velocity of particle. Calculate speed of the particle when it returns to the point from where it was thrown.

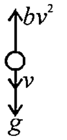
Soln. In presence of drag force only quantity that is common in upward and downward motion is the distance covered.

For upward motion, initial speed $= v_0$,

final speed = 0, let height reached = h



upward motion



downward motion

equation of motion $\frac{dv}{dt} = -g - bv^2$

$$\text{or}\quad \frac{dv}{dy}.\frac{dy}{dt} = -g - bv^3,\ \frac{dy}{dt} = v \quad \text{or} \quad \frac{vdv}{g+bv^2} = -dy$$

$$\therefore\ \int_{v_0}^{0} \frac{vdv}{g+bv^2} = -\int_0^h dy$$

$$\text{on integration we get,}\quad \frac{1}{2b}\ln\left(\frac{g+bv_0^2}{g}\right) = h \qquad \text{... (a)}$$

for downward motion, initial speed = 0, final speed = v(let), height descended = h

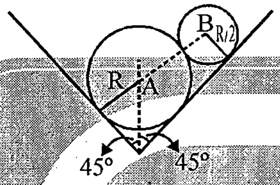
$$\text{equation of motion}\quad \frac{dv}{dt} = g - bv^2 \quad \text{or} \quad \frac{vdv}{g-bv^2} = dy \quad \text{or} \quad \int_0^{v} \frac{vdv}{g-bv^2} = \int_0^h dy$$

on integration we get, $\frac{1}{2b}\ln\left(\frac{g}{g-bv^2}\right)=h$ ... (b)

Comparing (a) and (b) we get, $\frac{g+bv_0^2}{g}=\frac{g}{g-bv^2}$, $g-bv^2=\frac{g^2}{g+bv_0^2}$ or $v=\frac{v_0}{\sqrt{1+\frac{bv_0^2}{g}}}\Rightarrow v<v_0$

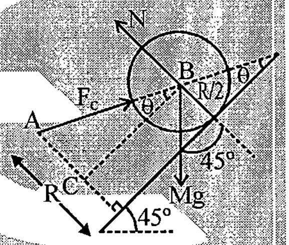
due to drag force the particle returns with a speed less than its initial speed. If drag force were absent $b=0$, then $v=v_0$. Therefore in absence of drag force particle returns with same speed as its initial value.

11. Two solid spheres each of mass M and radii R and $\frac{R}{2}$ stay in equilibrium between two perpendicular smooth planks as shown in the figure. Find the value of contact force between the two spheres.



Soln. Line of their contact force will pass through their centres.

Free body diagrams of smaller sphere is shown in figure.



In triangle ACB

$AC=R/2$

$AB=R+\frac{R}{2}=\frac{3R}{2}$

$\therefore\ CB=R\sqrt{2}$

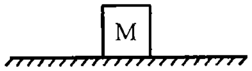
$\therefore\ \cos\theta=\frac{CB}{AB}=\frac{2\sqrt{2}}{3}$

Component of forces parallel to plank surfaces give

$F_c\cos\theta-Mg\sin 45°=0$

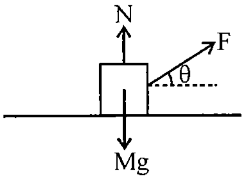
$\therefore\ F_c=\frac{Mg/\sqrt{2}}{\cos\theta}=\frac{3Mg}{4}$

12. A block of mass M lies on a rough surface. If μ_s be coefficient of static friction between the block and the surface. Calculate the minimum force required to move the block.



Soln. Let the force is applied at an angle θ with the horizontal

$\therefore$ For minimum value of F which can make the block move.



$N+F\sin\theta=Mg$

or $N=Mg-F\sin\theta$ ... (i)

$F\cos\theta=\mu N$

or $F\cos\theta = \mu(Mg - F\sin\theta)$

$$\therefore \quad F = \frac{\mu Mg}{\cos\theta + \mu\sin\theta} \qquad \text{... (ii)}$$

For F to be minimum $(\cos\theta + \mu\sin\theta)$ should be maximum

$$\therefore \quad \frac{d}{d\theta}(\cos\theta + \mu\sin\theta) = 0$$

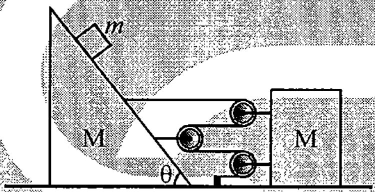
$$-\sin\theta + \mu\sin\theta = 0$$

$$\therefore \quad \tan\theta = u$$

$$\therefore \quad \sin\theta = \frac{\mu}{\sqrt{\mu^2+1}} \text{ and } \cos\theta = \frac{1}{\sqrt{1+\mu^2}}$$

$\therefore$ from (ii) we get $F = \dfrac{\mu Mg}{\sqrt{1+\mu^2}}$

13. In the figure shown, find the acceleration of triangular block. All the surfaces are smooth.



Soln. As block *m* slides triangular block will move towards left due to which rectangular block will also move towards left.

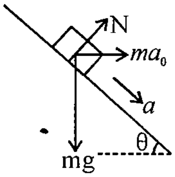
Let a_0 be acceleration of triangular block.

When triangular block moves, length of 3 segments of string changes whereas when rectangular block moves, length of four segments changes. Therefore, acceleration of rectangular block = $\frac{3}{4}a_0$.

Let T be tension in string.

$\therefore$ For rectangular block $4T = M\left(\frac{3}{4}a_0\right)$

$$\therefore \qquad T = \frac{3Ma_0}{16}$$



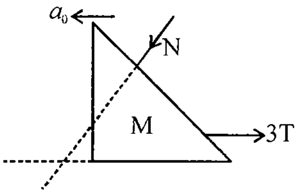
For *m*: (free body diagram is shown in figure)

$N = mg\cos\theta - ma_0\sin\theta$

For triangular block

$N\sin\theta - 3T = Ma_0$

$$m(g\cos\theta - a_0\sin\theta)\sin\theta - \frac{9Ma_0}{16} = Ma_0$$



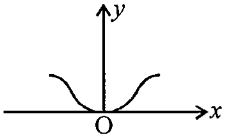
$$mg\sin\theta\cdot\cos\theta=\left(\frac{25}{16}M+m\sin^2\theta\right)a_0 \qquad \therefore \quad a_0=\frac{g\sin\theta\cdot\cos\theta}{\left(\sin^2\theta+\frac{25M}{16m}\right)}$$

14. A bead of mass m moves with constant speed v_0 along a wire having shape $y=b\cos\frac{2\pi x}{L}$ kept vertically. Calculate normal reaction due to the wire when bead is at $x=0$.

Soln. Equation of motion of bead at $x=0$

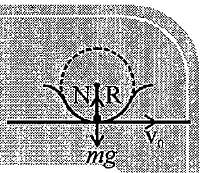
$$N-mg=\frac{mv_0^2}{R}$$

where N is normal reaction and R is radius of curvature.



$$\therefore \quad N=\frac{mv_0^2}{R}+mg \qquad \text{... (i)}$$

Here, $\frac{dx}{dt}=v_0$ and $\ddot{x}=0$ at $x=0$



and $\frac{d^2y}{dt^2}=-\frac{v_0^2}{R}$ at $x=0$

$$\therefore \quad \frac{d}{dt}\left(-\dot{x}\frac{2\pi b}{L}\sin\frac{2\pi}{L}x\right)_{x=0}=-\frac{v_0^2}{R}$$

$$\text{or} \quad \left[-\ddot{x}\frac{2\pi b}{L}\sin\frac{2\pi}{L}x-\dot{x}^2\left(\frac{2\pi}{L}\right)^2 b\cos\frac{2\pi}{L}x\right]_{x=0}=-\frac{v_0^2}{R}$$

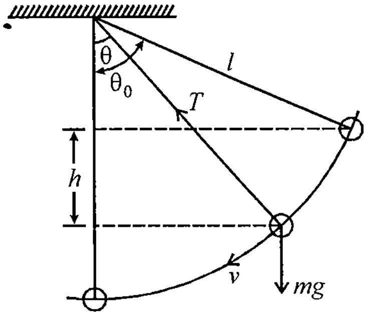
$$\Rightarrow R=\frac{L^2}{4\pi^2 b}$$

$\therefore$ from (i) we get

$$N=\frac{mv_0^2}{L^2}\cdot 4\pi^2 b+mg \quad =m\left[\left(\frac{2\pi}{L}v_0\right)^2 b+g\right]$$

15. A 40 kg mass, hanging at the end of a rope of length l, oscillates in a vertical plane with an angular amplitude of θ_0. What is the tension in the rope, when it makes an angle θ with the vertical ? If the breaking strength of the rope is 80 kg f, what is the maximum angular amplitude with which the mass can oscillate without the rope breaking ?

Soln. The situation is shown in figure below.



(a) From figure $h = l(\cos\theta - \cos\theta_0)$

and $v^2 = 2gh = 2gl(\cos\theta - \cos\theta_0)$ ... (i)

Again $T - mg\cos\theta = mv^2/l$ ... (ii)

Substituting the value of v^2 from equation (i) in equation (ii), we get

$$T - mg\cos\theta = m\{2gl(\cos\theta - \cos\theta_0)/l\}$$

or $T = mg\cos\theta + 2mg(\cos\theta - \cos\theta_0)$

or $T = mg(3\cos\theta - 2\cos\theta_0)$

or $T = 40g(3\cos\theta - 2\cos\theta_0)$ newton

or $T = 40(3\cos\theta - 2\cos\theta_0)$ kgf

(b) Let θ_0 be the maximum amplitude. The maximum tension T will be at mean position where $\theta = 0$.

$\therefore$ $T_{\max} = 40(3 - 2\cos\theta_0)$

But $T_{\max} = 80$

Solving we get $\theta_0 = 60°$.

16. A smooth table is placed horizontally and an ideal spring of spring constant $k = 1000$ N/m and unextended length of 0.5 m has one end fixed to its center. The other end is attached to a mass of 5 kg which is moving in a circle with constant speed 10 m/s. Find the tension in the spring and the extension of the spring beyond its normal length.

Soln. Let T be the tension produced in the streched string. The centripetal force required for the mass m to move in a circle is provided by the tension T. The streched length of the spring is r (= radius of the circle). Now,

Elongation produced in the spring $= (r - l_0)$

Tension produced in the spring $T = k(r - l_0)$ ... (i)

where k is spring constant.

Centripetal force $= \dfrac{mv}{r}$

Therefore, $k(r - l_0) = \dfrac{mv^2}{r}$ or $kr(r - l_0) = mv^2$ ... (ii)

Substituting the given values, we get

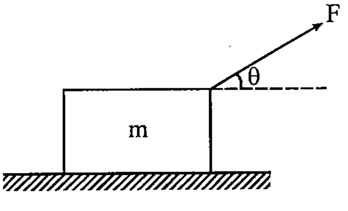
$$\begin{aligned} 1000\,r(r-0.5) &= 5\times 100 \\ r(r-0.5) &= 0.5 \\ \text{or} \quad 2r^2 - r - 1 &= 0 \end{aligned}$$

Solving, we get $t = 1.0$ m

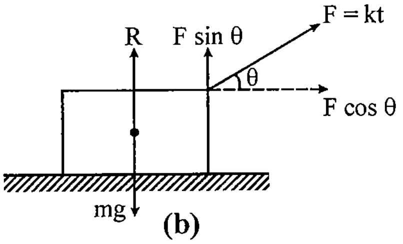
Now, the extension of the spring beyond its normal length is $r - l_0 = 1.0 - 0.5 = 0.5$ m

From equation (i), $T = 1000 \times 0.5 = 500$ N.

17. At the moment $t = 0$, the force $F = kt$ is applied to a small body of mass m resting on smooth horizontal plane (k = constant). The permanent direction of this force forms an angle θ with the horizontal (figure below). Find



(a)



(b)

(a) the velocity of the body at the moment of its breaking off the plane;
(b) the distance traversed by the body up to this moment.

Soln. The free body diagram is shown in figure.

From figure : $R + F\sin\theta = mg$

At breaking off, $R = 0$, $\therefore F = \dfrac{mg}{\sin\theta}$

Now $kt = \dfrac{mg}{\sin\theta}$ or $t = \dfrac{mg}{k\sin\theta}$

(a) If a be the acceleration, then $F\cos\theta = ma$

Therefore, $kt\cos\theta = m \times \left(\dfrac{dv}{dt}\right)$ $\left(\because a = \dfrac{dv}{dt}\right)$.

Integrating this expression within proper limits, we get $m\int_0^v dv = k\cos\theta \int_0^{mg/k\sin\theta} t\,dt$

or $mv = \dfrac{k\cos\theta}{2}\left[\dfrac{mg}{k\sin\theta}\right]^2$ or $v = \dfrac{mg^2}{2k}\left(\dfrac{\cos\theta}{\sin^2\theta}\right)$

(b) Without putting the limits, we have $v = \dfrac{k\cos\theta}{2m}t^2 + C$, where C = constant of integration.

When $t = 0$, $v = 0$ and hence $C = 0$.

Now, $\dfrac{ds}{dt} = \dfrac{k\cos\theta}{2m}t^2$ $\int_0^s ds = \dfrac{k\cos\theta}{2m}\int_0^{mg/k\sin\theta} t^2\,dt$

$s = \dfrac{k\cos\theta}{2m}\left[\dfrac{t^3}{3}\right]_0^{mg/k\sin\theta}$ or $s = \dfrac{m^2g^3\cos\theta}{6k^2\sin^3\theta}$.

18. A bar of mass m resting on a smooth horizontal plane starts moving due to a force $F = mg/3$ of constant magnitude. In the process of its rectilinear motion, the angle θ between the direction of this force and the horizontal varies as $\theta = ks$, where k is a constant. Find the velocity of the bar as a function of angle θ.

Soln. The situation is shown in figure below.

From free body diagram

$$F\sin\theta + R = mg \qquad \text{... (i)}$$

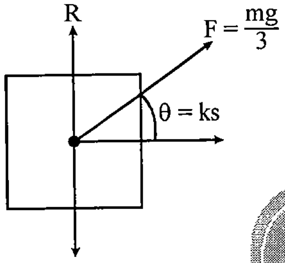
and $F\cos\theta = m\,a$ ... (ii)

From equation (ii),

$$\left(\frac{m\,g}{3}\right)\cos k\,s = m\left(\frac{dv}{dt}\right) = m\,v\left(\frac{dv}{ds}\right)$$

or $\int_0^v v\,dv = \frac{g}{3}\int \cos k\,s\;ds$

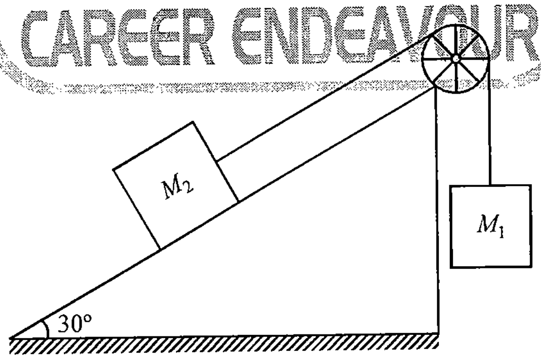
or $\frac{v^2}{2} = \frac{g}{3k}\sin k\,s + C$



But when $v = 0, s = 0\;\therefore C = 0$

Therefore, $v^2 = \frac{2g}{3k}\sin k\,s$ or $v = \sqrt{\left[\left(\frac{2g}{3a}\right)\sin\theta\right]}$.

19. (a) Two masses M_1 and M_2 are connected by light string, which passes over the top of a smooth plane inclined at 30° to the horizontal, so that one mass rests on the plane and the other hangs vertically as shown in figure. It is found that M_1, hanging vertically can draw M_2 up the full length of the plane in half the time in which M_2 hanging vertically draws M_1 up. Find M_1/M_2. Assume pulley to be smooth.



(b) Initially at time $t = 0$ smooth masses ($M_1 = 15$ kg and $M_2 = 10$ kg) are held at rest and then they released. If after one second, the string snaps, find the further time taken for the 15 kg mass to return to its original position on the plane.

(Take $g = 10$ m/s^2)

Soln. (a) Let T be the testing in the string. Then

$M_1 g - T = M_1 a_1$ ... (i)

and $T - M_2 g\sin 30° = M_2 a_1$ ... (ii)

Solving equations (i) and (ii), we get

$$a_1 = \frac{g(M_1 - M_2 \sin 30^\circ)}{M_1 + M_2} \qquad \text{... (iii)}$$

Further, $M_2 g - T = M_2 a_2$ ... (iv)

and $T - M_1 g \sin 30^\circ = M_1 a_2$ ... (v)

Solving equation (iv) and (v), we get

$$a_2 = \frac{g(M_2 - M_1 \sin 30^\circ)}{M_1 + M_2} \qquad \text{... (vi)}$$

If s be the distance travelled, then $s = \frac{1}{2} a_1 t_1^2 = \frac{1}{2} a_2 t_2^2$.

$$\therefore \quad \frac{a_1}{a_2} = \frac{t_2^2}{t_1^2}$$

$$\text{or} \quad \frac{a_1}{a_2} = 4 \quad \left(\because \frac{t_1}{t_2} = \frac{1}{2} \text{ given} \right) \qquad \text{... (vii)}$$

Dividing equations (iii) and (vi), we have

$$\frac{a_1}{a_2} = \frac{M_1 - M_2 \sin 30^\circ}{M_2 - M_1 \sin 30^\circ} \qquad \text{... (viii)}$$

Substituting the value of $\left(\frac{a_1}{a_2} \right)$ from equation (vii) in equation (viii), we get

$$4 = \frac{M_1 - M_2 \sin 30}{M_2 - M_1 \sin 30} = \frac{M_1 - (M_2/2)}{M_2 - (M_1/2)}$$

$$4 = \frac{2M_1 - M_2}{2M_2 - M_1}$$

$$\text{or} \quad 8M_2 - 4M_1 = 2M_1 - M_2 \quad \text{or} \quad 9M_2 = 6M_1 \quad \text{or} \quad \left(\frac{M_1}{M_2} \right) = \frac{3}{2}.$$

(b) The acceleration a before snapping of the string is given by

$$a = \frac{g(M_2 - M_1 \sin \theta)}{(M_1 + M_2)} = \frac{10(10 - 15/2)}{25} = 1 \text{ m/s}^2$$

Let s be the distance travelled in one second after the release of masses, then

$$s = 0 + \frac{1}{2} a t^2 = \frac{1}{2} \times 1 \times (1)^2 = 0.5 \text{ meter up the plane.}$$

The velocity v will be $v = at = 1 \times 1 = 1$ m/s.

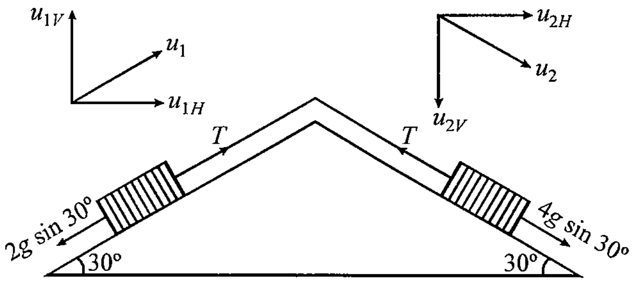
After snapping of the string, let t be the time taken by the mass 15 kg to return to its original position. The distance travelled would be -0.5 m. Its initial velocity u is 1 m/s and its acceleration a will be $-g \sin 30$.

Now, $s = ut + \frac{1}{2} a t^2$ or $-0.5 = 1 \times t - \frac{1}{2}\left(10 \times \frac{1}{2} \right) t^2$ or $5t^2 - 2t - 1 = 0$.

Solving we get $t = 0.69$ second.

20. A double inclined plane is in the form of a wedge with fixed horizontal base and smooth faces inclined at angle of 30° to the horizontal. A 4 gm particle is on one face and is connected by a light flexible, inextensible string over the smooth rigid of the wedge to a 2 gm particle on other face. If the particles start moving from the position with the string taut, find the direction and magnitude of the velocity of 2 gm particle relative to the 4 gm one second later, if the string is sufficiently long for the particles still to be on each face on the wedge.

Soln. The situation is shown in figure.



Let a be the acceleration produced. The equation of motion of two particles are written as

$$(4\times10^{-3})g\sin 30° - T = (4\times10^{-3})a \qquad ...(i)$$

$$T-(2\times10^{-3})g\sin 30° = (2\times10^{-3})a \qquad ...(ii)$$

Adding these two equations we get

$$2\times10^{-3}g\sin 30 = (6\times10^{-3})a$$

$$\therefore \quad a=\frac{g}{6}=\frac{9.8}{6}=1.63\ \text{m/s}^2$$

After one second, the speed of particle along the slopes will be 1.63 m/s. Resolving velocities horizontally and vertically, we have

For $2g$ mass,

$$u_{1H}=u_1\cos 30°=1.63\times\left(\frac{\sqrt{3}}{2}\right)\text{m/s}$$

$$u_{1V}=u_1\sin 30°=\left(\frac{1.63}{2}\right)\text{m/s}$$

For $4g$ mass,

$$u_{2H}=u_2\cos 30°=1.63\times\left(\frac{\sqrt{3}}{2}\right)\text{m/s}$$

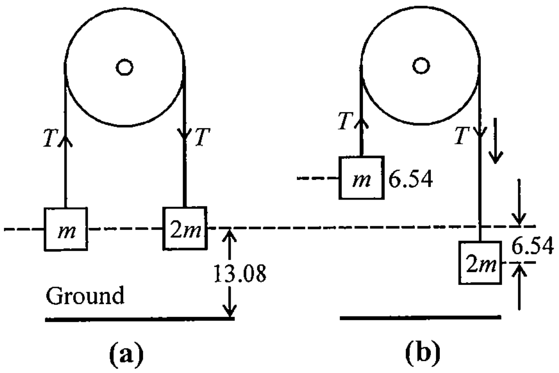
$$u_{2V}=u_2\sin 30°=\left(\frac{1.63}{2}\right)\text{m/s}$$

The horizontal components are equal and in the same direction while the vertical components are equal in magnitude but opposite in direction.

The relative vertical velocity is given by

$$\frac{1.63}{2}-\left(-\frac{1.63}{2}\right)=0.815+0.815=1.63\ \text{m/s upward.}$$

21. Two masses m and $3m$ are connected by a massless string which passes over a light frictionless pulley shown in figure. The masses are initially held with equal lengths of the strings on either side of the pulley. Find the velocity of the masses at the instant the lighter mass moves up a distance of 6.54 meter. This string is suddenly cut at that instant. Calculate the time taken by each mass to reach the ground. Take $g = 981$ cm/sec^2.



Soln. Let T be the tension in the string and a be the acceleration of the system. The mass m will move upward and mass $2m$ will go downward with same acceleration. Here we have

$$2mg - T = 2m \times a \qquad \text{...(i)}$$

and $T - mg = ma$ ...(ii)

Solving these equations,

$$a = \frac{g}{3} = \frac{981}{3} = 327 \text{ cm/sec}^2 \text{ or } a = 3.27 \text{ m/sec}^2.$$

For lighter mass, initial velocity $= u$, distance moved $= 6.54$ m, acceleration $a = 3.27$ m/sec^2, final velocity $v = ?$

Now $v^2 = u^2 + 2as = 0 + 2 \times 3.27 \times 6.54$ or $v^2 = 6.54 \times 6.54 \therefore v = 6.54$ m/sec.

Similarly, the velocity of heavior mass = 6.54 m/sec in downward direction.

When the string is cut, the distance of lighter mass from ground = 13.08 + 6.54 = 19.62 meter and the distance of heavier mass from the ground = 13.08 − 6.54 = 6.54 m. Let t_1 and t_2 be the time taken by the lighter and heavier masses to reach the ground respectively.

For lighter mass $s_1 = 19.62$ m, $u = -6.54$ m/sec, $g = 9.81$ m/sec^2, $t_1 = ?$

Using $s = ut + \frac{1}{2}gt^2$, we have $19.62 = \frac{1}{2} \times 9.8\, t_1^2 - 6.54\, t_1$

or $3t_1^2 + 4t_1 - 4 = 0$ or $(3t_1 - 2)(t_1 + 2) = 0$

As t can not be negative, hence $t_1 = \frac{2}{3}$ seconds.

For heavier mass, $s_2 = 6.54$ m, $u = 6.54$ m/sec.

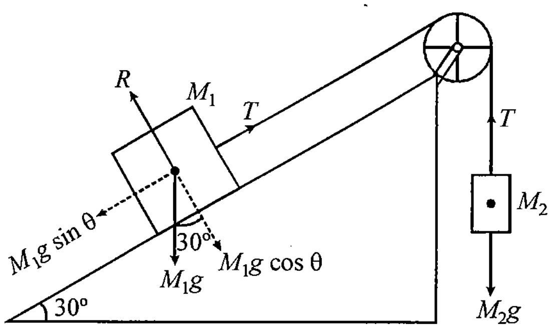
Therefore, $6.54 = 6.54 \times t_2 + \frac{1}{2} \times 9.81\, t_2^2$

$$3t_2^2 + 4t_2 - 4 = 0$$

Solving we get = 2.88 seconds.

22. A wire of 9.8×10^{-3} kg mass per meter meter passes over a frictionless pulley fixed on the top of an inclined frictionless plane which makes an angle of 30° with the horizontal. Masses M_1 and M_2 are tied at the two ends of the wire. The mass M_1 rests on the plane and the mass M_2 hangs freely vertically downwards. The whole system is in equilibrium. Now a transverse wave propagates along the wire with a velocity of 100 m/sec. Find M_1 and M_2.

Soln. The situation is shown in figure.



Here, we have

$$T = M_1 g \sin 30° \quad \text{...(i)}$$

$$R = M_1 g \cos\theta \quad \text{...(ii)}$$

and $$T = M_2 g \quad \text{...(iii)}$$

From equation (i) and (iii), we have

$$M_1 g \sin 30° = M_2 g \text{ or } \frac{M_1}{2} = M_2 \text{ or } M_1 = 2M_2$$

Velocity of transverse wave $v = \sqrt{\frac{T}{m}}$, where m is the mass per unit length.

$$\therefore \quad v^2 = \frac{T}{m} \text{ or } T = v^2 m$$

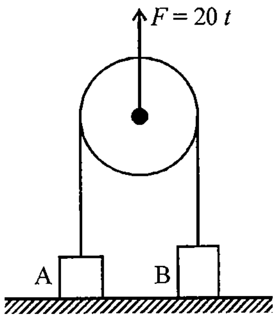
or $$T = (100 \text{ m/s})^2 \times (9.8 \times 10^{-3} \text{ kg/m}) = 98 \text{ newton} \quad \text{...(v)}$$

From equation (iii), $M_2 = \frac{T}{g} = \frac{98 \text{ newton}}{9.8 \text{ newton/kg}} = 10 \text{ kg}$

From equation (iv), $M_1 = 2 \times M_2 = 2 \times 10 \text{ kg} = 20 \text{ kg}$.

23. Two blocks A and B of mass 1 kg and 2 kg respectively are connected by a string, passing over a light frictionless pulley. Both the blocks are resting on a horizontal floor and the pulley is held such that string remains just taut.

At moment $t = 0$, a force $F = 20t$ newton starts acting on the pulley along vertically upward direction as shown in figure. Calculate



(a) Velocity of A and B when loses contact with the floor.
(b) Height raised by the pulley upto that instant and
(c) Workdone by the force F upto that instant $g = 10\text{ m/s}^2$.

Soln. (a) Let T be the tension in the string. Then, $2T = 20t$ or $T = 10t$ newton.

Let the block A loses its contact with the floor at time $t = t_1$. This happens when the tension in string becomes equal to the weight of A. Thus

$$T = mg \text{ or } 10t_1 = 1 \times 10 \text{ or } t_1 = 1\text{ sec.} \quad \text{... (i)}$$

Similarly, for block B, we have

$$10t_2 = 2 \times 10 \text{ or } t_2 = 2\text{ sec} \quad \text{... (ii)}$$

i.e., the block B loses contact after 2 seconds. For block A, at time t such that $t_1 \le t$, let a be the its acceleration in upward direction. Then

$$10t - 1 \times 10 = 1 \times a = \left(\frac{dv}{dt}\right)$$

or $$dv = 10(t-1)\,dt \quad \text{... (iii)}$$

Integrating this expression, we get

$$\int_0^v dv = 10 \int_{t_1}^{t} (t-1)\,dt$$

or $$v = 5t^2 - 10t + 5 \quad \text{... (iv)}$$

Now velocity v at $t = 2$ sec is $v = 20 - 20 + 5 = 5$ m/s. ... (v)

(b) From equation (i)

$$dy = (5t^2 - 10t + 5)\,dt \quad \text{... (vi)}$$

where y is the vertical displacement of block A at time $t\,(> t_1)$

Integrating equation (vi), we get

$$\int_{y=0}^{y=h} dy = \int_{t=1}^{t=2} (5t^2 - 10t + 5)\,dt$$

$$h = 5\left[\frac{t^3}{3}\right]_1^2 - 10\left[\frac{t^2}{2}\right]_1^2 + 5[t]_1^2 = \frac{5}{3}\text{ m}$$

Therefore, height raised by pulley upto that instant $= \frac{h}{2} = \frac{5}{6}$ m.

(c) The workdone by the force $W =$ increased in potential energy (mgh) of block $A +$ K.E. of the block

$$= mgh + \frac{1}{2}mv^2$$

$$= 1 \times 10 \times \left(\frac{5}{3}\right) + \frac{1}{2} \times 1 \times (25) = \left(\frac{175}{6}\right)\text{joule.}$$

24. A block is projected up with 10 m/s along a fixed inclined plane of inclination 37° with the horizontal. If the time of ascend from the point of projection is half the time of descend to the same point, find the distance travelled by the block during the up and down journey.

Soln. Let μ, t_1 and t_2 be the coefficient of friction between the plane and the block, time of ascend and time of descend respectively.

The retardation while going up

$$a_1 = g(\sin\theta + \mu\cos\theta) = 10\left(\frac{3}{5} + \mu\frac{4}{5}\right).$$

The acceleration while descending

$$a_2 = g(\sin\theta - \mu\cos\theta) = 10\left(\frac{3}{5} - \mu\frac{4}{5}\right).$$

Now, $s =$ distance of ascend $=$ distance of descend.

As final velocity is zero, we have

$$0 = u - a_1 t_1 \text{ or } u = a_1 t_1$$

Now, $s = a_1 t_1 - \frac{1}{2}a_1 t_1^2 = \frac{1}{2}a_1 t_1^2$

Then, $s = \frac{1}{2}a_1 t_1^2 = \frac{1}{2}a_2 t_2^2$ and $t_2 = 2t_1$

$$\therefore \quad \left(\frac{a_2}{a_1}\right) = \left(\frac{t_1}{t_2}\right)^2 = \left(\frac{1}{2}\right)^2 \text{ or } \frac{\left(\frac{3}{5} - \mu\frac{4}{5}\right)}{\left(\frac{3}{5} + \mu\frac{4}{5}\right)} = \left(\frac{1}{2}\right)^2$$

Solving we get $\mu = \left(\frac{9}{20}\right)$

Again $a_1 = 10\left(\frac{3}{5} + \frac{9}{20} \times \frac{4}{5}\right) = 9.6\text{ m/sec}^2$

$$\therefore \quad s = \frac{u^2}{2a_1} = \frac{(10)^2}{2 \times 9.6} = 5.21\text{ meter}$$

So total distance $= 2s = 10.42$ meter.

25. A block of mass 2 kg slides on an inclined plane which makes an angle of 30° with the horizontal. The coefficients of friction between the block and the surface is $\sqrt{(3/2)}$.

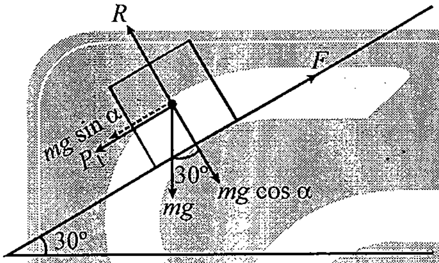
(i) What force should be applied to the block so that the block moves down without any acceleration ?
(ii) What force should be applied to the block so that it moves up without any acceleration ?
(iii) Calculate the ratio of power needed in the above two cases, if the block moves with the same speed in both the cases.

Soln. (i) The situation is shown in figure (a), acts upwards along the plane. Let a force P_1 be applied in this case. Resolving along and perpendicular to the plane, we have

$$P_1 + mg\sin\alpha = F \qquad \text{...(i)}$$

and $$mg\cos\alpha = R \qquad \text{...(ii)}$$

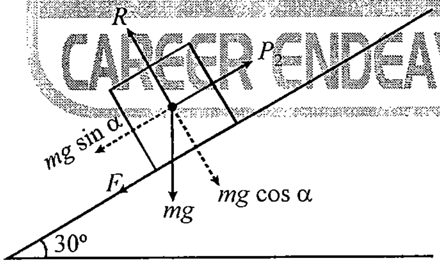
where $F = \mu R = \mu mg\cos\alpha = \frac{\sqrt{3}}{2} \times 2 \times 9.8 \times \frac{\sqrt{3}}{2} = \frac{29.4}{\sqrt{2}}$ N.



(a)

$$\therefore \quad P_1 = F - mg\sin\alpha = \frac{29.4}{\sqrt{2}} - \frac{2\times 9.8\times 1}{2} = \frac{29.4}{\sqrt{2}} - 9.8 = 11 \text{ newton.}$$

(ii) The situation is shown in figure (b).



(b)

Let P_2 be the force applied upwards in this case. Resolving along and perpendicular to the plane, we have

$$P_2 = mg\sin\alpha + F$$

$$R = mg\cos\alpha$$

$$\because \quad F = \frac{29.4}{\sqrt{2}} \text{ newton}$$

$$\therefore \quad P_2 = 2\times 9.8\times\frac{1}{2} + \frac{29.4}{\sqrt{2}} = 9.8 + \frac{29.4}{\sqrt{2}} = 30.6 \text{ newton.}$$

(iii) We know that, power = force × velocity.

Since velocity is same in both the cases, the ratio of powers will be equal to the ratio of forces applied.

Therefore, ratio $=\frac{P_1}{P_2}=\frac{11}{30.6}=0.36.$

26. A small bar starts sliding down an inclined plane forming an angle θ with the horizontal. The friction coefficient depends on the distance x covered as $\mu = kx$, where k is a constant. Find the distance covered by the bar till it stops and its maximum velocity over this distance.

Soln. The net acceleration of the body

$$a=\frac{mg\sin\theta-\mu\, mg\cos\theta}{m}=g\sin\theta-\mu g\cos\theta$$

$$=g\sin\theta-(kx)g\cos\theta \;=g(\sin\theta-kx\cos\theta) \qquad \text{...(i)}$$

We know that

$$a=\frac{dv}{dt}=\frac{dv}{dx}\times\frac{dx}{dt}=v\times\frac{dv}{dx}$$

$$\therefore \qquad v\times\frac{dv}{dx}=g(\sin\theta-kx\cos\theta)$$

or $\qquad v\,dv=g(\sin\theta-kx\cos\theta)\,dx \qquad \text{...(ii)}$

Integrating this expression, we get

$$\frac{v^2}{2}=(g\sin\theta)x-gk\frac{x^2}{2}\cos\theta+C \qquad \text{...(iii)}$$

where C is integration constant.

When $x=0,\ v=0$ and hence $C=0$

$$\frac{v^2}{2}=(g\sin\theta)x-gk\left(\frac{x^2}{2}\right)\cos\theta \qquad \text{...(iv)}$$

When $v=0$

$$x\left[(g\sin\theta)-gk\left(\frac{x}{2}\right)\cos\theta\right]=0$$

either $x=0$ or $g\sin\theta-gk\left(\frac{x}{2}\right)\cos\theta=0$

$$\therefore \qquad x=\frac{2\sin\theta}{k\cos\theta}=\frac{2}{k}\tan\theta \qquad \text{...(v)}$$

For maximum velocity, $\left(\frac{dv}{dx}\right)=0$

From equation (iv),

$$v\frac{dv}{dx}=g\sin\theta-gkx\cos\theta=0$$

or $\qquad x=\frac{\sin\theta}{k\cos\theta}=\frac{1}{k}\tan\theta \qquad \text{...(vi)}$

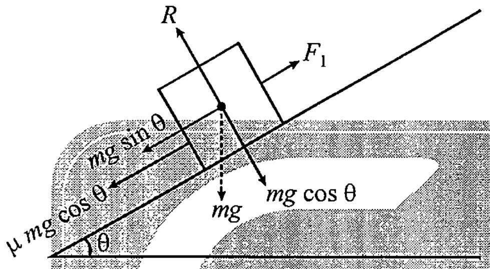
So, at a distance $x = \frac{1}{k}\tan\theta$, the velocity will be maximum. Putting this value in equation (iv), we get

$$\frac{v^2}{2} = (g\sin\theta)\,(\tan\frac{\theta}{k}) - gk\,(\tan^2\frac{\theta}{2k^2})\cos\theta$$

Solving we get $v_{\max} = \sqrt{\left[\left(\frac{g}{k}\right)\sin\theta\tan\theta\right]}$.

27. The force required just to move a body up an inclined plane is double the force required just to prevent the body sliding down it. The coefficient of friction is μ. Prove that the angle of inclination of the plane $\theta = \tan^{-1}(3\mu)$.

Soln. Let F_1 be the force required to move a body up the inclined plane with inclination θ (see figure (a)).

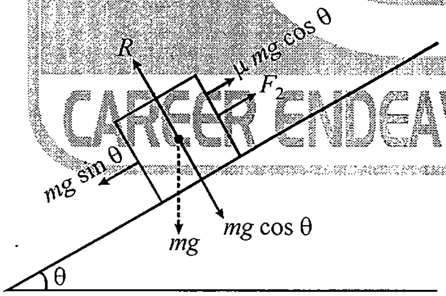


(a)

$$F_1 = mg\sin\theta + \mu mg\cos\theta \qquad \text{...(i)}$$

Let F_2 be the force required just to prevent the body sliding down. Then as shown in figure (b) we have

$$mg\sin\theta = F_2 + \mu\, mg\cos\theta \qquad \text{...(ii)}$$



(b)

According to the problem

$$mg\sin\theta + \mu mg\cos\theta = 2[mg\sin\theta - \mu\, mg\cos\theta]$$

or $\quad \sin\theta + \mu\cos\theta = 2\sin\theta - 2\mu\cos\theta$

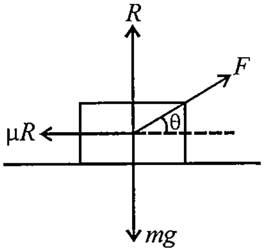
$$3\mu\cos\theta = \sin\theta$$

$$3\mu = \tan\theta$$

$\therefore \quad \theta = \tan^{-1}(3\mu)$.

28. A particle of mass m rests on a horizontal floor with it has a coefficient of static friction μ. It is desired to make the body move by applying the minimum possible force F. Find the magnitude of F and the direction in which it has to be applied.

Soln. The situation in shown in figure. Let F be the force applied at an angle θ with the horizontal.



The horizontal and vertical components of F are $F\cos\theta$ and $F\sin\theta$ respectively. For horizontal and vertical equilibriums, we have

$$R + F\sin\theta = mg, \quad \text{... (i)}$$

and $$F\cos\theta = \mu R, \quad \text{... (ii)}$$

From equation (i), we get $R = (mg - F\sin\theta)$.

Substituting in equation (ii), we get

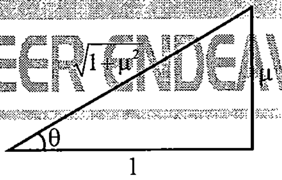
$$F\cos\theta = \mu(mg - F\sin\theta) = \mu mg - \mu F\sin\theta$$

$$F[\cos\theta + \mu\sin\theta] = \mu mg.$$

$$\therefore \quad F = \frac{\mu mg}{\cos\theta + \mu\sin\theta} \quad \text{... (iii)}$$

For the force to be minimum, the denominator $(\cos\theta + \mu\sin\theta)$ should be maximum.

i.e., $$\frac{d}{d\theta}(\cos\theta + \mu\sin\theta) = 0 - \sin\theta + \mu\cos\theta = 0 \text{ or } \tan\theta = \mu \quad \text{... (iv)}$$



Now, $\sin\theta = \dfrac{\mu}{\sqrt{(1+\mu^2)}}$ and $\cos\theta = \dfrac{1}{\sqrt{(1+\mu^2)}}$.

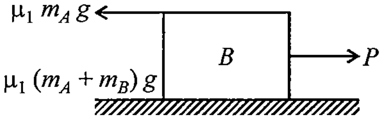
Substituting these values in equation (iii), we get

$$F_{\min} = \frac{\mu mg}{\dfrac{1}{\sqrt{(1+\mu^2)}} + \dfrac{\mu\cdot\mu}{\sqrt{(1+\mu^2)}}} = \frac{\mu mg}{\dfrac{1+\mu^2}{\sqrt{(1+\mu^2)}}} = \frac{\mu mg}{\sqrt{(1+\mu^2)}} \quad \text{... (v)}$$

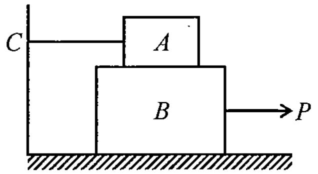
and the angle of direction of force with horizontal.

$\theta = \tan^{-1}\mu$ [from equation (iv)].

29. Block A (mass 200 kg) rests on block B (mass 300 kg) tied with a horizontal string to the wall at C shown in figure (a). The coefficient of friction between A and B is 1/4 and that between B and surface 1/5. What horizontal force P in necessary to move the block B.



(a)



(b)

Soln. Frictionless force on B due to A,

$$= \mu_1 m_A g = \frac{1}{4} \times 200 \times 10 = 500 \text{ newton.}$$

Frictional force on B due to surface,

$$= \mu_2 (m_A + m_B)\, g$$

$$= \frac{1}{5}(200 + 300) \times 10 = 1000 \text{ newton.}$$

$$\therefore \quad P = \mu_1 m_A g + \mu_2 (m_A + m_B)\, g = 500 + 1000 = 1500 \text{ newton.}$$

30. A body of mass m rests on a horizontal plane with the friction coefficient μ. At the moment $t = 0$, a horizontal force is applied to it, which varies with time $F = kt$, where k is a constant vector. Find the distance traversed by the body during first/seconds after the force action began.

Soln. Here just after applying the force, the motion does not start due to friction force. As the applied force is proportional to time, let after a time t_0, the motion starts.

Now, $F = k t_0 = \mu mg$ or $t_0 = \left(\frac{\mu mg}{k}\right)$

If $t \le t_0$, the distance traversed by the body $s = 0$.

When $t \le t_0$, then $F = k(t - t_0)$

$$\therefore \quad m\frac{dv}{dt} = k(t - t_0) \text{ or } m\,dv = k(t - t_0)\,dt \text{ ... (i)}$$

Integrating equation (i), we get $mv = \frac{k}{2}(t - t_0)^2 + C_1$

When $t = t_0, v = 0, \therefore C_1 = 0$

$$\therefore \quad mv = \frac{k}{2}(t - t_0)^2 \qquad \text{... (ii)}$$

$$\text{Again } m\frac{ds}{dt} = \frac{k}{2}(t - t_0)^2 \text{ or } m\,ds = \frac{k}{2}(t - t_0)^2\,dt \qquad \text{... (iii)}$$

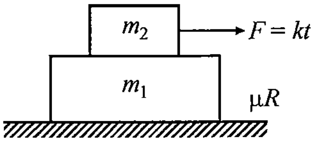
Integrating equation (iii), we get $s = \frac{k}{6m}(t - t_0)^3 + C_2$

Here $C_2 = 0$ because when $t = t_0, s = 0$

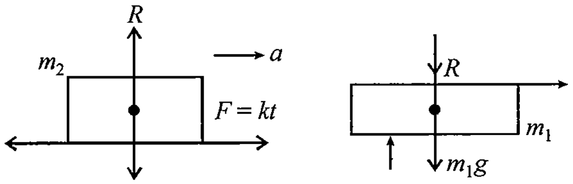
$$\therefore \quad s = \frac{k}{6m}(t - t_0)^3.$$

31. A plank of mass m_1 with a bar of mass m_2 placed on it lies on a smooth horizontal plane. A horizontal force growing with time t as $F = kt$ (k is constant) is applied to the bar. Find how the acceleration of the plank a_1 and of the bar a_2 depend on t, if the coefficient of friction between the plank and the bar is equal to μ.

Soln. The situation is shown in figure (a). Figure (b) shows the free body diagrams of m_1 and m_2.
It is obvious that if the frictional force is greater than the applied force, then both the blocks will move together. After some time, when the frictional force is less than applied force, then both the blocks will move with different accelerations.



(a)



(b)

(i) Let, both the blocks move together for a time t_0. In this case, acceleration $a = \dfrac{F}{m_1 + m_2}$ or $a = \dfrac{k t_0}{m_1 + m_2}$.

From free body diagram of m_2, we have

$$F - \mu m_2 g = m_2 a$$

$$k t_0 - \mu m_2 g = \frac{m_2 k t_0}{m_1 + m_2}$$

$$t_0\left(k - \frac{m_2 k}{m_1 + m_2}\right) = \mu m_2 g$$

$$\therefore \quad t_0 = \frac{\mu m_2 g (m_1 + m_2)}{m_1 k}$$

Hence for $t \le t_0$, both the blocks will move together with acceleration $= \dfrac{kt}{(m_1 + m_2)}$.

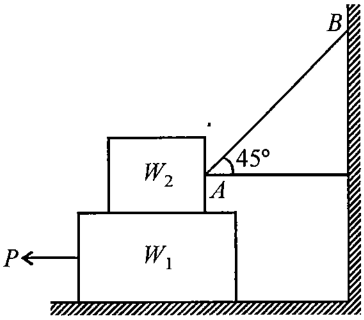
(ii) When applied force on m_2 is greater than frictional force :
In this case, both the blocks will move with different accelerations.
Consider the free diagram of body m_2. We have $(F - \mu m_2 g) = m_2 a_2$ or $(at - \mu m_2 g) = m_2 a_2$

$$\therefore \quad a_2 = \frac{(at - \mu m_2 g)}{m_2} \text{ where } t \ge t_0.$$

Similarly, for mass m_1

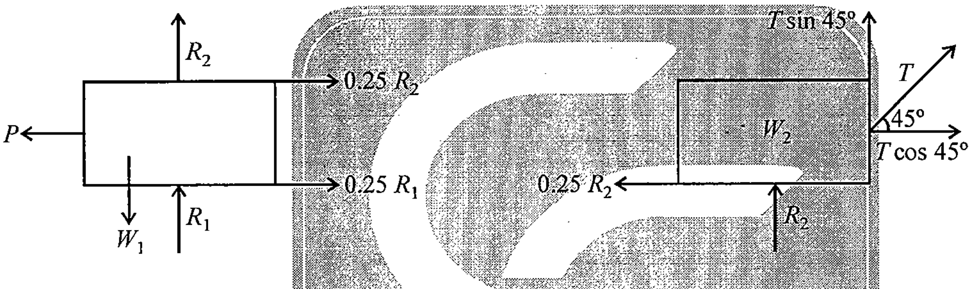
$$a_1 = \frac{(\mu m_2 g)}{m_1} \text{ where } t \ge t_0.$$

32. Calculate the force P required to cause the block of weight $W_1 = 200$ N just to slide the block of weight $W_2 = 100$ N. What is the tension in the string AB? Coefficient of friction $\mu = 0.25$ for all surfaces in contact.



(a)

Soln. The different forces are shown in figure (b).



(b)

For upper mass

$$R_2 + T\sin 45° = 100 \qquad \ldots(i)$$

and $$T\cos 45° = 0.25\,R_2 \qquad \ldots(ii)$$

Solving equations (i) and (ii), we get $R_2 = 80$ Newton.

For lower mass, $R_2 + W_1 = R_1$ or $80 + 200 = R_1$

Therefore, $R_1 = 280$ Newton

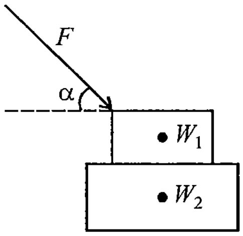
Now, $P = 0.25\,R_1 + 0.25\,R_2 = 0.25\,(R_1 + R_2) = 0.25\,(280 + 80) = 90$ Newton.

From equation (i), $R_2 + T\sin 45° = 100$ or $T\sin 45° = 100 - 80 = 20$

Therefore, $T = \dfrac{20}{\sin 45°} = 20 \times \sqrt{2}$ Newton.

33. In figure, weight W_1 is on top of weight W_2 which is turn on a horizontal plane. The coefficient of friction between W_1 and W_2 is μ_1 while that between W_2 and the plane is μ_2. Suppose that a force F inclined at an angle α to the horizontal is applied to weight W_1. Prove that if $\cot\alpha \geq \mu_1 > \mu_2$, then W_2 moves relative to the plane while W_1 does not move relative to W_2 is that

$$\frac{\mu_2 (W_1 + W_2)}{(\cos\alpha - \mu_2 \sin\alpha)} < F \leq \frac{\mu_1 W_1}{(\cos\alpha - \mu_1 \sin\alpha)}.$$



Soln. The following forces are acting on W_1:

(i) Weight W_1 acting vertically downwards.

(ii) Applied force F at an angle α.

(iii) Normal reaction R due to W_2.

(iv) Frictional force F_r along the horizontal direction towards left.

Since W_1 does not move with respect to W_2, hence

$$F_r = F\cos\alpha \quad \text{...(i)}$$

and $$R = W_1 + F\sin\alpha \quad \text{...(ii)}$$

Now $F_r = \mu_1 R = \mu_1(W_1 + F\sin\alpha) = F\cos\alpha$

or $\mu_1 W_1 + \mu_1 F\sin\alpha = F\cos\alpha$

$$\therefore \quad F = \frac{\mu_1 W_1}{\cos\alpha - \mu_1\sin\alpha} \quad \text{...(iii)}$$

If F is less than this, the block W_1 would still not move.

Let the block of mass W_2 moves horizontally with acceleration a. The following forces are acting on this block.

(i) The weight $(W_1 + W_2)$ acting vertically downward.

(ii) Normal reaction N due to floor.

(iii) Applied force F at an angle α.

(iv) Frictional force F_r

As there is no vertical motion

$$N = F\sin\alpha + W_1 + W_2 \quad \text{...(iv)}$$

As the weight is moving towards the right

$$F\cos\alpha - F_r > 0 \quad \text{...(v)}$$

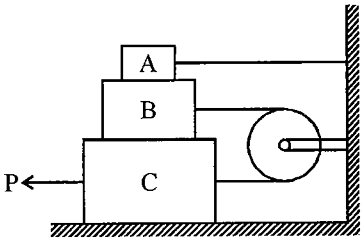
$$F\cos\alpha > F_r > \mu_2(F\sin\alpha + W_1 + W_2)$$

$$F > \frac{(W_1 + W_2)\mu_2}{\cos\alpha - \mu_2\sin\alpha} \quad \text{...(vi)}$$

For W_2 to move when W_1 is at rest w.r.t. W_2

$$\frac{(W_1 + W_2)\mu_2}{(\cos\alpha - \mu_2\sin\alpha)} < F \le \frac{\mu_1 W_1}{(\cos\alpha - \mu_1\sin\alpha)}.$$

34. In figure, the blocks A, B and C weight are 3 kg and 8 kg respectively. The coefficient of sliding friction between any two surfaces is 0.25. A is held at rest by a massless rigid rod fixed to the wall, while B and C are connected by a light flexible cord passing around a fixed frictionless pulley. Find the force P necessary to drag C along the horizontal surface to the left at a constant speed. Assume that the arrangement shown in the diagram, B on C and A on B is maintained all throughout.

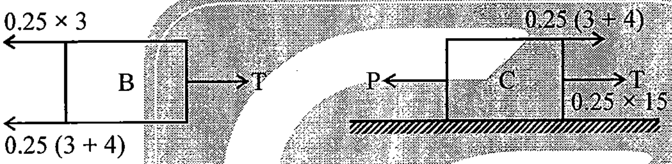


(a)

Soln. When block C moves towards left, B moves towards right, while A is fixed. There would be a tension T in the string. Under this condition, let us consider the frictional forces between different surfaces.

Frictional force between A and B $= \mu R = 0.25 \times 3$

Frictional force between C and surface $= 0.25\,(3+4+8) = 0.25 \times 15.$



(b)

Considering figure (b),

Tension in the string = Frictional forces at upper and lower surfaces of block B

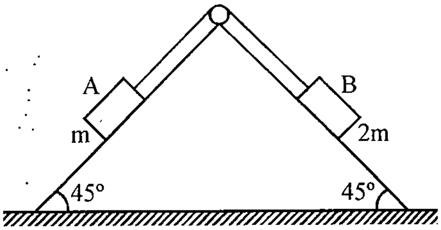
or $\quad T = 0.25 \times 3 + 0.25 \times 7 = 2.5$ kg wt.

35. Block A of mass m and block B and mass $2m$ are placed on a fixed triangular wedge by means of a massless, inextensible string and a frictionless pulley as shown in figure. The wedge is inclined at 45° to the horizontal on both sides. The coefficient of friction between block A and wedge is 2/3 and that between block B and the wedge is 1/3. If the system of A and B is released from rest, find

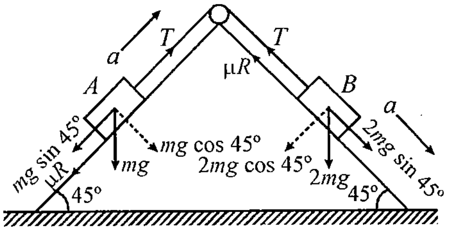
(i) the acceleration of A

(ii) tension in the string

(iii) the magnitude and direction of the force of friction acting on A.



Soln. (i) Let the acceleration of B is in the downward direction. The different forces on the masses are shown in figure.



From figure, for B

$$2\,mg\sin 45° - T - \frac{1}{3}\times 2\,mg\cos 45° = 2\,ma$$

or $$\sqrt{2}\,mg\sin 45° - T - \frac{\sqrt{2}\,mg}{3} = 2\,ma$$

or $$\frac{2\sqrt{2}\,mg}{3} - T = 2\,ma \qquad \text{... (i)}$$

For A, $T - mg\sin 45° - \frac{2}{3}mg\cos 45° = ma$

$$T - \frac{5}{3}\frac{mg}{\sqrt{2}} = ma \qquad \text{... (ii)}$$

Adding equations (i) and (ii), we get $\left(\frac{2\sqrt{2}}{3} - \frac{5}{3\sqrt{2}}\right)mg = 3\,ma$

Solving we get, $a = \frac{g}{9}\left[\frac{4-5}{\sqrt{2}}\right] = -\frac{g}{9\sqrt{2}}$.

Thus, a is negative.

Similarly, if we take the motion of A downward, then too a is negative. This is not possible. Here it should be noted that the body can moved only under the component of weight of the body parallel to the plane but here the resistive force is more than this force. So, acceleration of $A = 0$.

(ii) Now, $T + \frac{1}{3}\times(2\,mg\cos 45) = 2\,mg\sin 45$

$$T = \sqrt{2}\,mg - \frac{\sqrt{2}}{3}mg = \frac{2\sqrt{2}}{3}mg$$

(iii) If we take the frictional force in the upward $T + f = \frac{mg}{\sqrt{2}}$

or $$f = \frac{mg}{\sqrt{2}} - \frac{2\sqrt{2}}{3}mg = -\frac{mg}{3\sqrt{2}}.$$

So, the magnitude of frictional force will be $\frac{mg}{3\sqrt{2}}$ and it will act downward.

36. If the earth be one-half its present distance from the sun, how many days will be in one year ?

Soln. Let the radius of the earth's orbit round the sun be R. When it is reduced to half, the new distance becomes $R/2$. As the earth is moving round the sun, centripetal force is provided by the gravitational attraction of the sun on the earth.

Hence, $\frac{mv^2}{R} = \frac{GMm}{R^2}$, where the symbols have usual meanings.

$$\therefore \quad v^2 = \frac{Gm}{R} \qquad \text{... (i)}$$

If T be the time period of the earth to complete one revolution round the sun, then

$$v=\frac{2\pi R}{T} \qquad \text{... (ii)}$$

From equations (i) and (ii), we have

$$\frac{4\pi^2 R^2}{T^2}=\frac{GM}{R} \text{ or } T^2=\frac{4\pi^2 R^3}{GM} \qquad \text{... (iii)}$$

Let T_1 be the time period of revolution of earth round the sun when the distance between the two is $R/2$.

$$T_1^2=\frac{4\pi^2 R^3}{8GM} \qquad \text{... (iv)}$$

Dividing equation (iv) and equation (iii), we have

$$T_1^2=\frac{T^2}{8} \text{ or } T_1=\frac{T}{2\sqrt{2}}$$

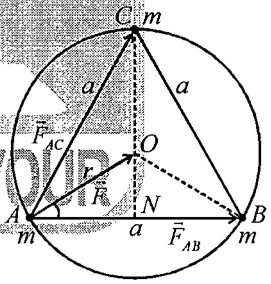
$$\therefore \quad T_1=\frac{365}{2\sqrt{2}} \quad (\because \text{ 1 year} = 365 \text{ days})$$

or $\quad T_1 =$ **129 days.**

37. Three particles, each of mass m are situated at the vertices of an equilateral triangle of side length a. The only forces acting on the particles are their mutual gravitational forces. It is desired that each particle moves in a circle while maintaining the original separation a. Find the initial velocity that should be given to each particle and also the time period of the circular motion.

Soln. The situation is shown in figure. The radius of circular path

$$r=OA=OB=OC=\left(\frac{AN}{\cos 30^\circ}\right)=\frac{a/2}{\sqrt{3}/2}=\frac{a}{\sqrt{3}}.$$



The gravitational force at A, due to mass m at C

$$F_{AC}=\frac{Gmm}{a^2} \text{along } AC$$

Similarly, the gravitational force at A, due to mass m at B

$$F_{AB}=\frac{Gmm}{a^2} \text{along } AB$$

The resultant F of these forces would be along AO. As the angle between the two forces is 60°, hence the resultant force F is given by

$$F=\sqrt{\left(F_{AC}^2+F_{AB}^2+2F_{AC}F_{AB}\cos 60^\circ\right)}$$

$$=\sqrt{\left[\left(\frac{Gmm}{a^2}\right)^2+\left(\frac{Gmm}{a^2}\right)^2+2\left(\frac{Gmm}{a^2}\right)\left(\frac{Gmm}{a^2}\right)\cdot\frac{1}{2}\right]}$$

$$=\frac{Gmm}{a^2}\left[\sqrt{3}\right].$$

This force provides the necessary centripetal force, i.e.,

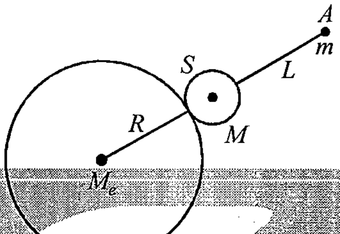
$$\frac{mv^2}{r}=\frac{Gmm}{a^2}\sqrt{3} \text{ or } \frac{mv^2}{\left(a/\sqrt{3}\right)}=\frac{Gmm}{a^2}\sqrt{3}$$

Therefore, required initial speed $v = \left\{\dfrac{Gm}{a}\right\}^{1/2}$

Time period $T = \dfrac{2\pi r}{v} = \dfrac{2\pi\left(a/\sqrt{2}\right)}{(Gm/a)^{1/2}} = \dfrac{2\pi a^{3/2}}{(3Gm)^{1/2}}$.

38. A cord of length 64 m is used to connect a 100 kg astronaut to spaceship whose mass is much larger than that of the astronaut. Estimate the value of the tension in the cord. Assume that the spaceship is orbiting near earth surface. Also assume that the spaceship and the astronaut fall on a straight line from the earth center. The radius of the earth is 6400 km.

Soln. See figure below. Given that $R = 6400$ km, $l = 64$ m and mass m of astronaut = 100 kg.



The gravitational attraction between earth and spaceship $S = \dfrac{GM_eM}{R^2}$.

This force supplies the necessary centripetal force to spaceship, i.e.,

$$\frac{GM_eM}{R^2} = \frac{Mv^2}{R} = M\omega^2R \text{ or } \omega^2 = \frac{GM_e}{R^3} \text{ or } \omega = \sqrt{\left(\frac{GM_e}{R^3}\right)}.$$

But, $\dfrac{GM_e}{R^2} = g$, hence $\omega = \sqrt{\left(\dfrac{g}{R}\right)}$ ... (i)

The centrifugal force on the astronaut $F_1 = m\omega^2(R+l)$ ... (ii)

The centripetal force on astronaut $F_2 = \dfrac{GM_em}{(R+l)^2}$ ... (iii)

The tension in the cord would be the difference of centrifugal force and centripetal force i.e., $T = F_1 - F_2$.

$$\therefore \quad T = m\omega^2(R+l) - \frac{GM_em}{(R+l)^2} \qquad \text{... (iv)}$$

Substituting the value of ω from equation (i) and $GM_e = gR^2$ in equation (iv), we get

$$T = \frac{mg(R+l)}{R} - \frac{gR^2m}{(R+l)^2} = mg + \frac{mgl}{R} - \frac{mg}{[1+(l/R)]^2} = mg + \frac{mgl}{R} - mg\left[1 - \frac{2l}{R}\right]$$

$$= mg + \frac{mgl}{R} - mg + \frac{2mgl}{R} = \frac{3mgl}{R} = \frac{3\times100\times10\times64}{6400\times10^3} = \frac{3}{10^2} = \mathbf{3\times10^{-2}\ N}.$$

Chapter 3

WORK ENERGY & POWER

Conservative force: A force is said to be conservative if curl of the force is zero.

i.e. if $\vec{\nabla}\times\vec{F}=0$ then, $\vec{F}$ is conservative force.

Work done by a conservative force is independent of path followed and work done along a closed path is zero. Gravitational and electrostatic forces are conservative. Friction, viscous forces are not conservative.

Potential energy: It is defined as work done by conservative force in taking the system from given configuration (position) to a reference position. Reference positions is that where the conservative force vanishes.

Potential energy can be directly obtained from relation

$U=-\int F dr$ or $U=-\int F dx$

for attractive force F is used with negative sign and for repulsive force F is used with positive sign.

Spring Force: When a spring is either stretched or compressed such that its length changes by x then restoring force or tension in the spring is $-kx$.

Spring potential energy: Particle attached to the spring is always attracted towards mean position. Therefore, $F=-kx$

$\therefore U=-\int(-kx)dx=\frac{1}{2}kx^2$

Gravitational potential energy for two particles:

We know that gravitational force between two particles of masses m_1 and m_2 at separation r is always attractive and is equal to

$$f=-\frac{Gm_1.m_2}{r^2} \qquad \therefore U=-\int\left(-\frac{Gm_1.m_2}{r^2}\right)dr=-\frac{Gm_1.m_2}{r}$$

Work: It is defined as $W=\int \vec{F}.\vec{dl}$

where $\vec{dl}$ is infinitesimal displacement of the object on which force is acting.

Power: It is defined as dot product of force and instantaneous velocity $P=\vec{F}.\vec{v}$

Average power is defined as total work done divided by total time $\langle P\rangle=\frac{W}{t}$

Kinetic Energy: For a particle of mass 'm' and speed v, kinetic energy is defined as $K=\frac{1}{2}mv^2$

In terms of momentum it is written as $K=\frac{p^2}{2m}$.

- **Work-energy theorem:** It states that total work done by all forces is equal to change in kinetic energy of system.

$$W = K_{final} - K_{initial}$$

Conservation of mechanical energy: If only conservative forces are acting on a system then total mechanical energy of system remains conserved. Therefore

$$E_{initial} = E_{final}$$

or $K_i + U_i = K_f + U_f \Rightarrow U_i - U_f = K_f - K_i$

Therefore, loss in potential energy = gain in kinetic energy (or vice versa).

SOLVED EXAMPLES

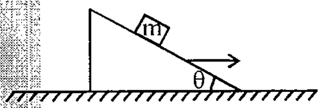
1. A particle moves from point (1, –1, 2) to (2, 3, 5) when a force $\vec{F} = 3\hat{i} + 5\hat{j} + 4\hat{k}$ acts on it. Calculate work done by the force.

Soln. The given force is constant. Therefore it is a conservative force. Therefore work done is independent of path followed.

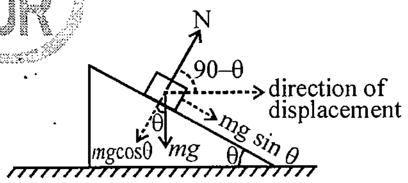
$$W = \int \vec{F}.d\vec{l} = \int F_x.dx + \int F_y dy + \int F_z dz = \int_1^2 3dx + \int_{-1}^3 5dy + \int_2^5 5dz$$

$$= 3(2 - 1) + 5(3 + 1) + 5(5 - 2) = 38 \text{ Joule}$$

2. A block of mass *m* is lying on a wedge as shown in figure. The block does not slide on the wedge while the wedge moves horizontally with uniform speed. Calculate work done by normal reaction on the block during the time wedge moves by a distance S in horizontal direction.



Soln. From the figure shown we find that $N = mg\cos\theta$ and angle between N and direction of displacement is $90 - \theta$. Therefore work done by normal reaction is,



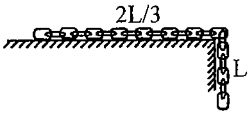
$$W = \int \vec{F}.d\vec{l} = \vec{N}.\int d\vec{l} = \vec{N}.\vec{S}$$

$$= N\,S\cos(90 - \theta) = mg\cos\theta.S\sin\theta$$

$$= mg\,S\sin\theta.\cos\theta$$

3. Two third length of a thin metallic chain of mass per unit length λ is held on a smooth table and remaining hanging off from its edge. If the chain is released what will be its speed at the moment the chain completely slips off the table.

Soln. There is no dissipative force acting on the system. Therefore total energy of the system remains conserved. Therefore we will use conservation of energy to find speed. Let us take potential energy to be zero at level of table top. Initially centre of mass of hanging part is L/6 below the table top and finally centre of mass of whole chain is L/2 below the table top. Therefore,



$$U_{initial} = -\left(\frac{\lambda L}{3}\right) g \left(\frac{L}{6}\right) = -\frac{\lambda g L^2}{18}, \quad U_{final} = -(\lambda L)\, g \left(\frac{L}{2}\right) = -\frac{\lambda g L^2}{2}$$

$K_{initial} = 0,\ K_{final} = \frac{1}{2}(\lambda L)v^2$

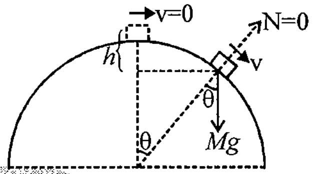
From conservation of energy $K_{initial} + U_{initial} = K_{final} + U_{final}$

$$0 + \left(-\frac{\lambda g L^2}{18}\right) = \frac{1}{2}(\lambda L)v^2 + \left(-\frac{\lambda g L^2}{2}\right) \quad \Rightarrow v = \sqrt{\frac{8gL}{9}}$$

4. A small object is released at the top of a smooth fixed sphere due to which it slides on the sphere for some time and then breaks off (leaves) the sphere. Calculate the angle with upward vertical at the position where object leaves the sphere.

Soln. Let the angle with upward vertical is θ. Therefore height descended by object from its initial position.

$h = R(1-\cos\theta)$



from conservation of energy

loss in potential energy = gain in kinetic energy

$$mgR(1-\cos\theta) = \frac{1}{2}mv^2$$

$$\therefore v^2 = 2Rg(1-\cos\theta) \qquad \text{... (a)}$$

At the moment of leaving the sphere the object is still moving on circular path of radius R. Therefore

Net force toward centre = $\frac{mv^2}{R}$, or $mg\cos\theta = \frac{mv^2}{R}$

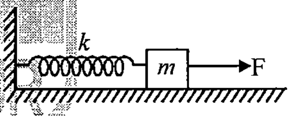
or $v^2 = Rg\cos\theta$ ... (b)

from (a) and (b) we get $\cos\theta = \frac{2}{3}$ or $\theta = \cos^{-1}\frac{2}{3}$

5. A block of mass *m* attached to a light spring of spring constant *k* lies in on a smooth horizontal plane as shown in figure. A horizontal force F is applied to the block.

Calculate maximum speed of the block.

Also calculate maximum elongation in spring.



Soln. When force is applied, the block accelerates for some time then it decelerates. Therefore acceleration of block is initially positive, then it momentarily becomes zero and finally becomes negative till the block stops. Speed of block is maximum at the instant when its acceleration is zero.

$$a_x = 0 \Rightarrow F_x = 0 \qquad \therefore F - kx = 0 \Rightarrow x = \frac{F}{k}$$

Thus speed is maximum when elongation in spring is equal to $\frac{F}{k}$

To find maximum speed let us use work energy theorem.

Total work done = change in kinetic energy

$$W_{Spring} + W_{Force} = \frac{1}{2}mv^2 - 0 \text{ or } \int_0^{F/k} kx\,dx\cos 180^\circ + \int_0^{F/k} F\,dx = \frac{1}{2}mv^2$$

$$-\frac{F^2}{2k} + \frac{F^2}{k} = \frac{1}{2}mv^2 \text{ or } v = \sqrt{\frac{F^2}{mk}}$$

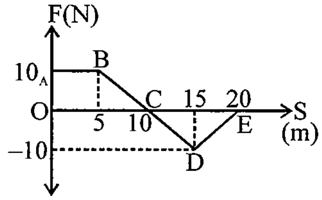
Let x_0 be the maximum elongation

When elongation becomes maximum the block again comes to rest.
Therefore from work energy theorem we get,

$$W_{Spring} + W_{Force} = 0 - 0, \quad \int_0^{x_0} kx dx \cos 180^\circ + \int_0^{x_0} F dx = 0$$

$$-\frac{1}{2}kx_0^2 + Fx_0 = 0, \qquad \therefore x_0 = \frac{2F}{k}$$

6. Force acting on a particle of mass 2kg varies with distance as shown in the figure. If speed of the particle at S = 0m be 5 m/s, calculate its speed at S = 20 m.



Soln. Total work done

= area OABCO − area CEDC

$$= 10\times5+\frac{1}{2}\times10\times5 - \frac{1}{2}\times10\times10 = 75 - 50 = 25 \text{ Joule}$$

Work energy theorem

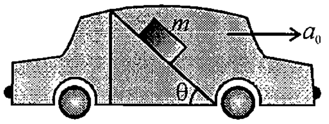
Total work done = change in K.E.

$$25 = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 \quad = \frac{1}{2}\times2\times v_2^2 - \frac{1}{2}\times2\times25$$

$$v_2^2 = 50 \quad \Rightarrow v_2 = 5\sqrt{2} \text{ m/s}$$

7. A block of mass m stays at rest on an inclined plane inside a car which is moving with constant acceleration a_0. Calculate work done by friction on block as the car moves by a distance x.

Soln. Free body diagram of block inside the car is shown in figure.

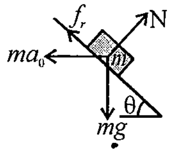


Block is in equilibrium.

$\therefore$ Along the plane

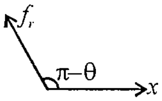
$$f_r + ma_0\cos\theta = mg\sin\theta$$

or $f_r = m(g\sin\theta - a_0\cos\theta)$



As the car moves through a distance 'x',
the block also moves the same distance.

$\therefore$ Work done by friction



$$= \vec{f}_r \cdot \vec{x} \quad = f_r x\cos(\pi - \theta)$$

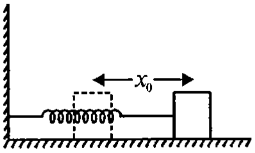
$$= -f_r x\cos\theta \quad = -mx(g\sin\theta - a_0\cos\theta)\cos\theta$$

8. A block of mass M is attached to one end of a spring of mass m and force constant K. The other end of spring is attached to a wall and the block lies on a smooth horizontal surface. The block is compressed against the spring through a distance x_0 and released. Calculate the speed of the block when it passes through the mean position.

Soln. Let the spring is uniformly stretched

$\therefore$ displacement of an element dx

which is at a distance x from the fixed end is



$$=\frac{x_0}{L}x; L \text{ is present length of spring.}$$

Therefore, if v be the speed of block at an instant then velocity of element dx.

$$=\frac{v}{L}x$$

$$\therefore \text{ K.E. of element} = \frac{1}{2}\cdot\left(\frac{m}{L}dx\right)\cdot\left(\frac{v}{L}x\right)^2$$

$$\therefore \text{ K.E. of spring} = \int_0^L \frac{1}{2}\left(\frac{m}{L}dx\right)\left(\frac{v}{L}x\right)^2 = \frac{1}{6}mv^2$$

Now, let v_0 be the speed of block when it passes through the origin.

$\therefore$ From conservation of energy we get, $\frac{1}{2}kx_0^2 = \frac{1}{2}Mv_0^2 + \frac{1}{6}mv_0^2$

$$\therefore \quad v_0 = x_0\sqrt{\frac{3K}{3M+m}}$$

9. A man is drawing water from a well with a bucket which leaks uniformly. The bucket when full weights 20 kg and when it arrives the top only half the water remains. The depth of the water is 20 meters. If $g = 10$ m/sec^2, what is the work done ?

Soln. When the bucket arrives at the top, the mass is 10 kg. Hence loss in mass = 20 – 10 = 10 kg. The depth of the well is 20 meters.

Therefore, mass lost per unit distance $=\frac{10}{20}=\frac{1}{2}$ kg.

Consider a point at a height x from the bottom of the well.

At height x from the bottom, the bucket weight $=\left(20-\frac{x}{2}\right)$ kg. The workdone against the force during elementary displacement $dx=\left(20-\frac{x}{2}\right)dx\cdot g$.

Therefore, total workdone

$$W=\int_0^{20}\left(20-\frac{x}{2}\right)g\,dx = g\int_0^{20}\left[20\,dx-\frac{1}{2}x\,dx\right] = g\left[\{20x\}_0^{20}-\left\{\frac{x^2}{4}\right\}_0^{20}\right]$$

$$=g[400-100]=300g=300\times10=\mathbf{3000\ J.}$$

10. A body of mass m is thrown at an angle α to the horizontal with an initial velocity v_0. Find the mean power developed by gravity over the whole time of motion of the body and the instantaneous power of gravity as a function of time.

Soln. We know that $P_{\text{instan.}} = F \cdot v$

The velocity of the particle after time t is given by $v = v_0 \cos\alpha\, i + (v_0 \sin\alpha - gt)\, j$ and $F = -mg\, j$.

Therefore, $P_{\text{instan.}} = (-mg\, j) \cdot \{v_0 \cos\alpha\, i + (v_0 \sin\alpha - gt) j\} = -mg\,(v_0 \sin\alpha - gt).$

The average power is given by

$$\langle P \rangle = \frac{\int_0^T P(t)\,dt}{\int_0^T dt} = \frac{-mg\int_0^T (v_0 \sin\alpha - gt)\,dt}{T} = \frac{-mg\left[v_0 \sin\alpha T - gT^2/2\right]}{T}.$$

Here T is total of flight $T = \dfrac{(2v_0 \sin\alpha)}{g}$.

Substituting this value, we have

$$\langle P \rangle = -mg\left[\frac{v_0 \sin\alpha\,(2v_0 \sin\alpha)/g - g\,(4v_0^2 \sin^2\alpha)/g^2}{(2v_0 \sin\alpha)/g}\right]$$

Solving we get $\langle P \rangle = 0$.

11. A particle of mass m moves along a circle of radius R with a normal acceleration varying with time as $a_n = kt^2$, where k is a constant. Find time dependence of power developed by all the forces acting on the particle and the mean value of this power averaged over the first t seconds after the beginning of the motion.

Soln. Normal acceleration $a_n = kt^2$

Therefore, $\dfrac{v^2}{R} = kt^2$ or $v^2 = kRt^2$ or $v = \sqrt{(kR)}\,t$ or $\left(\dfrac{dv}{dt}\right) = \sqrt{(kR)}$

We know that in circular motion, workdone by normal force is zero. For tangential force

$$F_t = m\left(\frac{dv}{dt}\right) = m\sqrt{(kR)}$$

Now power $P = F_t \cdot v = F_t \cdot v\cos\theta = F_t\, v \quad (\because\ \theta = 0)$

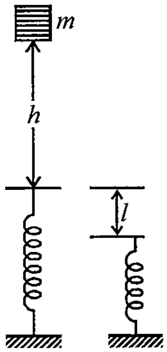
$= mkRt$

$$\text{Further, } \langle P \rangle = \frac{\int_0^T P(t)\,dt}{\int_0^T dt} = \frac{\int_0^T mkRt\,dt}{T} = \frac{mkRt\left[\dfrac{t^2}{2}\right]_0^T}{T} = \frac{mkRT}{2}.$$

12. A 2.0 kg block of mass, initially at rest, is dropped from a height of 0.40 meter onto a spring whose force constant is 1960 nt/meter. Find the maximum distance that the spring will be compressed.

Soln. The situation is shown in figure. Let m be the mass of the block and k be the force constant of the spring. Let l be the distance through which the spring is compressed. The total vertical fall of the block is $(h+l)$.

Loss of the gravitational potential energy of the block $= mg\,(h+l)$.



Elastic potential energy in the spring $= \left(\frac{1}{2}\right)kt^2$.

By the law of conservation of energy

$$mg\,(h+l) = \frac{1}{2}kt^2 \quad \text{or} \quad \frac{2mgh}{k} + \frac{2mg}{k}l = l^2 \quad \text{or} \quad l^2 - \frac{2mg}{k}l - \frac{2mgh}{k} = 0$$

$$\therefore \quad l = \frac{\left(\frac{2mg}{k}\right) \pm \sqrt{\left[\left(\frac{2mg}{k}\right)^2 + \left(\frac{8mgh}{k}\right)\right]}}{2}$$

According to given problem, $m = 2$ kg, $h = 0.40$ m and $k = 1960$ newton/meter. Hence,

$$l = \frac{(2\times2\times9.8/1960) \pm \sqrt{(2\times2\times9.8/1960)^2 + (8\times2\times9.8\times0.40/1960)}}{2}$$

$l =$ **0.1 meter.**

13. A particle of mass 3 kg is moving under the action of a central force whose potential energy is given by $U(r) = 10r^3$ joule. For what energy and angular momentum will the orbit be a circle of radius 10 m ? Calculate the time period of this motion.

Soln. Given that $U(r) = 10r^3$.

So the force F acting on the particle is given by

$$F = \frac{\partial U}{\partial r} = -\frac{\partial}{\partial r}(10r^3) = -10\times3r^2 = -30r^2$$

For circular motion of the particle $F = \frac{mv^2}{r} = 30r^2$.

Substituting the given values, we have $\frac{3\times v^2}{10} = 30\times(10)^2$ or $v = 100$ m/s.

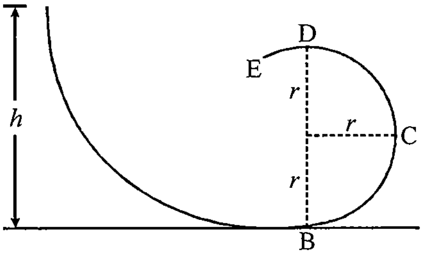
The total energy in circular motion

$$E = \text{K.E.} + \text{P.E.} = \frac{1}{2}mv^2 + U(r) = \frac{1}{2}\times3\times(100)^2 + 10\times(10)^3 = 2.5\times10^4 \text{ joule}$$

Angular momentum $= mvr = 3\times100\times10 = 3000\text{ kg-m}^2/\text{sec}$

Time period $T = \dfrac{2\pi r}{v} = \dfrac{2\times\pi\times10}{100} = \dfrac{\pi}{5}\text{ sec}.$

14. In figure, *ABCDE* is a channel in the vertical plane, part *BCDE* being circular with radius *r*. A ball is released from *A* and slides without friction and without rolling. Show that it will complete the loop path if *h* is greater than 5*r*/2.



Soln. Let *m* be the mass of the ball. When the ball comes down to *B*, its loses its potential energy *mgh* which is converted into kinetic energy. Let v_B, be the velocity of the ball at *B*. Then,

$$mgh = \frac{1}{2}mv_B^2 \quad \text{...(i)}$$

The ball now rises to a point *D*, where its potential energy is $mg(h-2r)$. If v_D be the velocity of the ball at *D*, then

$$mg(h-2r) = \frac{1}{2}mv_D^2 \quad \text{...(ii)}$$

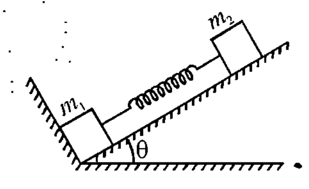
Now to complete the circular path, it is necessary that the centripetal force acting upward at point *D* should be equal or greater than the force *mg* acting downward. Therefore,

$$\frac{mv_D^2}{r} \geq mg \text{ or } v_D^2 \geq rg \quad \text{...(iii)}$$

From equation (ii) $v_D^2 = 2g(h-2r)$

$\therefore \quad 2g(h-2r) \geq rg, \; h \geq \dfrac{5}{2}r.$

15. Two blocks of masses m_1 and m_2 attached to the ends of a spring of spring constant K, lie on a smooth inclined surface as shown in the figure. Find initial compression in the spring so that m_1 just leaves the contact with the left wall (take $m_1 = m_2 = m$)



Soln. m_1 will leave contact because the spring will elongate and m_2 will rise up.

$\therefore \quad m_1 g\sin\theta = kx$

where *x* is elongation

$\therefore \quad x = \dfrac{m_1 g\sin\theta}{k}$

Let x_0 be initial compression in spring

$\therefore \quad m_2$ rises up the plane by a distance of $x_0 + x$

∴ from conservation of energy we get

$$\frac{1}{2}kx_0^2 = m_2g(x+x_0)\sin\theta + \frac{1}{2}kx^2 = m_2g(x+x_0)\sin\theta + \frac{1}{2}k.\frac{m_1^2g^2\sin^2\theta}{k^2}$$

$$= \frac{m_2g\cdot m_1g\sin^2\theta}{k} + m_2g\,x_0\sin\theta + \frac{m_1^2g^2\sin^2\theta}{2k}$$

$$\frac{1}{2}kx_0^2 = \frac{m_1\left(\frac{m_1}{2}+m_2\right)g^2\sin^2\theta}{k} + m_2g\,x_0\sin\theta$$

$$\therefore \quad x_0^2 - \left(\frac{2m_2g\sin\theta}{k}\right)x_0 - \frac{m_1(m_1+2m_2)g^2\sin^2\theta}{k^2}$$

for $m_1 = m_2 = m$

$$x_0^2 - \left(\frac{2mg\sin\theta}{k}\right)x_0 - \frac{3m^2g^2\sin^2\theta}{k^2} = 0$$

solving this quadratic equation we get

$$x_0 = \frac{3mg\sin\theta}{k}$$

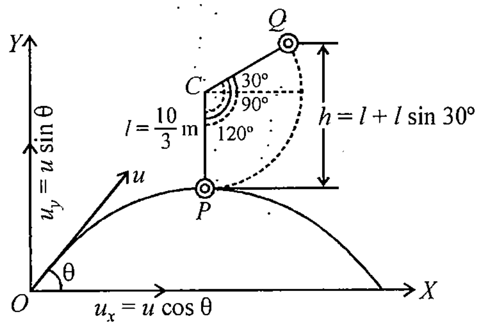
16. A bullet of mass M is fired with a velocity of 50 m/sec. at an angle θ with the horizontal. At the highest point of its trajectory, it collides head on with a bob of mass $3M$ suspended by a massless string of length 10/3 metres and gets embedded in the bob. After the collides, the string moves through an angle of 120°. Find (i) the angle θ; (ii) the vertical and horizontal co-ordinates of the initial position of the bob with respect to the point of firing of the bullet (take $g = 10$ m/sec²).

Soln. (i) The situation is shown in figure. At highest point, the vertical components becomes zero, while the horizontal component remains unchanged, i.e., $u_y = 0$ and $u_x = u\cos\theta$.

Now applying the law of conservation of linear momentum, we have

$$M\,u_x = (M+3M)\,v, \qquad \text{...(i)}$$

where v is the final velocity.



According to particle of conservation of energy

$$\frac{1}{2}(M+2M)v^2 = (M+3M)\,g\,h$$

or $\quad v^2 = 2gh = 2g(l + l\sin 30).$ ... (ii)

From equation (i),

$$v = \frac{u_x}{4} = \frac{u\cos\theta}{4} = \frac{50\cos\theta}{4}$$

Substituting this value of v in equation (ii), we get

$$\left(\frac{50\cos\theta}{4}\right)^2 = 2g\,(l + l\sin 30^\circ)$$

or $$\cos^2\theta = 2gl(1+\sin 30^\circ)\left(\frac{4}{50}\right)^2 = 2\times 10\times\left(\frac{10}{3}\right)\left(\frac{4}{50}\right)^2(1+\sin 30^\circ)$$

$$= 2\times 10\times\left(\frac{10}{3}\right)\left(\frac{4}{50}\right)^2\left(1+\frac{1}{2}\right) = 100\times\left(\frac{4}{50}\right)^2$$

$\therefore$ $$\cos\theta = 10\times\left(\frac{4}{50}\right) = \left(\frac{4}{5}\right)$$

$$= \cos^{-1}\left(\frac{4}{5}\right) = \cos^{-1}(0.8) = 37^\circ.$$

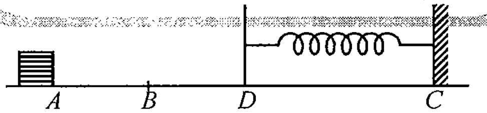
(ii) Time of ascent, $t = \dfrac{u\sin\theta}{g} = \dfrac{50\sin 37^\circ}{10} = 5\sin 37^\circ = 5\times 0.6 = 3\text{ sec}$

Therefore, horizontal coordinate $x = u\cos\theta\cdot t = 50\times 0.8\times 3 = 120\text{ m}$

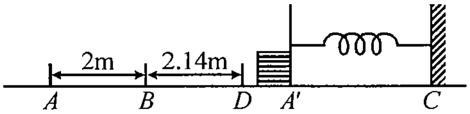
Vertical coordinate $y = u\sin\theta\cdot t - \dfrac{1}{2}gt^2 = 50\times 0.6\times 3 - \dfrac{1}{2}\times 10\times 9 = 45\text{ m}$.

17. A 0.5 kg block slides from the point A [see figure (a)] on a horizontal track with an initial speed of 3 m/s towards a weightless horizontal spring of length 1 meter and force constant 2 Nm^{-1}. The part AB of the track is frictionless and the part BC has the coefficients of static and kinetic friction as 0.22 and 0.2 respectively. If the distance AB and BD are 2 m and 2.14 m respectively, find the total distance through which the block moves before it comes to rest completely. Take $g = 10\text{ m/s}^2$.

Soln. The situation is shown in figure (b).



(a)



(b)

First of all the block moves the track AB without change in kinetic energy as it is frictionless. The initial kinetic energy of the block is given by $\frac{1}{2}mv^2$.

Therefore, initial kinetic energy of block $= \frac{1}{2}\times 0.5\times 3^2 = 2.25\text{ joule}$.

Now the block enters the track BC. This track has a coefficient of kinetic friction as 0.2. A part of the kinetic energy is used in workdone against frictional force. The workdone against friction over the distance BC is given by $= F \times s = \mu_k m g s = 0.2 \times 0.5 \times 10 \times 2.14$ joule.

This workdone is less than the initial kinetic energy of the block, hence the block moves further and compresses the spring by a distance say x. Hence, the initial kinetic energy of the block = workdone against the friction of track BD + workdone against the friction of distance x + potential energy of the compressed spring

$$\therefore \quad 2.25 = 2.14 + 0.2 \times 0.5 \times 10 \times x + \frac{1}{2}kx^2$$

where k is the force constant of the spring or $2.25 = 2.14 + x + x^2$

$$x^2 + x - 0.11 = 0 \qquad x = 0.1\text{m} \text{ or } (-1.1\text{m})$$

As the block compresses the spring, x can not be negative. Thus $x = 0.1$ meter

Moving through a distance of 0.1 m, the block comes to rest. The compressed spring now exerts a force on the block given by $F = kx = 2 \times 0.1$ newton.

The limiting value of the frictional force between block and track

$$BC = \mu_s mg = 0.22 \times 0.5 \times 10 = 1.1 \text{ newton.}$$

Since the limiting frictional force on the block by track is more than the force applied by the spring, the block comes to rest completely.

Total distances moved by the block,

$$= AB + BD + x$$

$$= 2 + 2.14 + 0.1 = \mathbf{4.24 \text{ metre.}}$$

18. A small ring of mass m can slide on a smooth circular wire of radius r and center O, which is fixed in a vertical plane. From a point on the wire at a vertical distance $r/2$ above O, the ring is given a velocity $\sqrt{(g\,r)}$ along the downward tangent to the wire. Show that it will just reach the highest point of the wire. Find the reaction between the ring and the wire when the ring is at a vertical distance $r/2$ below.

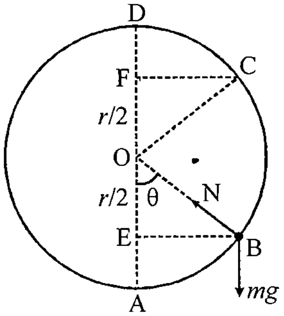
Soln. At point C, the velocity of the ring $= \sqrt{(r\,g)}$.

K.E. at $C = \frac{1}{2}mv^2 = \frac{1}{2}mrg$ and P.E. at $C = mgh = mg\,(AF) = mg\,(r + r/2) = \frac{3mgr}{2}$.

$$\text{Total energy} = mgr\left[\frac{1}{2} + \frac{3}{2}\right] = 2mgr. \qquad \text{...(i)}$$

If the ring has to reach at D, with kinetic energy zero, its potential energy $= m\,g\,(2r)$.

Becuase the particle at C has this amount of energy and it will just reach at D with zero velocity.



Now consider the ring at B, a distance $r/2$ below O. Resolving mg in two parts and considering the equilibrium, we get

$$N - mg\cos\theta = \frac{mv_1^2}{r}$$

or $\quad N = \dfrac{mv_1^2}{r} + mg\cos\theta \qquad$ ... (ii)

Falling from C, i.e., through a distance r, the ring has lost a potential energy mgr. This is the gain in kinetic energy.

Hence, $\frac{1}{2}mv_1^2 = \frac{1}{2}mgr + mgr = \frac{3}{2}mgr$ or $v_1^2 = 3gr$ or $\frac{v_1^2}{r} = 3g$.

From equation (ii), $N = m(3g + g \times \frac{1}{2})$ $\qquad (\because \cos\theta = \frac{1}{2})$

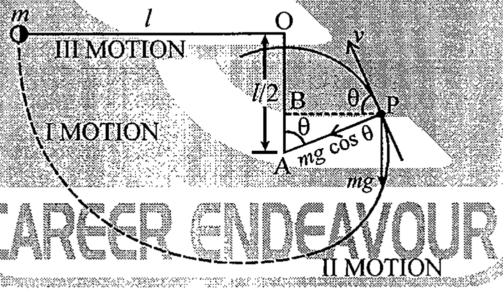
or $\quad N = \left(\dfrac{7}{2}\right)mg = 3.5\,mg.$

19. A small ball is suspended from point O by a thread of length l. A nail is driven into the wall at a distance of $l/2$ below O, at A. The ball is drawn aside so that the thread takes up a horizontal position. At what point to the ball trajectory, will the tension in the thread disappear ? How much further will the ball move ? What will be the highest point to which it will rise ?

Soln. The ball will perform the following :

(i) Firstly, the ball describes a quadrant of circle of radius equal to the length of the thread.

(ii) Secondly, the thread touches the nail and describes a circular motion till the tension in thread becomes zero [point P in figure below].



(iii) Thirdly, the ball continuous its path just like a body thrown at certain angle (θ in figure) with certain initial velocity. When the tension in the thread becomes zero at P, the component of mass along the radius provides centripetal force,

i.e., $\quad mg\cos\theta = \dfrac{mv^2}{r} \qquad$ ... (i)

where $r = \dfrac{l}{2}$.

To calculate the velocity of ball at P, we proceed as follows:

Kinetic energy at P = Decrease in potential energy due to a distance OB, $\frac{1}{2}mv^2 = mg\,(OB)$.

From figure, $OB = OA - AB$.

Now, $OB = \dfrac{l}{2} - \dfrac{l}{2}\cos\theta.$

Therefore, $\frac{1}{2}mv^2 = mg\left(\frac{l}{2} - \frac{l}{2}\cos\theta\right)$ or $v^2 = gl(1-\cos\theta)$ ... (ii)

From equation (i), $mg\cos\theta = \frac{mgl(1-\cos\theta)}{l/2}\cos\theta = \frac{2}{3}$.

Therefore, the tension in the thread becomes zero at height, $\frac{l}{2} + \frac{l\cos\theta}{2} = \frac{l}{2} + \frac{l}{2} \times \frac{2}{3} = \frac{5l}{6}$ from lowest point.

From equation (ii), $v^2 = gl\left(1 - \frac{2}{3}\right) = \frac{gl}{3}$ or $v = \sqrt{\left(\frac{gl}{3}\right)}$.

Now, the particle describes a parabolic path. The height point lies above the point of throwing at a distance, $\frac{(v\sin\theta)^2}{2g} = \frac{(gl/3)\cdot(5/9)}{2g} = \frac{5l}{54}$.

Chapter 4

MOMENTUM, COLLISION & CENTRE OF MASS

Conservation of momentum: Total momentum of a system remains conserved if there is no external force on the system. This conclusion comes directly from Newton's second law $\vec{F} = \frac{d\vec{p}}{dt}$, if $\vec{F} = 0$, $\frac{d\vec{p}}{dt} = 0$

$\Rightarrow \vec{p} =$ constant

Note: If two particles are interacting with each other, momentum of each particle changes with time, however sum of momenta of the two particles remains constant. For example in rocket propulsion, momentum of rocket increases with time, however sum of momenta of fuel and rocket remains constant.

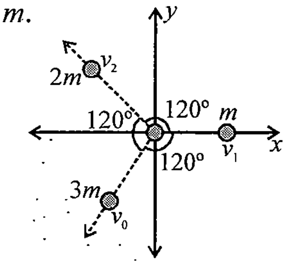
Example: An object initially kept at rest breaks up into three fragments of mass ratio 1 : 2 : 3. If speed of heaviest fragment just after explosion is v_0 what is speed of lightest fragment. It is given that the three fragments move in same plane and make equal angle with each other.

Soln. There is no external force on the object. Therefore its momentum remains constant. Initial momentum is zero. Therefore vector sum of momenta of three fragments must be zero.

$\vec{p} = 0 \Rightarrow p_x = 0, p_y = 0$. Let the masses of three fragments be m, $2m$ and $3m$.

Let v_1 and v_2 be speed of fragments of masses m and $2m$.

It is given that speed of $3m$ is v_0.



$p_x = 0$

$\Rightarrow mv_1 - 2mv_2 \cos 60 - 3mv_0 \cos 60 = 0$

$$\therefore\ v_1 - v_2 = \frac{3v_0}{2} \qquad \text{... (a)}$$

$$p_y = 0 \qquad \Rightarrow 2mv_2 \sin 60 - 3mv_0 \sin 60 = 0$$

$$\therefore\ v_2 = \frac{3v_0}{2} \qquad \text{... (b)}$$

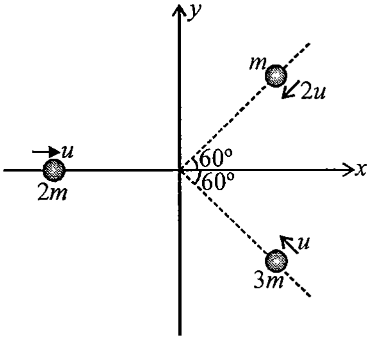
from (a) $v_1 = \frac{3v_0}{2} + v_2 = \frac{3v_0}{2} + \frac{3v_0}{2} = 3v_0$

Therefore speed of the lightest fragment is $3v_0$.

Example : Three particles moving as shown in figure, collide at origin and coalesce to form a single particle. Calculate speed of the new particle formed.

Soln. There is no external force on the system of three particles. Therefore momentum must be conserved or each component of moment must be conserved.

Let V_x and V_y be x and y components of velocity of new particle



Therefore, $p_{x_{initial}} = p_{x_{final}}$

$$2mu - m2u\cos 60 - 3mu\cos 60 = 6mV_x$$

$\therefore V_x = -\dfrac{u}{12}$ and $p_{y_{initial}} = p_{y_{final}}$

$-m2u\sin 60 + 3mu\sin 60 = 6mV_y \quad \therefore V_y = \dfrac{\sqrt{3}u}{12}$

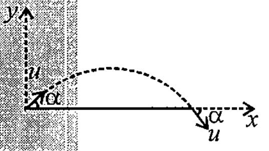
Therefore, velocity of new particle $= \sqrt{V_x^2 + V_y^2} = \dfrac{u}{12}\sqrt{3+1} = \dfrac{u}{6}$

Short time interaction between objects when they come in contact is called collision. If the colliding objects are moving with small speeds then collision can be studied using non-relativistic mechanics. However, if objects are moving with the large speed then relativitics mechanics is required. Here we will discuss collision at small speeds only.

Impulse: It is equal to change in momentum of object $\vec{I} = \vec{p}_{Final} - \vec{p}_{Initial}$ or $\vec{I} = \int \vec{F}dt$

Example: A projectile is thrown with initial velocity u at angle α with horizontal. Calculate total impulse on the projectile due to gravitational force:

Soln. Initial momentum of projectile $\vec{p}_{initial} = mu(\cos\alpha\hat{i} + \sin\alpha\hat{j})$



final momentum of projectile $\vec{p}_{final} = mu(\cos\alpha\hat{i} - \sin\alpha\hat{j})$

Impulse on the projectile, $\vec{I} = \vec{p}_{final} - \vec{p}_{initial} = -2mu\sin\alpha\hat{j}$

Types of Collision:

1. **Elastic:** Internal energy of colliding objects doesn't change. It implies that kinetic energy is not converted into internal energy. Therefore, in cae of elastic collision, we have following conservation laws

(a) Conservation of kinetic energy: Sum of kinetic energy of all colliding objects remains constant.

i.e. $\quad \text{K.E.}_{\text{initial}} = \text{K.E}_{\text{final}}$

(b) Convervation of momentum: Sum of momentum, of all colliding objects remains constant.

i.e. $\quad \vec{p}_{initial} = \vec{p}_{final}$

or, $\quad (p_{initial})_x = (p_{final})_x, (p_{initial})_y = (p_{final})_y$

2. **Inelastic:** Internal energy of colliding objects changes due to which kinetic energy doesn't remain conserved. Therefore, in case of inelastic collision we can only apply conservation of momentum. **Pefectly inelastic collision** is a special case of inelastic collision in which objects stick to each other.

How to distinguish elastic and inelastic collision:

If collision is elastic then the in the problem it is explicitly mentioned in most cases that collision is elastic. However, if collision is inelastic then it may be explicitly mentioned or phrases such as 'there is loss of kinetic energy' or 'breaking of objects' or 'objects stick to each other' etc are used.

Coefficient of restitution:

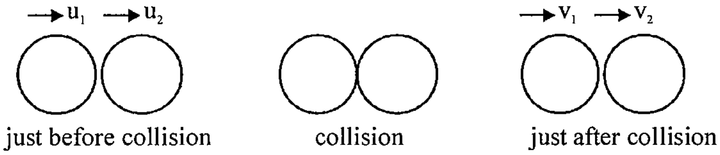
It is defined as

$$e = \frac{|\text{velocity of separation after collision}|}{|\text{velocity of approach before collision}|}$$

For elastic collision (involving no rotation) $e = 1$. For inelastic collision $0 \le e \le 1$. If objects stick to each other after collision. Then $e = 0$

Elastic collision in one dimension:

Let m_1 and m_2 be masses of two objects having velocities u_1 and u_2 just before collision and velocities v_1 and v_2 just after collision.



From conservation of momentum we get,

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2 \qquad \text{... (i)}$$

From conservation of kinetic energy we get,

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + m_2v_2^2 \qquad \text{... (ii)}$$

Solving (i) and (ii), we get

$$\boxed{\begin{aligned} v_1 &= \frac{u_1(m_1 - m_2) + 2m_2u_2}{m_1 + m_2} \\ v_2 &= \frac{u_2(m_2 - m_1) + 2m_1u_1}{m_1 + m_2} \end{aligned}}$$

Using above results we can calculate velocities of objects after collision in one dimensional elastic collision if velocities before collision is either given or can be determined by some method. The above results have been derived by assuming that objects are moving in same direction. If in a problem objects are moving in opposite direction then one velocity must be taken to be negative.

Example: Two objects of masses m and 4m are moving with speed u and 2u in opposite direction. What is their velocities after collision

Soln. Let us take, $m_1 = m, u_1 = u$ and $m_2 = 4m$, $u_2 = -2u$

Therefore, velocity of m after collision

$$v_1 = \frac{u_1(m_1 - m_2) + 2m_2u_2}{m_1 + m_2} = \frac{u(m - 4m) + 2(4m)(-2u)}{5m} = \frac{-19}{5}u$$

Velocity of 4m after collision,

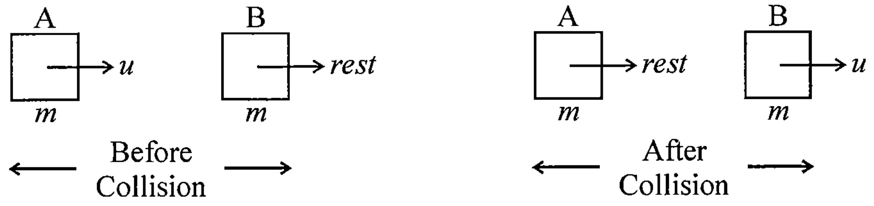
$$v_2 = \frac{u_2(m_2 - m_1) + 2m_1u_1}{m_1 + m_2} = \frac{-2u(4m - m) + 2mu}{5m} = -\frac{4}{5}u$$

Therefore, after collision both the masses move in the direction of velocity of 4m before collision.

Special case

- I-d Elastic collision of two equal masses

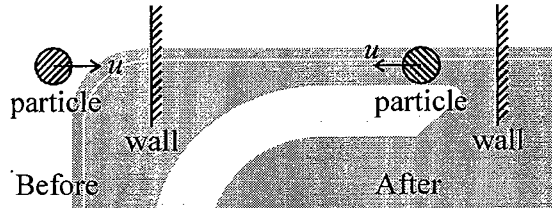
If $m_1 = m_2$ then, $v_1 = u_2$ and $v_2 = u_1$ i.e. the two objects exchange their velocities.


A
B
u
rest
m
m
Before Collision
A
B
rest
u
m
m
After Collision

- One object is much heavier than the other object and heavier object is initially at rest.

$m_1 >> m_2$ and $u_1 = 0$

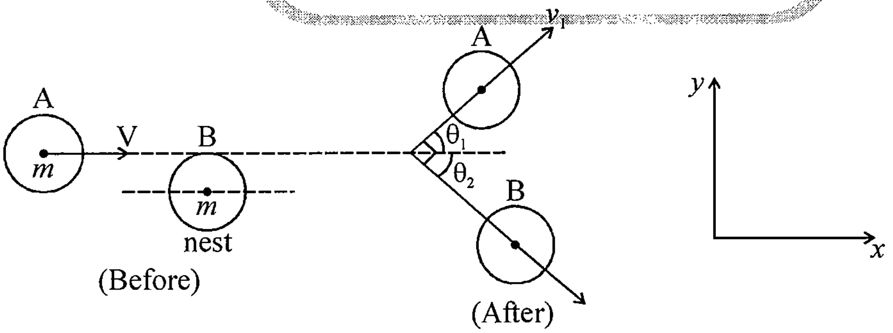
then, $v_1 = 0$ and $v_2 = -u_2$


u
particle
wall
Before
u
particle
wall
After

Elastic collision in two dimension:
In this case also we apply conservation of linear momentum and conservation of kinetic energy to solve problems. However momentum conservation (in most cases) has to be applied separately along two perpendicular directions (Viz X and Y directions).

Special case:
If elastic collision between two objects of equal masses is non-headon (not 1 dimensional) and one of them is initially at rest then after collision the two masses move at right angle.

Let us consider off center collision between two blocks A and B as shown in figure.


A
m
V
B
m
nest
(Before)
A
v_1
θ_1
θ_2
B
(After)
y
x

As shown in figure velocity vector $(\vec{v})$ of A before collision dosn't pass through center of B, therefore collision will definitely be two dimensional.

Let us consider off centre collision between two blocks A and B as shown in figure. As shown in figure velocity vector $(\vec{v})$ of vector A before collision doesn't pass through centre of B. Therefore, collision will definitely be two dimensional.

Let after collision A and B move with speed v_1 and v_2 and at angles θ_1 and θ_2 with direction of initial velocity $(\vec{v})$ of A.

Then as mentioned before $\theta_1 + \theta_2 = \frac{\pi}{2}$

Proof: From conservation of kinetic energy we get,

$$\frac{1}{2}mv^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2$$

or $\quad v^2 = v_1{}^2 + v_2{}^2$ —(*i*)

From converstion of linear momentum in x-direction we get,

$$mv = mv_1 \cos\theta_1 + mv_2 \cos\theta_2$$

or $\quad v = v_1 \cos\theta_1 + v_2 \cos\theta_2$ —(*ii*)

from conservation of linear momentum in y direction we get,

$$0 = mv_1 \sin\theta_1 - mv_2 \sin\theta_2$$

or $\quad v_1 = \dfrac{v_2 \sin\theta_2}{\sin\theta_1}$ —(*iii*)

Put v_1 in (*ii*) to get,

$$v_2 = \frac{v \sin\theta_1}{\sin(\theta_1 + \theta_2)}$$

use v_2 in (*iii*) to get,

$$v_1 = \frac{v \sin\theta_2}{\sin(\theta_1 + \theta_2)}$$

Now put v_1 and v_2 in (*i*) to get,

$$\sin^2(\theta_1 + \theta_2) = \sin^2\theta_1 + \sin^2\theta_2$$

or $\quad (\sin\theta_1 \cdot \cos\theta_2 + \cos\theta_1 \cdot \sin\theta_2)^2 = \sin^2\theta_1 + \sin^2\theta_2$

$$\sin^2\theta_1 \cdot \cos^2\theta_2 + \cos^2\theta_1 \cdot \sin^2\theta_2 + 2\sin\theta_1 \cdot \sin\theta_2 \cdot \cos\theta_1 \cdot \cos\theta_2 = \sin^2\theta_1 + \sin^2\theta_2$$

or $\quad \sin^2\theta_1(1 - \cos^2\theta_2) + \sin^2\theta_2(1 - \cos^2\theta_1) - 2\sin\theta_1 \cdot \sin\theta_2 \cdot \cos\theta_1 \cdot \cos\theta_2 = 0$

or $\quad 2\sin^2\theta_1 \cdot \sin^2\theta_2 - 2\sin\theta_1 \cdot \sin\theta_2 \cdot \cos\theta_1 \cdot \cos\theta_2 = 0$

or $\quad 2\sin\theta_1 \cdot \sin\theta_2 (\sin\theta_1 \cdot \sin\theta_2 - \cos\theta_1 \cdot \cos\theta_2) = 0$

Since $\theta_1 \neq 0, \ \ \theta_2 \neq 0$. Therefore

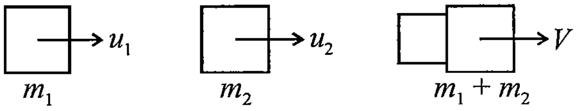
$$\sin\theta_1 \cdot \sin\theta_2 - \cos\theta_1 \cdot \cos\theta_2 = 0$$

or $\quad \cos(\theta_1 + \theta_2) = 0$

Therefore, $\theta_1 + \theta_2 = \frac{\pi}{2}$

Perfectly inelastic collision in one dimension:

Consider the situation shown in figure in which two blocks moving with speed u_1 and u_2 collide and stick to each other.



Before After

From conservation of momentum we get,

$$m_1 u_1 + m_2 u_2 = (m_1 + m_2)V$$

Therefore, $\boxed{V = \dfrac{m_1 u_1 + m_2 u_2}{m_1 + m_2}}$

This is common speed of two block after collision
Kinetic energy is not conserved in the process.
so we can calculate loss in kinetic energy as

$$\Delta K = K \cdot E_i - K \cdot E_f$$

$$= \frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 - \frac{1}{2}(m_1 + m_2)V^2$$

$$= \frac{1}{2}\left[m_1 u_2^2 + m_2 u_2^2 - (m_1 + m_2) \cdot \frac{(m_1 u_1 + m_2 u_2)^2}{(m_1 + m_2)^2}\right]$$

$$\boxed{\text{Loss in } K \cdot E = \frac{m_1 m_2 (u_1 - u_2)^2}{2(m_1 + m_2)}}$$

This formula can be used to calculate loss in $K \cdot E$ in collision of two paticles only.

Example: Two particles A and B of mass m and one particle C of mass M are kept on the x axis in the order ABC. Particle A is given a velocity $v\hat{i}$. Consequently there are two collisions, both of which are completely inelastic. If the net energy loss because of these collisions is $\frac{7}{8}$ of the initial energy, the value of M is (ignore frictional losses)

(a) 8 m (b) 6 m (c) 4 m (d) 2 m

Soln. Let v' be final speed of the joint (three) masses.
From conservation of momentum we get,

$$mv = (2m + M)v'$$

$$v' = \frac{mv}{2m + M}$$

Therefore, Loss in $K \cdot E = \dfrac{1}{2}mv^2 - \dfrac{1}{2}(M + 2m)v'^2$

According to question

$$\text{Loss in } K \cdot E = \frac{7}{8} \text{ of initial } K \cdot E$$

$$= \frac{7}{8} \cdot \frac{1}{2} mv^2$$

Therefore, $$\frac{7}{16}mv^2 = \frac{1}{2}mv^2 - \frac{1}{2}(M+2m)\frac{m^2v^2}{(2m+M)^2}$$

$$\frac{7}{16}mv^2 = \frac{1}{2}mv^2\left[1 - \frac{m}{2m+M}\right]$$

$$\therefore \quad \frac{7}{8} = 1 - \frac{m}{2m+M}$$

or $$\frac{m}{2m+M} = \frac{1}{8}$$

or $$M = 6m$$

Example: A particle is released form some height after time 't' it breaks up into two equal parts. One of which flies horizontally with speed equal to half of the speed of the whole particle just before breaking. What is speed of another part.

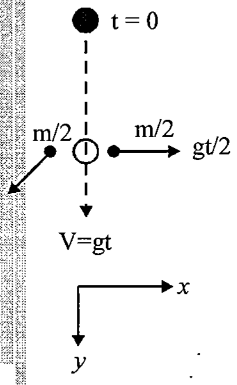
Soln. Let m be the mass of particle

Therefore its speed after time 't' will be $v = gt$.

Momentum just before and just after breaking must be same.

$$\therefore \quad \vec{p}_{\text{initial}} = \vec{p}_{\text{final}}$$

$$m\,gt\,\hat{j} = \frac{m}{2}\frac{gt}{2}\hat{i} + \frac{m}{2}\vec{V'}$$



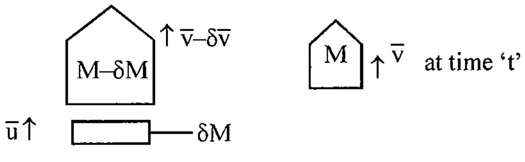
where $\vec{V'}$ is velocity of $2nd$ part

$$\therefore \quad \vec{V'} = 2gt\left(\hat{j} - \frac{\hat{i}}{4}\right)$$

$$\left|\vec{V'}\right| = \frac{\sqrt{17}}{2}gt$$

VARIABLE MASS PROBLEM:

Let a rocket of mass 'M' moving at velocity $\vec{v}$ at any time 't' in earth reference frame. Let at a latter instant $t + \delta t$. δM mass of ejected gas jet be moving with velocity $\vec{u}$ and the rocket of mass $M - \delta M$ with new velocity $\vec{v} + \delta\vec{v}$.



Initial momentum $= M\vec{v}$

Final momemtum $= (M - \delta M)(\vec{v} + \delta\vec{v}) + \delta M\vec{u}$

$$F_{\text{ext}} = \frac{\vec{P}_f - \vec{P}_i}{\delta t} = \frac{(M - \partial M)(\vec{v} + \delta\vec{v}) + \delta M\vec{u} - M\vec{V}}{\delta t}$$

$$= M\frac{\delta\vec{v}}{\delta t} - \frac{\vec{v}\delta M}{\delta t} - \delta\vec{v}\frac{\delta M}{\delta t} + \vec{u}\frac{\delta M}{\delta t}$$

Now, as $\delta t \to 0$, $\frac{\delta M}{\delta t} = -\frac{\delta M}{\delta t}, \frac{\delta v}{\delta t} = \frac{\delta v}{\delta t}; \delta v \to 0$

$$\vec{F}_{ext} = M\frac{d\vec{v}}{dt} + V\frac{dM}{dt} - \vec{u}\frac{dM}{dt}$$

$$M\frac{d\vec{v}}{dt} = F_{ext} + (\vec{u} - \vec{v})\frac{dM}{dt} = F_{ext} + V_{rel}\frac{dM}{dt}$$

$V_{rel} = \vec{u} - \vec{v}$ = Relative velocity of exhaust gas w.r.t. rocket.

F_{ext} is the 'external force' acting on the system and $V_{rel}\frac{dM}{dt}$ is the 'thrust'.

On gravitational field, $F_{ext} = -Mg$

$$M\frac{d\vec{v}}{dt} = -M\vec{g} + V_{rel}\frac{dM}{dt} = -Mg - V_{rel}\frac{dM}{dt}$$

As M is decreasing, $\frac{dM}{dt} = -\text{ive}$

Integrating, $\int_0^V dU = -g\int_0^t dt - V_{rel}\int_{M_0}^{M}\frac{dM}{dt}$

$$V = V_{rel}\log_e\left(\frac{M_0}{M}\right) - gt$$

Example: Find the velocity of the rocket assuming that gas is ejected at constant rate $\propto$ and a constant velocity with respect to it and that it moves vertically upwards in a constant gravitational field. Also calculate the total distance travelled by the rocket.

Soln. Gas ejected at constant rate $\alpha > 0$; then $m = m_0 - \alpha t$

$$F_{ext} = -mg, \quad \frac{dm}{dt} = -\alpha$$

General equation for rocket is

$$m\frac{dv}{dt} = F_{ext} + v_{rel}\frac{dM}{dt} = -mg + (-v_0)(-\alpha) = -mg + v_0\alpha$$

Or, $(m_0 - \alpha t)\frac{dv}{dt} = -(m_0 - \alpha t)g + \alpha v_0$

Or, $\frac{dv}{dt} = -g + \frac{\alpha v_0}{m_0 - \alpha t}$

$$V = -gt - V_0\ell n(m_0 - \alpha t) + C_1$$

At $t = 0$, $v = 0$; $\Rightarrow$ $0 = 0 - V_0\ell n m_0 + C_1$ $\Rightarrow$ $C_1 = V_0\ell n\, m_0$

Hence, $V = -gt + V_0\ell n\left(\frac{m_0}{m_0 - \alpha t}\right)$

$$\frac{ds}{dt} = -gt + V_0\ell n\left(\frac{m_0}{m_0 - \alpha t}\right)$$

And $$-\frac{gt^2}{2}+V_0\int\left(\ell n\, m_0-\ell n\left(m_0-\alpha t\right)\right)dt$$

$$-\frac{gt^2}{2}+V_0 t\ell n m_0+\frac{V_0}{\alpha}\left[(m_0-\alpha t)\ell n(m_0-\alpha t)-(m_0-\alpha t)\right]+C_2$$

$$\left(\int \ell n\, x\, dx = x\,\ell n\, x - x\right)$$

At $t=0,\ s=0$

$$0=0+0+\frac{v_0}{\alpha}\left(m_0\ell n\, m_0-m_0\right)+C_2$$

$$C_2=-\frac{v_0}{\alpha}\left(m_0\ell n m_0-m_0\right)$$

Then, $$S=-gt^2+v_0\ell n\, m_0+\frac{v_0}{\alpha}\left[(m_0-\alpha t)\ell n(m_0-\alpha t)-(m_0-\alpha t)-m_0\ell n m_0+m_0\right]$$

Or, $$S=-\frac{gt^2}{2}+v_0\left[(\ell n\, m_0)t+\left(\frac{m_0-\alpha t}{\alpha}\right)\ell n(m_0-\alpha t)+t-\frac{m_0}{\alpha}\ell n m_0\right]$$

$$=-\frac{gt^2}{2}+v_0\left[t+\left(\frac{m_0-\alpha t}{\alpha}\right)\ell n(m_0-\alpha t)-\left(\frac{m_0}{\alpha}-t\right)\ell n m_0\right]$$

$$=-\frac{gt^2}{2}+v_0\left[t+\left(\frac{m_0-\alpha t}{\alpha}\right)\ell n\left(\frac{m_0-\alpha t}{m_0}\right)\right]$$

Center of mass:

The point where whole mass an object or system of particle can be assumed to be concentrated is called center of mass. Concept of center of mass comes from Newton's equation of motion when dynamics of a system of particles subjected a net external force is collectively discussed.

Consider a system of N particles of masses $m_1, m_2 ......m_N$. Let $\vec{r}_1, \vec{r}_2 ......\vec{r}_N$ be instantaneous position vectors of particles and $\vec{F}_1^e, \vec{F}_2^e ....\vec{F}_N^e$ be external forces on $1^{st}, 2^{nd} ....N^{th}$ particles respectively. Then equation of motion of each particle can be written as

$$\vec{F}_1^e+\sum_{i+1}^{N}\vec{F}_{1j}=m_1\frac{d^2\vec{r}_1}{dt^2}$$

$$\vec{F}_2^e+\sum_{j\neq 2}^{N}\vec{F}_{2j}=m_2\frac{d^2\vec{r}_2}{dt^2}$$

$$\vec{F}_N^e+\sum_{i=1}^{N-1}\vec{F}_{Nj}=m_N\frac{d^2\vec{r}_N}{dt^2}$$

Here $\vec{F}_{ij}$ is internal force between i^{th} and i^{th} particle. Since according to Newton's third law sum of interval forces is zero therefore on adding the above equations we get

$$\sum_{i=1}^{N}\vec{F}_i^e=\frac{d^2}{dt^2}\sum_{i=1}^{N}m_i\vec{r}_i$$

or $$\sum_{i=1}^{N} \vec{F}_i^e = \sum_{i=1}^{N} m_i \frac{d^2}{dt^2}\left(\frac{\sum_{i=1}^{N} m_i \vec{r}_i}{\sum_{i=1}^{N} m_i}\right)$$

$$\vec{F}_{total}^e = M_{total} \frac{d^2}{dt^2} \vec{R}_{total}, \text{ where } \vec{R}_{total} = \frac{\sum m_i \vec{r}_i}{\sum m_i}$$

The above equation is a single equation describing the dynamics if all partcles. It is Newton's equation of motion for whole mass. Therefore $\vec{R}_{total}$ is posotion vector of a point in space where whole mass is assumed to be concentrated. We call it centre of mass. Therefore we write $\vec{R}_{com}$ in place of $\vec{R}_{total}$

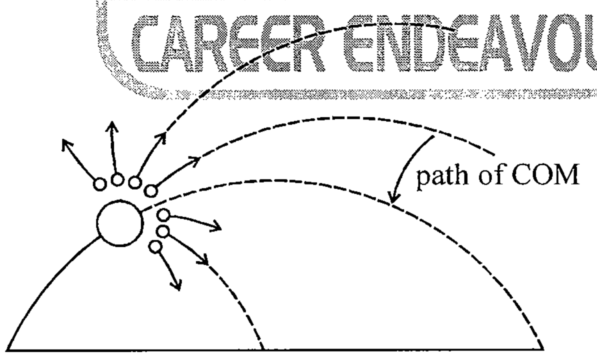
$$\vec{F}_{total}^e = M_{total} \frac{d^2 \vec{R}_{com}}{dt^2}$$

or $$\boxed{\vec{F}_{total}^e = M_{total} \vec{a}_{com}}$$

Thus for a system of particles, we can directly apply Newton's equation of motion to center of mass (COM)

Thus we note the following points about dynamics of COM.

- If net external force is zero then center of mass moves with constant velocity.
- If net external force is zero and center of mass is initially at rest then it will always remain at rest although particles of system may start moving due to their mutual internal forces.
- If an object is moving and breaks due to internal forces then center of mass follows the same path as the unbroken object would have followed. For example: If a projectile breaks into two or more particles then also path followed by COM of all particles is still parabolic although various particles may follow different paths.



Position of center of mass:

Discrete **Continuous**

$$\vec{R}_{com} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i} \qquad \vec{R}_{com} = \frac{\int \vec{r} dm}{\int dm}$$

Coordinates of center of mass:

$$X_{com}=\frac{\sum m_i x_i}{\sum m_i} \qquad X_{com}=\frac{\int x dm}{\int dm}$$

$$Y_{com}=\frac{\sum m_i y_i}{\sum m_i} \qquad Y_{com}=\frac{\int y dm}{\int dm}$$

$$Z_{com}=\frac{\sum m_i z_i}{\sum m_i} \qquad Z_{com}=\frac{\int z dm}{\int dm}$$

Note: Center of mass lies at point of symmetry or at line of symmetry.
Velocity of center of mass

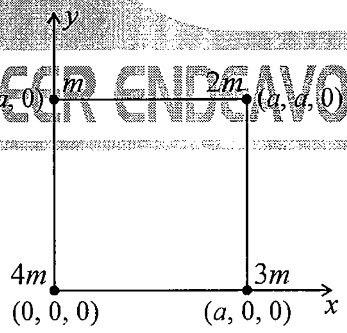
$$\vec{v}_{com}=\frac{\sum_i m_i \vec{v}_i}{\sum m_i}$$

Acceleration of center of mass

$$a_{com}=\frac{\sum_i m_i \vec{a}_i}{\sum m_i} \text{ or } \vec{a}_{com}=\frac{\vec{F}^{e}_{\text{total}}}{M_{\text{total}}}$$

Above formulae for velocity and acceleration of COM are used when velocity and acceleration of consituents of system are known.

Example: Four particles of masses m, $2m$, $3m$ and $4m$ are placed at the corners of a square of side 'a' as shown in figure. If coordinate system is taken as shown y. Find position of center of mass.



Soln. Coordinates of the masses $m, 2m, 3m$ and $4m$ are (0, a, 0), (a, a, a), (a, 0, 0) and (0, 0, 0) respectively.

Therefore, $$X_{com}=\frac{\sum m_i x_i}{\sum m_i}=\frac{m\times 0+2m\times a+3m\times a+4m\times 0}{m+2m+3m+4m}=\frac{a}{2}$$

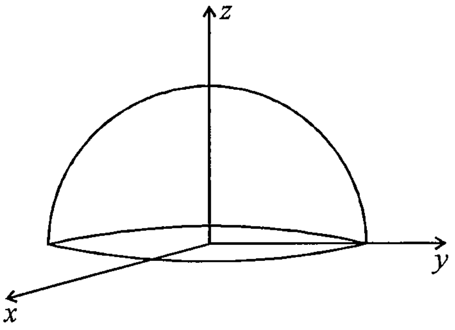
$$Y_{com}=\frac{\sum m_i y_i}{\sum m_i}=\frac{m\times a+2m\times a+3m\times 0+4m\times 0}{10m}=\frac{3}{10}a$$

Therefore position of center of mass is $\left(\frac{a}{2},\frac{3a}{10}10\right)$

Example: Find coordinate of center of mass of a thin uniform hollow hemisphere of radius R.

Soln. Let M be the mass of hollow hemisphere. Let us take coordinate system as shown in figure. Z axis. symmetric therefore center of mass must lie on Z axis.

$\therefore \quad X_{com} = 0$ and $Y_{com} = 0$



$$Z_{com} = \frac{\int zdm}{\int dm} = \frac{\int zdm}{M}$$

Here dm is mass of small element on surface of sphere

$$\therefore \quad dm = \frac{M}{A}dA = \frac{M}{2\pi R^2} \cdot R^2 \sin\theta \cdot d\theta d\phi$$

$$= \frac{M}{2\pi}\sin\theta d\theta d\phi$$

In spherical polar coordinate $Z = R\cos\theta,\ dA = R^2 \sin\theta d\theta d\phi$

$$\therefore \quad Z_{com} = \frac{1}{M}\iint R\cos\theta \frac{M}{2\pi}\sin\theta d\theta d\phi$$

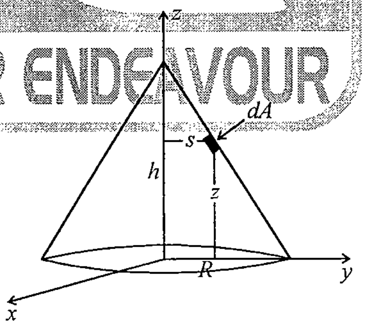
$$= \frac{R}{2\pi}\int_0^{\pi/2}\cos\theta\sin\theta\int_0^{2\pi}d\phi \quad = \frac{R}{2\pi}\times\frac{1}{2}\times 2\pi = \frac{R}{2}$$

Therefore center of mass of a thin hollow sphere is $\frac{R}{2}$ distance above its base center.

Example : Find position of center of mass of a thin hollow cone of height h and base radius R.

Soln. Let M be mass of the cone. Let us choose coordinate system as shown in figure. Therefore center of mass must lie somewhere on Z axis.

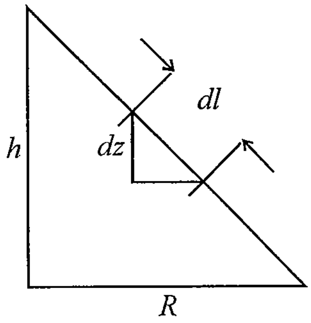
$$\therefore \quad Z_{com} = \frac{\int zdm}{\int dm} = \frac{1}{M}\int zdm$$



Let us use cylindrical coordinates.

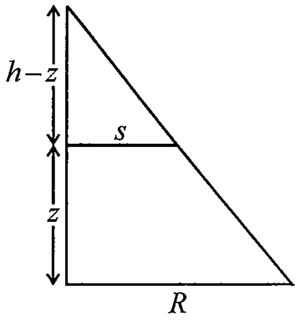
$$\therefore \quad dm = \frac{M}{A}dA = \frac{M}{\pi R\sqrt{R^2+h^2}} \cdot sd\phi\, dl$$

From similarity of small and big triangles shown in figure we get.



$$\frac{dl}{\sqrt{R^2+h}} = \frac{dz}{h} \quad \therefore \quad dl = \frac{\sqrt{R^2+h^2}}{h}\cdot dz$$

Also from similarity of two triangles shown in figure we get,



$$\frac{s}{R} = \frac{h-z}{h}$$

$$s = R\left(1-\frac{z}{h}\right)$$

$$\therefore \qquad dm = \frac{M}{\pi R\sqrt{R^2+h^2}} = \frac{M}{\pi R}\cdot\frac{R\left(1-\frac{z}{h}\right)d\phi}{\sqrt{R^2+h^2}}\cdot\frac{\sqrt{R^2+h^2}}{h}dz$$

or $$dm = \frac{M}{\pi h}\left(1-\frac{z}{h}\right)dz\,d\phi$$

$$\therefore \qquad Z_{com} = \frac{1}{M}\iint\frac{zM}{\pi h}\left(1-\frac{z}{h}\right)dz\,d\phi$$

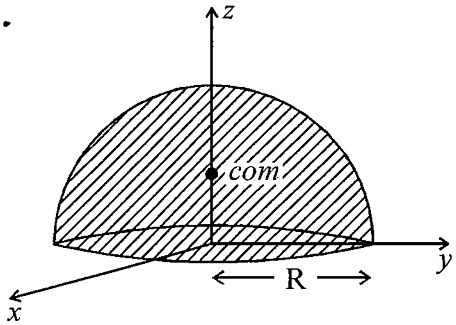
$$= \frac{1}{\pi h}\int_0^h z\left(1-\frac{z}{h}\right)dz\int_0^{2\pi}d\phi$$

$$= \frac{1}{\pi h}\cdot\frac{h^2}{6}\cdot 2\pi = \frac{h}{3}$$

Therefore center of mass of a thin hollow cone of height ‘h’ is $\frac{h}{3}$ distance above the base center of cone.

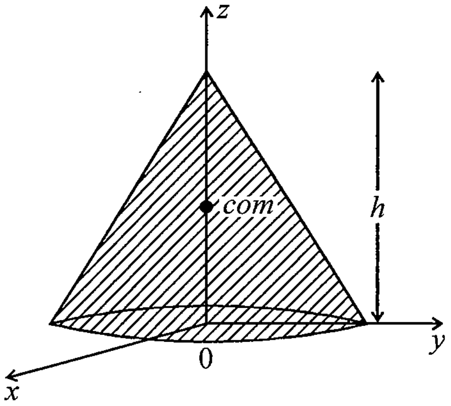
CENTER OF MASS OF SOME UNIFORM OBJECTS

- Solid hemisphere



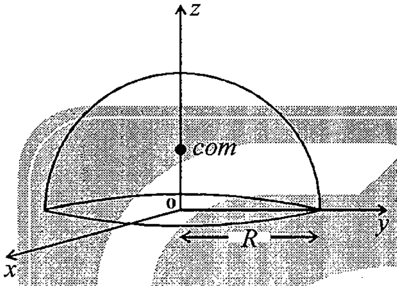
$$Z_{com} = \frac{3R}{8}$$

- Solid cone



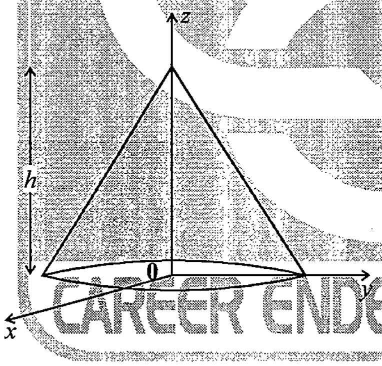
$$Z_{com} = \frac{h}{4}$$

- Thin hollow hemisphere



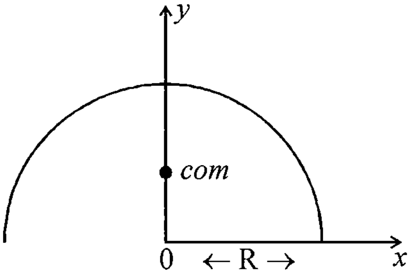
$$Z_{com} = \frac{R}{2}$$

- Thin hollow cone



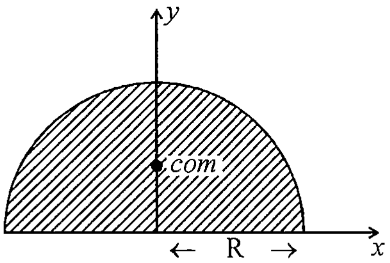
$$Z_{com} = \frac{h}{3}$$

- Semi ring



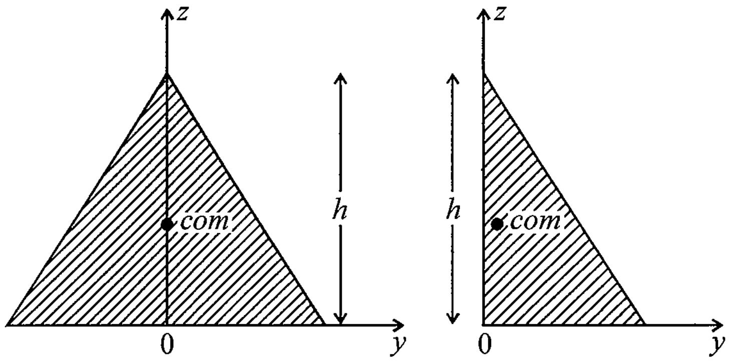
$$Y_{com} = \frac{2R}{\pi}$$

- Semi disc



$$Y_{com} = \frac{4R}{3\pi}$$

- Triangle



$Y_{com} = \frac{h}{3}$

SOLVED PROBLEMS

1. Show that two particles always move in opposite direction in their center of mass frame.

Soln. Let m_1 and m_2 be masses of two particles moving with velocities $\vec{v}_1$ and $\vec{v}_2$.

Therefore, velocity of center of mass is

$$\vec{v}_{com} = \frac{\sum m_i v_i}{\sum m_i} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2}$$

velocity of m_1 w.r.t. center of mass

$$\vec{v}_{1c} = \vec{v}_1 - \vec{v}_{com} = \vec{v}_1 - \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2}; \quad \vec{v}_{1c} = \frac{m_2(\vec{v}_1 - \vec{v}_2)}{m_1 + m_2}$$

velocity of m_2 w.r.t. center of mass

$$\vec{v}_{2c} = \vec{v}_2 - \vec{v}_{com}$$

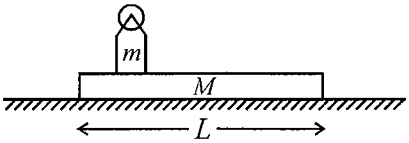
$$= \vec{v}_2 - \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2}$$

$$= \frac{m_1(\vec{v}_2 - \vec{v}_1)}{m_1 + m_2} = \frac{m_1(\vec{v}_1 - \vec{v}_2)}{m_1 + m_2}$$

or $\quad \vec{v}_{2c} = -\frac{m_1}{m_2}\vec{v}_{1c}$

Thus the two particles move in opposite direction as seen from center of mass frame.

2. A person of mass m_1 stands at one end of a plank of mass M and length L. The plank is lying on a smooth horizontal surface. What is distance moves by the plank as the person walks on it to reach the other end.



Soln. There is no horizontal force on person + plank system. Therefore center of mass of system will remain stationaly i.e. coordinate of center of mass will not change i.e. $\Delta X_{com} = 0$.

$$\Delta X_{com} = \frac{m_1 \Delta x_1 + m_2 \Delta x_2}{m_1 + m_2} = 0$$

$$\therefore \quad m_1 \Delta x_1 + m_2 \Delta x_2 = 0$$

Here $m_1 = m, \; m_2 = M$

Let plank move rightward by a distance x. Therefore distance moved by person is $L + x$.

$$\therefore \quad \Delta x_1 = L + x_1, \quad \Delta x_2 = x$$

$$\therefore \quad m_1(L + x) + Mx = 0$$

$$\therefore \quad x = -\frac{m_1 L}{M + m_1}$$

–ve sign shows that plank moves leftward as person moves rightward. Therefore distance moved by the plank is $\dfrac{mL}{m+L}$

3. A projectile is thrown to get a range R. At the highest point it splite into two parts having mass ratio 1:2. The smaller part returns to point of projection. At what distance from point of projection the bigger one lands.

Soln. Let m and $2m$ be masses of two fragments. For m, $\Delta x_1 = -R$. Since there is no external force in horizontal direction. Therefore,

$$m_1 \Delta x_1 + m_2 \Delta x_2 = 0$$

$$m(-R) + 2m\,\Delta x_2 = 0 \;\Rightarrow \Delta x_2 = \frac{R}{2}$$

Change in coordinate of bigger part is +ve so it falls furthur away from the point where projectile was expected to land.

So, distance of bigger part from point of projection is $R + \dfrac{R}{2}$ i.e. $\dfrac{3R}{2}$

4. A solid sphere of mass m collides head on with identical sphere at rest. If coefficient of restitution is e, calculate the loss in kinetic energy.

Soln. $e = \dfrac{\text{velocity of separation}}{\text{velocity of approach}}$

Let u be the velocity of a moving sphere before collision and v_1 and v_2 be the velocities of the two spheres after collision.

$$\therefore \quad e = \frac{v_1 - v_2}{u}$$

$$v_1 - v_2 = eu \qquad \text{... (i)}$$

from conservation of momentum we get

$$mu = mv_1 + mv_2$$

$$\therefore \quad v_1 + v_2 = u \qquad \text{... (ii)}$$

Solving (i) and (ii) we get $v_1 = \dfrac{u(1+e)}{2}$ and $v_2 = \dfrac{u(1-e)}{2}$

Loss of kinetic energy = $\mathrm{K \cdot E}_i - \mathrm{K \cdot E}_f$

$$\Delta K \cdot E = \frac{1}{2}mu^2 - \frac{1}{2}mv_1^2 - \frac{1}{2}mv_2^2$$

$$= \frac{1}{2}m\left[1 - \frac{(1+e)^2}{4} - \frac{(1-e)^2}{4}\right]u^2$$

$$= \frac{1}{2}mu^2\left[1 - \frac{1+e^2}{2}\right] = \frac{1}{4}mu^2(1-e^2)$$

5. Radius of a spherical water drop increases at the rate kt. Assuming its density to be uniform and initial radius r_0, calculate the velocity when it has fallen for a time t (neglect air drag and gravity).

Soln. $\frac{dr}{dt} = kt$ $\qquad\qquad$ $\therefore r = r_0 + \frac{kt^2}{2}$

Equation of motion $\frac{dp}{dt} = F$

In absence of air drag and gravity $F = 0$

$\therefore \frac{d(mv)}{dt} = 0$ $\qquad\qquad$ $v\frac{dm}{dt} + m\frac{dv}{dt} = 0$

or $\frac{dm}{m} = -\frac{dv}{v}$

Let m_0 and v_0 be respectively initial mass and speed.

Integrating we get $\frac{v}{v_0} = \frac{m_0}{m}$

or $v = v_0\frac{m_0}{m}$

since $m = \frac{4}{3}\pi r^3\rho$ $\qquad\qquad$ $\because$ ρ = density = constant

$\therefore \frac{m_0}{m} = \left(\frac{r_0}{r}\right)^3$ $\qquad\qquad$ $\therefore v = v_0 \cdot \left(\frac{r_0}{r}\right)^3 = \frac{r_0^3}{\left(r_0 + \frac{kt^2}{2}\right)^3}$

$$v = v_0\left[\frac{1}{\left(1 + \frac{kt^2}{2v_0}\right)^3}\right] \qquad \text{or} \qquad v = \frac{v_0}{\left(1 + \frac{2kt^2}{2v_0}\right)^3}$$

This shows that speed of drop decreases with time and drop comes to rest after a long time. However inreality rain drops attain a constant velocity, because of presence of air drag and gravity.

6. A falling rain drop accumulates moisture due to which its radius increases at a constant rate k. Neglecting air drag calculate the speed of rain drop after it has fallen for a time t.

Soln. Equation of motion $\frac{dp}{dt} = mg$ ... (i)

given $\frac{dr}{dt} = k \Rightarrow r = kt$ (if initial radius is zero)

since $m = \frac{4}{3}\pi r^3 \rho$ $\quad \therefore \quad \frac{dm}{dt} = 4\pi r^2 \cdot \frac{dr}{dt} \cdot \rho$

ρ is density, we assume it to be constant

$$\frac{dm}{dt} = \left(\frac{4}{3}\pi r^3 \rho\right) \cdot \frac{3k}{r}$$

$$\frac{dm}{dt} = \frac{3mk}{r} \quad \text{... (ii)}$$

Dividing (i) by (ii) we get

$$\frac{dp}{dm} = \frac{g}{3k} r \qquad dp = \left(\frac{g}{3k}\right) \cdot \left(\frac{3m}{4\pi\rho}\right)^{1/3} dm$$

on integration we get

$$\therefore \quad p = \left(\frac{g}{3k}\right)\left(\frac{3}{4\pi\rho}\right)^{1/3} \cdot \frac{m^{4/3}}{4/3} + c$$

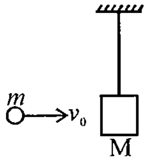
taking $p = m = 0$ at $t = 0$, we get $c = 0$

$$\therefore \quad p = \frac{g}{4k}\left(\frac{3}{4\pi\rho}\right)^{1/3} \cdot m^{4/3}$$

$$\text{or } v = \left(\frac{g}{4k}\right) \cdot \left(\frac{3m}{4\pi\rho}\right)^{1/3}$$

$$v = \frac{g}{4k} r \Rightarrow \boxed{v = \frac{g}{4} t}$$

7. Ballistic pendulum is a device for measuring speed of bullet. Consider a bullet of mass 'm' striking a ballistic pendulum as shown in the figure. The bullet gets embedded into the block and the system rises to a height h. Calculate the initial speed of the bullet.



Soln. Let v_0 be the initial speed of bullet

Here both momentum and mechanical energy are conserved.

$\therefore$ velocity of bullet + block system just after collision

$$v = \frac{mv_0}{(M+m)}$$

$\therefore$ From conservation of mechanical energy we get (apply after collision)

$$\frac{1}{2}(M+m)v^2 = (M+m)gh$$

or $$\frac{1}{2}(M+m)\cdot\frac{m^2v_0^2}{(M+m)^2} = (M+m)gh$$

$$v_0 = \left(1+\frac{M}{m}\right)\sqrt{2gh}$$

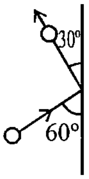
Thus by measuring height, speed of bullet can be determined.

8. A ball hits a wall with velocity 20 m/s at an angle of 60° and rebounds at an angle of 30° with the wall. Calculate the coefficient of restitution.

Soln. Since no force acts on the ball along the plane of wall.

Therefore, momentum along the wall remains conserved.

Let v be the velocity with which ball rebounds.



$\therefore\ m\cdot 20\cos 60 = mv\cos 30$

$\therefore\ v = \frac{20}{\sqrt{3}}$ m/s

coefficient of restitution $= \frac{\text{velocity of separation}}{\text{velocity of approach}}$

$$= \frac{v\sin 30°}{20\sin 60°} = \frac{\frac{20}{\sqrt{3}}\times\frac{1}{2}}{20\times\frac{\sqrt{3}}{2}} = \frac{1}{3}$$

9. Locate the centre of mass of area between the curves

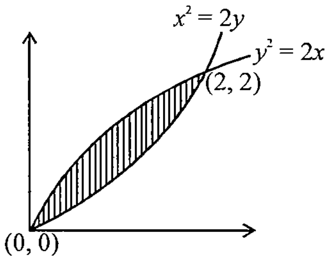
$$y^2 = 2x$$

$$x^2 = 2y$$

Soln. Area between the two curves is shown in figure.

Let A be the area between the two curves

$\therefore$ X co-ordinate of centre of mass is

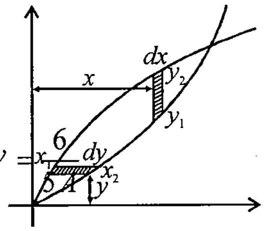


$$X_C = \frac{1}{A}\int_0^2 x\,dA = \frac{1}{A}\int_0^2 x(y_2 - y_1)dx$$

$$X_C = \frac{1}{A}\int_0^2 x\left(\sqrt{2}x - \frac{x^2}{2}\right)dx$$

$$= \frac{1}{A}\left[\sqrt{2}\frac{x^{5/2}}{5/2} - \frac{1}{2}\cdot\frac{x^4}{4}\right]_0^2$$

$$= \frac{1}{A}\left[\frac{\sqrt{2}\cdot 2^{5/2}}{5/2} - \frac{2^4}{8}\right] = \frac{6}{5A}$$



Y- co-ordinate of centroid is given by

$$Y_C = \frac{1}{A}\int_0^2 y\,dA = \frac{1}{A}\int_0^2 y(x_2 - x_1)dy = \frac{1}{A}\int_0^2 y\left(\sqrt{2y} - \frac{y^2}{2}\right)dy = \frac{6}{5A}$$

Area between the curves

$$A = \int dA = \int_0^2 (y_2 - y_1)dx = \int_0^2 (\sqrt{2x} - \frac{x^2}{2})dx = \left[\sqrt{2}\cdot\frac{x^{3/2}}{3/2} - \frac{1}{2}\cdot\frac{x^3}{3}\right]_0^2 = \frac{8}{3} - \frac{4}{3} = \frac{4}{3}$$

$$\therefore\ X_C = \frac{6}{5A} = \frac{6}{5}\times\frac{3}{4} = \frac{9}{10} \qquad Y_C = \frac{6}{5A} = \frac{9}{10}$$

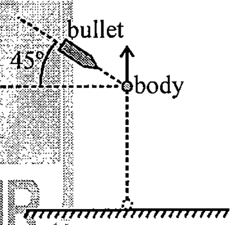
Therefore, coordinate of centroid is $= \left(\frac{9}{10}, \frac{9}{10}\right)$

10. A body of mass 9gm is thrown vertically upward with an initial velocity of 100 m/s. After one second, a bullet of mass 1gm hits the body at an angle of 45° with horizontal with a velocity of $90\sqrt{2}$ m/s and sticks to it. Calculate total horizontal distance travelled by the body bullet system. (take $g = 10\text{m/s}^2$).

Soln. Let the bullet strikes the body as shown in figure, if v be the horizontal component of velocity of body-bullet system after collision then from conservation of horizontal component of linear momentum we get,

$$(1\text{gm})\ (90\sqrt{2}\ \text{m/s})\cos 45 = 10(\text{gm})v$$

$$\therefore\ v = 9\text{m/s}$$



height of the body at the time of collision is

$$h = 100\times 1 + \frac{1}{2}(-10)\times 1^2 \qquad [\text{use } s = ut + \frac{1}{2}at^2]$$

$$h = 95 \text{ meter}$$

velocity of body just before collision $v = u + at = 100 - 10\times 1 = 90\text{m/s}$

let v' be vertical component of velocity of body bullet system just after collision.

$$\therefore\ 9\times 90 - 1\times 9\sqrt{2}\sin 45° = 10v'$$

$$890 = 10v' \Rightarrow v' = 89\text{m/s}$$

Time taken by the body-bullet system to reach the ground is obtained using

$$s = ut + \frac{1}{2}at^2,\ -95 = 89t - \frac{1}{2}\times 10t^2$$

$$5t^2 - 89t - 95 = 0$$

$$\therefore\ t = \frac{89 \pm \sqrt{(89)^2 + 4\times 5\times 95}}{2\times 5} = 18.81 \text{ sec}$$

Total horizontal distance moved by bullet-body system is

$$s = vt = 9\times 18.81 = 169.30 \text{ meter}$$

11. In an inelastic collision which one of the following is true? **[GP-2006]**

(a) Total energy is not conserved (b) Momentum is not conserved

(c) Kinetic energy is conserved (d) Kinetic energy is not conserved

Soln. (d)

12. A mass 0.5kg moving with a speed of 2m/s hits another mass 1kg moving in the same direction with a speed 1m/s, kinetic energy of centre of mass is **[GP-2008]**

(a) 4/3J (b) 3/4J (c) 0J (d) 8/3J

Soln. (a) Speed of centre of mass is

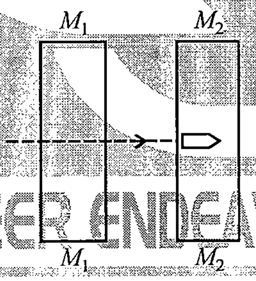
$$v_c = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} = \frac{0.5\times 2 + 1\times 1}{0.5+1} = \frac{2}{1.5}$$

$$v_c = \frac{2}{1.5} \text{ m/s}$$

kinetic energy of centre of mass

$$= \frac{1}{2}(m_1 + m_2)v_c^2 = \frac{1}{2}(0.5+1)\times\left(\frac{2}{1.5}\right)^2 = \frac{2}{1.5} = 4/3\text{Joule}$$

13. A 20 gm bullet pierces through a plate of mass $M_1 = 1$ kg and then comes to rest inside a second plate of mass $M_2 = 2.98$ kg as shown in figure. It is found that the two plates, initially at rest, now move with equal velocities. Find the percentage loss in the velocity of the bullet when it is between M_1 and M_2. Neglect any loss of material of the platem due to action of bullet.



Soln. Let v_1 m/sec be the initial velocity of the bullet and v_2 m/sec the velocity with which each plate moves. Here the momentum of the plate M_2 and the bullet because the bullet comes to rest inside the second plate M_2. Therefore, initial momentum = Final momentum

$$m v_1 = M_1 v_2 + (M_2 + m) v_2$$

$$0.02 v_1 = 1\times v_2 + (2.98 + 0.02) v_2$$

$$v_1 = 200 v_2 \qquad \text{... (i)}$$

Let the velocity of the bullet when it comes out of the plate M_1 be v_3. The momentum of the bullet between plates M_1 and M_2 is equal to the momentum of plates M_2 plus the bullet

$$0.02 v_3 = (0.02 + 2.98) v_2$$

$$v_3 = 150 v_2 \qquad \text{... (ii)}$$

Now % loss $= \dfrac{v_1 - v_3}{v_1}\times 100 = \dfrac{200 v_2 - 150 v_2}{200 v_2}\times 100 = \mathbf{25}$ %.

14. A projectile of mass 50 kg is shot vertically upwards with an initial velocity of 100 m/sec. After 5 seconds it explodes into two fragments, one of which having mass 20 kg travels vertically up with a velocity of 150 m/sec.

(i) What is the velocity of the other fragment at that instant ?

(ii) Calculate the sum of the momenta of the two fragments 3 second after the explosion. What would have been the momentum of the projectile at this instant if there had been no explosion ?

Soln. (i) After explosion, one fragment of mass 20 kg goes upward with velocity 150 m/sec. It is quite obvious that the second fragment of mass 30 kg will go downward with a velocity, say v.

The velocity of the projectile after 5 seconds

$$= u - gt$$

$$= 100 - 9.8 \times 5 = 51 \text{ m/sec.}$$

The momentum of projectile before explosion

$$= 50 \times 51 = 2550 \text{ kg m/sec.}$$

Momentum of projectile after explosion

$$= 20 \times 150 - 30v$$

$$2550 = 20 \times 150 - 30v \text{ or } v = \frac{3000 - 2550}{30} = \mathbf{15\ m/sec.}$$

(ii) After 3 seconds velocity of first fragment

$$= 150 - 9.8 \times 3 = 120.6 \text{ m/sec.}$$

Velocity of second fragment after 3 seconds

$$= 15 + 9.8 \times 3 = 44.4 \text{ m/sec.}$$

Sum of the momenta = Momentum of first fragment + Momentum of second fragment

$$= 20 + 120.6 + 30 \times 44.4$$

$$= 2412 + 1332 = \mathbf{3744\ kg\ m/sec}$$

Velocity of projectile after 8 second

$$= 100 - 9.8 \times 8$$

$$= 21.6 \text{ m/sec}$$

Momentum of projectile after 8 sec.

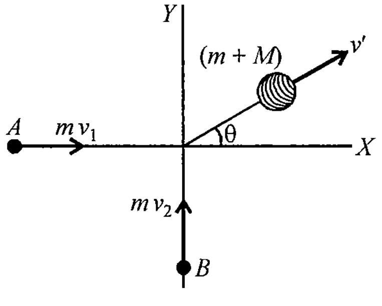
$$= 50 \times 21.6 = \mathbf{1080\ kg\ m/sec.}$$

15. A body of mass m moving with a velocity v_1 in the X-direction collides with another body of mass M moving in Y-direction with a velocity v_2. They coalesce into one body during collision. Calculate

(i) the direction and magnitude of the momentum of the final body,

(ii) the fraction of the initial kinetic energy transformed into heat during the collision in terms of the two masses.

Soln. The situation is shown in figure.



(i) Equating the total initial and final momentum along each axis, we get

$$mv_1 + 0 = (M + m)v' \cos\theta \quad \text{... (i)}$$

$$0+Mv_2=(M+m)v'\sin\theta \qquad \text{... (ii)}$$

Squaring and adding equations (i) and (ii), we get

$$(mv_1)^2+(Mv_2)^2=(M+m)^2v'^2 \qquad \text{... (iii)}$$

The final momentum P,

$$=(M+m)v'=\sqrt{[(mv_1)^2+(Mv_2)^2]} \qquad \text{from equation (iii)}$$

Dividing equation (ii) by equation (i), we have

$$\tan\theta=\frac{Mv_2}{mv_1} \text{ or } \theta=\tan^{-1}\left(\frac{Mv_2}{mv_1}\right)$$

(ii) The initial kinetic energy $=\frac{1}{2}mv_1^2+\frac{1}{2}Mv_2^2$

Final kinetic energy $=\frac{1}{2}(M+m)v'^2=\frac{1}{2}\cdot\frac{1}{(M+m)}\cdot[(mv_1)^2+(Mv_2)^2]$

Decrease in total kinetic energy is given by

$$\Delta E=\left(\frac{1}{2}mv_1^2+\frac{1}{2}Mv_2^2\right)-\frac{1}{2}\cdot\frac{1}{(M+m)}[(mv_1)^2+(Mv_2)^2]$$

$$=\frac{1}{2}\frac{Mm}{(M+m)}(v_1^2+v_2^2)$$

Fraction of kinetic energy transformed into heat

$$=\frac{\Delta E}{\frac{1}{2}mv_1^2+\frac{1}{2}Mv_2^2}=\left(\frac{Mm}{M+m}\right)\left(\frac{v_1^2+v_2^2}{mv_1^2+Mv_2^2}\right).$$

16. A horizontal spring-block-system of mass M executes simple harmonic motion. When the block is passing through its equilibrium position, an object of mass m is put on it and the two move together. Find the new amplitude and frequency of vibration.

Soln. Let A and ω be the initial amplitude and angular frequency of the spring-block-system. Then

$$\omega=\sqrt{\left(\frac{K}{M}\right)},$$

where K = spring constant.

When an object of mass m is placed on the spring, then the new angular frequency ω' is given by

$$\omega'=\sqrt{\left(\frac{K}{M+m}\right)}=\omega=\sqrt{\left(\frac{M}{M+m}\right)} \qquad \text{... (i)}$$

Let v and v' be the velocities of the system before and after putting the object, then

$$Mv=(M+m)v' \qquad \text{... (ii)}$$

$$\frac{1}{2}Mv^2=\frac{1}{2}KA^2 \qquad \text{... (iii)}$$

$$\frac{1}{2}(M+m)(v')^2=\frac{1}{2}KA'^2 \qquad \text{... (iv)}$$

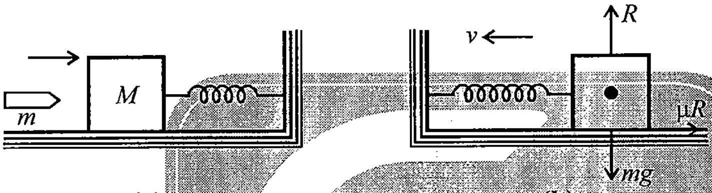
From eq. (ii) $\frac{v}{v'} = \left(\frac{M+m}{M}\right)$ ...(v)

Dividing eq. (iii) by eq. (iv), we get

$$\frac{M}{(m+M)}\cdot\frac{v^2}{v'^2} = \frac{A^2}{A'^2} \text{ or } \frac{A}{A'} = \frac{v'}{v}\left[\frac{(M+m)}{M}\right]^{1/2}$$

$$\therefore \quad A' = A\left(\frac{v'}{v}\right)\left[\frac{(M+m)}{M}\right]^{1/2} = A\left(\frac{M}{M+m}\right)\left[\frac{(M+m)}{M}\right]^{1/2} \text{ or } A' = A\left[\frac{M}{(M+m)}\right]^{1/2}.$$

17. (a) A rifle bullet of mass 0.1 kg strikes and embeds itself in a block of mass 0.9 kg. The block rests on a horizontal frictionless table figure (a) and is attached to a spring. The impact compresses the spring by 0.2 m. If a force of 1 N compresses the spring by 2 cm, calculate the initial velocity of the bullet. What is the velocity of the block just after the impact ?



(a) (b)

(b) A 2 kg block collides with a horizontal weightless spring of force constant 1.0 nt/m figure (b). The block compresses the spring 6 m from the rest position. Assuming the coefficient of kinetic friction between the block and horizontal surface as 0.25, what was the speed of the block at the instant of collision ?

Soln. (a) We know that the elastic potential energy of the spring $= \frac{1}{2}kx^2$, where k is the spring constant and x is the compression.

Here, $x = 0.2$ m and $k = \frac{1}{0.02} = 50$ N/m.

Therefore, potential energy of spring $= \frac{1}{2}kx^2 = \frac{1}{2}\times 50\times(0.2)^2 = 1$ joule.

Let u be the velocity of the bullet before collision and v the velocity of block and bullet after collision. Applying the law of conservation of momentum, we have $mu = (m+M)v$.

Therefore, $v = \frac{mu}{m+M} = \frac{0.1u}{(0.1+0.9)} = 0.1u$ ...(i)

After collision, K.E. of block and bullet = P.E. of the spring.

Therefore, $\frac{1}{2}(m+M)v^2 = 1$ or $v^2 = 2$ m/sec or $v = \sqrt{2}$ **m/sec.** ...(ii)

From eq. (i), $\sqrt{2} = 0.1u$ or $u = \mathbf{10\sqrt{2}}$ **m/sec.**

(b) When the spring is compressed the workdone on the spring is stored as elastic potential energy

Therefore, P.E. of the spring $= \frac{1}{2}kx^2$, where k is force constant and x is the compression.

Hence, P.E. $= \frac{1}{2}\times 1.0\times(6)^2 = 18$ joule.

Workdone against frictional force $= \mu \cdot mg \cdot x$

$$= 0.25 \times 2 \times 9.8 \times 6 = 29.4 \text{ joule.}$$

Let v be the initial velocity of the block, then $\frac{1}{2}mv^2 = 18 + 29.4 = 47.4.$

Therefore, $v^2 = \frac{47.4 \times 2}{m} = \frac{47.4 \times 2}{2} = 47.4$ or $v = \mathbf{6.89\ m/sec.}$

18. A moving particle of mass m collides head-on with a particle of mass $2m$ which is initially at rest. Show that the particle m will loose 8/9th part of its initial kinetic energy after the collision.

Soln. Let u_1 be the initial velocity of mass m before collision and v_1 and v_2 the velocities of masses m and $2m$ after collision respectively.

According to the law of conservation of kinetic energy, we have

$$\frac{1}{2}m u_1^2 = \frac{1}{2}m v_1^2 + \frac{1}{2}(2m)v_2^2$$

or $\quad u_1^2 - v_1^2 = 2v_2^2$

or $\quad (u_1 - v_1)(u_1 + v_1) = 2v_2^2 \qquad \text{...(i)}$

By the law of conservation of momentum, we have

$$m u_1 = m v_1 + (2m) v_2$$

$$(u_1 - v_1) = 2v_2 \qquad \text{...(ii)}$$

From eqs. (i) and (ii), we get $(u_1 + v_1) = v_2 \qquad \text{...(iii)}$

Substituting the value of v_2 from eq. (iii) in eq. (ii), we get

$$u_1 - v_1 = 2(u_1 + v_1) \text{ or } -3v_1 = u_1 \text{ or } v_1 = -\left(\frac{1}{3}\right)u_1 \qquad \text{...(iv)}$$

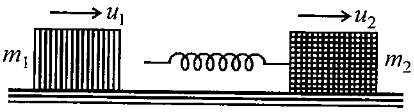
Now the initial and final kinetic energies of mass m are

$$K_i = \left(\frac{1}{2}\right)m u_1^2 \text{ and } K_f = \left(\frac{1}{2}\right)m v_1^2 = \left(\frac{1}{2}\right)m\left(\frac{u_1^2}{9}\right)$$

Therefore, fraction loss $= \frac{K_i - K_f}{K_i} = \frac{\frac{1}{2}m u_1^2\left(1 - \frac{1}{9}\right)}{\frac{1}{2}m u_1^2} = \mathbf{\frac{8}{9}}.$

19. Two blocks of masses $m_1 = 2$ kg and $m_2 = 5$ kg are moving in the same direction along a frictionless surface with speeds 10 m/s and 3 m/s respectively, m_2 being ahead of m_1. An ideal spring with $k = 1120$ nt/m is attached to the back side of m_2. Find the maximum compression of the spring when the blocks collide.

Soln. The situation is shown in figure.



Let v be the speed of the system after collision. Applying the law of conservation of energy, we have

$$m_1 u_1 + m_2 u_2 = (m_1 + m_2)v$$

Substituting the given values

$$(2\times 10)+(5\times 3)=(2+5)v$$

$$v=\frac{20+15}{7}=5\text{ m/s.}$$

Now applying the law of conservation of kinetic energy, we get

$$\frac{1}{2}m_1u_1^2+\frac{1}{2}m_2u_2^2=\frac{1}{2}(m_1+m_2)v^2+\frac{1}{2}kx^2$$

or $$m_1u_1^2+m_2u_2^2=(m_1+m_2)v^2+kx^2$$

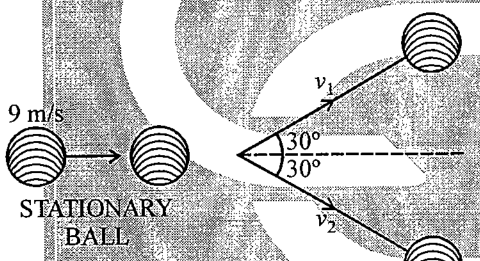
$$(2\times 100)+(5\times 9)=(7\times 25)+(1120\times x^2)$$

$$200+45=175+1120x^2$$

$$x^2=\frac{200+45-175}{1120}=\frac{1}{16}$$

$$x=\frac{1}{4}=\mathbf{0.25\ m.}$$

20. A ball moving with a speed of 9 m/s strikes an identical stationary ball such that after the collision the direction of each ball makes an angle of 30° with the original line of motion figure.



Find the speeds of the two balls after the collision. Is the kinetic energy conserved in this collision process ?

Soln. Initial momentum of the balls $= m\times 9+m\times 0=9m$ ... (i)

where m is the mass of each ball.

Let after collision their velocities are v_1 and v_2 respectively.

Final momentum of the balls after collision along the same line $= mv_1\cos 30+mv_2\cos 30$.

$$=\frac{mv_1\sqrt{3}}{2}+\frac{mv_2\sqrt{3}}{2} \qquad \text{... (ii)}$$

According to law of conservation of momentum

$$9m=\frac{mv_1\sqrt{3}}{2}+\frac{mv_2\sqrt{3}}{2}$$

$$\frac{9\times 2}{\sqrt{3}}=v_1+v_2 \qquad \text{... (iii)}$$

The initial momentum of the balls along perpendicular direction $=0$.

Final momentum of balls along the perpendicular direction $= mv_1\sin 30-mv_2\sin 30=\frac{m}{2}(v_1-v_2)$.

Again by the law of conservation of momentum $\left(\frac{m}{2}\right)(v_1 - v_2) = 0$.

Therefore, $(v_1 - v_2) = 0$ ... (iv)

Solving equations (iii) and (iv), we have $v_1 = 3\sqrt{3}$ m/s and $v_2 = 3\sqrt{3}$ m/s.

According to law of conservation of energy, Energy before collision = Energy after collision

$$\frac{1}{2}mu_1^2 + \frac{1}{2}mu_2^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2$$

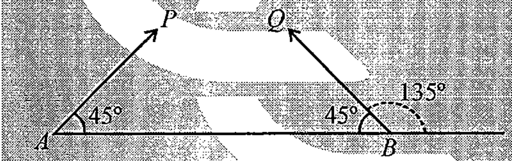
$$\frac{1}{2}m(9)^2 + 0 = \frac{1}{2}m(3\sqrt{3})^2 + \frac{1}{2}m(3\sqrt{3})^2$$

$$\frac{81\,m}{2} = \frac{54\,m}{2}$$

$$L.H.S. \neq R.H.S.$$

i.e., energy is not conserved in this collision or this is a case of inelastic collision.

21. Particles P and Q of mass 20 gm and 40 gm respectively are simultaneously projected from points A and B on the ground. The initial velocities of P and Q make 45° and 135° angles respectively with the horizontal as shown in the figure. Each particle has an initial speed of 49 m/sec. The separation AB is 245 m. Both particles travel in the same vertical plane and undergo a collision. After the collision P retraces its path. Determine the position of Q when it hits the ground. How much time after the collision does the particle Q take to reach the ground. Take $g = 9.8$ m/sec^2.



Soln. We know that horizontal range $R = u^2 \sin 2\theta / g$.

For particle P, $\theta = 45°$, $u = 49$ m/s

$$\therefore \quad R = \frac{(49)^2 \sin 90}{9.8} = \frac{(49)^2}{9.8} \text{ meter}$$

For particle Q, $\theta = 180 - 135 = 45°$, $u = 49$ m/s

$$\therefore \quad R = \frac{(4.9)^2}{9.8} \text{ meter}$$

Thus horizontal range is the same for both particles.

As the horizontal range is the same for both the particles, they will meet each other exactly midway between AB.

Therefore, horizontal distance $x = u\cos\theta \cdot t$, where t is the time after which the two particles meet.

$$x = \frac{AB}{2} = \frac{245}{2} = 122.5 \text{ meter}$$

$$\therefore \quad 122.5 = 49\cos 45° \cdot t$$

or $$t = \frac{122.5}{49\left(1/\sqrt{2}\right)} = 5/\sqrt{2} \text{ second} \qquad \text{... (i)}$$

At the instant of the collision, the vertical velocities of both particles are zero. The two particles have reached the maximum height. The horizontal velocity is given by

$$v = u\cos\theta = 49\cos 45° = 49/\sqrt{2} \text{ m/s}$$

Initial momentum of two particles

$$= m_1 v - m_2 v = 0.02v - 0.04v = -0.02v \qquad \text{... (ii)}$$

Momenta of two particles after collision $= m_1(-v) + m_2 V$, where V is the velocity of mass m_2 after collision.

$$= -0.02v + 0.04V \qquad \text{... (iii)}$$

Therefore, $V = 0$.

The velocity of particle Q becomes zero after collision. Now it falls vertically down midway between A and B.

$$\text{Maximum height} = \frac{u^2\sin^2\theta}{2g}$$

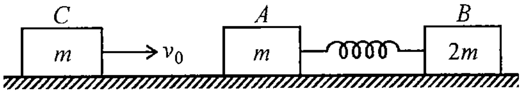
Here $\theta = 45°$, $u = 49$ m/s and $g = 9.8$ m/sec^2

$$\text{Therefore, maximum height} = \frac{(49)^2 \times \left(\frac{1}{\sqrt{2}}\right)^2}{2\times 9.8} = \frac{245}{4} \text{ m}$$

Time of free fall t, is given by $\frac{1}{2}gt_1^2 = \text{maximum height} = \frac{245}{4}$

$$t_1^2 = \frac{245\times 2}{4\times 9.8} \text{ or } t_1 = \frac{5}{\sqrt{2}} \text{ sec.}$$

22. Two bodies A and B of masses m and $2m$ respectively are placed on a smooth floor. They are connected by a spring. A third body C of mass m moves with a velocity v_0 along the line joining A and B and collides elastically with A, as shown in figure. At a certain instant of time t_0 after collision, it is found that the instantaneous velocities of A and B are the same. Further, at this instant the compression of the spring is found to be x_0. Determine (i) the common velocity of A and B at time t_0; and (ii) the spring constant.



Soln. The masses of C and A are same. Initially the body A is not in motion. When the body C collides elastically to body A, it stops (masses of two bodies are same) and transfers its momentum mv_0 and kinetic energy $(1/2)mv_0^2$ to body A. In this position, B is stationary and the spring is in its normal state. Thus, the instantaneous momentum of the system is mv_0 and kinetic energy is $(1/2)mv_0^2$. Now the spring compresses and B comes in motion.

Let the instantaneous velocity of A and B after collision be v and the compression of spring x_0 (given). As no external force acts on the system, we have, applying the law of conservation of momentum and energy:

$$mv_0 = mv + (2m)v \qquad \text{... (i)}$$

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 + \frac{1}{2}(2m)v^2 + \frac{1}{2}kx_0^2 \qquad \text{... (ii)}$$

From equation (i), $v = \frac{1}{3}v_0$ ... (iii)

Substituting the value of v in equation (ii) from equation (iii), we have

$$\frac{1}{2}mv_0^2 = \frac{1}{2}m\left(\frac{v_0}{3}\right)^2 + \frac{1}{2}(2m)\left(\frac{v_0}{3}\right)^2 + \frac{1}{2}kx_0^2$$

Solving we get $k = \dfrac{2mv_0^2}{3x_0^2}$. ... (iv)

23. A ball of mass 100 gm is projected vertically upwards from the ground with a velocity of 49 m/sec. At the same time another identical ball is dropped from a height of 98 m to fall freely along the same path as that followed by the first ball. After some time the two balls collide and stick together and finally fall to the ground. Find the time of flight of the masses.

Soln. Let the two balls meet after a time of t second. The distance s_1 covered by the ball projected vertically upwards in time t is given by

$$s_1 = ut - \frac{1}{2}gt^2 = 49t - \frac{1}{2}gt^2 \qquad \text{... (i)}$$

The distance s_2 covered by the ball dropped from a height 98 m in t second is given by

$$s_2 = 0 + \frac{1}{2}gt^2 \qquad \text{... (ii)}$$

Adding equations (i) and (ii), we get

$$s_1 + s_2 = 49t = 98 \text{ m} \quad (\because s_1 + s_2 = 98 \text{ m})$$

$\therefore \quad t = \mathbf{2}$ **sec.**

So the two balls collide after 2 seconds.

Before collision, velocity of the ball thrown upwards

$$v_1 = 49 - 9.8 \times 2 = 29.4 \text{ m/s}$$

Velocity of the ball coming downwards

$$v_2 = 0 + gt = 9.8 \times 2 = 19.6 \text{ m/s}.$$

Let after collision, v be the velocity of combined mass.

Applying the law of conservation of linear momentum

$$m_1v_1 + m_2v_2 = (m_1 + m_2)v$$

$$-100 \times 29.4 + 100 \times 19.6 = 200v$$

$\therefore \quad v = -4.9$ m/s.

The negative sign shows that the velocity of the combined mass is in the upward direction.

The height to which the combined mass rises after impact

$$= \frac{v^2}{2g} = \frac{(4.9)^2}{2 \times 9.8} = 1.225.$$

Time taken to cover this distance is given by $v = u - gt$ or $0 = 4.9 - 9.8t_1$ or $t_1 = \mathbf{0.5}$ **sec.**

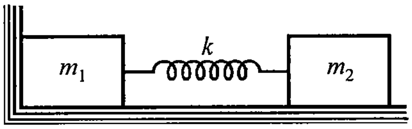
The distance of the combined mass from the ground at this time is given by $s_1 + 1.225 = 78.4 + 1.225 = 79.625$.

The time t_2 taken by the combined mass to reach ground $s = ut + \frac{1}{2}gt^2$ or $79.625 = 0 + \frac{1}{2} \times 9.8 \times t_2^2$.

$t_2 = \mathbf{4.03}$ **sec.**

Therefore, total time of flight of the masses $= t + t_1 + t_2 = \mathbf{6.53}$ **sec.**

24. Two blocks of masses m_1 and m_2 connected by a weightless spring of stiffness k rest on a smooth horizontal plane. Block 2 is shifted a small distance x to be left and then released. Find the velocity of center of mass of the system after block 1 breaks of the wall.



Soln. We know that the potential energy of compression $=\dfrac{1}{2}kx^2$.

When the block m_1 breaks off from the wall, the spring has its unstretched length, and the kinetic energy of block m_2 is given by $\dfrac{1}{2}m_2 v_2^2 = \dfrac{1}{2}kx^2$.

$$\therefore \qquad v_2^2 = \frac{kx^2}{m_2} = x\sqrt{\left(\frac{k}{m_2}\right)}$$

For center of mass $\bar{x} = \dfrac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$

The distances x_1 and x_2 are measured from the wall.

Again $\dfrac{d\bar{x}}{dt} = \dfrac{m_1}{m_1+m_2}\dfrac{dx_1}{dt} + \dfrac{m_2}{m_1+m_2}\dfrac{dx_2}{dt}$

At start $\dfrac{dx_1}{dt} = 0$

$$\frac{d\bar{x}}{dt} = \frac{m_2 v_2}{m_1+m_2} = \left(\frac{m_2 x}{m_1+m_2}\right)\sqrt{\left(\frac{k}{m_2}\right)} = \frac{x\sqrt{(k\,m_2)}}{m_1+m_2}.$$

Chapter 5

Stability Analysis

By stability analysis we mean finding equilibrium positions and investigating whether the given equilibrium is stable or unstable. It is easier to do stability analysis through potential rather than force. Therefore we will try to write potential of given system to discuss equilibrium whenever required.

Equilibrium criteria in one dimension:

If $V(x)$ be the potential under which a particle is moving then force acting on the particle is $F_x = -\frac{dV(x)}{dx}$

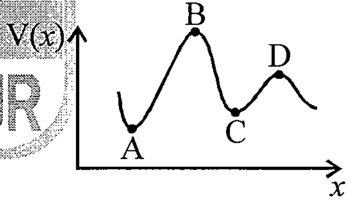
at equilibrium point $F_x = 0$. Therefore, condition for equilibrium in terms of potential becomes, $\frac{dV(x)}{dx} = 0$

In $V(x)$ versus x graph $\frac{dV}{dx} = 0$ at the points where tangent to the curve is parallel to x axis. Therefore in the figure shown below point A, B, C and D are equilibrium points.

Stable equilibrium point: A point is stable equilibrium point if, a particle at this point when displaced towards right experiences force towards left and vice versa. That is the force tries to bring it back.

Therefore at stable equilibrium point $\frac{dF_x}{dx} < 0$

$$\therefore \frac{d^2V(x)}{dx^2} > 0 \text{ (condition for minimum)}$$



Thus, a stable equilibrium point is a minimum on $V(x)$ versus x plot.
Therefore points A and C in the figure shown above are stable point.

Unstable equilibrium point: In this case a particle at equilibrium point is when displaced towards right it experiences force also in rightward direction. That is the force tries to displace the particle away from equilibrium point. Therefore at unstable equilibrium point, $\frac{dF_x}{dx} > 0$

$$\therefore \frac{d^2V(x)}{dx^2} < 0 \text{ (condition of maximum)}$$

Thus, an unstable point is a maximum on $V(x)$ versus x plot. Therefore points B and D in the figure shown above are unstable points.

Frequency of oscillation: We know a particle of mass m moving under potential $V(x)=V_0+\frac{1}{2}kx^2$ has frequency of oscillation $\omega=\sqrt{\frac{k}{m}}$, about stable equilibrium point. For a particle moving under some arbitrary potential $V(x)$, if x_0 be the stable equilibrium point, then we can expand $V(x)$ about x_0 using Taylor's series expansion as,

$$V(x)=V(x_0)+(x-x_0)\left.\frac{dV}{dx}\right|_{x=x_0}+\frac{(x-x_0)^2}{2}\left.\frac{d^2V}{dx^2}\right|_{x=x_0}+\frac{(x-x_0)^3}{\underline{|3}}\left.\frac{d^3V}{dx^3}\right|_{x=x_0}+\ldots\ldots$$

where $(x-x_0)$ is displacement from stable equilibrium point.

At this point, $\left.\frac{dV}{dx}\right|_{x=x_0}=0$, therefore, if $(x-x_0)$ be small then

$$V(x)=V(x_0)+\frac{(x-x_0)^2}{2}\left.\frac{d^2V}{dx^2}\right|_{x=x_0} \text{ (higher power terms neglected)}$$

or $V(x)=V(x_0)+\frac{1}{2}k(x-x_0)^2$

Therefore frequency of oscillation about stable equilibrium is

$$\omega=\sqrt{\frac{k}{m}} \quad \text{where, } k=\left.\frac{d^2V}{dx^2}\right|_{x=x_0}, \ k \text{ is called force constant.}$$

Equilibrium criteria in two dimension: If a particle is moving under a two dimensional potential $V(x,y)$ then, at equilibrium point,

$$\left.\frac{\partial}{\partial x}V(x,y)\right|_{x_0,y_0}=0,\ \left.\frac{\partial}{\partial y}V(x,y)\right|_{x_0,y_0}=0$$

for stable equilibrium point (minimum)

$$\left.\frac{\partial^2}{\partial x^2}V(x,y)\right|_{x_0,y_0}>0,\ \left.\frac{\partial^2}{\partial y^2}V(x,y)\right|_{x_0,y_0}>0$$

$$\left.\frac{\partial^2}{\partial x^2}V(x,y)\right|_{x_0,y_0}\cdot\left.\frac{\partial^2}{\partial y^2}V(x,y)\right|_{x_0,y_0}>\left[\left.\frac{\partial^2}{\partial x\partial y}V(x,y)\right|_{x_0,y_0}\right]^2$$

For unstable equilibrium point (maximum)

$$\left.\frac{\partial^2}{\partial x^2}V(x,y)\right|_{x_0,y_0}<0,\ \left.\frac{\partial^2}{\partial y^2}V(x,y)\right|_{x_0,y_0}<0$$

$$\left.\frac{\partial^2}{\partial x^2}V(x,y)\right|_{x_0,y_0}\cdot\left.\frac{\partial^2}{\partial y^2}V(x,y)\right|_{x_0,y_0}>\left[\left.\frac{\partial^2}{\partial x\partial y}V(x,y)\right|_{x_0,y_0}\right]^2$$

for saddle point:

$$\left.\frac{\partial^2}{\partial x^2}V(x,y)\right|_{x_0,y_0}\cdot\left.\frac{\partial^2}{\partial y^2}V(x,y)\right|_{x_0,y_0}<\left[\left.\frac{\partial^2}{\partial x\partial y}V(x,y)\right|_{x_0,y_0}\right]^2$$

SOLVED PROBLEMS

1. A particle of mass m is moving under a one dimensional potential $V(x)=-ax+bx^2$ where $a>0$ and $b>0$. Find the equilibrium points and find frequency of oscillation about the stable equilibrium.

Soln. At equilibrium point, $\left.\frac{dV}{dx}\right|_{x=x_0}=0$

$$\therefore\ -a+2bx_0=0\quad\text{or}\quad x_0=\frac{a}{2b}$$

Thus, $x_0=\frac{a}{2b}$ is an equilibrium point.

To know whether it is stable or unstable equilibrium point let us calculate second derivative of potential at equilibrium point.

$$\left.\frac{d^2V}{dx^2}\right|_{x=x_0}=2b>0$$

Therefore $x_0=\frac{a}{2b}$ is stable equilibrium point, force constant $k=\left.\frac{d^2V}{dx^2}\right|_{x=x_0}=2b$

Therefore, frequency of oscillation is

$$\omega=\sqrt{\frac{k}{m}}=\sqrt{\frac{2b}{m}}$$

2. Potential corresponding to force between the atoms of a diatomic molecule is $V(r)=\frac{a}{r^{12}}-\frac{b}{r^6}$ where a and b are positive constants and r is separation between the atoms. Calculate bond length for stable configuration and also calculate frequency of oscillation of atoms if mass of each atom be m.

Soln. For stable configuration $\left.\frac{dV}{dr}\right|_{r=r_0}=0$

$$-\frac{12a}{r_0^{13}}+\frac{6b}{r_0^7}=0\Rightarrow r_0=\left(\frac{2a}{b}\right)^{1/6}$$

Therefore bond length for stable configuration is $\left(\frac{2a}{b}\right)^{1/6}$

$$\text{Force constant } k=\left.\frac{d^2V}{dr^2}\right|_{r=r_0}=\frac{12\times13a}{r_0^{14}}-\frac{6\times7b}{r_0^8}=\frac{1}{r_0^8}\left(\frac{12\times13a}{r_0^6}-6\times7b\right)$$

$$=\left(\frac{b}{2a}\right)^{8/6}\left(\frac{12\times 13a}{2a/b}-42b\right)=\left(\frac{b}{2a}\right)^{4/3}.36b=\frac{18}{2^{1/3}}.\frac{b^{7/3}}{a^{4/3}}$$

Reduced mass of system $\mu=\frac{m.m}{m+m}=m/2$

Frequency of oscillation $\omega=\sqrt{\frac{k}{\mu}}=\sqrt{\frac{18}{2^{1/3}}.\frac{b^{7/3}}{m/2a^{4/3}}}=6\left(\frac{b^7}{2m^3a^4}\right)^{1/6}$

3. A particle of mass 'm' is moving under potential $V(x)=ax^3-bx^2$. Initially the particle is at rest at stable point. What minimum speed be given to the particle so that it reaches unstable point. Plot the potential versus x.

Soln. For equilibrium point $\left.\frac{dV}{dx}\right|_{x=x_0}=0$

$$3ax_0^2-2bx_0=0\ \Rightarrow x_0=0,\frac{2b}{3a}$$

$$\left.\frac{d^2V}{dx^2}\right|_{x=x_0}=6ax_0-2b,\ \left.\frac{d^2V}{dx^2}\right|_{x=0}=-2b<0,\ \therefore x_0=0 \text{ is unstable point}$$

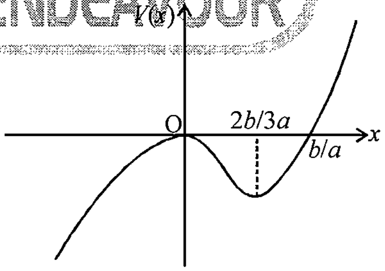
$$\left.\frac{d^2V}{dx^2}\right|_{x=\frac{2b}{3a}}=2b>0,\ \therefore\ x_0=\frac{2b}{3a} \text{ is stable point}$$

To calculate speed let us apply conservation of energy.
Total energy initial = Total energy final

(Kinetic Energy + Potential Energy) at $x=\frac{2b}{a}$ = (Kinetic Energy + Potential Energy) at $x=0$

for minimum speed (u) at stable point the particle will just reach unstable point and stops there.

$$\therefore\ \frac{1}{2}mu^2+V\left(x=\frac{2b}{3a}\right)=\frac{1}{2}m.0^2+V(x=0)$$



$$\frac{1}{2}mu^2+a\left(\frac{2b}{3a}\right)^3-b\left(\frac{2b}{3a}\right)^2=0+0$$

$$u^2=-\frac{2}{m}\left(\frac{2b}{3a}\right)^2\left(\frac{2b}{3}-b\right)=\frac{2}{m}.\frac{4b^2}{9a^2}.\frac{b}{3}=\frac{8b^3}{27ma^2}$$

$$\therefore u=\sqrt{\frac{8b^3}{27ma^2}}$$

4. A particle is moving under potential $V(r)=\frac{a}{r^2}-\frac{b}{r}$. Calculate the minimum value of potential energy.

Soln. For potential to be minimum $\left.\frac{dV}{dr}\right|_{r=r_0}=0$

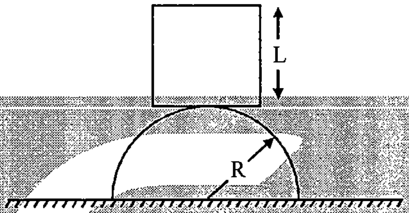
$$-\frac{2a}{r_0^3}+\frac{b}{r_0^2}=0 \qquad \therefore\ r_0=\frac{2a}{b}$$

$$\left.\frac{d^2V}{dr^2}\right|_{r=r_0}=\frac{6a}{r_0^4}-\frac{2b}{r_0^3}=\frac{1}{r_0^3}\left(\frac{6a}{r_0}-2b\right)=\frac{1}{r_0^3}\left(\frac{6a}{2a/b}-2b\right)=\frac{b}{r_0^3}>0$$

Therefore at $r=r_0$ potential is minimum

$$\therefore \qquad V_{\min}=V(r_0)=\frac{1}{r_0}\left(\frac{a}{r_0}-b\right)=-\frac{b^2}{4a}$$

5. A cube is placed on the top of a fixed hemisphere as shown in figure. What should be relation between length of side of cube and radius of hemisphere so that cube has stable equilibrium.



Soln. To discuss equilibrium of cube we first write its potential energy as function of angle from its equilibrium position. As shown in the figure below, initially point A was in contact with spherical surface but now in displaced position point B is in contact. Therefore,

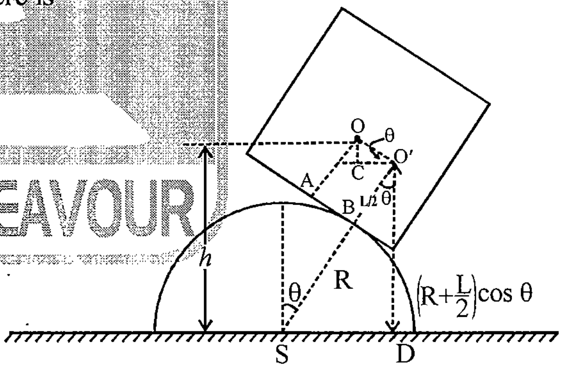
$AB=R\theta$

Height of centre of cube from centre level of hemisphere is

$$h=OC+O'D$$
$$=OO'\sin\theta+O'S\cos\theta$$
$$=AB\sin\theta+(O'B+BS)\cos\theta$$
$$=R\theta\sin\theta+\left(\frac{L}{2}+R\right)\cos\theta$$

Potential energy of the cube

$$V(\theta)=Mgh \quad =Mg\left[R\theta\sin\theta+\left(\frac{L}{2}+R\right)\cos\theta\right]$$



$\theta=0$ is equilibrium position of the cube. For this position to be stable equilibrium position, $\left.\frac{d^2V}{d\theta^2}\right|_{\theta=0}>0$

$$\therefore\ \frac{d^2}{d\theta^2}Mg\left[R\theta\sin\theta+\left(\frac{L}{2}+R\right)\cos\theta\right]_{\theta=0}>0$$

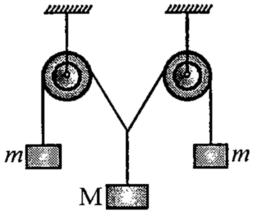
$$\text{or } \frac{d}{d\theta}\left[R\theta\cos\theta+R\sin\theta-\left(\frac{L}{2}+R\right)\sin\theta\right]_{\theta=0}>0$$

$$\text{or } \left[R\cos\theta-R\theta\sin\theta+R\cos\theta-\left(\frac{L}{2}+R\right)\cos\theta\right]_{\theta=0}>0$$

$$\text{or}\quad \left[2R-\left(\frac{L}{2}+R\right)\right]>0 \quad \therefore R-\frac{L}{2}>0 \quad \text{or} \quad 2R>L$$

Thus, cube can be in stable equilibrium position if its side length is less than diameter of hemisphere.

6. For the mass pulley system shown in figure, what should be relation between m and M so that system remains in stable equilibrium position. Pulley are smooth and strings are tight and inextensible.



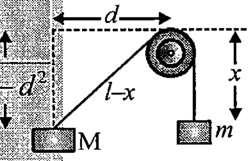
Soln. The pulleys are fixed. Therefore we can write potential energy of the system by specifying position of blocks with respect to pulleys.

Let 'l' be length of string and 'd' be the half distance between two pulleys. Therefore, l and d are constants. If x be distance of m below the pulley as shown in figure then potential energy of system is

$$V(x)=-2mgx-Mg\sqrt{(l-x)^2-d^2} \quad \therefore \frac{dV}{dx}=-2mg+\frac{Mg(l-x)}{\sqrt{(l-x)^2-d^2}}$$

$$\text{for equilibrium } \left.\frac{dV}{dx}\right|_{x=x_0}=0 \qquad \therefore -2mg+\frac{Mg(l-x_0)}{\sqrt{(l-x_0)^2-d^2}}=0$$

$$\text{or } \frac{2m}{M}=\frac{l-x_0}{\sqrt{(l-x_0)^2-d^2}} \qquad \ldots(a)$$



$$\left.\frac{d^2V}{dx^2}\right|_{x=x_0}=\frac{Mgd^2}{\left[(l-x_0)^2-d^2\right]^{3/2}}>0 \text{ for all values of } d>0$$

$$\text{since, } \frac{l-x_0}{\sqrt{(l-x_0)^2-d^2}}>1. \quad \therefore \text{ from (a) } \frac{2m}{M}>1 \text{ or } 2m>M$$

7. The potential energy between two atoms are given $v(r)=\frac{a}{r^{12}}-\frac{b}{r^6}$

where a, b positive constants.

(i) Find the equilibrium distance of two atoms.

(ii) Plot the potential

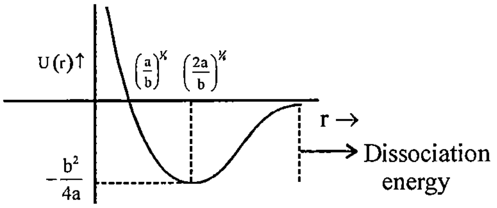
(iii) Calculate the frequency of small oscillation.

Soln. (i) For equilibrium potential energy should be minimum.

$$\frac{dU}{dr}=-\frac{12a}{r^{13}}+\frac{6b}{r^7}=0 \Rightarrow r^6=\frac{2a}{b}$$

$$r=\left(\frac{20}{b}\right)^{1/6}=r_0 \qquad \text{[Position of stability equilibrium]}$$

$$U(r_0)=\frac{ab^2}{4a^2}-\frac{b.b}{2a}=\frac{b^2}{4a}-\frac{b^2}{2a}=-\frac{b^2}{4a}$$

(ii) 

(iii) Approximate to harmonic oscillator

$$U(r_0)=\frac{-b^2}{4a};\ U(r)=U(r_0)+(r-r_0)\left.\frac{dU}{dr}\right|_{r_0}+\frac{(r-r_0)^2}{2}\left.\frac{d^2U}{dr^2}\right|_{r_0}$$

$$\frac{d^2U}{dr^2}=\frac{12\times 13a}{r^{14}}-\frac{42b}{r^8};\ \left.\frac{d^2U}{dr^2}\right|_{r_0}=\frac{156a}{r^{14}}-\frac{42b}{r^8}=\frac{156a}{(2a)^{14/6}}-\frac{42b}{\left(\frac{2a}{h}\right)^{8/6}}$$

$$=\frac{156\,a}{\left(\frac{2a}{b}\right)^{7/3}}-\frac{42b}{\left(\frac{2a}{b}\right)^{4/3}}=\text{constant}=c$$

(Force constant equivalent to spring constant

$$U(r)=U(r_0)+\frac{c}{2}(r-r_0)^2$$

So, the force $=-\frac{dU}{dr}=-\frac{c}{2}.2(r-r_0)^2=-c(r-r_0)$

Force $=-\nabla U;\ U=U(r)$ only

$\vec{F}=m\,\vec{a}$ (Here introduced to harmonic oscillator may necessary).

Equation of motion, $m\frac{d^2r}{dt^2}=-c(r-r_0);\ \frac{d^2r}{dt^2}+\frac{c}{m}(r-r_0)=0$

Take $c=m\omega^2;\ \omega=\sqrt{\frac{c}{m}}$

8. A particle of mass 'm' moving in a potential

$V(x)=\frac{1}{2}m\omega_0^2x^2+\frac{a}{2mx^2}$ $\quad(\omega_0$ & are constant$)$

Find the angular frequency of small oscillation.

$$\frac{dv}{dx}=m\omega_0^2x-\frac{a}{m\,x^3}=0$$

$$m\omega_0^2x_0-\frac{a}{mx^3}=0;\ x_0^4=\frac{a}{m^2\omega_0^2};\ \chi_0=\left(\frac{a}{m^2\omega_0^2}\right)^{1/4}$$

Equilibrium distance, $\frac{d^2z}{dx^2}=m\omega_0^2+\frac{3a}{mx^4}$

$$\left.\frac{d^2r}{dx^2}\right|_{x=x_0} = m\omega_0^2 + \frac{3a}{ma}m^2\omega^2 = 4m\omega_0^2 \qquad \left(x_0^4 = \frac{a}{m^2\omega_0^2}\right)$$

$$U(x) = U(x_0) + (x-x_0)\left.\frac{dU}{dx}\right|_{x=x_0} + \frac{(x-x_0)^2}{2}\left.\frac{d^2U}{dx^2}\right|_{x=x_0}$$

$$= U(x_0) + (x-x_0)^2\, 2m\omega_0^2$$

$$\text{Force} = -\frac{dU}{dx} = -4m\omega_0^2(x-x_0)$$

Equation of motion, $m\dfrac{d^2x}{dt^2} = F = -4\omega_0^2(x-x_0)$

$$\frac{d^2x}{dt^2} + 4\omega_0^2(x-x_0) = 0$$

Hence, the frequency of small oscillation, $\omega = \sqrt{4\omega_0^2} = 2\omega_0$.

9. A particle of mass 'm' is constrained to move in one dimension under a potential $u(x) = \dfrac{\alpha}{x^2} - \dfrac{\beta}{x}$, where α and β are positive constants. Show that period of small oscillations about the equilibrium point is, $T = 4\pi\sqrt{\dfrac{2\alpha^3 m}{\beta^4}}$.

Soln. At equilibrium point $(x = x_0)$

$$\left.\frac{du}{dx}\right|_{x=x_0} = 0$$

$$-\frac{2\alpha}{x_0^3} + \frac{\beta}{x_0^2} = 0 \qquad \therefore\ x_0 = \frac{2\alpha}{\beta}$$

time period of oscillation is given by

$$T = 2\pi\sqrt{\frac{m}{k}}$$

where k is force constant and is given by

$$k = \left.\frac{d^2u}{dx^2}\right|_{x=x_0} = \frac{6\alpha}{x_0^4} - \frac{2\beta}{x_0^3} = 6\alpha\cdot\frac{\beta^4}{16\alpha^4} - \frac{2\beta^4}{8\alpha^3}$$

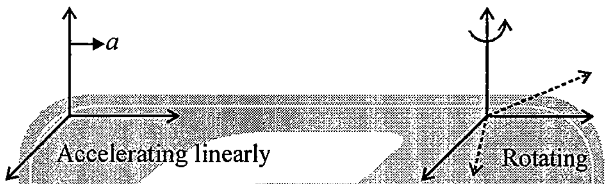
$$= \frac{3}{8}\cdot\frac{\beta^4}{\alpha^3} - \frac{2}{8}\cdot\frac{\beta^4}{\alpha^3} \quad \Rightarrow k = \frac{\beta^4}{8\alpha^3}$$

$$\therefore\ T = 2\pi\sqrt{\frac{8\alpha^3}{\beta^4}\cdot m} = 4\pi\sqrt{\frac{2\alpha^3 m}{\beta^4}}$$

Chapter 6

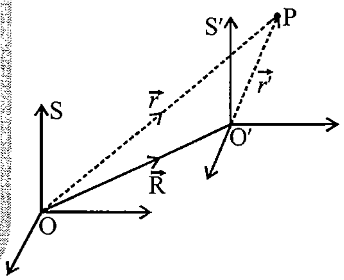
Non-inertial frames and pseudoforces

A non-inertial frame is that which is accelerated. It might be accelerating translationally or might be rotating. In the figure shown below two non-inertial fames are shown.



When observation is made from non-inertial frame then it appears that Newton's laws of motion are not being obeyed. The reason is that it appears as if in addition to the known forces some unknown forces are also acting the system. The unknown forces are called pseudoforces or fictitious forces.

Linearly accelerated frame: Let us consider a frame S' which is linearly accelerated with acceleration $\vec{a}_0$ with respect to an inertial frame S.



From figure, $\vec{R}$ = position vector of S' w.r.t. S

$$\therefore\ \vec{a}_0 = \frac{d^2\vec{R}}{dt^2}$$

If $\vec{F}$ be the known forces acting on a particle as measured in S frame

then, $\vec{F} = \dfrac{md^2\vec{r}}{dt^2} = m\vec{a}$, where $\vec{a}$ is acceleration of particle in S frame.

From figure, $\vec{r}' = \vec{r} - \vec{R}$ or $\dfrac{d^2\vec{r}'}{dt^2} = \dfrac{d^2\vec{r}}{dt^2} - \dfrac{d^2\vec{R}}{dt^2}$

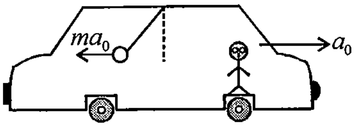
If $\vec{a}'$ be acceleration of particle in $\vec{S}'$ then from above relation we get $\vec{a}' = \vec{a} - \vec{a}_0$

Force on the particle as seen from S'

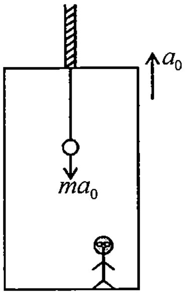
$\vec{F}' = m\vec{a}' = m(\vec{a} - \vec{a}_0)$ $\qquad \therefore\ \vec{F}' = \vec{F} - m\vec{a}_0$

from above relation we see that net force in S' frame consists of a term $-m\vec{a}_0$ in addition to the known forces $\vec{F}$, clearly $-m\vec{a}_0$ is coming into picture because of acceleration of $\vec{S}'$ frame. Therefore in linearly accelerated frame, pseudo force = $-m\vec{a}_0$

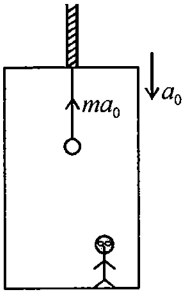
minus sign shows that pseudo force acts opposite to the direction of acceleration of the frame.
Pseudo force in some linearly accelerated frames is shown in figure.



Horizontally accelerating car



Upward accelerating lift



Downward accelerating lift

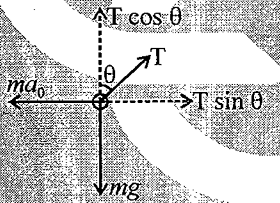
Note: The word pseudo force is used by an observer standing on the non-inertial frame. An observer in inertial frame cannot use the word pseudo force. However observer on non-inertial frame finds that pseudo force acts on all objects including the observer itself irrespective of location of the objects. That is objects may or may not be on non-inertial frame.

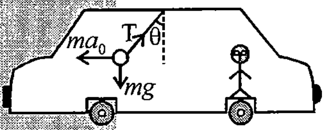
Example: A simple pendulum consisting of a point mass m and a light string of length 'l' is suspended from the roof of a car which is moving horizontally with constant acceleration a_0. Find the angle made by string with vertical.

Soln. As seen from car the pseudo force ma_0 will act on the bob opposite to the direction of acceleration of car due to which string will deflect from its vertically downward position.

Let θ be the angle made by string with downward vertical in the equilibrium position. The known forces acting on the bob are mg and tension T as shown in figure.

For equilibrium of bob, we must have,



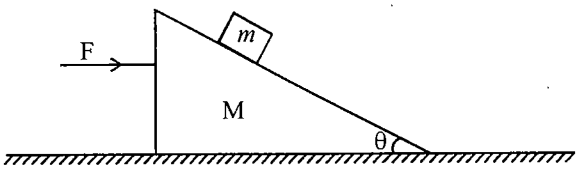


$$T\sin\theta = ma_0,\quad T\cos\theta = mg$$

Dividing the above two relations we get.

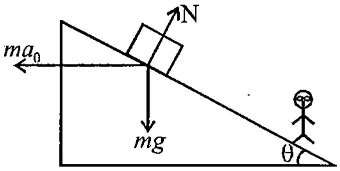
$$\tan\theta = \frac{a_0}{g} \text{ or } \theta = \tan^{-1}\frac{a_0}{g}$$

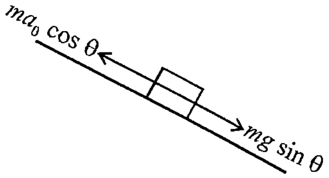
Example: A small block of mass 'm' lies on a wedge of mass M as shown in figure. All the contact surfaces are smooth. When a horizontal force F is applied to the wedge, the block does not slide on the wedge. What must be value of F.



Soln. Horizontal acceleration of the system is $a_0 = \dfrac{F}{M+m}$

According to question, the block does not slide on the wedge, therefore if block is seen from the reference frame of the wedge it will appear stationary. The wedge has a linear acceleration a_0 therefore if observation is made from the wedge frame a pseudo force $-ma_0$ must be applied on the block as shown in figure. Block is stationary on the wedge so component of forces parallel to the inclined plane must cancel.

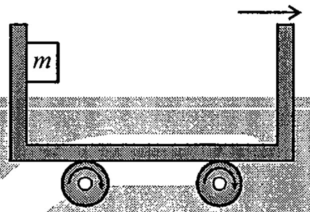




Therefore, $mg\sin\theta = ma_0\cos\theta$ or $\tan\theta = \dfrac{a_0}{g} = \dfrac{F}{(M+m)g}$

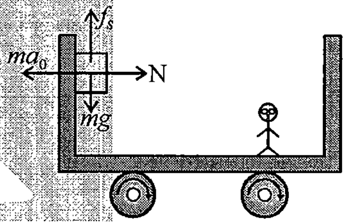
or $F = (M+m)g\tan\theta$

Example: A trolley is moving horizontally with a constant acceleration as shown in figure. A block remains stationary on the vertical wall of the trolley. If coefficients of static friction between the wall and block is μ what must be acceleration of the trolley.



Soln. As seen from the trolley's frame the different forces acting on the block are shown in figure. Let a_0 be acceleration of trolley. f_S and N are static friction and normal reaction respectively. For equilibrium of block we must have

$f_S = mg$ and $N = ma_0$



We know that $f_S \le \mu N$ $\therefore mg \le \mu ma_0$ $\therefore a_0 \ge \dfrac{g}{\mu}$

Therefore acceleration of trolley must either be equal to or greater than g/μ.

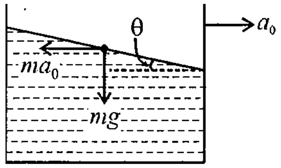
Example: A vessel containing water is linearly accelerated due to which surface of water does not remain horizontal. If a_0 be acceleration of vessel what is angle made by water surface with horizontal.

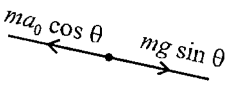
Soln. Due to pseudo force water will move opposite to direction of acceleration. Therefore surface of water becomes inclined as shown in figure. Consider a particle of water of mass m right at the surface. If θ be the angle made by water surface with horizontal then component of pseudo force $-ma_0$ and mg parallel to the surface are as shown in figure.

For equilibrium these forces must cancel.

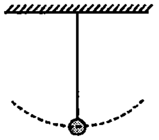
Therefore $ma_0\cos\theta = mg\sin\theta$ or $\tan\theta = \dfrac{a_0}{g}$

$\theta = \tan^{-1}\dfrac{a_0}{g}$





Time period of simple pendulum in linearly accelerated frame: We know that time period of simple pendulum in an inertial frame when point of suspension is stationary is given by



$$T = 2\pi\sqrt{\frac{l}{g}}$$

when simple pendulum is suspended in such a way that point of suspension moves with acceleration $\vec{a}_0$ then time period for small oscillation about the equilibrium point is given as

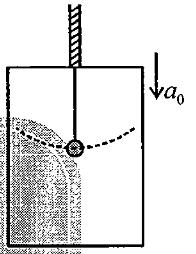
$$T = 2\pi\sqrt{\frac{l}{g_{eff}}}$$

where g_{eff} is effective value of acceleration due to gravity and it is given as $g_{eff} = |\vec{g} - \vec{a}_0|$

Special cases:

(i) Lift accelerating downward.

In this case $\vec{a}_0$ is parallel to $\vec{g}$.

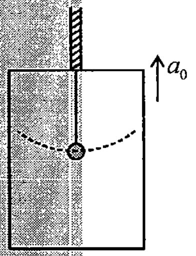


Therefore $\therefore g_{eff} = g - a_0,\ a_0 < g$

$$\therefore T = 2\pi\sqrt{\frac{l}{g - a_0}}$$

(ii) Lift accelerating upward

In this case $\vec{a}_0$ is opposite to $\vec{g}$

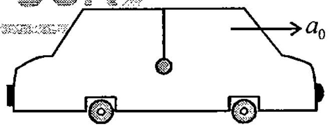


$$\therefore\ g_{eff} = g + a_0$$

$$\therefore T = 2\pi\sqrt{\frac{l}{g + a_0}}$$

(iii) Car accelerating horizontally.

In this case $\vec{a}_0$ is perpendicular to $\vec{g}$. Therefore



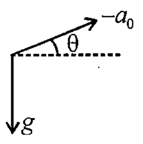
$$g_{eff} = \sqrt{g^2 + a_0^2}$$

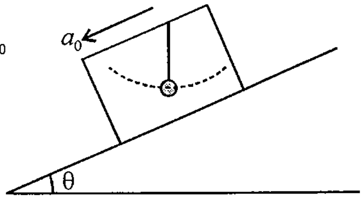
$$\therefore T = 2\pi\sqrt{\frac{l}{\sqrt{g^2 + a_0^2}}}$$

Example: A large box is placed on a smooth inclined plane of inclination θ due to which the box slides down. Simple pendulum is suspended from the top of the box inside it. Calculate time period of small oscillation of the pendulum.

Soln. Acceleration of box down the inclined plane is $a_0 = g\sin\theta$

Angle between $-a_0$ and g in this case is $90 + \theta$





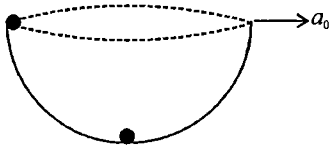
$$g_{eff} = |\vec{g} - \vec{a}_0|$$

$$= \sqrt{g^2 + a_0^2 + 2ga_0\cos(90 + \theta)}$$

$$= \sqrt{g^2 + g^2\sin^2\theta - 2g^2\sin^2\theta} = g\cos\theta$$

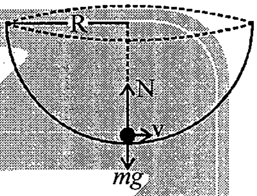
$$\therefore T = 2\pi\sqrt{\frac{l}{g_{eff}}} = 2\pi\sqrt{\frac{l}{g\cos\theta}}$$

Example: A small particle of mass m is released at the edge of a smooth bowl as shown in figure. The bowl is accelerated horizontally with constant acceleration a_0. The particle slides along the surface of bowl. Calculate normal reaction on the particle when it reaches the bottom.



Soln. This problem gets simplified if we observe motion of the particle with respect to the bowl. In bowl's frame particle moves simply in vertical circle of radius equal to radius of bowl. And it is acted upon by mg, $-ma_0$ and normal reaction.

Let v be the speed of particle w.r.t. bowl at the lowest point. Therefore



$$N - mg = \frac{mv^2}{R}$$

or $N = mg + \dfrac{mv^2}{R}$ ....(a)

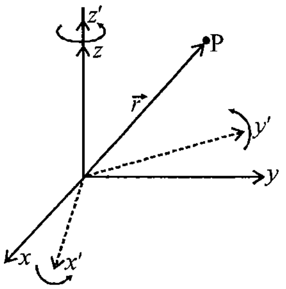
To calculate v let us apply work energy theorem.

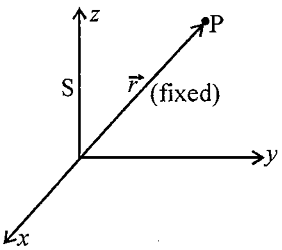
Total work done by all forces = Change in kinetic energy

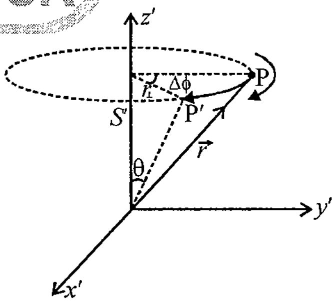
$$mgR - ma_0R = \frac{1}{2}mv^2 \Rightarrow v = \sqrt{2(g-a_0)R}$$

from (a) $N = mg + \dfrac{m2(g-a_0)R}{R}$, $N = m(3g - a_0)$

Rotating Frame: Let S' frame is rotating with respect to S frame. Therefore a vector which is fixed in S frame will appear to be rotating in S' in a direction opposite to the rotation of S'.







For convenience we have assumed that origin and z-axes of S and S' coincide. In figures above we have shown the two frames together and a fixed vector as seen from the two frames separately.

As seen from S' a vector which is fixed in S frame describes a cone in S' frame. Therefore change in vector $\vec{r}$ in small time Δt as seen frame S' is

$= p\vec{p}'$

$= r_\perp \Delta\phi\hat{e} = r\sin\theta\omega\Delta t\hat{e}$ { $\hat{e}$ is unit vector parallel to $p\vec{p}'$ }

$= (r\omega\sin\theta\hat{e})\Delta t = -\vec{\omega}\times\vec{r}\Delta t$

Thus a fixed vector $\vec{r}$ appears to change by an amount $-\vec{\omega}\times\vec{r}\Delta t$ in time Δt when looked at from a rotating frame. However if the vector $\vec{r}$ is not fixed in S frame and if it changes by $\Delta\vec{r}$ in S frame in time Δt then change in vector as seen frame S' will include extra change due to rotation also. Therefore in S' frame change in vector will be $\Delta\vec{r}' = \Delta\vec{r} - \vec{\omega}\times\vec{r}\Delta t$

$$\therefore (\Delta\vec{r})_{\text{in } S'} = (\Delta\vec{r})_{\text{in } S} - \vec{\omega}\times\vec{r}\Delta t$$

or $$\left(\frac{\Delta\vec{r}}{\Delta t}\right)_{\text{in } S'} = \left(\frac{\Delta\vec{r}}{\Delta t}\right)_{\text{in } S} - \vec{\omega}\times\vec{r}$$

In the limit $\Delta t \to 0$

$$\left(\frac{d\vec{r}}{dt}\right)_{\text{in } S'} = \left(\frac{d\vec{r}}{dt}\right)_{\text{in } S} - \vec{\omega}\times\vec{r}$$

The same concept can be applied to all vectors because all vectors will change in similar way when observed from rotating frame. For example when applied to angular momentum we get

$$\left(\frac{d\vec{L}}{dt}\right)_{\text{in } S'} = \left(\frac{d\vec{L}}{dt}\right)_{\text{in } S} - \vec{\omega}\times\vec{L}$$

Pseudo forces in rotating frame: Let us consider dynamics of a particle of mass m in S and S' frames. Let $\vec{r}(t)$ be instantaneous position vector of particle. Therefore in S frame (inertial frame)

Velocity of particle $\vec{v} = \left(\frac{d\vec{r}}{dt}\right)_{\text{in } S}$

Acceleration of particle $\vec{a} = \left(\frac{d\vec{v}}{dt}\right)_{\text{in } S}$

Net force on the particle $\vec{F} = m\left(\frac{d\vec{v}}{dt}\right)_{\text{in } S}$ where $\vec{F}$ is resultant of all known forces.

In S' frame (rotating frame)

Net force on the particle is $\vec{F}' = m\left(\frac{d\vec{v}'}{dt}\right)_{\text{in } S'}$

From relation $\left(\frac{d\vec{r}}{dt}\right)_{\text{in } S'} = \left(\frac{d\vec{r}}{dt}\right)_{\text{in } S} - \vec{\omega}\times\vec{r}$

We get $\vec{v}' = \vec{v} - \vec{\omega}\times\vec{r}$ and $\left(\frac{dv'}{dt}\right)_{\text{in } S'} = \left(\frac{dv'}{dt}\right)_{\text{in } S} - \vec{\omega}\times\vec{v}'$

$$\therefore \vec{F}' = m\left[\left(\frac{d\vec{v}'}{dt}\right)_{\text{in } S} - \vec{\omega}\times\vec{v}'\right] = m\left[\frac{d}{dt}(\vec{v} - \vec{\omega}\times\vec{r})_{\text{in } S} - \vec{\omega}\times\vec{v}'\right]$$

$$= m\left(\frac{dv}{dt}\right)_{\text{in } S} - m\left(\frac{d\omega}{dt}\times\vec{r}\right)_{\text{in } S} - m\vec{\omega}\times\left(\frac{d\vec{r}}{dt}\right)_{\text{in } S} - m\vec{\omega}\times\vec{v}'$$

$$\vec{F}' = \vec{F} - m\frac{d\vec{\omega}}{dt}\times\vec{r} - m\vec{\omega}\times\vec{v} - m\vec{\omega}\times\vec{v}'$$

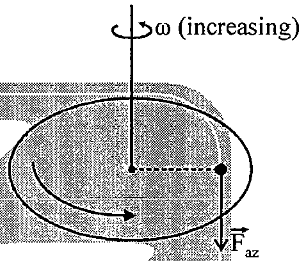
Since observation is being made with respect to the rotating frame therefore we would like to express $\vec{F}'$ in terms of velocity w.r.t. rotating frame i.e. $\vec{v}'$, therefore let us remove $\vec{v}$ by using $\vec{v} = \vec{v}' + \vec{\omega} \times \vec{r}$
Therefore

$$\vec{F}' = \vec{F} - m\frac{d\vec{\omega}}{dt} \times \vec{r} - m\vec{\omega} \times (\vec{v}' + \vec{\omega} \times \vec{r}) - m\vec{\omega} \times \vec{v}'$$

$$\boxed{\vec{F}' = \vec{F} - m\frac{d\vec{\omega}}{dt} \times \vec{r} - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) - 2m\vec{\omega} \times \vec{v}'}$$

Thus, as seen from S', in addition to the a known force $\vec{F}$ some unknown forces also appear to be acting on the particle. These forces are called pseudo forces. In the following we discuss each term of pseudo forces one by one.

Azimuthal Force: It appears to act on the particles which are being observed from a rotating frame which has non-uniform angular velocity i.e. $\frac{d\omega}{dt} \neq 0$.



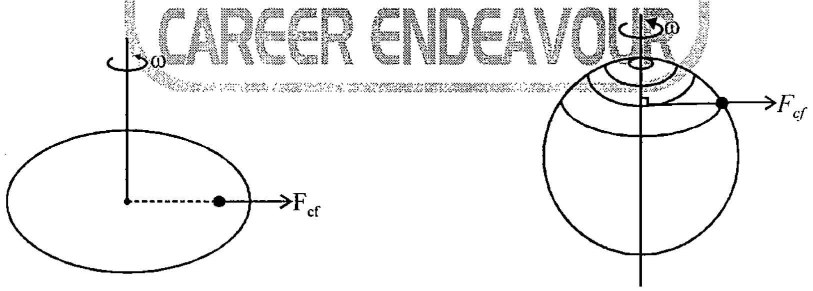
Its direction is tangential to the rotation of frame.

Azimuthal Force $= -m\frac{d\vec{\omega}}{dt} \times \vec{r} = \vec{F}_{az}$

In the figure direction of azimuthal force on a particle lying on a non-uniformly rotating disc is shown.

Centrifugal Force: It acts on all particles which are being observed from the rotating frame, irrespective of state of the particle. Direction of centrifugal force is perpendicular to the axes of rotation and away from it.

Centrifugal Force $= -m\vec{\omega} \times (\vec{\omega} \times \vec{r}) = \vec{F}_{cf}$ or $|\vec{F}_{cf}| = m\omega^2 r_\perp$,



where $r_\perp$ is ⊥ distance from axis of rotation

In the figure shown direction of centrifugal force on a particle lying on a disc and one lying surface of a rotating sphere is shown.

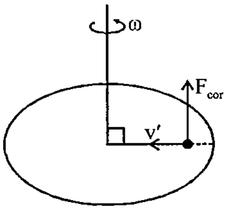
Coriolis Force: It acts on all particles which are being observed from the rotating frame provided that particles have non-zero velocity with respect to the rotating frame. That is, coriolis force acts only on those particles which are moving with respect to the rotating frame. Direction of coriolis force depends on direction of velocity $\vec{v}'$.

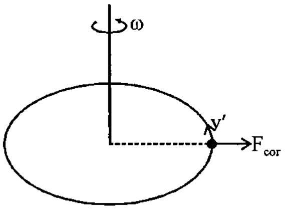
Coriolis force $= -2m\vec{\omega} \times \vec{v}' = \vec{F}_{cor}$

or $\vec{F}_{cor} = 2m\vec{v}' \times \vec{\omega}$

$\vec{F}_{cor}$ is $\perp$ to both $\vec{\omega}$ and $\vec{v}'$ and its direction can be found by using right hand thumb rule of cross product of two vectors.

Direction of coriolis force on an insect which is moving along the radius of a rotating disc and along rim is shown in figure.





Earth: A non-inertial (rotating) frame: Earth spins about its axis, therefore it is a non-inertial frame.

Angular velocity of earth's spinning motion is $\omega = \dfrac{2\pi}{T} = \dfrac{2\pi}{24h} = 7.27 \times 10^{-5}$ rad/sec

Due to rotations of earth following effects arise.

(a) All objects on earth's surface experience centrifugal force which is directed away from the axis of rotation.

(b) All objects moving on earth's surface experience coriolis force in addition to centrifugal force.

(c) An observer standing on earth's surface notices that on all objects a centrifugal and a coriolis force (if $v' \neq 0$) act irrespective of the location of the object. That is the object which is being observed by the observer may or may not be on the earth's surface. For example we on the earth, see that the sun rises and sets. If we analyse the dynamics of the sun we will find that sun experiences a centrifugal force away from the earth and a coriolis force towards the earth.

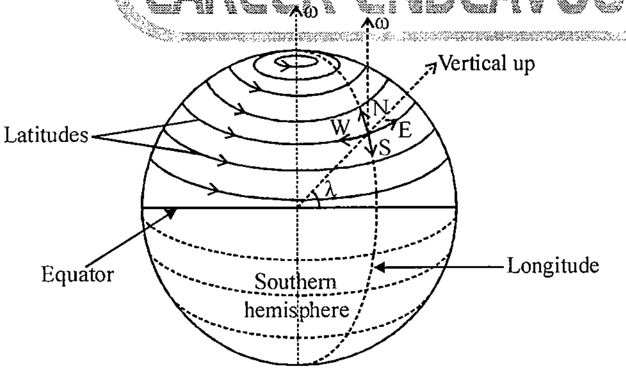
Coordinate system on earth's surface: To analyse dynamics of objects from earth's surface we define local cartesian coordinate system as follows:

+X-axis : Along local east, –X axis: Along local west

+Y-axis : Along local north, –Y axis: Along local south

+Z-axis : Along local vertical upward, –Z axis: Along local vertical downward

Therefore angular velocity vector of earth's spinning motion lies in Y-Z plane as shown in figure.



A point in northern hemisphere

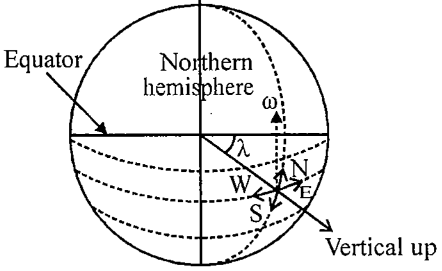
Therefore, at any point in northern hemisphere,

$\vec{\omega} = \omega(\cos\lambda\hat{j} + \sin\lambda\hat{k})$

where λ is latitude of the point.

And at any point in southern hemisphere

$\vec{\omega} = \omega(\cos\lambda\hat{j} - \sin\lambda\hat{k})$



A point in southern hemisphere

Coriolis force on a freely falling object: An object falls vertically downward therefore velocity of object is in negative z-direction,

$\therefore \vec{v}' = -v'\hat{k}$

In northern hemisphere: $\vec{F}_{cor} = 2m\vec{v}' \times \vec{\omega} \; = -2mv'\omega\hat{k} \times (\cos\lambda\hat{j} + \sin\lambda\hat{k}) \; = +2mv'\omega\cos\lambda\hat{i}$

Coriolic force is directed along +X direction, i.e. towards east. Therefore a freely falling object deviates towards east in northern hemisphere.

In southern hemisphere: $\vec{F}_{cor} = 2m\vec{v}' \times \vec{\omega} \; = -2mv'\omega\hat{k} \times (\cos\lambda\hat{j} - \sin\lambda\hat{k}) \; = 2mv'\omega\cos\lambda\hat{i}$

In southern hemisphere also, a freely falling object deviates towards east.

Coriolis force on river water: Consider a river flowing from north to south direction in northern hemisphere.

Therefore, velocity of water, $\vec{v}' = -v'\hat{j}$

Coriolis force on water, $\vec{F}_{cor} = 2m\vec{v}' \times \vec{\omega} \; = -2mv'\omega\hat{j} \times (\cos\lambda\hat{j} + \sin\lambda\hat{k}) \; = -2mv'\omega\sin\lambda\hat{i}$

Coriolis force on water is directed towards west. Due to which west bank of river flowing from north to south in northern hemisphere undergoes more erosion than east bank.

Coriolis force on a train moving from west to east:

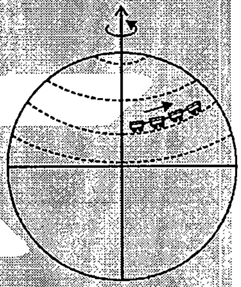
Velocity of train, $\vec{v}' = v'\hat{i}$

Fcoriolis force on train,

$\vec{F}_{cor} = 2m\vec{v}' \times \vec{\omega}$

$= 2mv'\omega\hat{i} \times (\cos\lambda\hat{j} + \sin\lambda\hat{k})$

$= 2mv'\omega(\cos\lambda\hat{k} - \sin\lambda\hat{j})$



Therefore lateral force on the track of train due to coriolis force is $2mv\omega\sin\lambda$.

Coriolis and centrifugal force on the sun as seen from the earth: The sun appears to be rising and setting because of rotation of earth about its own axis. A person on the earth finds the sun to be moving. This is exactly the same effect as seen by a person sitting in a moving car who sees everything moving past the car.

Let us assume that axis of rotation of earth shown here is perpendicular to the plane of paper. Therefore as seen from the earth the sun appears to be rotating as shown in figure. And velocity of the sun with respect to the earth is

$v' = \omega r$, r = distance of sun from centre of earth

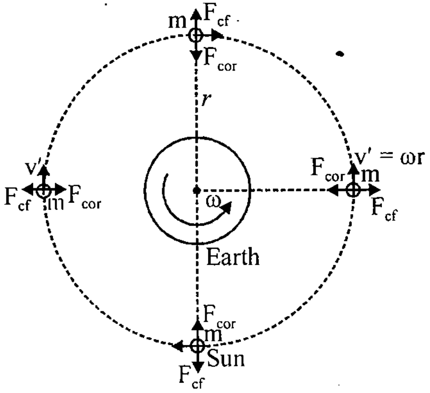
$\therefore$ Centrifugal force is always away from the axis of rotation. The direction of coriolis force can be found using right hand thumb rule and it is found to be towards the centre of earth.

$F_{cor} = |2m\vec{v}' \times \vec{\omega}| = 2m\vec{v}'\omega\sin 90° = 2mv'\omega$

$= 2m\omega^2 r$

$F_{cf} = m\omega^2 r_\perp, \; r_\perp = r$

$\therefore F_{cf} = m\omega^2 r$



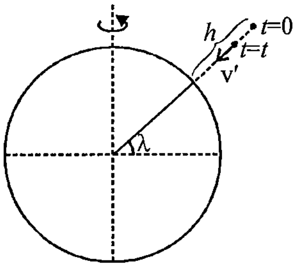
Eastward deviation of a freely falling object in northern hemisphere:

Suppose a particle falls from a height h at a place where latitude is λ.

Here we assume that $h \ll R_e$

Velocity of particle at time t is $\vec{v}' = -gt\hat{k}$

Coriolis force on the particle



$$\vec{F}_{cor} = 2m\vec{v}' \times \vec{\omega} \quad = -2m\omega gt\hat{k} \times (\cos\lambda\hat{j} + \sin\lambda\hat{k})$$

$$= 2m\omega g\cos\lambda t\hat{i}$$

Force is towards east so particle will deviate towards east, equation of motion of the particle in eastward direction is

$$F_{cor} = \frac{md^2x}{dt^2} \text{ or } \frac{d^2x}{dt^2} = 2\omega g\cos\lambda t$$

We have following initial condition which we use to solve above equation,

at $t = 0, x = 0, y = 0, z = h$ ... (a)

at $t = 0, \frac{dx}{dt} = 0, \frac{dy}{dt} = 0, \frac{dz}{dt} = 0$ ... (b)

integrating the above differential equation we get, $\frac{dx}{dt} = \omega g\cos\lambda t^2 + c_1$

using (b) we get $c_1 = 0$. Therefore $\frac{dx}{dt} = \omega g\cos\lambda t^2$

Integrating again we get, $x = \frac{1}{3}\omega g\cos\lambda t^3 + c_2$, using (a) we get, $c_2 = 0$

$$\therefore \boxed{x = \frac{1}{3}\omega g\cos\lambda t^3}$$

where t is total time taken to reach earth surface

$$\therefore \; t = \sqrt{\frac{2h}{g}} \qquad \therefore \boxed{x = \frac{1}{3}\omega g\cos\lambda\left(\frac{2h}{g}\right)^{3/2}}$$

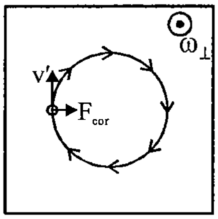
If a particle falls from a height of 100 metre at equator $(\lambda = 0)$ then using above equation we get.

$$x = \frac{1}{3}\times 7.27\times 10^{-5}\times 9.8\times\left(\frac{2\times 100}{9.8}\right)^{3/2} \simeq 2.19 \approx 2.2\text{cm}$$

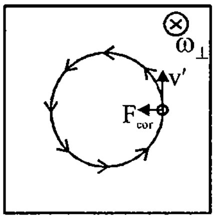
Note: The expression derived for eastward deviation is an approximate one. In the derivation we have assumed that particle is uniformly accelerated in downward direction which is not true, when particle attains a velocity in X(each direction) there arises a component of coriolis force in vertical direction, due to which downward acceleration becomes a function of v'_x. Because of v'_x component of velocity, particle also experiences a force in southward direction. Therefore a freely falling object in northern hemisphere actually deviates in south-east direction.

Circular motion on earth's surface due to coriolis force: Direction of coriolis force is perpendicular to the velocity of particle, therefore in absence of any other force like gravity, friction etc. a particle will move in circular path in the same manner as a charge particle does due to magnetic force.

Let us suppose that there is no friction and a particle is thrown horizontally on the surface of earth with speed v'. Let λ be the latitude of place. Therefore, $\vec{\omega} = \omega(\cos\lambda\hat{j} \pm \sin\lambda\hat{k})$ we use + or – sign depending on the case whether we are in northern or southern hemisphere. Component of $\vec{\omega}$ perpendicular the plane is $\pm\omega\sin\lambda$. Therefore a particle thrown horizontally will deviate in opposite direction in the two hemispheres, as shown below:



Horizontal plane in N-hemisphere



Horizontal plane in S-hemisphere

Thus, circular path is clock wise in N-hemisphere and anticlockwise in S-hemisphere.

If r be the radius of circle then,

horizontal component of $F_{cor} = \dfrac{mv'^2}{r}$ or $2mv'\omega_\perp \sin 90° = \dfrac{mv'^2}{r}$

or $2mv'\omega\sin\lambda = \dfrac{mv'^2}{r}$ $\quad\therefore \boxed{r = \dfrac{v'}{2\omega\sin\lambda}}$

If the velocity with which particle is thrown is very small say 1mm/sec then radius of circular path near pole $(\lambda = 90°)$ will be

$$r = \frac{1\times10^{-3}}{2\times7.27\times10^{-5}\times1} \simeq 6.87\text{ metres}$$

A stone thrown horizontally does not move in circle because it falls down due to gravity and then stops because of friction.

SOLVED PROBLEMS

1. An insect of mass m crawls on earth's surface with velocity v_0 at ϕ angle north of east. The latitude of place where insect is situated is λ. If ω be angular velocity of earth's spinning motion and R be radius of earth. Calculate centrifugal and coriolis force on the insect.

Soln. Velocity of insect w.r.t. earth

$$\vec{v}' = v_0(\cos\phi\hat{i} + \sin\phi\hat{j})$$

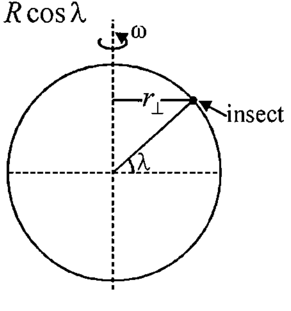
perpendicular distance of insect from axis of rotation is $r_\perp = R\cos\lambda$

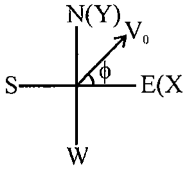
$\therefore$ centrifugal force $= m\omega^2 r_\perp = m\omega^2 R\cos\lambda$

coriolis force $\vec{F}_{cor} = 2m\vec{v}'\times\vec{\omega}$

$$= 2mv_0\omega(\cos\phi\hat{i} + \sin\phi\hat{j})\times(\cos\lambda\hat{j} + \sin\lambda\hat{k})$$

$$= 2mv_0\omega(\cos\phi\cos\lambda\hat{k} - \cos\phi\sin\lambda\hat{j} + \sin\phi\sin\lambda\hat{i})$$





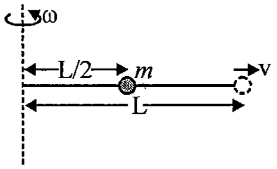
$$\therefore\ |\vec{F}_{cor}| = 2mv_0\omega\sqrt{\cos^2\phi\cos^2\lambda+\cos^2\phi\sin^2.\lambda+\sin^2\phi\sin^2\lambda}$$

$$= 2mv_0\omega\sqrt{\cos^2\phi+\sin^2\phi.\sin^2\lambda}$$

2. A bead of mass 'm' lies at the centre of a thin wire of length 'L'. The wire is then rotated with uniform angular velocity ω about its one end due to which bead slides away from the axis of rotation. Find the velocity of bead with respect to the rod at the moment when the bead is about to leave the free end of the wire.

Soln. Due to rotation of the wire the bead is acted upon by centrifugal as well as coriolis force (while moving). However it is centrifugal force which accelerates the bead away from the axis of rotation. Let v be the speed of bead w.r.t. the wire when it leaves the other end. Let us apply work energy theorem to find v.

Work done by centrifugal force = change in kinetic energy



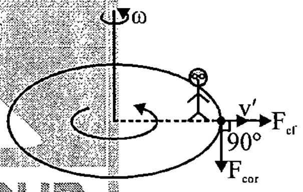
$$\int_{L/2}^{L} m\omega^2 r dr = \frac{1}{2}mv^2 - 0,\quad \frac{m\omega^2}{2}\left[L^2-\frac{L^2}{4}\right]=\frac{1}{2}mv^2$$

$$\therefore\ v=\frac{\sqrt{3}}{2}\omega L$$

3. A person is standing at the edge of a disc of radius R. The disc is rotating about its axis with uniform angular velocity ω. The person throws a stone in radially outward direction with speed $\frac{\omega R}{2}$ relative for the disc. Calculate acceleration of stone as seen by the person soon after throwing. (neglect gravity).

Soln. As seen by the person the stone is acted upon by centrifugal and coriolis forces. The direction of the two forces soon after throwing is shown in the figure.

Net force on the stone is



$$F' = \sqrt{F_{cor}^2+F_{cf}^2}$$

$$= \sqrt{|2m\vec{v}'\times\vec{\omega}|^2+(m\omega^2 r_\perp)^2} = \sqrt{\left(2m.\frac{\omega R}{2}.\omega\sin 90^\circ\right)^2+(m\omega^2 R)^2}$$

$$F' = \sqrt{2}m\omega^2 R$$

Therefore acceleration of stone as seen by the person is

$$a' = \frac{F'}{m} = \sqrt{2}\omega^2 R$$

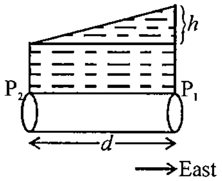
4. Show that when a tidal current runs due north in the northern hemisphere at an altitude of λ with velocity v in a channel of width d, level of water on east coast is raised by $(2v\omega d\sin\lambda)/g$ above that on the west coast. Here ω is the angular velocity of the earth.

Soln. Water flowing towards the north experiences a coriolis force directed towards east. Due to which water accumulates at the east coast. Thus water level at the east coast is raised above in comparison to the west coast.

In the figure below we show a cross-section of water let its mass be m.

$\therefore$ Net coriolis force on it = $2m\omega v\sin\lambda$ (towards east)

This force on the given portion of water is balanced by pressure difference arising due difference in water level



$\therefore\ (P_1 - P_2)A = 2m\omega v\sin\lambda$

$\rho ghA = 2(\rho Ad)\omega v\sin\lambda$

$\therefore\ h = 2v\omega d\sin\lambda / g$

5. Show that lateral force on the train rails differ in magnitude by $4mv_0\omega\sin\lambda$ when train travels towards the east and when it travels towards the west with speed v_0 and at a latitude λ.

Soln. When train reverses its direction of motion the direction of coriolis force also gets reversed. But the magnitude remains the same. Therefore, magnitude of change in coriolis force is equal to twice the magnitude of coriolis force itself.

$\vec{\omega} = \omega(\cos\lambda\hat{j} + \sin\lambda\hat{k})$

$\vec{v} = v\hat{i}$ (eastward)

$\therefore$ Coriolis force $\vec{F}_{1C} = -2m\vec{\omega}\times\vec{v} = -2m(-\hat{k}\cos\lambda + \hat{j}\sin\lambda)\omega v$

$\vec{v} = -v\hat{i}$ (westward)

$\therefore\ \vec{F}_{2C} = +2m(-\hat{k}\cos\lambda + \hat{j}\sin\lambda)\omega v$

Difference in force $= \vec{F}_{2C} - \vec{F}_{1C} = 2m\omega v\sin\lambda\hat{j}$

lateral force is given by $\hat{j}$ component of coriolis force.

$\therefore$ Difference in lateral force is $= 4m\omega v\sin\lambda$

Its direction is northwards.

6. A car running at uniform speed v_0 along the equator has a simple pendulum suspended from its roof. Calculate the ratio of time period of small oscillation when the car moves from the west to the east and when it moves from east to west.

Soln. Besides gravity, centrifugal and coriolis forces will also act on the pendulum.

$\therefore$ Net force on pendulum when car is going west to east.

$F = mg - m\omega^2R - 2mv_0\omega$

$\therefore$ effective acceleration of pendulum

$g' = g - \omega^2R - 2v_0\omega$

$\therefore$ Time period $T_1 = 2\pi\sqrt{\dfrac{l}{g - \omega^2R - 2v_0\omega}}$

and when car is moving east to west

$g' = g - \omega^2R + 2v_0\omega$

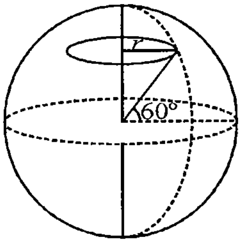
$\therefore$ Time period $T_2 = 2\pi\sqrt{\dfrac{l}{g - \omega^2R + 2v_0\omega}}$

$\therefore$ ratio of time period $T_1/T_2 = \sqrt{\dfrac{g - \omega^2R + 2v_0\omega}{g - \omega^2R - 2v_0\omega}}$

7. Assuming earth to be a sphere, calculate the linear speed of an object lying on the earth's surface at an altitude of $\lambda = 60°$ and also calculate the centrifugal force experienced by an object of mass 50 kg. Compare this force with gravitational force.

Soln. Angular velocity of earth about its axis due to its spinning motion is

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{(24\times 60\times 60)} \text{ rad/sec}$$



Radius of the earth R = 6400 × 10^3 m.

At latitude $\lambda = 60°$ objects move in circle of radius $r = R\cos\lambda$

$\therefore$ linear speed $= \omega r$

$$= \frac{2\pi}{24\times 60\times 60}\times 6400\times 10^3 \cos 60° \text{ m/s}$$

$$= \frac{2\pi\times 10^3}{27} \text{ m/s} \quad \simeq 233 \text{ m/s}$$

Centrifugal force = $m\omega^2 r$

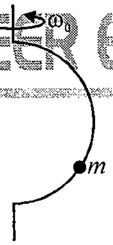
$$= 50\times\left(\frac{2\pi}{24\times 60\times 60}\right)^2 \times 6400\times 10^3$$

$$\simeq 1.70\text{N}$$

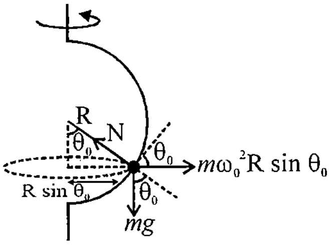
Gravitational force = mg = 50 × 9.8 = 490N

Since centrifugal force due to spinning of earth is very small compared to the gravitational force due to this reason we do not feel the rotation of earth.

8. A semi-circular wire rotating with uniform angular velocity ω_0 has a small bead of mass m threaded into it. Calculate the time period of small oscillation of bead about the equilibrium point.



Soln. At equilibrium position, tangential component of weight is balanced by tangential component of centrifugal force.



$$\therefore \; mg\sin\theta_0 = m\omega_0^2 R\sin\theta_0\cdot\cos\theta_0$$

$$\therefore \; \cos\theta_0 = \frac{g}{\omega_0^2 R}$$

$$\text{or } \theta_0 = \cos^{-1}\left[\frac{g}{\omega_0^2 R}\right] \quad \text{... (i)}$$

Now, let the particle is displaced by angle θ (which is very small) from its equilibrium position.

Therefore, now line joining the particle to the centre of wire makes $(\theta_0+\theta)$ with the vertical.

Therefore, net torque on the bead (particle)

$$= mg\cdot R\sin(\theta+\theta_0) - m\omega_0^2 R\sin(\theta+\theta_0)\cdot R\cos(\theta+\theta_0)$$

$$= m\omega_0^2 R^2 \sin(\theta_0+\theta)\left\{\frac{g}{\omega_0^2 R} - \cos(\theta+\theta_0)\right\}$$

$$= m\omega_0^2 R^2 \sin(\theta_0')\left\{\frac{g}{\omega_0^2 R} - \cos\theta\cdot\cos\theta_0 + \sin\theta\cdot\sin\theta_0)\right\}$$

from (i) $$= m\omega_0^2 R^2 \sin\theta_0\left\{\frac{g}{\omega_0^2 R} - 1\cdot\frac{g}{\omega_0^2 R} + \theta\cdot\sin\theta_0\right\} = m\omega_0^2 R^2 \sin^2\theta_0\cdot\theta$$

This is the restoring torque

$$\therefore \quad -mR^2\ddot{\theta} = mR^2\omega_0^2\sin^2\theta_0\cdot\theta$$

or $\quad \ddot{\theta} = -\omega_0^2\sin^2\theta_0\cdot\theta$

or $\quad \ddot{\theta} + \omega_0^2\sin^2\theta_0\cdot\theta = 0$

This is the equation of S.H.M. with frequency

$$\omega = \omega_0\sin\theta_0 = \omega_0\sqrt{1-\cos^2\theta_0} \quad \Rightarrow \omega = \omega_0\sqrt{1-\left(\frac{g}{\omega_0^2 R}\right)^2}$$

9. A particle of mass m moves radially outward with a constant velocity v_0, on a turn table rotating with uniform angular velocity ω. Find co-efficients of friction.

Soln. On the turn table particle experiences coriolis and centrifugal forces.

$$\vec{F}_{\text{coriolis}} = -2m\,\vec{\omega}\times\vec{v}_0$$

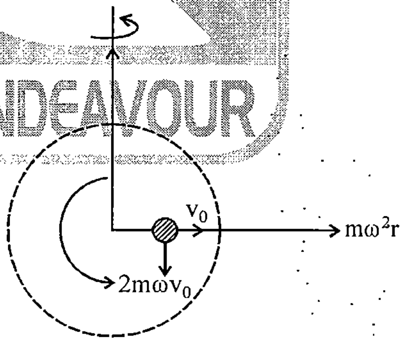
$$= -2m\,\omega\,v_0\,\hat{z}\times\hat{r}$$

$$= -2m\,\omega\,v_0\,\hat{\theta}$$

$$\vec{F}_{\text{centrifugal}} = -m\,\vec{\omega}\times(\vec{\omega}\times\vec{r})$$

$$= -m\,\vec{\omega}\times(\omega\,r\,\hat{\theta})$$

$$= m\,\omega^2\,\hat{r}$$



Net force on the particle

$$F = \sqrt{4m^2\omega^2 v_0^2 + m^2\omega^4 r^2} \quad \Rightarrow F = m\,\omega\,v_0\sqrt{4+\left(\frac{\omega r}{v_0}\right)^2}$$

Since, v_0 is constant, the net force is balanced by friction force.

$$\therefore \quad f_r = F$$

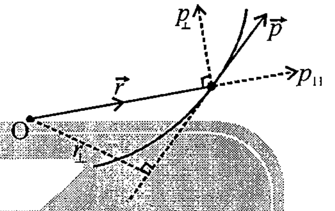
$$\mu m g = m\,\omega\,v_0\sqrt{4+\left(\frac{\omega r}{v_0}\right)^2}$$

$$\therefore \quad \mu = \frac{\omega\,v_0}{g}\sqrt{4+\left(\frac{\omega r}{v_0}\right)^2}$$

Chapter 7

Angular momentum and torque

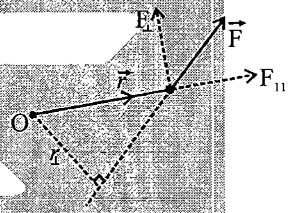
Angular momentum: If $\vec{p}$ be momentum of a particle and $\vec{r}$ be its position vector w.r.t. a point 'o' then angular momentum about point 'o' is defined as



$$\vec{L} = \vec{r} \times \vec{p}$$

Magnitude of angular momentum is written as $L = rp_\perp$ or $L = r_\perp p$

Torque: If $\vec{F}$ be force acting at some point and $\vec{r}$ be the position vector of the point w.r.t. some origin 'O' then torque about 'O' is defined as $\vec{\tau} = \vec{r} \times \vec{F}$



Magnitude of force is written as $\tau = rF_\perp$ or $\tau = r_\perp F$

If a system is in translational equilibrium then net force is zero however if system is in rotational equilibrium then net torque is zero about any point.

Relation between torque and angular momentum:

$$\vec{L} = \vec{r} \times \vec{p}$$

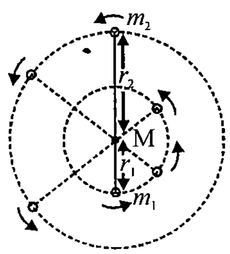
Differentiate w.r.t. time to get $\boxed{\frac{d\vec{L}}{dt} = \vec{\tau}}$

If $\vec{\tau} = 0$ then $\vec{L}$ = constant. This is called conservation of angular momentum.

Double star problem: A double star consists of two stars revolving in circular orbits about their common centre of mass. The necessary centripetal force for circular motion is provided by their mutual gravitational attraction. Let us consider a double star which consists of two star of masses m_1 and m_2 separated by a distance r.

There is no external force on the system therefore centre of mass of the system does not move.

Let r_1 and r_2 be radii of the orbits of the two stars.



$$\therefore \quad r_1 = \frac{m_2 r}{m_1 + m_2} \Rightarrow \frac{r_1}{r_2} = \frac{m_2}{m_1},$$

$$r_2 = \frac{m_1 r}{m_1 + m_2}$$

Since centre of mass is stationary. Therefore, the angular velocity of the two stars must be same.

$\therefore$ For star 1 $\quad m_1\omega^2 r_1 = \dfrac{Gm_1.m_2}{r^2} \Rightarrow \omega = \sqrt{\dfrac{G(m_1+m_2)}{r^3}}$

For start 2 $\quad m_2\omega^2 r_2 = \dfrac{Gm_1.m_2}{r^2} \Rightarrow \omega = \sqrt{\dfrac{G(m_1+m_2)}{r^3}}$

Ratio of linear speeds of two stars:

$$\frac{v_1}{v_2} = \frac{\omega r_1}{\omega r_2} = \frac{r_1}{r_2} = \frac{m_2}{m_1} \qquad \therefore \boxed{\frac{v_1}{v_2} = \frac{m_2}{m_1}}$$

Ratio of kinetic energy of two stars

$$\frac{K_1}{K_2} = \frac{\frac{1}{2}m_1v_1^2}{\frac{1}{2}m_2v_2^2} = \frac{m_1(\omega r_1)^2}{m_2(\omega r_2)^2} = \frac{m_1}{m_2}.\left(\frac{r_1}{r_2}\right)^2 = \frac{m_2}{m_1} \qquad \therefore \boxed{\frac{K_1}{K_2} = \frac{m_2}{m_1}}$$

Ratio of angular momentum of two stars

$$\frac{L_1}{L_2} = \frac{m_1v_1r_1}{m_2v_2r_2} = \frac{m_1\omega r_1^2}{m_2\omega r_2^2} = \frac{m_2}{m_1} \qquad \therefore \boxed{\frac{L_1}{L_2} = \frac{m_2}{m_1}}$$

ratio of magnitude of linear momentum $\dfrac{P_1}{P_2} = \dfrac{m_1v_1}{m_2v_2} = \dfrac{m_1\omega r_1}{m_2\omega r_2} = \dfrac{m_1}{m_2}.\dfrac{r_1}{r_2} = \dfrac{m_1}{m_2}.\dfrac{m_2}{m_1} \quad \therefore \boxed{\dfrac{P_1}{P_2} = 1}$

Total energy of double star system $E = \dfrac{1}{2}m_1v_1^2 + \dfrac{1}{2}m_2v_2^2 - \dfrac{Gm_1.m_2}{r} = \dfrac{1}{2}m_1\omega^2 r_1^2 + \dfrac{1}{2}m_2\omega^2 r_2^2 - \dfrac{Gm_1.m_2}{r}$

put values of r_1, r_2 and ω to get

$$E = \frac{1}{2}\frac{G(m_1+m_2)}{2r^3}\left[\frac{m_1m_2^2r^2 + m_1^2r^2m_2}{(m_1+m_2)^2}\right] - \frac{Gm_1m_2}{r} = \frac{Gm_1.m_2}{2r} - \frac{Gm_1.m_2}{r} \qquad \boxed{E = \frac{-Gm_1.m_2}{2r}}$$

SOLVED PROBLEMS

1. Three particles are moving in x-y plane with constant velocities as shown in the figure. Calculate angular momentum of each particle about the origin.

Soln. Angular momentum can be calculated in various ways.

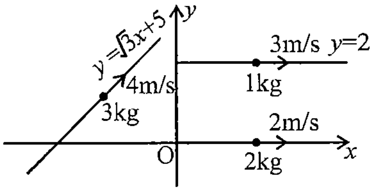
Let us use $L = r_\perp p$

For 1 kg particle:

momentum $(p) = mv = 1\times 3 = 3\,\text{kg m/s}$

perpendicular distance of line of momentum from origin $(r_\perp) = 2m$

$\therefore L = r_\perp p = 2\times 3 = 6\,\text{kg m}^2/\text{s}$



For 2kg particle: Line of momentum of this particle passes through origin therefore $r_\perp = 0 \;\therefore L = 0$

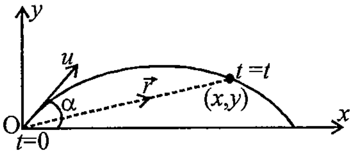
For 3kg particle: $r_\perp = \dfrac{|y-\sqrt{3}x-5|_{x=0,y=0}}{\sqrt{1^2+(\sqrt{3})^2}} = \dfrac{|0-\sqrt{3}\times 0-5|}{\sqrt{4}} = \dfrac{5}{2}$

$p = mv = 3\times 4 = 12\text{kg m/s}$

$\therefore \text{L} = r_\perp p = \dfrac{5}{2}\times 12 = 30\text{kg m}^2/\text{sec}$

2. A particle of mass m is thrown with initial speed u at angle α with horizontal. If maximum height reached is much small compared to radius of earth. Calculate angular momentum of particle about point of projection after time t.

Soln. Position vector of particle after time t



$$\vec r = x\hat i + y\hat j$$

$$\vec r = u\cos\alpha t\hat i + (u\sin\alpha t - \frac{1}{2}gt^2)\hat j$$

velocity of particle after time t

$$\vec v = \frac{d\vec r}{dt} = u\cos\alpha\hat i + (u\sin\alpha - gt)\hat j$$

$$\therefore\ \vec L = \vec r\times\vec p = m\vec r\times\vec v\ = m\begin{vmatrix}\hat i & \hat j & \hat k\\ u\cos\alpha t & u\sin\alpha t - \frac{1}{2}gt^2 & 0\\ u\cos\alpha & u\sin\alpha - gt & 0\end{vmatrix}$$

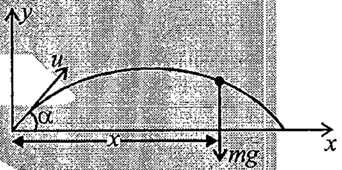
$$= \hat k[u^2\cos\alpha\sin\alpha t - gt^2u\cos\alpha - u^2\cos\alpha\sin\alpha t + \frac{1}{2}gu\cos\alpha t^2] = -\frac{1}{2}gt^2u\cos\alpha\hat k$$

Another method: At time t, torque on the particle about point of projection is

$$\tau = r_\perp F = xmg$$

or $\vec\tau = -mg\,x\,\hat k$

$$\vec\tau = -mg\,u\cos\alpha t\,\hat k$$



We know that $\vec\tau = \dfrac{d\vec L}{dt}$

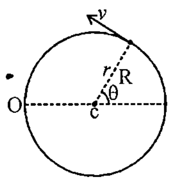
$$\therefore\ \frac{d\vec L}{dt} = -mgu\cos\alpha t\hat k \quad\text{or}\quad \int_0^L d\vec L = -mgu\cos\alpha\int_0^t t\,dt\,\hat k$$

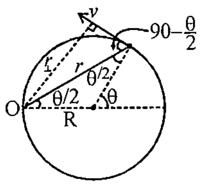
$$\therefore\ \vec L = -\frac{1}{2}mgut^2\cos\alpha\,\hat k$$

3. A particle of mass 'm' moves with constant speed v along a circle of radius R. Calculate angular momentum about centre of circle. Also calculate angular momentum about a point on the circumference of circle.

Soln. About centre $L_c = r_\perp.p = Rmv$

About a point on circumference



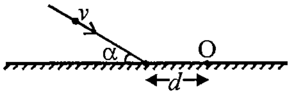


$$L_O = r_\perp p\ = r\sin\left(90 - \frac{\theta}{2}\right)mv \text{ (from figure)}$$

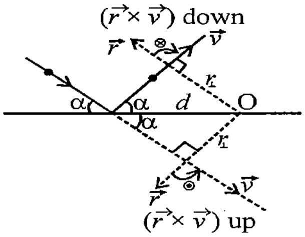
$$= r\cos\frac{\theta}{2}mv\ = \left(2R\cos\frac{\theta}{2}\right)\cos\frac{\theta}{2}mv \text{ (use sine rule to get } r = 2R\cos\frac{\theta}{2})$$

$$= 2Rmv\cos^2\frac{\theta}{2}$$

4. A particle of mass m moving with constant speed v strikes a floor elastically. Calculate change in angular momentum about a point which is at a distance 'd' from the point of impact as shown in figure.



Soln. The collision is elastic therefore particle will rebound with same speed and at same angle with the floor as shown in the figure.



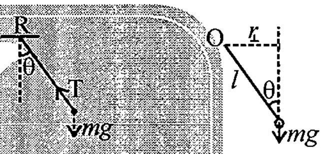
$\therefore r_\perp = d\sin\alpha$

Angular momentum before impact

$L_1 = r_\perp p = d\sin\alpha\, mv, \quad L_2 = r_\perp p = -d\sin\alpha\, mv$

$\therefore$ Change in angular momentum $\Delta L = L_2 - L_1 = -2mvd\sin\alpha$

5. A simple pendulum consisting of a small bob of mass m and light string of length 'l' oscillates in vertical plane. Calculate torque due to gravity and tension in the string at the instant string makes θ angle with downward vertical.



Soln. Line of action of tension passes through point of suspension. Therefore $r_\perp = 0$

Thus, torque due to tension = 0

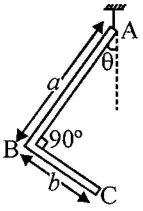
For mg, $r_\perp = l\sin\theta$

$\therefore \tau = r_\perp F$

$= l\sin\theta mg = mgl\sin\theta$

6. An L-shaped uniform wire is suspended from a string as shown in figure. If length of the two sections of wire be 'a' and 'b' then what is value of angle θ made by upper section with vertical.

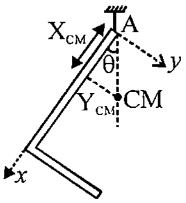
Soln. The wire is in rotational equilibrium therefore centre of mass of the rod must lie on the vertical line through point A so that gravity does not produce any torque about point A.



To find the position of centre of mass let us choose coordinate system as shown in figure.

Let λ be mass per unit length of the wire.

$$\therefore X_{CM} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} = \frac{\lambda a.\frac{a}{2} + \lambda ba}{\lambda a + \lambda b} = \frac{a(a+2b)}{2(a+b)}$$

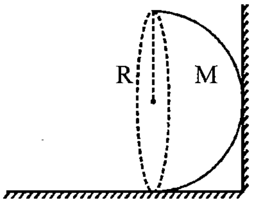


$$Y_{CM} = \frac{m_1y_1 + m_2y_2}{m_1 + m_2} = \frac{\lambda a.0 + \lambda b.\frac{b}{2}}{\lambda a + \lambda b} = \frac{b^2}{2(a+b)}$$

$$\therefore \tan\theta = \frac{Y_{CM}}{X_{CM}} = \frac{b^2}{a(a+2b)}$$

or $\theta = \tan^{-1}\left(\dfrac{b^2}{a(a+2b)}\right)$ if $a = b$ then $\theta = \tan^{-1}\left(\dfrac{1}{3}\right)$

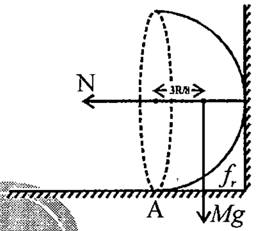
7. A solid hemisphere of mass M and radius R is placed against a smooth wall as shown in the figure. What is normal reaction on the sphere due to the wall.



Soln: Centre of mass of the hemisphere lies at a distance $\frac{3R}{8}$ from its base centre. The sphere is in equilibrium therefore torque about point A must be zero.

$$\therefore NR - Mg.\frac{3R}{8} = 0$$

or, $N = \frac{3Mg}{8}$



8. A thin rod of mass M and length L is suspended from one end while its other end lies on a smooth horizontal plane and makes an angle α with the plane. Find the normal reaction from the plane and tension in the string from which it is suspended.

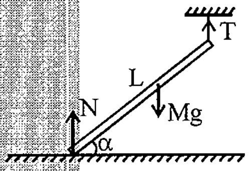
For translational equilibrium $T + N = Mg$ ... (a)

For rotational equilibrium

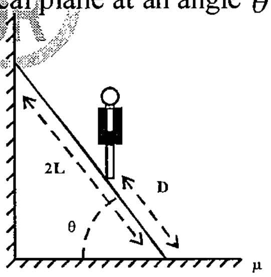
Torque about lower end of the rod is zero

$$\therefore \quad T.L\cos\alpha - Mg\frac{L}{2}\cos\alpha = 0 \qquad \therefore \quad T = \frac{Mg}{2}$$

from (a) $N = Mg - T = Mg - \frac{Mg}{2} = \frac{Mg}{2}$



9. A uniform ladder of length 2L and mass 'm' leans against a wall in a vertical plane at an angle θ to the horizontal. The floor is rough, having a coefficient of static friction μ. A person of mass M stands on the ladder at a distance D from its base (see figure). If the wall is frictionless, the maximum distance (D_{max}) up the ladder that the person can reach before the ladder slips is



Soln. Let us take ladder plus person as system. Various forces acting on the system are shown in the figure. For translational equilibrium we must have

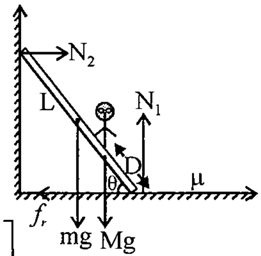
$N_1 = (M + m)g$ ... (a)

$N_2 = f_r$ ... (b)

For rotational equilibrium torque about lower end of the ladder must be zero. Therefore,

$$MgD\cos\theta + mgL\cos\theta - N_2.2L\sin\theta = 0$$

$$\therefore N_2 = \frac{g\cot\theta}{2}\left[\frac{MD}{L} + m\right]$$



$\therefore$ from (b) we get, $f_r = \dfrac{g\cot\theta}{2}\left[\dfrac{D}{L}M + m\right]$

since $f_r \le \mu N_1$, $\therefore \dfrac{g\cot\theta}{2}\left[\dfrac{D}{L}M + m\right] \le \mu(M+m)g$

or $D \le \dfrac{L}{M}[2\mu(M+m)\tan\theta - m]$ $\therefore D_{\max} = L\left[2\mu\left(1+\dfrac{m}{M}\right)\tan\theta - \dfrac{m}{M}\right]$

10. Angular momentum of an object about a point varies with time as $\vec{L} = \alpha\hat{i} + \beta t^2\hat{j}$. At a certain instant of time, torque acting on the object about that point makes 60° with $\vec{L}$. Calculate magnitude of torque at that instant.

Soln. Torque is related to angular momentum as

$$\vec{\tau} = \frac{d\vec{L}}{dt}, \quad \vec{\tau} = \frac{d}{dt}(\alpha\hat{i} + \beta t^2\hat{j})$$

$$\vec{\tau} = 2\beta t\hat{j} \qquad \text{... (i)}$$

given, $\theta = \cos^{-1}\left[\dfrac{\vec{L}\cdot\vec{\tau}}{|\vec{L}||\vec{\tau}|}\right] = 60^\circ \Rightarrow \cos^{-1}\left[\dfrac{2\beta^2t^3}{2\beta t(\alpha^2+\beta t^2)^{1/2}}\right] = 60^\circ$

or $\dfrac{2\beta^2t^3}{2\beta t(\alpha^2+\beta^2t^4)^{1/2}} = \dfrac{1}{2}$

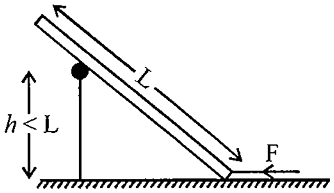
or $\beta^2t^4 = \dfrac{1}{4}(\alpha^2+\beta^2+4)$ or $4\beta^2t^4 - \beta^2t^4 - \alpha^2 = 0$

$3\beta^2t^4 = \alpha^2$ or $t = \left[\dfrac{\alpha^2}{3\beta^2}\right]^{1/4}$

$\therefore$ from (i)

$$\therefore \tau = 2\beta\left[\frac{\alpha^2}{3\beta^2}\right]^{1/4}\hat{j}$$

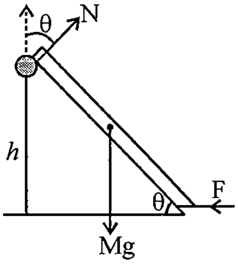
11. Consider the situation shown in the figure. All the surfaces are smooth and the rod has mass M.



What is the minimum value of F required to keep the rod in equilibrium?

Soln. Clearly, the force will be minimum when upper end of the rod lies on the round support. We are excluding the case when rod is vertical.

In figure, $\sin\theta = \frac{h}{L} \Rightarrow \tan\theta = \frac{h}{\sqrt{L^2 - h^2}}$



$N\cos\theta = Mg$

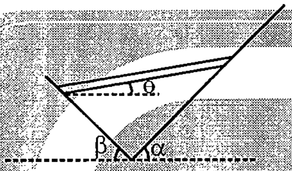
$N\sin\theta = F$

$\therefore \ \tan\theta = \frac{F}{Mg}$

or $F = Mg\tan\theta$

$F = Mg \cdot \frac{h}{\sqrt{h^2 - L^2}}$

12. A thin rod of mass M and length L is lying on two inclined smooth planes having angles of inclination α and β as shown in the figure. Calculate the angle θ that the rod makes with horizontal in equilibrium.



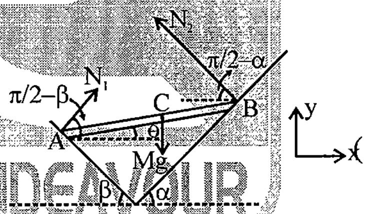
Soln. The forces acting on the rod are shown in the figure.

In equilibrium,

$\Sigma F_x = 0, \Sigma F_y = 0, \Sigma\tau = 0$

From $\Sigma F_x = 0$, we get

$N_1 \sin\beta = N_2 \sin\alpha$



or $N_1 = \frac{N_2 \sin\alpha}{\sin\beta}$ ... (i)

From $\Sigma F_y = 0$, we get

$N_1\cos\beta + N_2\cos\alpha = Mg$

using (1) we get

$\frac{N_2 \sin\alpha \cdot \cos\beta}{\sin\beta} + N_2\cos\alpha = Mg$

or $N_2 = \frac{Mg \cdot \sin\beta}{\sin(\alpha + \beta)}$ ... (ii)

Now, from $\Sigma\tau_A = 0$, we get

$\overrightarrow{AB} \times \overrightarrow{N}_2 + \overrightarrow{AC} \times (-Mg\hat{j}) = 0$ ... (iii)

Here $\overrightarrow{AB} = L(\cos\theta\hat{i} + \sin\theta\hat{j})$

$\overrightarrow{N}_2 = N_2(-\sin\alpha\hat{i} + \cos\alpha\hat{j})$

$$\overrightarrow{AC} = \frac{L}{2}(\cos\theta\hat{i} + \sin\theta\hat{j})$$

$\therefore$ from (iii) we get

$$LN_2(\cos\alpha\cos\theta + \sin\theta\sin\alpha)\hat{k} - \frac{MgL}{2}\cos\theta\hat{k} = 0$$

use N_2 to get

$$\text{or}\quad \left[\frac{1}{2} - \frac{\sin\beta\cos\alpha}{\sin(\alpha+\beta)}\right]\cos\theta + \frac{\sin\alpha\cdot\sin\beta}{(\sin\alpha+\beta)}\sin\theta = 0$$

$$\text{or}\quad \tan\theta = \frac{2\sin\beta\cdot\cos\alpha - \sin(\alpha+\beta)}{2\sin\alpha\cdot\sin\beta)} = \frac{1}{2}(\cot\alpha - \cot\beta)$$

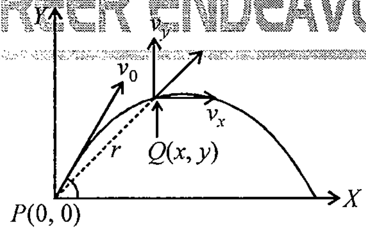
$$\therefore\ \theta = \tan^{-1}\left[\frac{1}{2}(\cot\alpha - \cot\beta)\right]$$

13. A particle is projected at time $t = 0$ from a point P on the ground with a speed v_0 at an angle 45° to the horizontal. Find the magnitude and direction of the angular momentum of the particle about the point P, at time $t = \frac{v_0}{g}$.

Soln. In figure, P is the origin (0, 0) from where the particle is projected with velocity v_0 at time $t = 0$. Let after time t, $Q(x, y)$ be the co-ordinates of the particle, where the resultant velocity of the particle is v. The distance of Q from P is r.

The rectangular components v_x and v_y along the x and y axes are $v_x = v_0\cos 45°$ and $v_y = v_0\sin 45° - gt$.

The co-ordinates (x, y) of the point Q after time t are given by $x = v_0 t\cos 45°$ and $y = v_0 t\sin 45° - \frac{1}{2}gt^2$.



Angular momentum of particle $Q(x, y)$ about point $P(0, 0) = mv_y x - mv_x y$

Substituting the values, we get

$$= m(v_0\sin 45° - gt)(v_0 t\cos 45°) - m(v_0\cos 45°)(v_0 t\sin 45° - \frac{1}{2}gt^2)$$

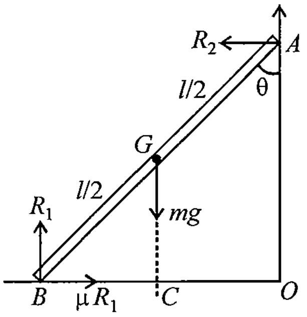
$$= -\frac{1}{2\sqrt{2}}\frac{mv_0^3}{g}.$$

Solving it we get the magnitude of angular momentum of the particle about $P = \frac{1}{2\sqrt{2}}\frac{mv_0^3}{g}$.

The direction of angular momentum will be mutually perpendicular to distance r and momentum vector. The direction will be along the inward normal to the plane of the paper.

14. A uniform ladder AB of length 3 meters and weight 20 kg. rests with the end A against rough vertical wall and the end B on level ground. If the wall and the ground are equally rough and the coefficient of friction is 0.5, find the limiting position of equilibrium.

Soln. The different forces on the ladder are shown in figure. Let in the limiting position of equilibrium, the ladder makes an angle θ with the wall.



Equating the horizontal and vertical forces, we get

$$R_2 = \mu R_1 \quad \text{...(i)}$$

$$R_1 + \mu R_2 = mg \quad \text{...(ii)}$$

From equating (ii),

$$R_1 + \mu^2 R_1 = mg \text{ or } R_1 = \frac{mg}{(1+\mu^2)} \quad \text{...(iii)}$$

From equation (i),

$$R_2 = \left(\frac{\mu}{1+\mu^2}\right) mg \quad \text{...(iv)}$$

Taking the moment of forces about B, we get

$$mg \times BC = R_2 \times OA + \mu R_2 \times OB.$$

From figure,

$$OA = l\cos\theta,\ OB = l\sin\theta \text{ and } BC = \frac{l}{2}\sin\theta$$

$$\therefore \quad mg\frac{l}{2}\sin\theta = R_2\, l\cos\theta + \mu R_2\, l\sin\theta$$

$$\sin\theta\left[\frac{mg}{2} - \mu R_2\right] = R_2\cos\theta$$

$$\tan\theta = \frac{R_2}{\left[\frac{mg}{2} - \mu R_2\right]} = \frac{\left(\frac{\mu}{1+\mu^2}\right)mg}{\left[\frac{mg}{2} - \frac{\mu^2}{1-\mu^2}mg\right]}$$

$$\tan\theta = \frac{2\mu}{1-\mu^2} = \frac{2\times 0.5}{1-(0.5)^2} = 1.33 \quad (\because \mu = 0.5)$$

$$= \tan^{-1}(1.33)$$

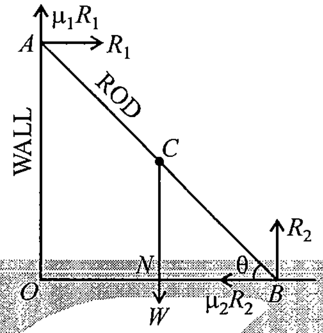
$$\theta = 53^\circ 4'.$$

15. A uniform rod is made to lean between a rough vertical wall and the ground. Show that the least angle at which the rod can be leaned without slipping is given by

$$\theta = \tan^{-1}\left(\frac{1-\mu_1\mu_2}{2\mu_2}\right),$$

where μ_1 and μ_2 stand for the coefficient of friction between (i) the rod and the wall, and (ii) the rod and the ground.

Soln. The situation is shown in figure.



Considering the equilibrium of the rod, we have

$$R_1 + \mu_2 R_2 \qquad \text{...(i)}$$

and $R_2 + \mu_1 R_1 = W$ ...(ii)

Taking the moments of the forces about A, we have

$$R_2 \times OB = W \times ON + \mu_2 R_2 \times OA$$

$$R_2 \times AB\cos\theta = W \times \frac{AB\cos\theta}{2} + \mu_2 R_2 \times AB\sin\theta$$

$$\cos\theta\left(R_2 - \frac{W}{2}\right) = \sin\theta\,(\mu_2 R_2).$$

$$\therefore \quad \tan\theta = \frac{\left(R_2 - \frac{W}{2}\right)}{\mu_2 R_2} \qquad \text{...(iii)}$$

From equations (i) and (ii), we have

$$W = R_2 + \mu_1 R_1 = R_2 + \mu_1\mu_2 R_2 = R_2(1+\mu_1\mu_2)$$

Now $R_2 - \frac{W}{2} = R_2 - \frac{R_2}{2}[(1+\mu_1\mu_2)] = R_2\left(1 - \frac{1}{2} - \frac{\mu_1\mu_2}{2}\right) = \frac{R_2}{2}(1-\mu_1\mu_2).$

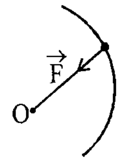
$$\therefore \quad \tan\theta = \frac{\frac{R_2}{2}(1-\mu_1\mu_2)}{\mu_2 R_2} = \frac{1-\mu_1\mu_2}{2\mu_2}$$

$$\theta = \tan^{-1}\left(\frac{1-\mu_1\mu_2}{2\mu_2}\right).$$

Chapter 8

Central force motion and Gravitation

A central force is expressed as $\vec{F} = \pm f(r)\hat{r}$, positive sign is used for repulsive and negative sign for attractive force. Central force acting on a particle is always directed towards a point which is called centre of force.



Features of central force motion:

- Central forces are conservative, therefore total energy of a particle moving under central force is conserved.
- Torque about centre of force is zero, therefore angular momentum about centre of force is conserved. This implies that central force motion is a planar motion.
- Areal velocity is constant, that is area swept by line joining the particle to the centre of force per unit time is constant.

$$\frac{\Delta A}{\Delta t} = \text{constant}$$

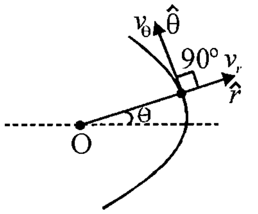
Relation between angular momentum and areal velocity: Central force motion is a planar motion, therefore we can use plane polar coordinate to describe dynamics of particle.

Therefore velocity of particle is $\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$

Angular momentum of particle about centre of force 'O' is

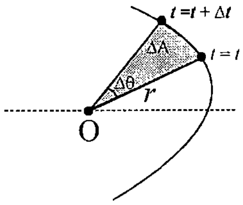
$L = mv_{\theta}r = mr\dot{\theta}r = mr^2\dot{\theta}$

$\boxed{L = mr^2\dot{\theta}}$



Areal velocity: $\dfrac{\Delta A}{\Delta t} = \dfrac{\frac{1}{2}r^2\Delta\theta}{\Delta t} = \dfrac{1}{2}r^2\dfrac{\Delta\theta}{\Delta t} = \dfrac{1}{2}r^2\dot{\theta}$

from expression of angular momentum $r^2\dot{\theta} = \dfrac{1}{m}$



$\therefore \boxed{\text{Areal velocity} = \dfrac{L}{2m}}$

Differential equation of orbit: Acceleration of a particle moving in a plane can be expressed in plane polar coordinate as

$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$

Therefore in central force motion Newton's equation is

$\vec{F} = m\vec{a}$, therefore, $f(r)\hat{r} = m[(\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}]$

Therefore, equating coefficient of $\hat{r}$ and $\hat{\theta}$ we get

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0 \qquad \text{... (i)}$$

and $m(\ddot{r} - r\dot{\theta}^2) = f(r)$ ... (ii)

Equation (i) can also be obtained by differentiating $L = mr^2\dot{\theta}$ therefore it implies constancy of angular momentum. Equation (ii) is differential equation for the dynamics of particle under central force. To obtain differential equation of orbit we must remove time from equation (ii). To accomplish this we introduce a new variable $u = 1/r$ and use $\dot{\theta} = L/mr^2 = Lu^2/m$. This leads to,

$$\boxed{\frac{\partial^2 u}{\partial \theta^2} + u = -\frac{mf(r)}{L^2u^2}} \qquad \text{... (iii)}$$

equation (iii) is differential equation of orbit. The above equation is mostly used for following purposes.

(a) To find $f(r)$ when equation of orbit is given. That is a relation between r and θ is given then $f(r)$ can be found.

(b) If $f(r)$ is given then to find equation of orbit.

Example: Equation of the orbit of a particle moving under central force is $r\theta = \beta$, where β is a constant. Find the force acting on the particle.

Soln. $r\theta = \beta$, therefore, $u = \frac{1}{r} = \frac{\theta}{\beta}$ $\therefore \frac{\partial^2 u}{\partial \theta^2} = 0$

Differential equation of orbit, $\frac{\partial^2 u}{\partial \theta^2} + u = \frac{-mf(r)}{L^2u^2}$

$\therefore \quad 0 + u = \frac{-mf(r)}{L^2u^2}$ $\quad \therefore f(r) = \frac{-L^2u^3}{m} = \frac{-L^2}{mr^3}$

Example: Equation of orbit of a particle moving under central force is $r^n = a\cos n\theta$. Find the force on the particle.

Soln. Given, $r^n = a\cos n\theta$, therefore, $u^n = \frac{1}{a\cos n\theta} = \frac{1}{a}\sec n\theta$ ... (a)

Taking ln both sides we get, $n\ln u = \ln\left(\frac{1}{a}\right) + \ln\sec n\theta$

Differentiating w.r.t. θ we get, $\frac{n}{u}\frac{\partial u}{\partial \theta} = n\tan n\theta$ $\quad \therefore \frac{\partial u}{\partial \theta} = u\tan n\theta$

Differentiating again w.r.t. θ we get $\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial u}{\partial \theta}\tan n\theta + u\,n\sec^2 n\theta$ $= u\tan^2 n\theta + u\,n\sec^2 n\theta$

Differential equation of orbit is $\frac{\partial^2 u}{\partial \theta^2} + u = \frac{-mf(r)}{L^2u^2}$

$$u\tan^2 n\theta + u\,n\sec^2 n\theta + u = \frac{-mf(r)}{L^2u^2},\quad u\sec^2 n\theta + u\,n\sec^2 n\theta = \frac{-mf(r)}{L^2u^2}$$

$$\therefore f(r) = -\frac{L^2u^3(1+n)\sec^2 n\theta}{m}$$

from (a) $\sec^2 n\theta = a^2u^{2n}$

$$\therefore \qquad f(r) = \frac{-L^2(n+1)a^2u^{2n+3}}{m}$$

$$\therefore \qquad f(r) = \frac{-L^2(n+1)a^2}{m}\cdot\frac{1}{r^{2n+3}}$$

Or $\boxed{f(r) \propto \frac{1}{r^{2n+3}}}$, the force is attractive in nature.

Example: A particle of mass m is moving under a central force. The angular momentum of the particle is L and its equation of orbit is $r = Ae^{k\theta}$. Calculate potential energy of the particle.

Soln. Potential energy is given as $V(r) = -\int f(r)dr$ ... (a)

Therefore, let us first find $f(r)$.

Given $r = Ae^{k\theta}$, therefore, $u = \frac{1}{A}e^{-k\theta}$ $\therefore \frac{\partial^2 u}{\partial\theta^2} = k^2u$

Differential equation of orbit is $\frac{\partial^2 u}{\partial\theta^2} + u = \frac{-mf(r)}{L^2u^2}$ $\therefore (k^2+1)u = \frac{-mf(r)}{L^2u^2}$

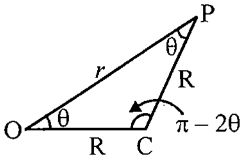
$\therefore f(r) = \frac{-L^2(k^2+1)u^3}{m}$ or $f(r) = \frac{-L^2(k^2+1)}{mr^3}$

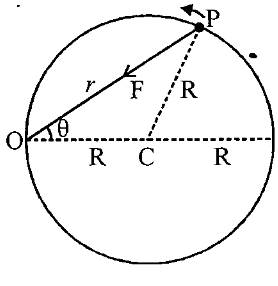
Therefore, from (a) $V(r) = -\int f(r)dr = \frac{-L^2(k^2+1)}{2mr^2}$

Example: A particle is moving in a circle under a force which is always directed towards a point on the circumference of the circle. Find the force law.

Soln. Let R be the radius of the circle and r be distance of prticle from the centre of force. Let θ be the angular coordinate of the particle as shown in the figure.

Therefore using sine rule we get,





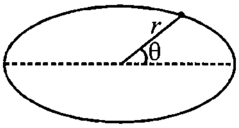
$$\frac{r}{\sin(\pi-2\theta)} = \frac{R}{\sin\theta} \qquad \therefore r = 2R\cos\theta$$

In previous example we have seen that for $r^n = a\cos n\theta$, $F \propto \frac{1}{r^{2n+1}}$. In present case $n=1$, therefore, force law is $F \propto \frac{1}{r^5}$.

Polar equation of orbit under $F = -\frac{k}{r^2}$ force: Potential energy corresponding to this force is $V(r) = -\int F dr = -\frac{k}{r}$. Let L be the angular momentum of particle and E be total energy. Therefore,

$L = mr^2\dot{\theta}$ and $E = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{k}{r}$

Differential equation of orbit is $\frac{\partial^2 u}{\partial\theta^2} + u = \frac{-m f(r)}{L^2u^2}$



Using $f(r) = -\frac{k}{r^2} = -ku^2$ we get $\frac{\partial^2 u}{\partial\theta^2} + u = +\frac{mk}{L^2}$

Solution of above differential equation gives $\boxed{r = \frac{L^2/mk}{1+\sqrt{1+2E.\frac{L^2}{mk^2}}\cos\theta}}$

This is polar equation of orbit under $F = -\frac{k}{r^2}$ force. The above is rewritten as,

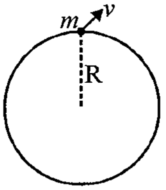
$\boxed{r = \frac{l}{1+e\cos\theta}}$ where $l = \frac{L^2}{mk}$ = Latus rectum of the conic section (curve) represented by above equation and $e = \sqrt{1+2E\frac{L^2}{mk^2}}$ is eccentricity of the orbit.

Classification of orbits: From polar equation of orbit we see that under a given force law orbit of a particle depends on its energy. Therefore, in present case,

Energy	**Eccentricity**	**Orbit**
$\frac{-mk^2}{2L^2} < E < 0$	e < 1	Elliptical
$E = 0$	e = 1	Parabola
$E > 0$	e > 1	Hyperbola
$E = \frac{-mk^2}{2L^2}$	e = 0	Circle

Therefore if nature of orbit is to be found for $F = -\frac{k}{r^2}$ we can just check if its energy satisfies any of the above energy conditions.

Example: A particle is thrown from earth's surface with speed $v=\sqrt{\dfrac{1.5GM}{R}}$, where M and R are mass and radius of the earth respectively. What is the nature of the path of the particle.



Soln. Force on the particle is $F=\dfrac{-GMm}{r^2}$

Therefore potential energy of the particle is, $V=-\int F dr=\dfrac{-GMm}{r}$

Near earth's surface $V=\dfrac{-GMm}{R}$

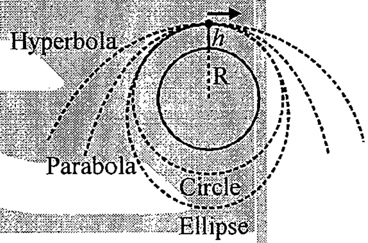
Total energy of the particle is $E=\dfrac{1}{2}mv^2-\dfrac{GMm}{R}=\dfrac{1}{2}m.\left(\dfrac{1.5GM}{R}\right)-\dfrac{GMm}{R}=\dfrac{-GMm}{4R}<0$

Since $E<0$, therefore path of the particle will be elliptical.

Nature of orbit from velocity of projectile: In the previous example we have seen that the orbit of projectile depends on the initial velocity with which it is thrown. In the following we give nature of orbit for given initial velocity.

Suppose a particle is thrown from some height above the earth's surface as shown in figure.

Velocity	**Path**
$\sqrt{\dfrac{GM}{R+h}}$	Circle
$\sqrt{\dfrac{GM}{R+h}}<\text{speed}<\sqrt{\dfrac{2GM}{R+h}}$	Ellipse
$\sqrt{\dfrac{2GM}{R+h}}$	Parabola
Speed $>\sqrt{\dfrac{2GM}{R+h}}$	Hyperbola



Example: For a particle moving under gravitational force pericentre distance in parabolic orbit is r_p while the radius of the circular orbit with same angular momentum is r_c. What is relation between r_p and r_c?

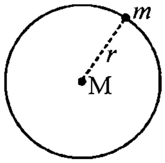
Soln. Pericentre distance is the minimum distance $(\cos\theta=\max=1)$ and for parabolic orbit $e=1$.

Therefore, $r_{\min}=\dfrac{l}{1+e\cos\theta}=\dfrac{l}{2}=r_p$

for circular orbit $e=0$, therefore $r_c=\dfrac{l}{1+e\cos\theta}=l \quad \therefore r_p=\dfrac{r_c}{2}$

Energy and angular momentum of planet: Let m be the mass of planet and M be the mass of the sun. Let us assume that sun is stationary and planet revolves around the sun. Here we will use polar equation of orbit to get the expression.

Circular orbit: Let the orbit is circular and its radius is r, eccentricity of circular orbit is zero.



$$\therefore \quad e=\sqrt{1+2E.\frac{L^2}{k^2}}=0$$

$$\therefore \quad E=\frac{-mk^2}{2L^2} \qquad \text{... (i)}$$

From polar equation of orbit $r=\dfrac{L^2/mk}{1+0}$

$$\therefore \quad L^2=mkr \qquad \text{... (ii)}$$

for gravitational force $k=GMm$, $L=\sqrt{Gm^2Mr}$

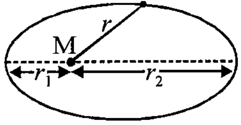
from (i), $E=\dfrac{-m(GMm)^2}{2.Gm^2Mr}=\dfrac{-GMm}{2r}$

if v be the orbital speed then $L=mvr \Rightarrow v=\dfrac{L}{mr}=\sqrt{\dfrac{GM}{r}}$

Therefore, for the planet in circular orbit angular momentum energy and speed are given as

and $\boxed{L=m\sqrt{GMr},\ E=-\dfrac{GMm}{2r},\ v=\sqrt{\dfrac{GM}{r}}}$

Elliptical orbit: Let r_1 and r_2 be minimum and maximum distance of planet from the sun



From polar equation of orbit $r=\dfrac{L^2/mk}{1+\sqrt{1+\dfrac{2E.L^2}{mk^2}}\cos\theta}$

for minimum distance $\cos\theta=1$

$$\therefore \quad r_1=\frac{L^2/mk}{1+\sqrt{1+\dfrac{2E.L^2}{mk^2}}} \quad \text{or} \quad 1+\sqrt{1+\frac{2E.L^2}{mk^2}}=\frac{L^2}{mkr_1} \qquad \text{... (a)}$$

Similarly for maximum distance $(\cos\theta=-1)$

$$1-\sqrt{1+\frac{2E.L^2}{mk^2}}=\frac{L^2}{mkr_2} \qquad \text{... (b)}$$

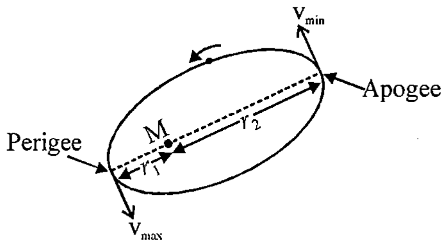
Add equations (a) and (b) and use $k=GMm$ to get $L=m\sqrt{\dfrac{2GMr_1.r_2}{r_1+r_2}}$

Put value of L in any of the two equations to get $E=-\dfrac{GMm}{r_1+r_2}$

Therefore, for a planet in elliptical orbit energy and angular momentum are given as

$$\boxed{E=-\frac{GMm}{r_1+r_2},\ L=m\sqrt{\frac{2GMr_1.r_2}{r_1+r_2}}}$$

Maximum and minimum velocity in elliptical orbit: Angular momentum of planet is conserved therefore its velocity is maximum at perigee and minimum at apogee.



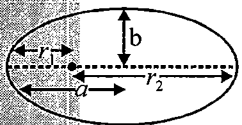
Therefore, $m v_{\max}.r_{\min}=L$ or $v_{\max}=\dfrac{L}{m r_{\min}}=\dfrac{L}{m r_1}$

Using value $L=m\sqrt{\dfrac{2GMr_1.r_2}{r_1+r_2}}$ we get $v_{\max}=\sqrt{\dfrac{2GM r_2}{r_1(r_1+r_2)}}$

Similarly $m v_{\min}.r_{\max}=L$ gives $v_{\min}=\dfrac{L}{mr_2}=\sqrt{\dfrac{2GMr_1}{r_2(r_1+r_2)}}$

Therefore, $\boxed{v_{\max}=\sqrt{\dfrac{2GMr_2}{r_1(r_1+r_2)}},\ v_{\min}=\sqrt{\dfrac{2GMr_1}{r_2(r_1+r_2)}}}$

E, L, v_{max}, v_{min} in terms of ellipse parameters: Let a, b and e be the semi-major axis, semi-major axis and eccentricity of ellipse.



$\therefore\ r_1=a(1-e),\ r_2=a(1+e),\ e=\sqrt{1-\dfrac{b^2}{a^2}}$

Using result of previous section we get

$E=\dfrac{-GMm}{2a}$ (E depends only on length of major axis)

$L=m\sqrt{GMa(1-e^2)}$, $v_{\max}=\sqrt{\dfrac{GM}{a}\left(\dfrac{1+e}{1-e}\right)}$, $v_{\min}=\sqrt{\dfrac{GM}{a}\left(\dfrac{1-e}{1+e}\right)}$

Example: Ratio of maximum to minimum speed of a planet revolving around the sun in an elliptical orbit is 2:1, What is eccentricity of the orbit?

Soln. Given $\dfrac{v_{\max}}{v_{\min}}=\dfrac{2}{1}$ $\therefore$ $\dfrac{\sqrt{\dfrac{GM}{a}\left(\dfrac{1+e}{1-e}\right)}}{\sqrt{\dfrac{GM}{a}\left(\dfrac{1-e}{1+e}\right)}}=\dfrac{2}{1}$ or $\dfrac{1+e}{1-e}=\dfrac{2}{1}$ $\therefore e=\dfrac{1}{3}$

Example: An object is thrown from the earth's surface with initial speed $\sqrt{\dfrac{1.5GM}{R}}$ due to which it moves in an elliptical orbit. What is the semi-major axis of the ellipse.

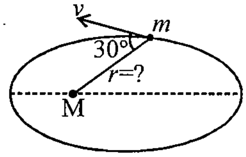
Soln. Total energy of the particle

$$E=\frac{-GMm}{R}+\frac{1}{2}mv^2=\frac{-GMm}{R}+\frac{1}{2}m.\frac{1.5GM}{R}=\frac{-GMm}{4R}$$

In central force motion energy of the particle is conserved. Therefore its energy in elliptical orbit will have the same value. Let a be semi-major axis of the ellipse.

Therefore, $E=\dfrac{-GMm}{2a}=\dfrac{-GMm}{4R}$ $\therefore a=2R$

Example: A planet is revolving around the sun in elliptical orbit of semi-major axis 'a' and eccentricity $\sqrt{3}/2$. At certain instant of time velocity vector of planet makes 30° with the radial line. What is distance of planet from the sun at this instant.



Soln. Energy of the planet is

$$E=\frac{-GMm}{r}+\frac{1}{2}mv^2 \qquad \text{... (i)}$$

Angular momentum of the planet is

$$L=mvr\sin 30°=\frac{mvr}{2} \qquad \text{... (ii)}$$

We know that $L=m\sqrt{GMa(1-e^2)},\ e=\frac{\sqrt{3}}{2}$

$$\therefore\ \frac{mvr}{2}=m\sqrt{GMa\frac{1}{4}} \quad \text{or} \quad v=\frac{\sqrt{GMa}}{r}$$

We know that $E=\frac{-GMm}{2a}$.

Therefore from (i) we get, $\frac{-GMm}{2a}=\frac{-GMm}{r}+\frac{1}{2}m.\frac{GMa}{r^2}$

or $-\frac{1}{2a}=-\frac{1}{r}+\frac{a}{2r^2}$ or $r^2-2ar+a^2=0 \quad \therefore r=a$

Example: A planet is revolving around the sun in a circular orbit. Due to some reason the speed of the planet suddenly becomes double. What is new orbit of the planet.

Soln. Orbital speed of the planet is $\sqrt{\frac{GM}{r}}$

New speed of the planet $=2\sqrt{\frac{GM}{r}}$

Therefore, new energy of the planet $=\frac{1}{2}mv^2-\frac{GMm}{r}=\frac{1}{2}m.\frac{4GM}{r}-\frac{GMm}{r}=\frac{2GMm}{r}>0$

Total energy of the planet becomes positive on doubling its speed therefore new orbit of the planet will be hyperbolic.

Effective potential (Reducing 2-d problem into 1-d problem): Under central force, a particle has planar motion. That is motion of the particle is 2-dimensional. Therefore total energy is written as $E=\frac{1}{2}mv^2+V(r)$

or, $E=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)+V(r)$

Angular momentum of the particle is constant of motion and it is given as, $L=mr^2\dot{\theta}$

$$\therefore\ \dot{\theta}=\frac{L}{mr^2}$$

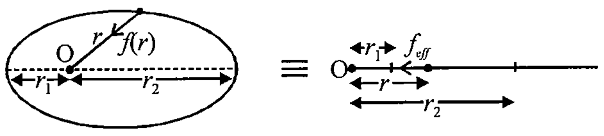
Let us put the value of $\dot{\theta}$ in the energy expression to get $E=\frac{1}{2}m\dot{r}^2+\frac{L^2}{2mr^2}+V(r)$

or $$\boxed{E=\frac{1}{2}m\dot{r}^2+V_{eff}(r)} \qquad \text{... (a)}$$

where $$\boxed{V_{eff}(r)=\frac{L^2}{2mr^2}+V(r)}$$

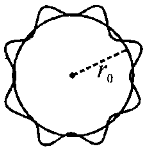
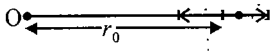
Equation (a) can be interpreted as follows.

Two dimensional motion under a central potential $V(r)$ is equivalent to radial motion under effective potential $V_{eff}(r)$.



$$f=\frac{-\partial V}{\partial r} \qquad f_{eff}=\frac{-\partial V_{eff}}{\partial r}$$

Therefore, reason for defining an effective potential is that it is easier to discuss 1-d problem than 2-d problem. Before we start dealing with problem let us first understand how the interpretation in one dimension is different from that in two dimension.

Actual 2-D problem	**Equivalent 1-D problem**
• Energy $E=\frac{1}{2}mv^2+V(r)$	• Energy $E=\frac{1}{2}m\dot{r}^2+V_{eff}(r)$
• At turning points velocity is ⊥ to the radial line	• At turning points $\dot{r}=0$. Therefore it appears as if particle stops there.
• Circular motion under a central force is uniform motion and $f(r)=\frac{mv^2}{r}$	• In 1-d description of circular motion particle appears stationary, $\dot{r}=0$ and $f_{eff}=0$.
• Oscillation about stable radius is shown as  	• An equivalent oscillation in $1-d$ is linear oscillation 

Finding the nature of possible orbits under a given central force / potential: In previous section we have seen that various types of orbits are possible under $F=-k/r^2$ force and the nature of orbit depends on energy of particle. It is tedious task to find polar equation of orbit for various forces and then conclude about the nature of orbit for a given energy as we did in previous section. Therefore instead of discussing the problem in 2-d we discuss equivalent problem in 1-d.

In equivalent 1-d problem total energy is written as $E=\frac{1}{2}m\dot{r}^2+V_{eff}$

or $\frac{1}{2}m\dot{r}^2=E-V_{eff}$

Therefore, E must be greater or equal to V_{eff}. Thus we conclude that if we plot V_{eff} versus r, movement of particle will be limited to that region only where V_{eff} line lies below the total energy line. That is particle cannot go in the region where V_{eff} line is above the total energy (E) line.

Dynamics under Inverse square force $F = -k/r^2$ or $V(r) = -k/r$:

Let m be mass of particle and L be its angular momentum.

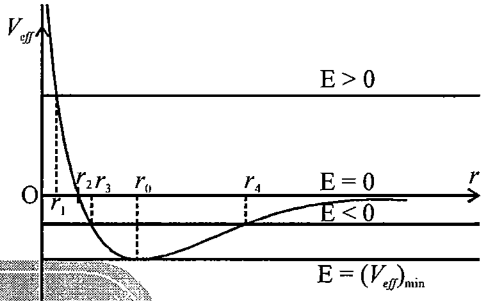
Therefore, $V_{eff}(r) = V(r) + \dfrac{L^2}{2mr^2} = \dfrac{-k}{r} + \dfrac{L^2}{2mr^2}$

The plot of $V_{eff}(r)$ versus r is shown in the figure below. To plot the graph consider the following

If $f(r) = ar^m - br^n$ where $m > n$ then

for $r \to 0$ lower power term (br^n) dominates

for $r \to \infty$ higher power term (ar^m) dominates



From the plot of V_{eff} versus r we draw following conclusion.

(i) There is a minimum at $r = r_0$. Therefore if particle maintains its distance from centre of force to be equal to r_0 then its orbit is stable. i.e. if particle moves in a circle of radius r_0 then its orbit is stable.

Therefore to find radius (r_0) of stable orbit we use $\left.\dfrac{dV_{eff}}{dr}\right|_{r=r_0} = 0$

(ii) For E > 0 minimum distance of the particle from centre of force is r_1 and maximum distance is infinity that is $r_1 \le r \le \infty$. Therefore, for E > 0 orbit is unbounded.

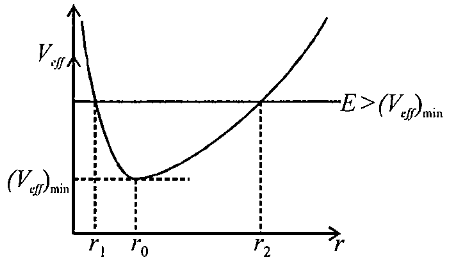
(iii) For E = 0, $r_{min} = r_2$, $r_{max} = \infty$, therefore $r_1 \le r \le \infty$ and orbit is unbounded.

(iv) For E < 0, $r_{min} = r_3$, $r_{max} = r_4$, therefore $r_3 \le r \le r_4$ orbit is bounded.

(v) $E = (V_{eff})_{min}$, only $r = r_0$ is possible, therefore orbit is a circle of radius r_0.

(vi) $E < (V_{eff})_{min}$ is not allowed

Dynamics under Hookes force: $F = -kr$, $V(r) = \dfrac{1}{2}kr^2$:



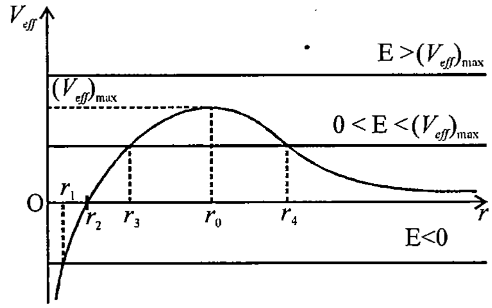
$$V_{eff} = \frac{1}{2}kr^2 + \frac{L^2}{2mr^2}$$

Plot of V_{eff} versus r is shown in the figure and from graph we draw following conclusion.

(i) For $E = (V_{eff})_{min}$ only $r = r_0$ is possible therefore orbit is circular.

(ii) For $E > (V_{eff})_{min}$, $r_{min} = r_1, r_{max} = r_2$ that is $r_1 \le r \le r_2$. Therefore orbit is bounded. Thus we see that under Hooke's force orbit is bounded for all positive energies which are greater than the minimum value of effective potential.

(iii) $E < (V_{eff})_{min}$ is not allowed



Dynamics under $F = \frac{-k}{r^4}$ force or $V(r) = \frac{-k}{3r^3}$:

$$V_{eff}(r) = \frac{-k}{3r^2} + \frac{L^2}{2mr^2}$$

Plot of $V_{eff}(r)$ versus r is shown in the figure.

From the plot we draw following conclusion.

(i) For E < 0, $r_{min} = 0$, $r_{max} = r_1$ that is $0 \le r \le r_1$. Therefore motion is bounded.

(ii) For E = 0, $0 \le r \le r_2$. Therefore motion is bounded.

(iii) For $0 < E < (V_{eff})_{max}$, $0 \le r \le r_3$ or $r_4 \le r \le \infty$. Therefore for such value of energy, motion may be bounded or unbounded.

(iv) For $E \ge (V_{eff})_{max}$, $0 \le r \le \infty$. Therefore motion is unbounded.

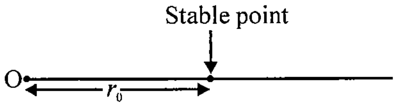
(v) This type of potential has a special feature.

- For $r > r_0$ $\frac{\partial V_{eff}}{\partial r} < 0$. Therefore $f_{eff} = \frac{-\partial V_{eff}}{\partial r} > 0$ i.e. force on the particle is away from the centre of force.
- For $r < r_0$ $\frac{\partial V_{eff}}{\partial r} > 0$. Therefore $f_{eff} = \frac{-\partial V_{eff}}{\partial r} < 0$ i.e. force on the particle is towards the centre of force.
- Due to attractive nature of force for $r < r_0$ any object coming closer to the centre of force gets captured if it is able to cross the $r = r_0$ point.

Example: A planet of mass m and angular momentum L moves in a circular orbit in a potential, $V(r) = -k/r$ where k is a positive constant find the radius of stable circular orbit. If the planet is slightly perturbed, find angular frequency of radial oscillation.

Soln. $V_{eff} = \frac{L^2}{2mr^2} + V(r) = \frac{L^2}{2mr^2} - \frac{k}{r}$

For stable point (orbit) $\left.\frac{\partial V_{eff}}{\partial r}\right|_{r=r_0} = 0$

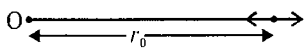


$$\therefore -\frac{L^2}{mr_0^3} + \frac{k}{r_0^2} = 0 \text{ or } r_0 = \frac{L^2}{mk}$$

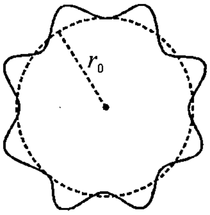
Therefore radius of stable circular orbit is L^2/mk

Angular frequency of oscillation about stable point:

$$\omega = \sqrt{\frac{\left.\frac{\partial^2 V_{eff}}{\partial r^2}\right|_{r=r_0}}{m}} = \sqrt{\frac{\left(\frac{3L^2}{mr_0^4} - \frac{2k}{r_0^3}\right)}{m}}$$

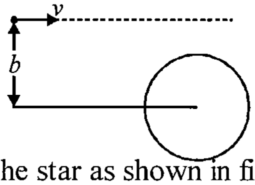


$$= \sqrt{\frac{\frac{1}{r_0^3}\left(\frac{3L^2}{m.\frac{L^2}{mk}} - 2k\right)}{m}} = \sqrt{\frac{k}{mr_0^3}} = \sqrt{\frac{k^4m^2}{L^6}} = \frac{mk^2}{L^3}$$



actual radial oscillation is as shown in figure.

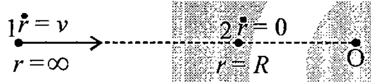
Example : A small asteroid is approaching a massive star with a speed v from very large distance, at an impact parameter 'b' as shown in figure. If the mass of the star is M and its radius is R, then what is the minimum value of b such that the asteroid will miss the star?

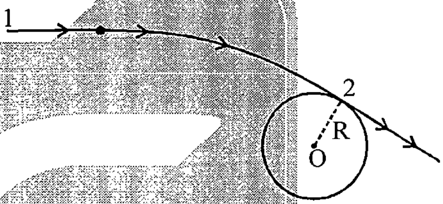


Soln. Angular momentum of asteroid about centre of star is $L = mvb$

Effective potential energy $V_{eff} = \frac{-GMm}{r} + \frac{L^2}{2mr^2} = \frac{-GMm}{r} + \frac{mv^2b^2}{2r^2}$

For minimum value of b we will have to assume that asteroid just misses the star as shown in figure. Equivalent 1-d problem is also shown in figure





Total energy is conserved in central force motion

Therefore, $E_1 = E_2$

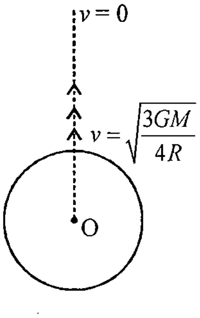
or $\left(\frac{1}{2}m\dot{r}^2 + V_{eff}\right)_{\text{at } r=\infty} = \left(\frac{1}{2}m\dot{r}^2 + V_{eff}\right)_{\text{at } r=R}$

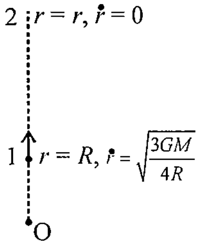
$$\frac{1}{2}mv^2 - \frac{GMm}{\infty} + \frac{mv^2b^2}{\infty} = \left(\frac{1}{2}m0^2 - \frac{GMm}{R} + \frac{mv^2b^2}{2R^2}\right)$$

$$\therefore\ v^2 + \frac{2GM}{R} = \frac{v^2b^2}{R^2} \qquad \text{or } b = R\sqrt{1 + \frac{2GM}{Rv^2}}$$

Example: A particle is thrown in radially outward direction from earth's surface with initial speed $\sqrt{\frac{3GM}{4R}}$ where M is mass of earth R is radius of earth. Find the height upto which particle goes.

Soln. Particle has been thrown radially outward





therefore $L = 0$, $V_{eff} = V(r) + \frac{L^2}{2mr^2}$

$\therefore\ V_{eff} = \frac{-GMm}{r} + 0$

from conservation of energy we get $E_1 = E_2$

or $\left(\frac{1}{2}m\dot{r}^2+V_{eff}\right)_{r=R}=\left(\frac{1}{2}m\dot{r}^2+V_{eff}\right)_{r=r}$

$$\frac{1}{2}m.\frac{3GM}{4R}-\frac{GMm}{R}=0-\frac{GMm}{r}$$

$$\frac{3}{8R}-\frac{1}{R}=-\frac{1}{r}\qquad\therefore\ r=\frac{8R}{5}$$

Therefore, maximum height from earth's surface $= r-R=\frac{3R}{5}$

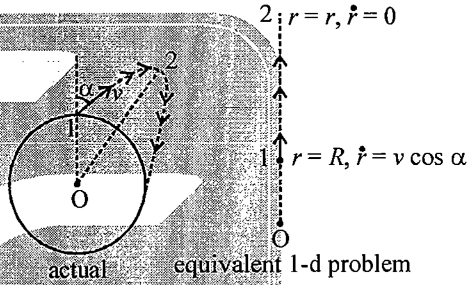
Example: A particle is thrown from the earth's surface with speed $\sqrt{\frac{GM}{R}}$ where M = mass of earth and R = radius of earth. If direction of initial velocity makes an angle α with the outward radial direction, find the maximum distance of the particle from centre of earth.

Soln. Angular momentum of the particle is

$$L=mv\sin\alpha R=m\sqrt{\frac{GM}{R}}\sin\alpha\, R$$

$$=m\sqrt{GMR}\sin\alpha$$

$$\therefore\ V_{eff}=V(r)+\frac{L^2}{2mr^2}=\frac{-GMm}{r}+\frac{GMmR}{2r^2}\sin^2\alpha$$



From conservation of energy we get $E_1=E_2$

or $\left(\frac{1}{2}m\dot{r}^2+V_{eff}\right)_{r=R}=\left(\frac{1}{2}m\dot{r}^2+V_{eff}\right)_{r=r}$ (see the figure)

$$\frac{1}{2}m\frac{GM}{R}\cos^2\alpha-\frac{GMm}{R}+\frac{GMm}{2R}\sin^2\alpha=\frac{-GMm}{r}+\frac{GMmR}{2r^2}\sin^2\alpha$$

Divide both sides by $\frac{GMm}{2R}$ to get $\cos^2\alpha-2+\sin^2\alpha=-2\frac{R}{r}+\frac{R^2}{r^2}\sin^2\alpha$

or $-1=-2\left(\frac{R}{r}\right)+\left(\frac{R}{r}\right)^2\sin^2\alpha\qquad\therefore\ \left(\frac{R}{r}\right)^2\sin^2\alpha-2\left(\frac{R}{r}\right)+1=0$

$$\therefore\ \frac{R}{r}=\frac{+2\pm\sqrt{4-4\sin^2\alpha}}{2\sin^2\alpha}\qquad\frac{R}{r}=\frac{+1\pm\cos\alpha}{\sin^2\alpha}$$

$$\therefore\ r=\frac{R\sin^2\alpha}{1\pm\cos\alpha}=\frac{R(1-\cos^2\alpha)}{(1\pm\cos\alpha)}\qquad\therefore\ r=R(1-\cos\alpha)\quad\text{or}\quad r=R(1+\cos\alpha)$$

Since r must be greater than r. Therefore, $r=R(1+\cos\alpha)$

Condition for stable orbit: If V_{eff} versus r plot has a minimum then stable circular orbit can exist.

Let the central force is $f(r)=-kr^n$

Therefore, $V_{eff} = V(r) + \frac{L^2}{2mr^2} = -\int f(r)dr + \frac{L^2}{2mr^2} = \frac{kr^{n+1}}{n+1} + \frac{L^2}{2mr^2}$

Condition for existence of minimum is

$$\left.\frac{\partial V_{eff}}{\partial r}\right|_{r=r_0} = 0 \qquad \text{... (i)}$$

and $$\left.\frac{\partial^2 V_{eff}}{\partial r^2}\right|_{r=r_0} > 0 \qquad \text{... (ii)}$$

using first condition we get $+kr_0^n - \frac{L^2}{mr_0^3} = 0$ or $r_0^{n+3} = \frac{L^2}{mk}$ ... (iii)

using second condition we get, $nkr_0^{n-1} + \frac{3L^2}{mr_0^4} > 0$ or $\frac{k}{r_0^4}\left(nr_0^{n+3} + \frac{3L^2}{mk}\right) > 0$

using equation (iii) we get, $\frac{k}{r_0^4}\left(nr_0^{n+3} + 3r_0^{n+3}\right) > 0$ or $kr_0^{n-1}(n+3) > 0$

Since k and r_0 are positive therefore we must have $n+3>0$ or $\boxed{n>-3}$

Thus, if $F = -kr^n$ then n must be greater than -3 for orbit to be stable.

Condition for closed orbit (Bertrand's theorem): If $F(r) = -kr^n$ then closed orbits are possible only for $n=1$ and $n=-2$ i.e. for $f=-kr$, Hooke's force and for $f(r) = \frac{-k}{r^2}$, inverse square force (gravitation).

Virial theorem: If $f \propto r^n$ then average value of potential energy and kinetic energy are related as

$$\langle K.E\rangle = \frac{n+1}{2}\langle P.E\rangle$$

for gravitational force $n=-2$ $\quad \therefore \langle K.E\rangle = -\frac{1}{2}\langle P.E\rangle$

for Hooke's force $n = 1$ $\quad \therefore \langle K.E\rangle = \langle P.E\rangle$

Example: A particle moves under a central potential $V(r) = \frac{-k}{r^m}$. What should be value of m for its orbit to be stable.

Soln. Corresponding is $f(r) = -\frac{\partial V}{\partial r} = \frac{-mk}{r^{m+1}} = -mk\,r^{-(m+1)}$

We know that for $f = -kr^n$ condition for stability is $n>-3$.

Therefore for orbit to be stable under given potential we must have.

$-(m+1) > -3$ or $m+1<3$ $\therefore m<2$

Example: A particle is moving under a central force $f(r) = \frac{-k}{r^{3-n}}$ for what values of n orbit of the particle can be closed.

Soln. We know that for $f(r) = -kr^m$ orbit is closed if $m = -2$ or 1.

Therefore, $-(3-n) = -2 \Rightarrow n = 1$ and $-(3-n) = 1 \Rightarrow n = 4$.

Therefore, values of n for which orbit can be closed are 1 and 4.

Keppler's law of planetary motion: Keppler had given three empirical laws of planetary motion on the basis of analysis of astronomical data collected by his master Tycho Brahe.

First Law: All the planets move in elliptical orbit around the sun with sun being at one focus of the ellipse.

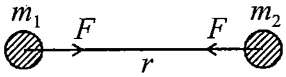
Second Law: Line joining the centre of planet to the centre of the sun sweeps equal area in equal interval of time. That is areal velocity of a planet is constant.

Third Law: Square of time of revolution is proportional to cube of semi-major axis of elliptical orbit.

i.e. $T^2 \propto a^3$

Newton's law of gravitation :

It states that every matter particle attracts every other matter particle with a force which is proportional to product of masses and inversely proportional to square of distance between them.



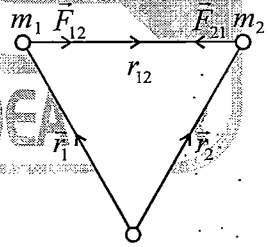
$$F = \frac{G m_1 \cdot m_2}{r^2}$$

G is universal gravitational constant. Its value is $6.67 \times 10^{-11}\ N m^2 / Kg^2$

The above expression for gravitational force is strictly valid for point masses and for two spherically symmetric masses.

If $\vec{r}_1$ and $\vec{r}_2$ be position vectors of m_1 and m_2 then $r = |\vec{r}_1 - \vec{r}_2|$ Therefore,

$$F = \frac{G m_1 m_2}{|\vec{r}_1 - \vec{r}_2|^2}$$



or, vector form of force on m_1 due to m_2 is

$$\vec{F}_{12} = \frac{G m_1 \cdot m_2}{|\vec{r}_1 - \vec{r}_2|} \hat{r}_{12} = \frac{G m_1 \cdot m_2}{|\vec{r}_1 - \vec{r}_2|^2} \cdot \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|}$$

$$\vec{F}_{12} = \frac{-G m_1 \cdot m_2}{|\vec{r}_1 - \vec{r}_2|^3} (\vec{r}_1 - \vec{r}_2)$$

and $$\vec{F}_{21} = \frac{-G m_1 \cdot m_2}{|\vec{r}_1 - \vec{r}_2|^3} (\vec{r}_2 - \vec{r}_1)$$

Note: All matter particles attracts each other due to gravitational force however if an air bubble and a particle is lying inside water they appear to repel each other.

Gravitational field: Gravitational force per unit test mass is called gravitational field $(\vec{E})$

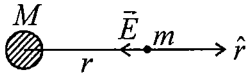
$$\boxed{\vec{E} = \frac{\vec{F}}{m}}$$ unit :N/Kg

Gravitational field due to a point mass:

Let M be mass of a point particle and at a distance r form it we have to find gravitational field.

Let us place a test mass there.

$$\therefore \qquad E = \frac{F}{m} = \frac{\frac{GMm}{r^2}}{m}$$



$$E = \frac{GM}{r^2}$$

or $$\boxed{\vec{E} = -\frac{GM}{r^2}\hat{r}}$$

$\vec{E}$ is always directed towards the particle.

From Newton's 2^{nd} law $\vec{F}/m = \vec{a}$. Therefore gravitational field is equal to acceleration of test mass due to gravitational force i.e. gravitational field = acceleration due to gravity.

Gauss law for garvitational field:

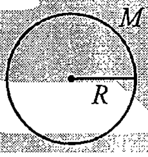
$$\boxed{\oint \vec{E}\cdot d\vec{s} = -4\pi G\,\mathrm{Men}_{\mathrm{c}}}$$

Total flux of gravitational field through a closed surface is equal to $-4\pi G$ times the total mass enclosed by the surface.

Its applicable in all cases. However it cab be used to calculate gravitational field only in the cases involving spherical, cylindrical or planar symmetry.

In gravitation we get result similar to electrostatic results. We can easily obtain gravitational results from electrostatic results and vice versa if we make following changes.

$$\frac{1}{4\pi E_0} \leftrightarrow G,\ Q \leftrightarrow M$$



Some important results in gravitational

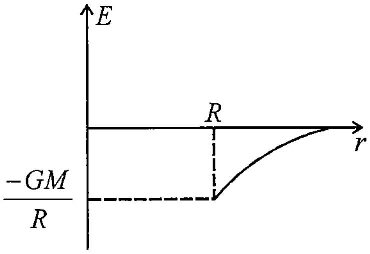
Uniform thin hollow shell (mass = M, radius = R)

Field inside is zero

i.e. $E = 0$ for $r < R$

Field outside is inversely proportional to r^2

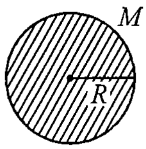
i.e. $$E = \frac{-GM}{r^2} \quad r > R$$



Uniform solid sphere (mass = M, radius = R)

Field inside is proportional to distance from center

i.e. $$E = \frac{-GM}{R^3} r \quad \text{for} \quad r < R$$



Field outside is inversely proportional to r^2

i.e. $$E = \frac{-GM}{r^2} r > R$$

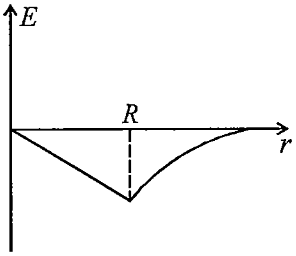
Acceleration to gravity near Earth:

Since acceleration due to garvity is equal to gravitational field, therefore considering earth as a uniform solid sphere we get

$$g = \frac{GM}{R^2} \text{ on earth's surface}$$

$$g = \frac{GM}{r^2} = \frac{GM}{(R+h)^2} \text{ at '}h\text{' height above earth's surface}$$

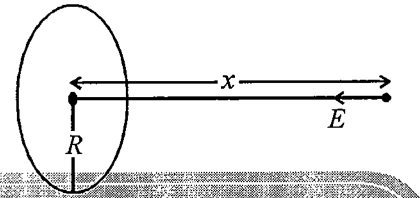
$$g = \frac{GMr}{R^3} = \frac{GM}{R^3}(R-d) \text{ at '}d\text{' depth below earth's surface}$$



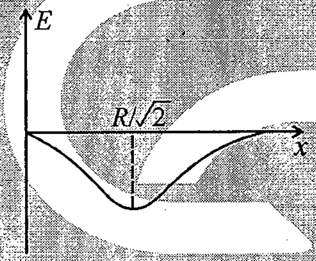
Uniform thin ring

Gravitational field at an axial point

$$E = \frac{-GMx}{(R^2+x^2)^{3/2}}$$

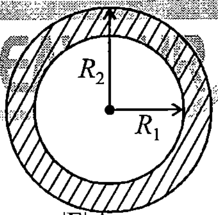


Field is maximum at a distance $x = \frac{R}{\sqrt{2}}$ from center



Example: A thick shell of inner and outer radius R_1 and R_2 has uniform mass density P. Calculate gravitational field and plot it verses radius distance.

Soln. Gravitational field due to spherically symmetric change distribution at a distance r from center is given as



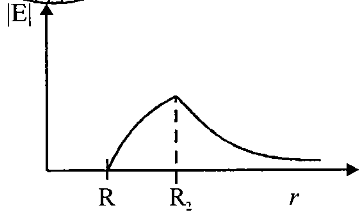
$$E = \frac{-GMen_c}{r^2}$$

Where Men_c is mass enclosed by sphere of radius r.

Therefore, for $r < R$, $Men_c = 0$

for $R_1 < r < R_2$ $Men_c = \rho\frac{4}{3}\pi(r^3 - R_1^3)$

for $r > R_2$ $Men_c = \rho\frac{4}{3}\pi(R_2^3 - R_1^3)$



Thus, $E = 0$ for $r < R_1$

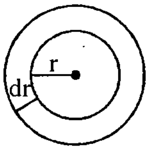
$$E = \frac{-4\pi\rho G}{3}\left(r - \frac{R_1^3}{r}\right) \text{ for } R_1 < r\ R_2$$

$$E = \frac{-4\pi\rho G}{3}\left(\frac{R_2^3 - R_1^3}{r^2}\right) \text{ for } r > R_2$$

radial variation of E is shown above.

Gravitational self energy:

Solid sphere,



$$\rho_m = \frac{M}{\frac{4}{3}\pi R^3}$$

Mass inside the sphere of radius 'r' = $\rho_m\left(\frac{4}{3}\pi r^3\right) = M\left(\frac{r^3}{R^3}\right)$

dm = Mass in the ring of rod but r and r + dr $= \rho_m 4\pi r^2 dr = \frac{3Mr^2dr}{R^3}$

Work done bringing dm from ∞ to r is:

$$dV = -\frac{Gdm}{r}\left(\frac{Mr^3}{R^3}\right) = -G\frac{Mr^3}{rR^3}\frac{3Mr^2}{R^3}dr = -\left(G\frac{3M^2}{R^6}\right)rdr$$

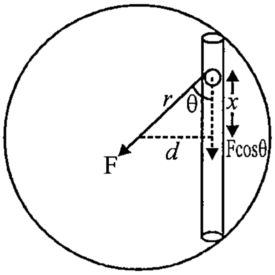
$$U = -\int_0^R \frac{G3M^2}{R^6}r^4dr = -\frac{3GM^2}{R}\frac{R^5}{5} = \boxed{-\frac{3}{5}\frac{GM^2}{R}}$$

SOLVED PROBLEMS

1. An object of mass '*m*' is dropped in a tunnel dug across a chord of earth. If *d* be the distance of the centre of chord from the earth's centre, calculate the period of oscillation of the object.

Soln. The situation is shown in the figure.

Let *r* be the distance of the object from the centre of the sphere at certain instant of time.



$\therefore$ Force on the object

$$F = \frac{GMm}{R^3}\cdot r$$

$\therefore$ Restoring force towards the centre of the tunnel = $F\cos\theta$

$$= F\cdot\frac{x}{r} \qquad = \frac{GMm}{R^3}\cdot x$$

$\therefore$ acceleration of object along the tunnel

$$\vec{a} = \frac{\vec{F}_{\text{restoring}}}{m} \qquad \vec{a} = \frac{GM}{R^3}(-\vec{x})$$

or $\vec{a} = \left(\frac{GM}{R^2}\right)\cdot\frac{1}{R}(-\vec{x})$

since $g = \frac{GM}{R^2}$ (acceleration due to gravity)

$\therefore \vec{a} = \left(\frac{g}{R}\right)(-\vec{x})$

comparing with equation of S.H.M. we get

$\omega^2 = g/R \Rightarrow T = 2\pi\sqrt{R/g}$

2. Show that the vector, $\vec{A}=\vec{p}\times\vec{L}-mk\frac{\vec{r}}{r}$

for planetary system is a constant of motion where $\vec{r}$ is position vector, $\vec{p}$ is momentum and $\vec{L}$ is angular momentum.

Soln. We have to show that $\frac{d\vec{A}}{dt}=0$

$$\frac{d\vec{A}}{dt}=\frac{d\vec{p}}{dt}\times\vec{L}+\vec{p}\times\frac{d\vec{L}}{dt}-\frac{mk}{r}\frac{d\vec{r}}{dt}-mk\vec{r}\frac{d(1/r)}{dt}$$

Here $\frac{d\vec{p}}{dt}=\vec{F}=-\frac{k}{r^2}\hat{r}$

Force is radial, therefore, $\frac{d\vec{L}}{dt}=0$

$$\therefore\ \frac{d\vec{A}}{dt}=-\frac{k}{r^2}\hat{r}\times(\vec{r}\times\vec{p})-\frac{mk}{r}\frac{d\vec{r}}{dt}+\frac{mk}{r^2}\vec{r}\frac{dr}{dt}$$

$$=\frac{-k\vec{r}}{r^2}\hat{r}\cdot\vec{p}+\frac{k}{r^2}\vec{p}(\vec{r}\cdot\hat{r})-\frac{mk}{r}\cdot\frac{d\vec{r}}{dt}+\frac{mk}{r^2}\vec{r}\frac{dr}{dt}$$

As, $\hat{r}\cdot\vec{p}=\frac{mdr}{dt}$ and $\vec{r}\cdot\hat{r}=r$ and $\vec{p}=m\frac{d\vec{r}}{dt}$

$\therefore$ first and fourth terms and second and third terms cancel out.

$\therefore\ \frac{d\vec{A}}{dt}=0\Rightarrow\vec{A}$ = constant

3. A satellite is moving around earth in a circular orbit of radius R. The time period T of the satellite is **[GP-2005]**

(a) proportional to R (b) proportional to R^2

(c) proportional to $R^{3/2}$ (d) independent of R

Soln. (c) According to Kepler's third law $T\propto R^{3/2}$

4. The escape velocity from the earth is v_0. For a planet with radius three times and density twice the earth the escape velocity will be **[GP-2007]**

(a) $v_0\sqrt{2}$ (b) $3v_0\sqrt{2}$ (c) $2v_0\sqrt{2}$ (d) $v_0\sqrt{6}$

Soln. (b)

Escape velocity from a planet of radius R and mass M is given as

$$v_{\text{escape}}=\sqrt{\frac{2GM}{R}}=\sqrt{\frac{2G\frac{4}{3}\pi R^3\rho}{R}}\qquad\therefore\ \frac{v_1}{v_2}=\sqrt{\frac{R_1^2\rho_1}{R_2^2\rho_2}}$$

If $v_1=v_0$ and $R_2=3R_1$, $\rho_2=2\rho_1$

then $\frac{v_0}{v_2}=\sqrt{\frac{1}{9\times 2}}$ $\qquad\therefore\ v_2=3v_0\sqrt{2}$

5. A satellite moves in elliptical orbit around the earth. The minimum and the maximum distances of satellite from the earth are 6.3×10^5m and 3.63×10^6m, respectively. The radius of earth is 6.37×10^6m. The ratio of speed of the satellite at apogee at its speed at perigee is **[GP-2012]**

(a) 0.3 (b) 0.35 (c) 0.6 (d) 0.7

Soln. (d) Maximum distance of satellite from earth's centre $= 3.63 \times 10^6 + 6.37 \times 10^6$

$= 10.00 \times 10^6$m

Minimum distance of satellite from earth's centre $= 0.63 \times 10^6 + 6.37 \times 10^6$

$= 7.00 \times 10^6$m

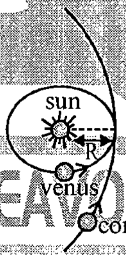
$$\therefore \quad \frac{v_{apogee}}{v_{perigee}} = \frac{\text{minimum distance}}{\text{maximum distance}} = \frac{7.00\times10^6}{10.00\times10^6} = 0.7$$

6. Suppose a comet of mass $\sqrt{20}\times10^{13}$ kg is moving in a parabolic orbit around the sun and has a possibility of colliding at its perihelion with venus. Assuming that venus is moving in a circular orbit around the sun and takes 225 days to complete a revolution around the sun. Calculate the angular momentum of comet about the sun. Neglect the interaction between the comet and venus. (Mass of sun $= 2\times10^{30}$kg, $G = \frac{20}{3}\times10^{-11}$ 5 unit, $\pi^2 = 10$) **[GP-2010]**

Soln. For venus (Mass $= M_v$)

$$\frac{GM_vM_s}{R^2} = \frac{M_v v_0^2}{R}, v_0 = \text{ orbital speed}$$

$$v_0 = \sqrt{\frac{GM_s}{R}}$$



$$\therefore \quad T = \frac{2\pi R}{v_0} = 2\pi\sqrt{\frac{R^3}{GM_s}}$$

$$\therefore \quad R^3 = \frac{T^2GM_s}{4\pi^2}$$

For comet (Mass = M_c)

Total energy = 0 ($\because$ orbit is parabolic)

$$\therefore \quad \frac{1}{2}M_cv_p^2 - \frac{GM_cM_s}{R} = 0, \qquad v_p = \text{velocity at perihelion}$$

$$\therefore \quad v_p = \sqrt{\frac{2GM_s}{R}}$$

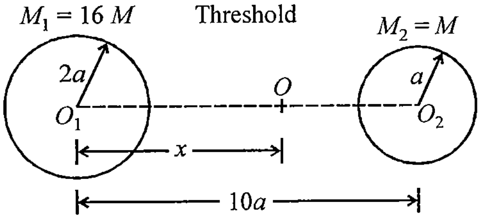
$\therefore$ Angular momentum of comet is

$$L = M_cv_pR = M_c\sqrt{\frac{2GM_s}{R}}R = \sqrt{2GM_sM_c^2R}$$

$$= \sqrt{2GM_sM_c^2}\cdot\left[\frac{T^2GM_s}{4\pi^2}\right]^{1/6}$$

7. Distance between the centers of two stars is $10a$. The masses of these stars are M and $16\,M$ and their radii a and $2a$ respectively. A body of mass m is fired straight from the surface of the larger star towards the smaller star. What should be its minimum initial speed to reach the surface of the smaller star ? Obtain the expression in terms of G, M and a.

Soln. The situation is shown in figure.



Let O be the point along O_1O_2 where gravitational intensities due to both the stars balance each other. Let $O_1O = x \cdot x$ is in a units.

Now, $-\frac{GM_1}{x^2} = -\frac{GM_2}{(10-x)^2}$ or $16(10-x)^2 = x^2$ or $x = 8a$.

Potential energy of the body on surface of larger star

$$U_1 = -\frac{Gm(16M)}{2a} - \frac{GmM}{8a} = -\frac{65GMm}{8a}$$

$$U_{\text{Threshold}} = -\frac{GMm}{2a} - \frac{G(16M)m}{8a} = -\frac{5GMm}{2a}$$

Now, $U_1 + \frac{1}{2}mv_{\min}^2 = U_{\text{Threshold}}$ or $-\frac{65GMm}{8a} + \frac{1}{2}mv_{\min}^2 = -\frac{5GMm}{2a}$

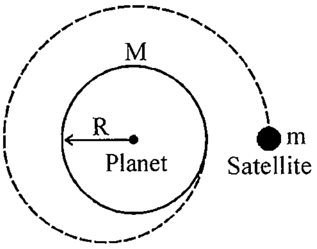
or $\frac{1}{2}mv_{\min}^2 = \frac{65GMm}{8a} - \frac{5GMm}{a} = \frac{45}{8}\frac{GMm}{a}$

$$v_{\min}^2 = \frac{45}{4}\frac{GM}{a} = \frac{9\times 5}{4}\times\frac{GM}{a}$$

$$v_{\min} = \frac{3}{2}\sqrt{\left(\frac{5GM}{a}\right)}.$$

8. An artificial satellite of mass m of a planet of mass M, revolves in a circular orbit whose radius is n times the radius R of the planet. In the process of motion, the satellite experiences a slight resistance due to cosmic dust. Assuming resistance force on satellite to depend on velocity as $F = av^2$ where a is constant. Calculate how long the satellite will stay in orbit before its falls onto the planet's surface.

Soln. The situation is shown in figure.



Let an instant, the distance of the satellite from the center of the planet be x.

Velocity of satellite, $v=\sqrt{\left(\frac{GM}{x}\right)}$ ... (i)

Given that resistive force $F=av^2$

Therefore, $F=a\left(\frac{GM}{x}\right)$ ... (ii)

During a very small time interval dt, distance travelled by the satellite is

$$v\,dt=\sqrt{\left(\frac{GM}{x}\right)}\,dt \qquad \text{... (iii)}$$

So, the workdone by satellite against resistance of cosmic dust

$$dW=F\cdot(v\,dt)=a\sqrt{\left(\frac{GM}{x}\right)}\times\sqrt{\left(\frac{GM}{x}\right)}\,dt$$

or $$dW=a\left(\frac{GM}{x}\right)^{3/2}dt \qquad \text{... (iv)}$$

Due to this work done, the energy of satellite decreases by the same amount and consequently the radius x also decreases simultaneously.

We know that $E=-\frac{1}{2}\frac{GMm}{x}$ or $dE=\frac{1}{2}\frac{GMm}{x^2}dx$ ... (v)

But $dE=-dW$

$$\therefore \quad \frac{1}{2}\frac{GmM}{x^2}dx=-a\left(\frac{GM}{x}\right)^{3/2}dt$$

$$dt=-\frac{m}{2a\sqrt{GM}}\times\frac{dx}{\sqrt{x}} \qquad \text{... (vi)}$$

Integrating equation (vi), we get

$$\int_{t=0}^{t=t}dt=-\frac{m}{2a\sqrt{GM}}\int_{x=nR}^{x=R}\frac{dx}{\sqrt{x}} \qquad (\because \text{ when } t=0,\ x=nR \text{ and at } t=t,\ x=R)$$

Solving we get, $t=\frac{m\sqrt{R}\left(\sqrt{n}-1\right)}{a\sqrt{(GM)}}$.

Chapter 9

Rigid Body Dynamics

A rigid body is collection of large number of particles. Distance between each pair of particle remains constant. i.e. $|\vec{r}_i - \vec{r}_j|$ = constant. Rotational dynamics of rigid body involves basically dynamics of large number of particles. Therefore it is not possible to describe the dynamics in terms of movement of each particle. However, by introducing concept of moment of inertia or inertia tensor rotational dynamics of a rigid body can be systematically discussed. Moment of inertia or inertia tensor collectively takes into account the movement of each particle.

Moment of inertia: Moment of inertia of a system of particles about an axis of rotation is defined as

$I = \sum_i m_i r_i^2$ (for discrete case)

where m_i = mass of each point particle

r_i = perpendicular distance from the axis of rotation

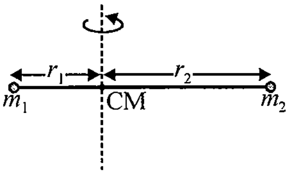
or $I = \int r_\perp^2 dm$ (for continuous case)

where dm = elementary mass

$r_\perp$ = perpendicular distance of elementary mass from the axis of rotation.

Example: Two particles of masses m_1 and m_2 are kept 'r' distance apart. Calculate moment of inertial of the system about an axis passing through the centre of mass and perpendicular to the line joining the masses.

Soln. Let $m_1 > m_2$, therefore centre of mass of the system lies nearer to m_1.

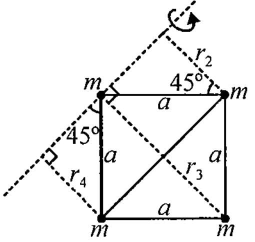


$$\therefore \quad I = \sum m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2$$

Since, $r_1 = \dfrac{m_2 r}{m_1 + m_2}$ and $r_2 = \dfrac{m_1 r}{m_1 + m_2}$

$$\therefore \; I = m.\left(\frac{m_2 r}{m_1 + m_2}\right)^2 + m_2\left(\frac{mr}{m_1 + m_2}\right)^2 = \frac{m_1.m_2}{m_1 + m_2} r^2$$

Example: Four particles each of mass 'm' are kept at the corners of a square of side 'a'. Calculate moment of inertia about an axis passing through one corner. The axis is parallel to diagonal of square.



Soln. $I = \sum_i m_i r_i^2 = m(r_1^2 + r_2^2 + r_3^2 + r_4^2)$

from figure shown, $r_1 = 0, r_2 = a\cos 45°, r_3 = a\sqrt{2}, r_4 = a\sin 45°$

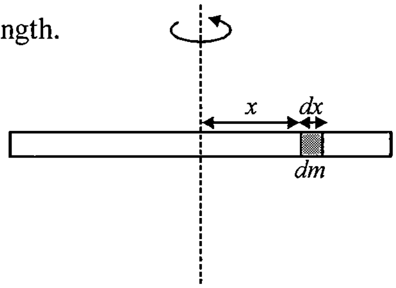
$$\therefore \; I = m\left[0 + \left(\frac{a}{\sqrt{2}}\right)^2 + (a\sqrt{2})^2 + \left(\frac{a}{\sqrt{2}}\right)^2\right] = 3ma^2$$

Moment of inertia of a uniform thin rod: Let M = mass of rod and L = length of the rod.

Moment of inertia of a continuous mass distribution is defined as $I=\int r_{\perp}^{2}dm$

Let m take an element of width dx at a distance x from the axis of rotation. Therefore $dm=\frac{M}{L}dx$

Case 1: About an axis passing through its centre and $\perp$ to its length.



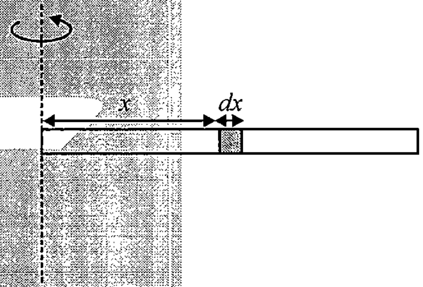
Here, $r_{\perp}=x$

$$\therefore \quad I=\int_{-L/2}^{+L/2}x^{2}\frac{M}{L}dx=\frac{M}{L}\int_{-L/2}^{L/2}x^{2}dx$$

$$=\frac{ML^{2}}{12} \quad \therefore \boxed{I=\frac{ML^{2}}{12}}$$

Case 2: About an axis through its end and perpendicular to its length

Here, $r_{\perp}=x$

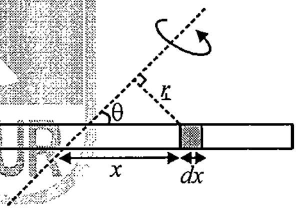


$$\therefore \quad I=\int_{0}^{L}x^{2}\frac{M}{L}dx$$

$$=\frac{ML^{2}}{3} \quad \therefore \boxed{I=\frac{ML^{2}}{3}}$$

Case 3: About an axis passing through its centre and making an angle θ with its length.

Here, $r_{\perp}=x\sin\theta$

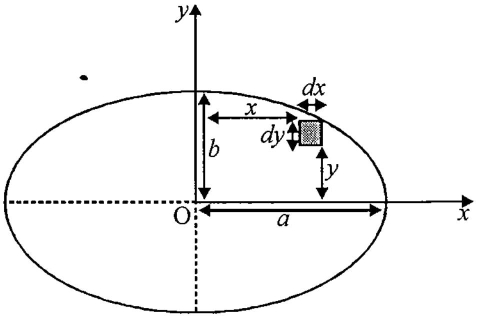


$$\therefore \quad I=\int_{-L/2}^{L/2}x^{2}\sin^{2}\theta.\frac{M}{L}dx=\frac{M}{L}\sin^{2}\theta\int_{-L/2}^{L/2}x^{2}dx$$

$$=\frac{ML^{2}}{12}\sin^{2}\theta \quad \therefore \quad \boxed{I=\frac{ML^{2}}{12}\sin^{2}\theta}$$

Moment of inertia of uniform elliptical thin plate: Let M = mass of plate and a, b be the semi-major and minor axes respectively.

If centre of ellipse is take as origin and major and minor axes as x and y axes respectively then equation of ellipse is given as $\frac{x^{2}}{a^{2}}+\frac{y^{2}}{b^{2}}=1$



Let us take an element at point (x, y). Let its area be $dx.dy$

therefore, $dm=\frac{M}{A}.dx.dy=\frac{M}{\pi ab}dx.dy$

where $A=\pi ab$ = area of ellipse.

Case 1: About the major axis

Here, $r_{\perp}=y$

$$\therefore \quad I=\int r_\perp^2 dm=\iint y^2 .\frac{M}{\pi ab}dxdy \;=\frac{M}{\pi ab}\iint y^2 dx.dy=\frac{M}{\pi ab}\int dx\int y^2 dy$$

From equation of ellipse $y=\pm b\sqrt{1-\frac{x^2}{a^2}}$

$$\therefore I=\frac{M}{\pi ab}\int_{-a}^{a}dx\int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}}y^2dy=\frac{2M}{3\pi ab}\int_{-a}^{a}dx\left(b\sqrt{1-\frac{x^2}{a^2}}\right)^3=\frac{2Mb^2}{3\pi a}\int_{-a}^{a}\left(1-\frac{x^2}{a^2}\right)^{3/2}dx=\frac{4Mb^2}{3\pi a}\int_{0}^{a}\left(1-\frac{x^2}{a^2}\right)^{3/2}dx$$

The value of integral is $\frac{3\pi}{16}a$

Therefore, $I=\frac{4Mb^2}{3\pi a}.\frac{3\pi}{16}a=\frac{Mb^2}{4}$ or $\boxed{I=\frac{Mb^2}{4}}$ about major axis

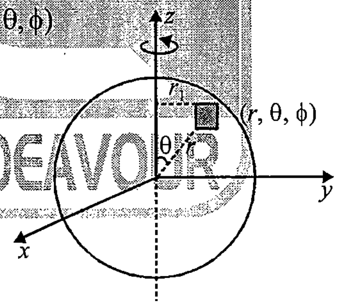
Case 2: About the minor axis

Here $r_\perp = x \quad \therefore \; I=\int r_\perp^2 d_m=\iint x^2\frac{M}{\pi ab}dx\,dy$

A similar calculation gives $\boxed{I=\frac{Ma^2}{4}}$ about minor axis

Note: It is easier to use plane polar, spherical polar and cylindrical coordinates for objects having circular symmetry, spherical symmetry and cylindrical symmetry. Therefore depending on the symmetry we will use convenient coordinate system.

Moment of inertia of a uniform solid sphere about its diameter: Let M = mass of the sphere and R = radius of sphere. Let us take an element at a point (r,θ,ϕ)



Volume of element (dv) = $r^2\sin\theta d\theta d\phi dr$

Therefore, $dm=\frac{M}{V}dV \;=\frac{M}{\frac{4}{3}\pi R^3}r^2dr\sin\theta d\theta d\phi$

Here $r_\perp = r\sin\theta$ (as shown in figure)

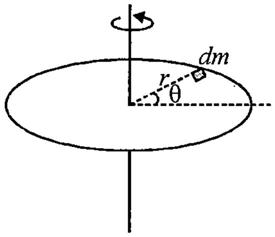
$$\therefore \quad I=\int r_\perp^2+dm=\iiint r^2\sin^2\theta.\frac{M}{\frac{4}{3}\pi R^3}r^2dr\sin\theta d\theta d\phi \;=\frac{3M}{4\pi R^3}\int_0^R r^4dr\int_0^\pi \sin^3\theta d\theta\int_0^{2\pi}d\phi$$

$$=\frac{3M}{4\pi R^3}.\frac{R^5}{5}.\frac{4}{3}.2\pi=\frac{2}{5}MR^2 \qquad \therefore \boxed{I=\frac{2}{5}MR^2}$$

Moment of inertia of uniform circular disc: Let M = mass of disc, R = radius of the disc, let us take an element at (r,θ). Area of element $(dA)=rdrd\theta$

Mass of element (dm) = $\frac{M}{A}.dA=\frac{M}{\pi R^2}rdrd\theta$

here $r_{\perp} = r$

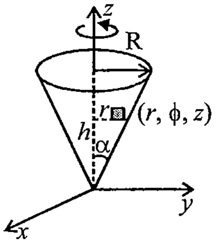


Therefore, $I = \int r_{\perp}^2 dm = \iint r^2 . \frac{M}{\pi R^2} r dr d\theta = \frac{M}{\pi R^2} \int_0^R r^3 dr \int_0^{2\pi} d\theta$

$= \frac{M}{\pi R^2} . \frac{R^4}{4} . 2\pi \qquad \therefore \boxed{I = \frac{MR^2}{2}}$

Moment of inertia of uniform right circular cone about its axis: Let M = mass, R = base radius, h = height of the cone. Let us take an element at (r, ϕ, z) in cylindrical coordinate system.

Volume of element $(dV) = r dr d\phi dz$



Mass of the element $(dm) = \frac{M}{\frac{1}{3}\pi R^2 h} . r dr dz d\phi$

Here $r_{\perp} = r$

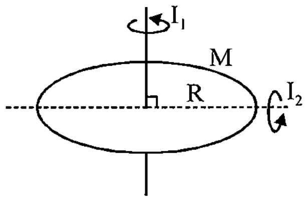
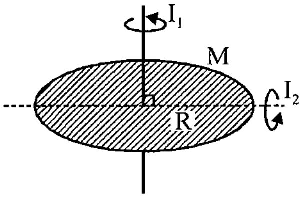
Therefore, $I = \int r_{\perp}^2 dm = \iiint r^2 . \frac{M}{\frac{1}{3}\pi R^2 h} r dr dz \phi = \frac{3M}{\pi R^2 h} \int dz \int r^3 dr \int_0^{2\pi} d\phi$

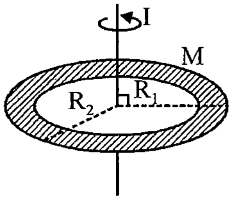
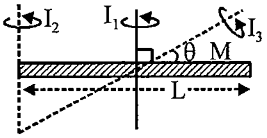
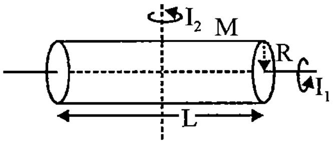
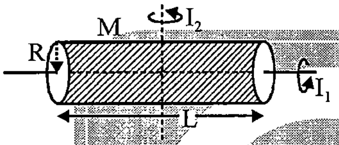
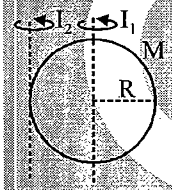
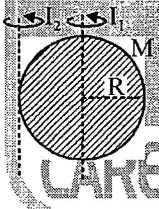
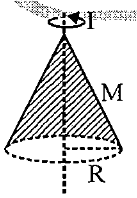
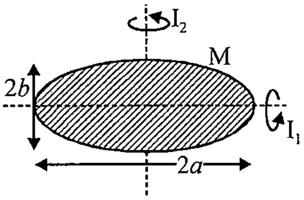
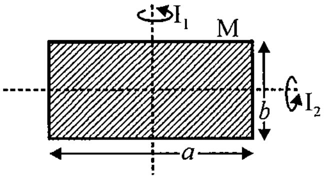
At the slanting surface of cylinder $\tan\alpha = \frac{r}{z}$ or $r = z \tan\alpha$

$\therefore \; I = \frac{3M}{\pi R^2 h} \int_0^h dz \int_0^{z\tan\alpha} r^3 dr \int_0^{2\pi} d\phi = \frac{3M}{\pi R^2 h} . \int_0^h dz . \frac{z^4 \tan^4\alpha}{4} . 2\pi$

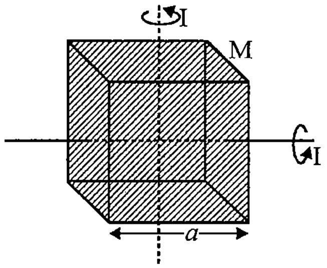
$= \frac{3M}{2R^2 h} \tan^4\alpha \int_0^h z^4 dz = \frac{3M}{2R^2 h} . \frac{R^4}{h^4} . \frac{h^5}{5} = \frac{3}{10} MR^2 \qquad \therefore \boxed{I = \frac{3}{10} MR^2}$

List of moment of inertia of some objects:

	Object	**Moment of inertia**
Thin circular ring		$I_1 = MR^2, I_2 = \frac{MR^2}{2}$
Thin circular disc		$I_1 = \frac{MR^2}{2}, I_2 = \frac{MR^2}{4}$

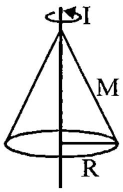
Annular ring		$I=\frac{1}{2}M(R_1^2+R_2^2)$
Thin rod		$I_1\frac{ML^2}{12},I_2=\frac{ML^2}{3},I_3=\frac{ML^2}{12}\sin^2\theta$
Thin hollow cylinder		$I_1=MR^2,I_2=M\left(\frac{R^2}{2}+\frac{L^2}{12}\right)$
Solid cylinder		$I_1=\frac{MR^2}{2},I_2=M\left(\frac{R^2}{4}+\frac{L^2}{12}\right)$
Thin hollow sphere		$I_1=\frac{2}{3}MR^2,I_2=\frac{5}{3}MR^2$
Solid sphere		$I_1=\frac{2}{5}MR^2,I_2=\frac{7}{5}MR^2$
Right circular solid cone		$I=\frac{3}{10}MR^2$
Elliptical plate		$I_1=\frac{Mb^2}{4},I_2=\frac{Ma^2}{4}$
Rectangular plate		$I_1=\frac{Ma^2}{12},I_2=\frac{Mb^2}{12}$

Solid cube



$I = \dfrac{Ma^2}{6}$

Right circular hollow cone



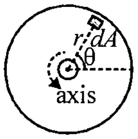
$I = \dfrac{MR^2}{2}$

Example : A thin circular disc of mass 'M' and radius 'R' has non-uniform mass density which varies linearly with distance from centre. Calculate moment of inertia of the disc about its axis.

Soln. Let us taken an element at (r, θ)

Area of element $(dA) = r dr d\theta$

Mass of element $(dm) = \sigma dA$



where σ = surface mass density. According to question $\sigma \alpha r$.

Therefore, $\sigma = kr$, k = constant

$\therefore \quad dm = \sigma dA = kr^2 dr d\theta$

$$\therefore \quad M = \iint kr^2 dr d\theta = k\int_0^R r^2 dr \int_0^{2\pi} d\theta = \frac{2\pi k R^3}{3} \quad \text{or} \quad k = \frac{3M}{2\pi R^3} \qquad \text{... (a)}$$

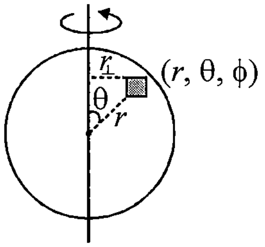
Moment of inertia about the axis of disc (axis passing through centre and perpendicular to its plane)

$$I = \int r_\perp^2 dm = \iint r^2 kr^2 dr d\theta = k\int_0^R r^4 dr \int_0^{2\pi} d\theta = 2\pi k \frac{R^5}{5}$$

Using value of k from (a) we get, $I = \dfrac{3}{5}MR^2$

Example: A solid sphere of mass M and radius R has non-uniform mass density which varies linearly with distance from centre. Calculate moment of inertia of the sphere about its diameter.

Soln. According to question, $\rho \alpha r$ or $\rho = kr$



Let us take an element at (r, θ, ϕ)

Mass of the element $dm = \rho dv = kr.r^2 dr \sin\theta d\theta d\phi$

$$\therefore M = \iiint kr^3 dr \sin\theta d\theta d\phi = k\int_0^R r^3 dr \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi = \pi k R^4$$

$$\therefore \quad k = \frac{M}{\pi R^4} \qquad \text{... (a)}$$

Moment of inertia about its diameter

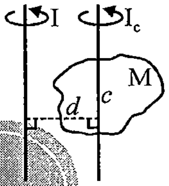
Here $r_\perp = r\sin\theta$

$$I = \int r_\perp^2 dm = \iiint r^2 \sin^2\theta . kr^3 dr \sin\theta . d\theta d\phi$$

$$= k\int_0^R r^5 dr \int_0^\pi \sin^3\theta d\theta . \int_0^{2\pi} d\phi \;=\; k\frac{R^6}{6}.\frac{4}{3}.2\pi \;=\; \frac{4\pi k R^6}{9}$$

using value of k from (a) we get, $I = \dfrac{4MR^2}{9}$

Parallel axes theorem: It states that if I_c be moment of inertia of an object about an axis passing through its centre of mass then moment of inertia about any other parallel axis which is at a separation 'd' is given as



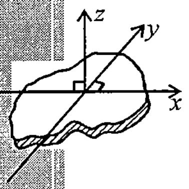
$$I = I_c + Md^2$$

where M = mass of object

This theorem is valid for all types of objects ($1d$, $2d$, $3d$)

Perpendicular axes theorem: This theorem is applicable for $1d$ or $2d$ objects. It states that if x and y axes lie in the plane of object and z axis is perpendicular to its plane then

$$I_{zz} = I_{xx} + I_{yy}$$

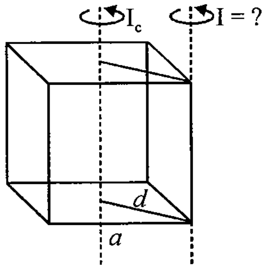


The three axes must intersect at common point, however there is no condition as such that the axes should pass through centre of mass. Therefore the axes may or may not pass through centre of mass.

Example: A uniform solid cube has mass M and side 'a'. Calculate moment of inertia of the cube about an axis which coincides with one of its sides.

Soln. The axis given in question and a parallel axis through centre of mass are shown in the figure.

separation between two axes is $d = \dfrac{a}{\sqrt{2}}$



we know that moment of inertia of cube about an axis through its centre and perpendicular to its face is $I_c = \dfrac{Ma^2}{6}$

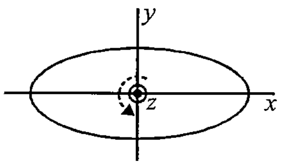
Therefore using parallel axes theorem we get

$$I = I_c + Md^2 = \frac{Ma^2}{6} + \frac{Ma^2}{2} = \frac{2}{3}Ma^2$$

Example: An elliptical plate has mass M and semi-major and minor axes 'a' and 'b' respectively. Calculate moment of inertia of the plate about an axis perpendicular to its plane and passing through its centre.

Soln. Let us take semi-major and minor axes as x and y axes and the axis given in the question as z axis.

$$\therefore\quad I_{xx}=\frac{Mb^2}{4},I_{yy}=\frac{Ma^2}{4}$$



From perpendicular axes theorem

$$I_{zz}=I_{xx}+I_{yy}=\frac{M}{4}(a^2+b^2)$$

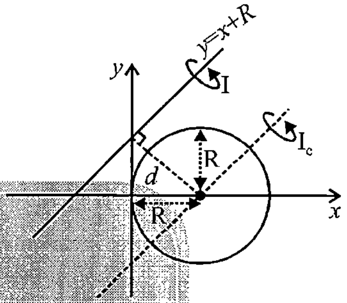
Example: A thin circular ring of mass M and radius R lies in *x*-*y* plane. Equation of the ring is $(x-R)^2+y^2=R^2$. Calculate moment of inertia of the ring about a straight line $y=x+R$.

Soln. The ring and the straight line are shown in the figure. To calculate moment of inertia about the given axis we draw a parallel line through centre of ring and apply parallel axes theorem.

$$\therefore I=I_c+Md^2$$

$$=\frac{MR^2}{2}+M(R\sqrt{2})^2$$

$$=MR^2+2MR^2=3MR^2$$



Example: Moment of inertia of a thin circular ring of unit mass lying in *x*-*y* plane is 2 unit and 3 unit respectively about the straight lines $y=x+3$ and $y=-x+2$. If centre of the ring lies at (1, 1) calculate moment of inertia of the ring about the axis perpendicular to its plane and passing through its centre.

Soln. Point of intersection of the two straight lines is $P\left(-\frac{1}{2},\frac{5}{2}\right)$. Distance of point of intersection from centre of the circle is $d=\sqrt{\left(1+\frac{1}{2}\right)^2+\left(1-\frac{5}{2}\right)^2}=\frac{3}{\sqrt{2}}$. Moment of inertia of the ring about an axis passing through point P and perpendicular to *x*-*y* plane is (use perpendicular axes theorem)

I = 2 + 3 = 5

The perpendicular axis through P is parallel to perpendicular axis through the centre of the ring. Therefore using parallel axis theorem we get

$$I=I_c+Md^2$$

$$\therefore\quad 5=I_c+1\times\frac{9}{2}\quad\text{or}\quad I_c=0.5$$

Dynamics of rigid body: A rigid body can have following types of dynamics

(i) Pure translation: If all the points of an object undergo equal displacement in equal interval of time then dynamics is called pure translation. Pure translational motion of a rigid body is equivalent to translation of centre of mass of the object. Therefore,

Kinetic energy = $\frac{1}{2}MV_c^2$, momentum = MV_c

Equation of motion $\vec{F}=M\vec{a}_c$

where v_c and a_c are velocity and acceleration of centre of mass.

(ii) Pure rotation: In this type of motion at least one point of rigid body remains stationary. Rotational motion of a rigid body cannot be described in terms of its outer of mass. Instead dynamics of each particle has to be considered. That is why concept of moment of inertia comes into play in rotational motion.

In pure rotation distance of each particle of rigid body remains constant from the point or axis about which rotation takes place.

In pure rotational motion we use following terms for a rigid body.

Kinetic energy $= \frac{1}{2}I\omega^2$, where I = moment of inertia about given axis

Angular momentum about the axis $L = I\omega$

Equation of motion $\tau = I\alpha$ (about the axis)

where $\alpha = \frac{d\omega}{dt}$, α = angular acceleration, ω = angular velocity

and $\omega = \frac{d\theta}{dt}$, θ = angular displacement

If α = constant then $\omega = \omega_0 + \alpha t,\ \omega^2 = \omega_0^2 + 2\alpha\theta,\ \theta = \omega_0 t + \frac{1}{2}\alpha t^2$

where ω_0 = initial angular velocity, ω = final angular velocity

(iii) General motion: In this case object undergoes translation as well rotation. There is no point which remains stationary for all the time. And different points of the object undergo different displacement. According to Chasle's theorem a general motion is equivalent to translation of centre of mass plus pure rotation about centre of mass i.e.

General motion = Pure translation of centre of mass + Pure rotation about centre of mass

Therefore in general motion we use following terms for a rigid body.

Kinetic energy $= \frac{1}{2}MV_c^2 + \frac{1}{2}I_c\omega^2$

Equation of motion $F = Ma_c$, $\tau = I_c\alpha_c$

Angular momentum $\vec{L} = I_c\vec{\omega} + M\vec{r}\times\vec{v}_c$. If the rigid body is round in shape and it is moving on a surface then about any point on the path of the object angular momentum is written as $L = I_c\omega + Mv_cR$ where R is radius of object.

(iv) Pure rolling: It is a special kind of general motion of round objects in which following relations are satisfied.

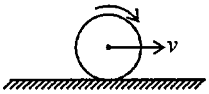
$$\left.\begin{aligned} v_c &= \omega R \\ a_c &= \alpha R \end{aligned}\right\} \text{condition for rolling}$$

In most cases friction is necessary for rolling motion. However there is no loss of energy against friction in rolling motion. Therefore total mechanical energy remains conserved in rolling motion all though friction is present.

Example: A hollow sphere of mass M and radius R rolls on a horizontal surface such that centre of the sphere moves with speed v. Calculate kinetic energy and angular momentum of the sphere.

Soln. Given, $v_c = v$ and sphere is rolling

Therefore we use condition of rolling $(v_c = \omega R)$



To find its angular velocity

$$\therefore \omega = \frac{v_c}{R} = \frac{v}{R}$$

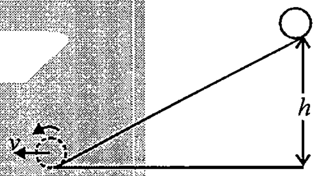
$$\text{Kinetic energy} = \frac{1}{2}Mv_c^2 + \frac{1}{2}I_c\omega^2 = \frac{1}{2}Mv^2 + \frac{1}{2}.\frac{2}{3}MR^2.\frac{v^2}{R^2} = \frac{5}{6}Mv^2$$

$$\text{Angular momentum} = I_c\omega + Mv_cR = \frac{2}{3}MR^2.\frac{v}{R} + MvR = \frac{5}{3}MvR$$

Example: A hollow sphere rolls down an inclined plane from a height 'h' what is speed of the sphere when it reaches the bottom of inclined plane.

Soln. There must be friction on the plane. However total mechanical energy is conserved in rolling motion. Therefore, loss in potential energy = gain in kinetic energy

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

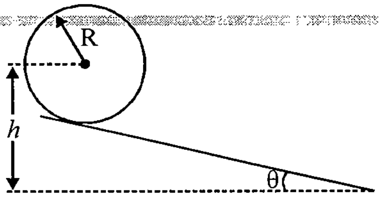


for rolling $\omega = v/r$, for hollow sphere $I = \frac{2}{3}mr^2$

$$\therefore\ mgh = \frac{1}{2}mv^2 + \frac{1}{2}.\frac{2}{3}mr^2.\frac{v^2}{r^2} \Rightarrow mgh = \frac{5}{6}mv^2$$

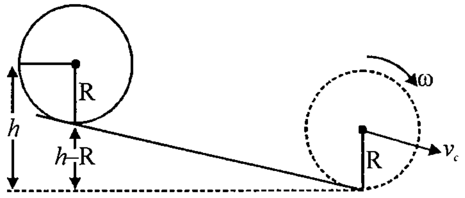
$$\therefore v = \sqrt{\frac{6gh}{5}}$$

Example: A ring of mass m and radius R rolls (without slipping) down an inclined plane starting from rest. If the centre of the ring is initially at a height h, the angular velocity when the ring reaches the base is



Soln. Height descended by centre of mass of the ring is $h - R$.
From conservation of energy we get,
loss in potential energy = gain in K.E.

$$mg(h-R) = \frac{1}{2}mv_c^2 + \frac{1}{2}I\omega^2$$



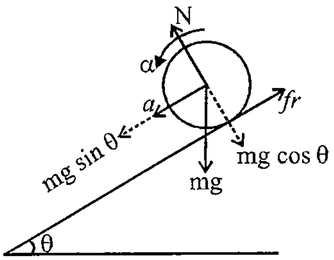
for rolling $v_c = \omega R, I = mR^2$

$$\therefore\ mg(h-R) = \frac{1}{2}m\omega^2R^2 + \frac{1}{2}mR^2\omega^2 \Rightarrow mg(h-R) = m\omega^2R^2, \quad \therefore \omega = \sqrt{\frac{g(h-R)}{R^2}}$$

Acceleration of rolling object on inclined plane: Let us consider an object of mass M, radius R and moment of inertia I about the axis about which it rotates while rolling.

Let a and α be linear and angular acceleration of the object. Therefore for rolling

$a = \alpha R$ ... (i)



Equation of translational motion $F = ma$

or $mg\sin\theta - f_r = ma$ ... (ii)

Equation of rotational motion $\tau = I\alpha$

or $f_r R = I.\alpha$

or $f_r = \dfrac{Ia}{R^2}$ ... (iii)

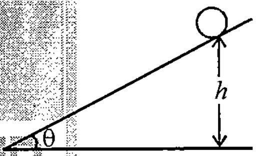
(ii) + (iii) gives $mg\sin\theta = a\left(m + \dfrac{I}{R^2}\right)$ or $\boxed{a = \dfrac{g\sin\theta}{1+\dfrac{I}{mR^2}}}$

from (iii) we get $\boxed{f_r = \dfrac{I}{R^2}.\dfrac{g\sin\theta}{1+\dfrac{I}{mR^2}}}$

friction force in rolling motion is static in nature. Therefore if μ be coefficient of static friction then

$f_r \le \mu N$ or $\dfrac{I}{R^2}.\dfrac{g\sin\theta}{1+\dfrac{I}{mR^2}} \le \mu.mg\cos\theta$ or $\boxed{\mu \ge \dfrac{\tan\theta}{1+\dfrac{mR^2}{I}}}$

Example: A solid sphere rolls down on inclined plane from a height 'h' as shown in figure. Calculate time taken by sphere to reach the bottom. Also calculate the minimum value of coefficient of friction necessary for rolling.



Soln. Moment of inertia of solid sphere $I = \dfrac{2}{5}mr^2$

acceleration of sphere $a = \dfrac{g\sin\theta}{1+\dfrac{I}{mr^2}} = \dfrac{g\sin\theta}{1+\dfrac{\frac{2}{5}mr^2}{mr^2}} = \dfrac{5g\sin\theta}{7}$

distance covered along the inclined plane $= \dfrac{h}{\sin\theta}$

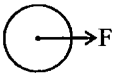
$\therefore$ If t be time taken then, $S = \dfrac{1}{2}at^2$

or $\dfrac{h}{\sin\theta} = \dfrac{1}{2}.\dfrac{5g\sin\theta}{7}.t^2 \quad \therefore t = \sqrt{\dfrac{14h}{5g\sin^2\theta}}$

minimum value of coefficient of friction necessary to support rolling is

$$\mu = \frac{\tan\theta}{1+\frac{mr^2}{I}} = \frac{\tan\theta}{1+\frac{mr^2}{\frac{2}{5}mr^2}} = \frac{2}{7}\tan\theta$$

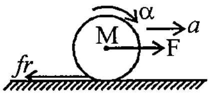
Example : A constant force F is applied horizontally at the centre of a circular disc of radius R and mass M as shown in the figure. If the disc rolls on the surface without sliding then. Calculate its acceleration.



Soln. There must be friction between the disc and surface to support rolling. Let *fr* be force of friction. Therefore, translational equation of motion. $\vec{F}_{net} = m\vec{a}$

$$f - f_r = Ma \qquad \text{... (a)}$$

Rotational equation of motion $\tau = I.\alpha$



$$f_r R = \frac{2}{3}MR^2.\alpha$$

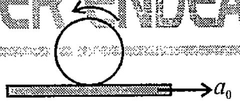
for pure rolling $a = \alpha R$

$$\therefore \; f_r R = \frac{2}{3}MR^2.\frac{a}{R}$$

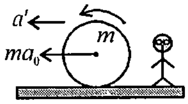
$$\text{or} \quad f_r = \frac{2}{3}Ma \qquad \text{... (b)}$$

(a) + (b) gives, $F = \frac{5}{3}Ma \quad \therefore \; a = \frac{3F}{5M}$

Example: A hollow sphere is kept on a plank which moves horizontally as shown in the figure. The sphere rolls on the plank. Calculate acceleration of the sphere with respect to ground.



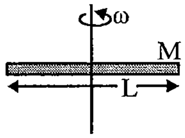
Soln. Let us first find acceleration of sphere with respect to the plank. For this we observe motion of sphere from the plank's frame. Therefore a pseudo force has to be applied on the sphere. Now this is exactly the same problem as the previous one with applied force being equal to ma_0 i.e. $F = ma_0$. Therefore acceleration of sphere with respect to the plank will be

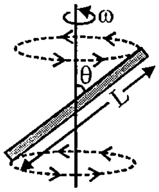


$$a' = \frac{3F}{5m} = \frac{3ma_0}{5m} = \frac{3}{5}a_0$$

Therefore acceleration of sphere with respect to the ground is $a = a_0 - \frac{3}{5}a_0 = \frac{2}{5}a_0$

Rotational about a fixed point: The formulae for angular momentum and kinetic energy $L = I\omega$ and kinetic energy = $\frac{1}{2}I\omega^2$ can be used if rotation is taking place about a fixed axis. In fact in some cases $L = I\omega$ represents angular momentum about the axis only, it is not equal to total angular momentum. One such situation is shown in figure below. In the figure we have shown two different rotational motion of a thin rod about a fixed axis





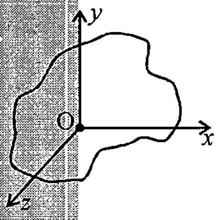
Angular momentum about the axis =

$$I\omega = \frac{ML^2}{12}\omega = \text{Total angular momentum}$$

Angular momentum about the axis

$$= I\omega = \frac{ML^2}{12}\sin^2\theta\omega \neq \text{total angular momentum}$$

However, if rotation is taking place about a fixed point (not about fixed axis) then above mentioned formulae for angular momentum and kinetic energy can not be used. Let O be a fixed point about which a rigid body is rotating. Let us take a set of three perpendicular axes through O as shown in the figure. The axes taken are supposed to be fixed with the rigid body. Therefore w.r.t. to lab the orientation of axes keep changing. These are called body axes and are taken as instantaneous axes of rotation. If ω_x, ω_y and ω_z be components of angular velocity and L_x, L_y and L_z components of angular momentum at some instant of time then value of angular momentum is calculated from following matrix relation.



$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

or $\boxed{\bar{L} = \overline{II}.\vec{\omega}}$

and rotational kinetic energy is written as

$$K = \frac{1}{2}(\omega_x \omega_y \omega_z) \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \text{ or } \boxed{K = \frac{1}{2}\vec{\omega}.\overleftrightarrow{II}.\vec{\omega}} \therefore \boxed{K = \frac{1}{2}\vec{\omega}.\vec{L}}$$

$$\text{where } \overleftrightarrow{II} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} = \text{inertia tensor}$$

The element of inertia tensor are written as

	Elements	Discrete case	Continuous case
Moment of inertia	$I_{xx} =$	$\sum_i m_i(y_i^2 + z_i^2)$	$\int (y^2 + z^2)dm$
	$I_{yy} =$	$\sum_i m_i(x_i^2 + z_i^2)$	$\int (x^2 + z^2)dm$
	$I_{zz} =$	$\sum_i m_i(x_i^2 + y_i^2)$	$\int (x^2 + y^2)dm$

	Elements	Discrete case	Continuous case
Product of inertia	$I_{xy} = I_{yx} =$	$-\sum_i m_i x_i y_i$	$-\int xydm$
	$I_{xz} = I_{zx} =$	$-\sum_i m_i x_i z_i$	$-\int xzdm$
	$I_{yz} = I_{zy} =$	$-\sum_i m_i y_i z_i$	$-\int yzdm$

Principal axes: A set of three axes so chosen that all the product of inertia becomes zero, are called principal axes. Therefore inertia tensor becomes diagonal if axes chosen happen to be principal axes.

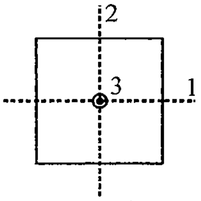
Thus, if x, y, z are principal axes then $I_{ij} = 0$ if $i \neq j$

Therefore, $\overleftrightarrow{I} = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}$

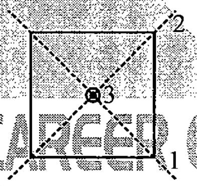
In this case I_{xx}, I_{yy}, I_{zz} are called principal moment of inertia. Thus principal moment of inertia are moment of inertia about principal axes. Principal axes are usually denoted by numbers 1, 2 and 3. Accordingly principal moment of inertia are denoted as I_1, I_2 and I_3.

- Symmetric axes of an object are always principal axes.
- Principal axes may or may not be symmetric axes.
- Through each point of an object a set of three principal axes can be defined. Therefore an object can have infinite number of principal axes.

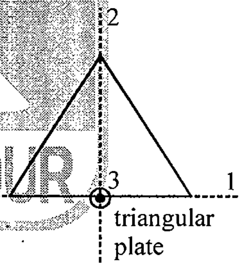
In the figure below principal axes of some objects are shown



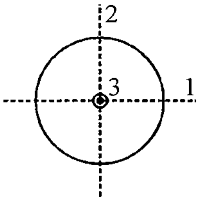
Square plate



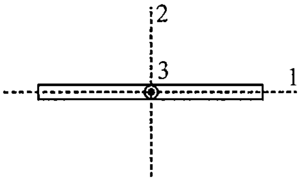
Square plate



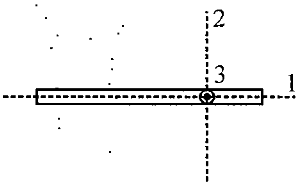
triangular plate



Sphere



Thin rod

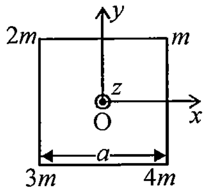


Thin rod

Finding principal moment of inertia and principal axes: Eigen values of inertia tensor are equal to principal moment of inertia and eigen vectors of inertia tensor give orientation of principal axes.

Therefore, to find principal axes through a point of an asymmetric object we can choose any three convenient axes and write inertia tensor and to get principal axes we need to calculate eigen vectors.

Example: Four point masses $m, 2m, 3m$ and $4m$ are placed at the corners of a square of side 'a' as shown in the figure. Write inertia tensor for the axes shown in figure and find the principal axes passing through point O.



Soln. Coordinates of masses are, $m\left(\frac{a}{2},\frac{a}{2},0\right)$, $2m\left(\frac{-a}{2},\frac{a}{2},0\right)$, $3m\left(\frac{-a}{2},\frac{-a}{2},0\right)$, $4m\left(\frac{a}{2},\frac{-a}{2},0\right)$

therefore, $I_{xx}=\sum_i m_i(y_i^2+z_i^2)=m\left(\frac{a^2}{4}\right)+2m\left(\frac{a^2}{4}\right)+3m\left(\frac{a^2}{4}\right)+4m\left(\frac{a^2}{4}\right)=\frac{5}{2}ma^2$

$$I_{yy}=\sum m_i(x_i^2+z_i^2)=m\left(\frac{a^2}{4}\right)+2m\left(\frac{a^2}{4}\right)+3m\left(\frac{a^2}{4}\right)+4m\left(\frac{a^2}{4}\right)=\frac{5}{2}ma^2$$

$$I_{zz}=\sum m_i(x_i^2+y_i^2)=5ma^2$$

$$I_{xy}=-\sum m_ix_iy_i=-m\left(\frac{a}{2}\right).\left(\frac{a}{2}\right)-2m\left(\frac{-a}{2}\right).\left(\frac{a}{2}\right)-3m\left(\frac{-a}{2}\right)\left(\frac{-a}{2}\right)-4m\left(\frac{a}{2}\right)\left(\frac{-a}{2}\right)$$

$$=\frac{ma^2}{4}(-1+2-3+4)=\frac{ma^2}{2}=I_{yx}$$

$$I_{xz}=I_{zx}=-\sum m_ix_iz_i=0\ (\because z_i=0),\quad I_{yz}=I_{zy}=-\sum m_iy_iz_i=0\ (\because z_i=0)$$

therefore, $II=ma^2\begin{pmatrix}5/2 & 1/2 & 0\\ 1/2 & 5/2 & 0\\ 0 & 0 & 5\end{pmatrix}$

Principal moment of inertia: Let λ be eigen value of inertia tensor.

$$\therefore\begin{vmatrix}\frac{5ma^2}{2}-\lambda & \frac{ma^2}{2} & 0\\ \frac{ma^2}{2} & \frac{5ma^2}{2}-\lambda & 0\\ 0 & 0 & \frac{5ma^2}{2}-\lambda\end{vmatrix}=0\qquad\therefore\left(\frac{5ma^2}{2}-\lambda\right)\left[\left(\frac{5ma^2}{2}-\lambda\right)^2-\left(\frac{ma^2}{2}\right)^2\right]=0$$

$$\therefore\ \frac{5ma^2}{2}-\lambda=0\quad\text{or}\quad\left(\frac{5ma^2}{2}-\lambda\right)^2-\left(\frac{ma^2}{2}\right)^2=0$$

$$\therefore \quad \lambda = \frac{5ma^2}{2} \qquad \text{or} \qquad \frac{5ma^2}{2} - \lambda = \pm\frac{ma^2}{2} \Rightarrow \lambda = 2ma^2, 3ma^2$$

$\therefore \ \lambda = 2ma^2, 3ma^2, 5ma^2$ (Principal moment of inertia). Let $\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$ be eigen vector corresponding to eigen value $2ma^2$.

Therefore,

$$\begin{pmatrix} \frac{5ma^2}{2} & \frac{ma^2}{2} & 0 \\ \frac{ma^2}{2} & \frac{5ma^2}{2} & 0 \\ 0 & 0 & 5ma^2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = 2ma^2 \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}, \quad \begin{pmatrix} \frac{5ma^2}{2}\alpha + \frac{ma^2}{2}\beta \\ \frac{ma^2}{2}\alpha + \frac{5ma^2}{2}\beta \\ 5ma^2\gamma \end{pmatrix} = \begin{pmatrix} 2ma^2\alpha \\ 2ma^2\beta \\ 2ma^2\gamma \end{pmatrix}$$

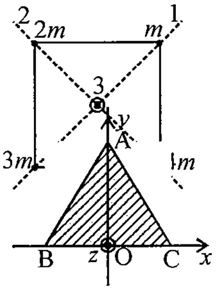
$\therefore \quad 5\alpha + \beta = 4\alpha \quad$ or $\quad \alpha = -\beta$

If $\alpha = 1$ then $\beta = -1$

Therefore, normalized eigen vector corresponding to $\lambda = 2ma^2$ is $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$.

Similarly we find that normalized eigen vector corresponding to $\lambda = 3ma^2$ is $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

Thus principal axes for given mass distribution are as shown in figure:



Example: A uniform equilateral triangular plate of mass 'M' and side 'a' lies in x-y plane as shown in figure. Write inertia tensor.

Soln. y-axis is a symmetric axis therefore it must be a principal axis. And x-being perpendicular to y, is also a principal axis.

Therefore, $x-y-z$ axes given in figure are principal axes. Therefore all product of inertia are zero.

And, $I_{xx} = \int (y^2 + z^2)dm = \int y^2 dm \quad (\because z = 0)$

$$= \iint y^2 . \frac{M}{\frac{\sqrt{3}}{4}a^2} . dxdy = \frac{4M}{\sqrt{3}a^2} \int dx \int y^2 dy$$

$$=\frac{8M}{\sqrt{3}\,a^2}\int_0^{a/2}dx\int_0^{\frac{\sqrt{3}}{2}(a-2x)}y^2dy \qquad \{\because \text{ equation of AC is } y=-\sqrt{3}x+\frac{\sqrt{3}}{2}a=\frac{\sqrt{3}}{2}(a-2x)\}$$

$$=\frac{M}{a^2}\int_0^{a/2}dx(a-2x)^3 \;=\frac{M}{a^2}\left[\frac{-\frac{1}{2}(a-2x)^4}{4}\right]_0^{a/2}=\frac{Ma^2}{8}$$

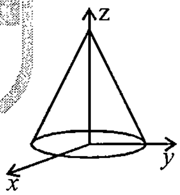
$$I_{yy}=\int(x^2+z^2)dm=\int x^2dm=\iint x^2.\frac{M}{\frac{\sqrt{3}}{4}a^2}dx.dy=\frac{4M}{\sqrt{3}a^2}\iint x^2dxdy=\frac{8M}{\sqrt{3}a^2}\int_0^{a/2}x^2dx\int_0^{\frac{\sqrt{3}}{2}(a-2x)}dy$$

$$=\frac{4M}{a^2}\int_0^{a/2}x^2(a-2x)dx=\frac{4M}{a^2}\left[\frac{a.\left(\frac{a}{2}\right)^3}{3}-\frac{2.\left(\frac{a}{2}\right)^4}{4}\right]=\frac{4M}{a^2}\left[\frac{a^4}{24}-\frac{a^4}{32}\right]=\frac{Ma^2}{24}$$

$$I_{zz}=\int(x^2+y^2)dm=I_{xx}+I_{yy}=\frac{Ma^2}{8}+\frac{Ma^2}{24}=\frac{Ma^2}{6}$$

Therefore, $II=Ma^2\begin{pmatrix}1/8 & 0 & 0\\ 0 & 1/24 & 0\\ 0 & 0 & 1/6\end{pmatrix}$

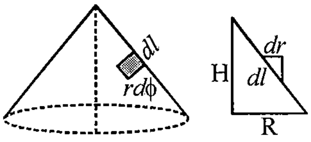
Example : A hollow right circular cone has mass M, radius R, height H. Find inertia tensor of the cone about the axes shown in the figure.



Soln. x, y, z are symmetric axes. Therefore, these are principal axes.
So the inertia tensor is diagonal i.e. all product of inertia must be zero.
let us use cylindrical coordinate system

$\therefore\; x=r\cos\phi, y=r\sin\phi, z=z$

Area element on the surface of cone $= rd\phi dl = rd\phi.\frac{\sqrt{R^2+H^2}}{R}dr$

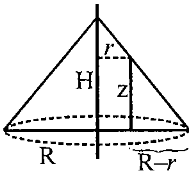


Area of slanting surface of cone $=\pi R\sqrt{R^2+H^2}$

$$\therefore\; dm=\frac{M}{\pi R\sqrt{H^2+R^2}}.rd\phi\frac{\sqrt{R^2+H^2}}{R}dr=\frac{M}{\pi R^2}.rdrd\phi$$

$$I_{xx}=\int(y^2+z^2)dm=\iint(r^2\sin^2\phi+z^2).\frac{M}{\pi R^2}rdrd\phi$$

Consider a point on surface of cone as shown in figure



using similarity of triangles we get

$$\frac{z}{R-r}=\frac{H}{R} \quad \therefore\ z=H\left(1-\frac{r}{R}\right)$$

$$\therefore\ I_{xx}=\iint\left[r^2\sin^2\phi+H^2\left(1-\frac{r}{R}\right)^2\right]\frac{M}{\pi R^2}.rdrd\phi$$

$$=\frac{M}{\pi R^2}\left[\int_0^R r^3dr\int_0^{2\pi}\sin^2\phi d\phi+H^2\int_0^R\left(1-\frac{r}{R}\right)^2 rdr\int_0^{2\pi}d\phi\right]=\frac{M}{\pi R^2}\left[\frac{R^4}{4}.\pi+H^2\frac{R^2}{12}.2\pi\right]=\frac{M}{4}\left[R^2+\frac{2H^2}{3}\right]$$

x and y axes are equivalent. Therefore,

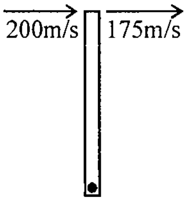
$$I_{yy}=\frac{M}{4}\left(R^2+\frac{2H^2}{3}\right)$$

$$I_{zz}=\int(x^2+y^2)dm=\iint(r^2\cos^2\phi+r^2\sin^2\phi).\frac{M}{\pi R^2}rdrd\phi$$

$$=\frac{M}{\pi R^2}\int_0^R r^3dr\int_0^{2\pi}d\phi=\frac{MR^2}{2} \qquad \therefore\ II=\begin{pmatrix}\frac{M}{4}\left(R^2+\frac{2H^2}{3}\right) & 0 & 0\\ 0 & \frac{M}{4}\left(R^2+\frac{2H^2}{3}\right) & 0\\ 0 & 0 & \frac{MR^2}{2}\end{pmatrix}$$

SOLVED PROBLEMS

1. A bullet of mass 0.1kg travelling at a speed of 200m/s hits one end of a stick of length 1.0m and mass 0.5kg lying on a horizontal table pivoted at the other end as shown in the figure. After hitting the stick the bullet continues to move along its original path but with a reduced speed of 175 m/s. Assume there is no friction.



(a) Find angular speed of stick after it is hit

(b) Find the magnitude of the impulse on the pivot because of the hit.

Soln. (a) If ω be the angular velocity of the stick just after collision then from conservation of angular momentum about the pivot we get,

$$0.1\times200\times1=0.1\times175\times1+\frac{0.5\times1^2}{3}\omega$$

$\therefore\ \omega=15$ rad/sec

(b) Let J be impulse on the rod due pivot

Therefore, using total impulse on rod = change in momentum of the rod we get $0.1\times(200-175)+J=0.5\times v_c$ (i)

$$2.5+J=0.5v_c \qquad \text{(ii)}$$

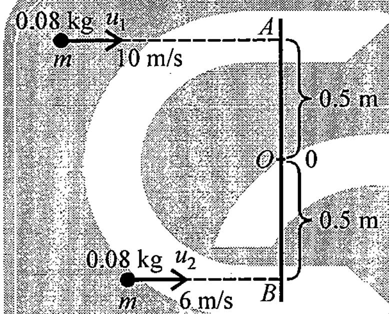
where v_c is velocity of centre of mass of rod after being hit.

$$\therefore \quad v_c=\omega\times\frac{1}{2}=\frac{15}{2}$$

$$\therefore \quad \text{from (i)} \quad 2.5+J=\frac{7.5}{2}\Rightarrow J=1.25$$

Therefore, impulse on pivot is –1.25 kgm/sec

2. A thin uniform bar lies on a frictionless horizontal surface and is free to move in any way on the surface. Its mass is 0.16 kg and length is $\sqrt{3}$ m. Two particles, each of mass 0.08 kg are moving on the same surface and towards the bar in a direction perpendicular to the bar, one with a velocity of 10 m/sec and other with 6 m/sec as shown in figure. The first particle strikes the bar at a point A and the other at point B. Points A and B are at a distance of 0.5 m from the center of the bar. The particles strike the bar at the same instant of time and stick to the bar no collision. Calculate the loss of kinetic energy of the system in the above collision process.



Soln. As the particles after the collision stick to the rod, the center of mass of the system coincides with the mid-point of the rod, because $OA=OB=0.5$ m as the masses of the particles being equal. During collision pair forces develop and the rod is free to rotate as well as translate, both total momentum and total angular momentum will be conserved.

Applying the conservation of linear momentum

$$mu_1+mu_2=(M+2m)v_{CM}, \qquad \text{...(i)}$$

where M is the mass of the bar.

Substituting the given values, we have

$$(0.08\times10+0.08\times6)=(0.16+2\times0.08)v_{CM}$$

Solving we get, $v_{CM}=4$ m/sec.

Taking angular momentum about center of bar, we get

$$mv_1\times y-mv_2\times y=I\omega. \qquad \text{...(ii)}$$

$$\text{Here } I=\left[\frac{Ml^2}{12}+2my^2\right]=\left[\frac{0.16\times\left(\sqrt{3}\right)^2}{12}+2\times0.08\times(0.5)^2\right]$$

$$\therefore \quad I=0.08\text{ kg m}^2$$

Substituting the given values in equation (ii), we get

$$0.08\times10\times0.5-0.08\times6\times0.5=0.08\,\omega.$$

Solving we get, $\omega = 2\ \text{rad/sec}^2$.

Loss of kinetic energy, $\Delta E = \frac{1}{2}mu_1^2 + \frac{1}{2}mu_2^2 - \frac{1}{2}(M+2m)v_{CM}^2 + \frac{1}{2}I\omega^2$

$$\therefore \quad \Delta E = \frac{1}{2}\times 0.08\times(10)^2 + \frac{1}{2}\times 0.08\times(6)^2 - \frac{1}{2}(0.16+2\times 0.08)\times(4)^2 + \frac{1}{2}(0.08)\times(2)^2$$

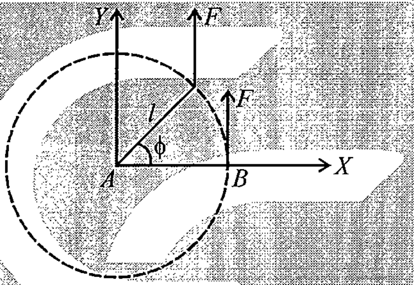
$= \mathbf{2.72\ joule.}$

3. A thin horizontal uniform rod AB of mass m and length l can rotate freely about a vertical axis passing through its end A. At a certain moment the end B starts experiencing a constant force F which is always perpendicular to the original position of the stationary rod and directed in the horizontal plane. Find the angular velocity of the rod as a function of its rotation angle ϕ counted relative to the initial position.

Soln. The situation of the rod at an angle ϕ is shown in figure.

Here, $r = il\cos\phi + jl\sin\phi$ and $F = jF$ (Force is always perpendicular to rod)

$\vec{\tau} = r\times F = (il\cos\phi + jl\sin\phi)\times(jF) = lF\cos\phi\, k$ or $|\vec{\tau}| = lF\cos\phi$



We know that $\tau = I\alpha$

Here, $I = \frac{1}{3}ml^2$ (for rod) and $\alpha = \omega\left(\frac{d\omega}{d\phi}\right)$

Therefore, $lF\cos\phi = \frac{1}{3}ml^2\cdot\omega\left(\frac{d\omega}{d\phi}\right)$ or $lF\cos\phi\, d\phi = \frac{1}{3}ml^2\cdot\omega\, d\omega$

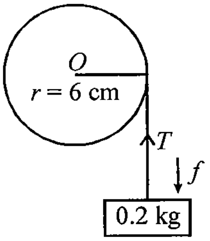
Integrating within proper limits, we have

$$lF\int_0^{\phi}\cos\phi\, d\phi = \frac{1}{3}ml^2\int_0^{\omega}\omega\, d\omega$$

$$lF\sin\phi = \frac{1}{3}ml^2\left(\frac{\omega^2}{2}\right)$$

$$\therefore \quad \omega = \sqrt{\left[\frac{6F\sin\phi}{ml}\right]}.$$

4. A wheel of radius 6 cm is mounted so as to rotate about a horizontal axis through its center. A string of negligible mass wrapped round its circumference carries a mass of 0.2 kg attached to its free end. When let fall, the mass descends through one meter in 5 seconds. Calculate the angular acceleration of the wheel, its moment of inertia and tension in the cord.



Soln. Let f be the acceleration of the falling mass, then using the formula $h = ut + \frac{1}{2}at^2$, we have $1 = 0 \cdot t + \frac{1}{2}f(5)^2$

Therefore, $f = \frac{2}{25} = 0.08$ m/sec^2.

Now angular acceleration, $\alpha = \frac{\text{linear acceleration}}{r} = \frac{0.08}{6 \times 10^{-2}} = \mathbf{1.33\ rad./sec^2}$.

After 5 seconds,
The velocity of wheel $v = ft = 0.08 \times 5$ m/sec. $= 0.4$ m/sec.

Angular velocity of wheel $\omega = \frac{v}{r} = \frac{0.4}{6 \times 10^{-2}} = 6.66$ rad/sec.

Potential energy lost by falling mass $= mgh$.

Increase in translational kinetic energy $= \frac{1}{2}mv^2$.

Increase in rotational energy $= \frac{1}{2}I\omega^2$. (where I is the moment of inertia)

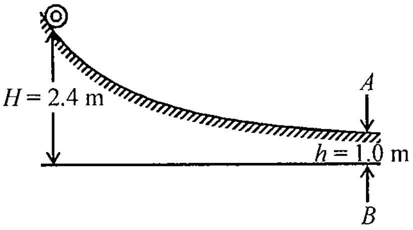
Therefore, $\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = mgh$ or $\frac{1}{2} \times 0.2 \times (0.4)^2 + \frac{1}{2}I(6.66)^2 = 0.2 \times 9.8 \times 1$.

Solving we get $I = \mathbf{8.747 \times 10^{-2}\ kg\text{-}m^2/sec.}$

Let T be the tension in the string. Hence, $mf = mg - T$ or $T = m(g - f)$

Therefore, $T = 0.2(9.8 - 0.08) = \mathbf{1.9445.}$

5. A small sphere rolls down without slipping from the top of a track in a vertical plane. The track has an elevated section and a horizontal part. The horizontal part is 1.0 meter above the ground level and the top of the track is 2.4 meters above the ground. Find the distance on the ground with respect to point B [which is vertically below the end of the track as shown in figure, where the sphere lands. During its flight as a projectile, does the sphere continue to rotate about its center of mass ? Explain.



Soln. The loss of potential energy when the cylinder comes from the top of the track to point A

$$= mg\,(H-h) = mg\,(2.4-1.0) = 1.4\,mg.$$

The gain in its energy $= \frac{1}{2}mv^2 + \frac{1}{2}I\,\omega^2$, where v and ω are the linear and angular velocities of cylinder.

Here $I = \frac{2}{5}mr^2$ and $\omega = \left(\frac{v}{r}\right)$.

Therefore, gain in its kinetic energy $= \frac{1}{2}mv^2 + \frac{1}{2}\times\left(\frac{2}{5}mr^2\right)\left(\frac{v}{r}\right)^2 = \frac{7}{10}mv^2$.

Now, loss of potential energy = gain in kinetic energy $1.4\,mg = \frac{7}{10}mv^2$.

Therefore, $v^2 = \frac{10\times1.4\times9.8}{7} = 19.6$ m/sec or $v = \sqrt{(19.6)}$ m/sec.

This is the horizontal velocity at point A. The vertical component is zero.

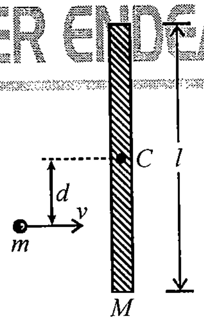
Now, time taken to fall a distance 1 m is $t = \sqrt{\left(\frac{2h}{g}\right)} = \sqrt{\left(\frac{2\times1}{9.8}\right)}$.

Distance of the position of landing the ball on ground from B.

$$s = vt = \sqrt{(19.6)}\times\sqrt{\left(\frac{2\times1}{9.8}\right)} = 2\,\text{m}$$

Yes, The sphere rotates about its center of mass so that the law of conservation of angular momentum holds good during its flight.

6. A meter stick lies on a frictionless horizontal table. It has a mass M and is free to move in any way on the table. A small body of mass m moving with speed v as shown in figure collides elastically with the stick. What must be the value of m if it is to remain at rest after the collision ?



Soln. The collision is elastic. Therefore, linear momentum, angular momentum about a point (say center of mass C) and mechanical energy should be conserved. Let v' be the linear velocity acquired by the stick after collision and ω be the angular velocity about C.

By conservation of linear momentum $mv = Mv'$ ... (i)

By conservation of angular momentum

Angular momentum of body at C = Angular momentum of stick about C

$$md^2\cdot\left(\frac{v}{d}\right) = \frac{Ml^2}{12}\cdot\omega \qquad \text{... (ii)}$$

[where angular momentum of body about $C = I\omega = md^2\cdot\left(\frac{v}{d}\right)$ and $\frac{Ml^2}{12}$ is the moment of inertia of stick about C].

By the conservation of mechanical energy

$$\frac{1}{2}mv^2 = \frac{1}{2}Mv'^2 + \frac{1}{2}\left(\frac{Ml^2}{12}\right)\omega^2 \qquad \text{... (iii)}$$

From equation (i), $v' = \left(\frac{mv}{M}\right)$

From equation (ii), $\omega = (12\,mvd/Ml^2)$

Substituting these values in equation (iii), we get

$$mv^2 = M\left(\frac{mv}{M}\right)^2 + \left(\frac{Ml^2}{12}\right)\left(\frac{12\,mvd}{Ml^2}\right)^2$$

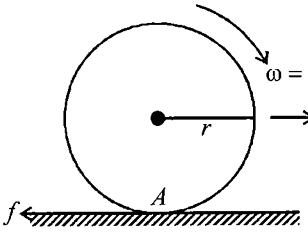
$$1 = \frac{m}{M} + \left(\frac{Ml^2}{12}\right)\left(\frac{12d}{Ml^2}\right)^2 m$$

or $$1 = \frac{m}{M} + md^2\left(\frac{12}{Ml^2}\right)$$

or $$1 = \frac{m}{M}\left(1 + \frac{12d^2}{l^2}\right) = \frac{m}{M}\left(\frac{l^2 + 12d^2}{l^2}\right)$$

$$m = \frac{Ml^2}{(l^2 + 12d^2)}.$$

7. A sphere of mass M and radius r shown in figure slips on a rough horizontal plane. At some instant, it has translational velocity v_0 and rotational velocity about the center $v_0/2r$. Find the translational velocity after the sphere starts pure rolling.



Soln. The velocity of the center $= v_0$

Angular velocity about center $= \frac{v_0}{2r}$.

Here $v_0 > \omega_0 r$. Hence the sphere slips forward. The friction by the plane on the sphere will act backward. The frictional force $= \mu Mg$. The deceleration of the sphere $a_{CM} = \frac{f}{M}$.

$$\therefore \quad v(t)=v_0-\frac{f}{M}t \qquad \text{... (i)}$$

Torque of frictional force $\tau = f \times r$ (clockwise)

Angular acceleration about center $\alpha = f\dfrac{r}{\left(\frac{2}{5}\right)Mr^2}=\dfrac{5f}{2Mr}$

Angular velocity at time t

i.e., $\omega(t)=\omega_0+\dfrac{5f}{2Mr}t$ or $\omega(t)=\dfrac{v_0}{2r}+\dfrac{5f}{2Mr}$ or $\left[\dfrac{v_0}{r}\right]=\dfrac{v_0}{2r}+\dfrac{5ft}{2Mr}$

$$v(t)=\frac{v_0}{2}+\frac{5ft}{2M} \qquad \text{... (ii)}$$

From equations (i) and (ii), eliminating t, we get

$$\frac{5}{2}v(t)+v(t)=\frac{5}{2}v_0+\frac{v_0}{2};\ v(t)=\frac{6}{7}v_0$$

Thus, the sphere rolls with translational velocity $\frac{6v_0}{7}$ in the forward direction.

8. A solid cylinder (a) rolls (b) slides from rest down an inclined plane. Neglect friction and compare the velocities in both cases when cylinder reaches the bottom of the incline.

Soln. (a) We know that the acceleration a of the body rolling down an inclined plane is given by $a=\dfrac{g\sin\theta}{(1+K^2/R^2)}$.

For a solid cylinder $K^2=\dfrac{R^2}{2}$, hence $a=\dfrac{2g\sin\theta}{3}$.

Let v_1 be the velocity acquired by the cylinder on reaching the bottom of the incline of length l, then

$$v_1^2=0+2al=\frac{4g\sin\theta\cdot l}{3} \qquad \text{... (i)}$$

(b) When the cylinder slides down the plane, the acceleration $a=g\sin\theta$.

If v_2 be the velocity acquired by the cylinder on reaching the bottom, then

$$v_2^2=0+2al=2g\sin\theta\cdot l \qquad \text{... (ii)}$$

Now, $\dfrac{v_2^2}{v_1^2}=\dfrac{2g\cdot\sin\theta\cdot l}{\left(4g\cdot\sin\theta\cdot\frac{l}{3}\right)}=\dfrac{3}{2}$ or $\dfrac{v_2}{v_1}=\sqrt{\left(\dfrac{3}{2}\right)}=\mathbf{1.225}.$

9. A sphere, a disc and a ring of the same mass and radius are allowed to roll down an inclined plane simultaneously from the same height without slipping. Prove that the sphere reaches down first, the disc next and the ring, the last.

Soln. We know that acceleration a of the body rolling down an inclined plane is given by

$$a=\frac{g\sin\theta}{\left(1+\frac{K^2}{R^2}\right)} \qquad \text{... (i)}$$

We also know that, Moment of inertia of solid sphere $I_1 = M\,K_1^2 = \dfrac{2}{5}M\,R^2$.

$$\therefore \qquad K_1^2 = \frac{2}{5}R^2 \qquad \text{... (ii)}$$

Moment of inertia of disc $I_2 = M\,K_2^2 = \dfrac{M\,R^2}{2}$.

$$\therefore \qquad K_2^2 = \frac{R^2}{2} \qquad \text{... (iii)}$$

Moment of inertia of ring $I_3 = M\,K_3^2 = M\,R^2$.

$$\therefore \qquad K_3^2 = R^2 \qquad \text{... (iv)}$$

Substituting these values in equation (i), we get

Acceleration a_1 for solid sphere $= \dfrac{g\sin\theta}{(1+2/5)} = \dfrac{5}{7}g\sin\theta$.

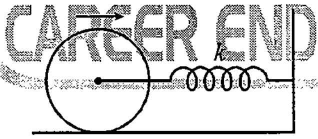
Acceleration a_2 for disc $= \dfrac{g\sin\theta}{(1+2/5)} = \dfrac{2}{3}g\sin\theta$.

Acceleration a_3 for ring $= \dfrac{g\sin\theta}{(1+1)} = \dfrac{g}{2}\sin\theta$.

$$\therefore \qquad a_1 > a_2 > a_3$$

Thus, solid sphere reaches down first, the disc next and the ring, the last.

10. A solid cylinder is attached to a horizontal massless spring, so that it can roll without slipping along a horizontal surface. The force constant k of the spring is 3.0 nt/meter. If the system is released from rest at a position in which the spring is stretched 0.25 meter, find the translational kinetic energy and the rotational kinetic energy of the cylinder as it passes through the equilibrium position.



Soln. (a) Let the linear speed of the center of mass at the equilibrium position be v and angular speed ω. Since the cylinder rolls without slipping, $v = r\,\omega$, where r is the radius of cylinder.

Translational kinetic energy $= \dfrac{1}{2}m\,v^2$.

Rotational kinetic energy $= \dfrac{1}{2}I\,\omega^2 = \dfrac{1}{2}\cdot\left(\dfrac{1}{2}m\,r^2\right)\left(\dfrac{v^2}{r^2}\right) = \dfrac{1}{4}m\,v^2$.

Thus, rotational K.E. $= \dfrac{1}{2}$ translational K.E.

Total energy $= \dfrac{1}{2}m\,v^2 + \dfrac{1}{4}m\,v^2 = \dfrac{3}{4}m\,v^2$.

This should be equal to the total potential energy stored in the spring when it is stretched 0.25 meter and the cylinder released from rest.

Therefore, $\frac{3}{4}mv^2=\frac{1}{2}k(0.25)^2=\frac{1}{2}\times 3.0\times(0.25)^2$ or $mv^2=\frac{1}{2}\times 3.0\times(0.25)^2\times\left(\frac{4}{3}\right)=0.125$ joules.

So, Translational kinetic energy $=\frac{1}{2}mv^2=\frac{1}{2}(0.125)=$ **0.0625 joules**

Rotational kinetic energy $=\frac{1}{4}mv^2=\frac{1}{4}(0.125)=$ **0.03125 joules.**

11. A carpet of mass M made of inextensible material is rolled along its length in the form of a cylinder of radius R and is kept on a rough floor. The carpet starts unrolling without sliding on the floor when a negligibly small push is given to it. Calculate the horizontal velocity of the axis of the cylindrical part of the carpet when its radius reduces to $R/2$.

Soln. Let l be the length of rolled carpet. The mass M of the rolled carpet is given by

$$M=\text{Volume}\times\text{density}=\pi R^2 l\times\rho$$

Let M_1 be the mass of the rolled carpet when its radius is reduced to $(R/2)$. Thus,

$$M_1=\frac{M}{\pi R^2 l\rho}\times\pi\left(\frac{R}{2}\right)^2 l\rho=\frac{M}{4}$$

Potential energy of cylinder before rolling $=MgR$.

For the mass $\left(\frac{M}{4}\right)$, the center of gravity is at $\frac{R}{2}$.

Hence change in potential energy $=MgR-\left(\frac{M}{4}\right)g\left(\frac{R}{2}\right)=\frac{7}{8}MgR$

Now, change in P.E. = linear kinetic energy + rotational K.E.

$$\therefore\quad \frac{7}{8}MgR=\frac{1}{2}M_1v^2+\frac{1}{2}I\omega^2$$

where v is the horizontal velocity of the cylindrical part of the carpet. Now,

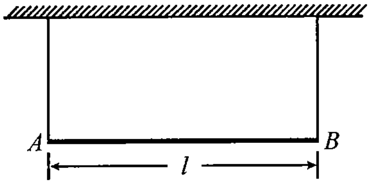
$$\frac{7}{8}MgR=\frac{1}{2}\left(\frac{M}{4}\right)v^2+\frac{1}{2}\left[\frac{1}{2}M_1\left(\frac{R}{2}\right)^2\right]\left[\frac{v}{R/2}\right]^2$$

or $$\frac{7}{8}MgR=\frac{1}{2}\left(\frac{M}{4}\right)v^2+\frac{1}{64}MR^2\times\frac{4v^2}{R^2}$$

or $$\frac{7}{8}gR=\frac{v^2}{8}+\frac{v^2}{16}=\frac{3v^2}{16}$$

or $$v=\left[\left(\frac{14}{3}\right)gR\right]^{1/2}=\mathbf{6.623\sqrt{R}\ m/s.}$$

12. A uniform rod of length l and mass M is suspended on two vertical inextensible strings as shown in figure. Calculate tension T in left string at the instant, when right string snaps.

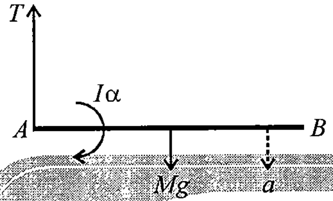


Soln. When the right string snaps, the following two forces are acting on the rod :

(i) weight Mg and

(ii) tension T in left string.

These two forces produce a moment and the rod rotates about its left end as shown in figure.



For vertical forces, we have $Mg - T = Ma$ ... (i)

If α be the angular acceleration of the rod, then the downward acceleration of its center a is given by

$$a = \left(\frac{l}{2}\right)\alpha$$

Substituting the value of a in equation (i), we get $Mg - T = M\left(\frac{l}{2}\right)\alpha$ ... (ii)

Taking moments about A

$$Mg = \left(\frac{l}{2}\right) = I\alpha = \left(\frac{Ml^2}{3}\right)\alpha$$

$$\alpha = \frac{3g}{2l} \quad \text{... (iii)}$$

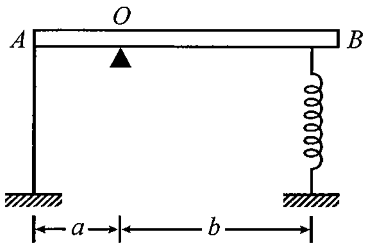
Substituting the value of α from equation (iii) in equation (ii), we get

$$Mg - T = M\left(\frac{l}{2}\right)\frac{3g}{2l}$$

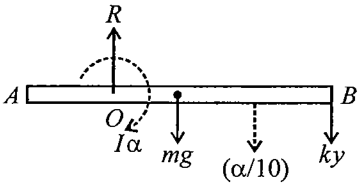
$$Mg - \frac{3Mg}{4} = T$$

$$\therefore \quad T = \frac{Mg}{4}.$$

13. A uniform rod AB of mass 2 kg and length $l = 100$ cm is placed on a sharp support O such that $AO = a = 40$ cm and $OB = b = 60$ cm. A spring of force constant $k = 600$ N/m is attached to end B as shown in figure. To keep the rod horizontal, its end A is tied with a thread such that the spring is elongated by $y = 1$ cm. Calculate reaction of support on the rod when the thread is burnt.



Soln. The different forces on rod are shown in figure.



Due to these forces (Mg and ky) the rod is subjected to a clockwise angular acceleration. As a result, the center of mass of the rod accelerates vertically downwards. If α be the angular acceleration of the rod, then downward acceleration of center of mass will be

$$\left(\frac{l}{2}-a\right)\alpha = 0.1\alpha = \left(\frac{\alpha}{10}\right) \quad \text{...(i)}$$

Taking moments about O, we have $mg\times(0.10)+ky(0.60)=I\alpha$ ...(ii)
where I = moment of inertia of the rod about O

$$=\frac{ml^2}{12}+m\left(\frac{l}{2}-a\right)^2=\frac{2}{12}+2\times(0.1)^2=0.167+0.02\approx 0.19\text{ kg-m}^2$$

From equation (ii)

$$2\times10\times0.10+600\times(0.01)\,(0.60)=0.19\,\alpha$$

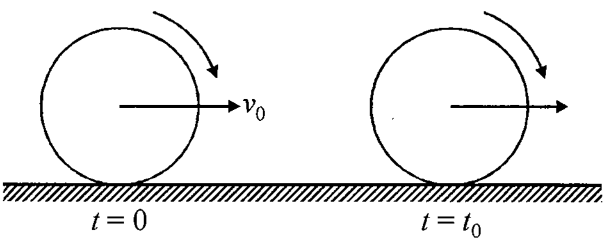
Solving we get $\alpha = 30\text{ rad/s}^2$

For rod, $mg+ky-N=m\left(\frac{\alpha}{10}\right)$

$$2\times10+600(0.01)-R=2\times3$$

$R =$ **20 newton.**

14. A uniform disc of mass m and radius R is projected horizontally with velocity v_0 on a rough horizontal floor so that it starts with a purely sliding motion at $t=0$. After t_0 seconds it acquires a purely rolling motion as shown in figure.



(i) Calculate the velocity of the center of mass of the disc at t_0

(ii) Assuming the coefficient of friction to be μ, calculate t_0. Also calculate the workdone by the frictional force as a function of time and the total workdone by it over a time t much longer than t_0

Soln. (i) Maximum frictional force $= \mu mg$, therefore acceleration $= \mu g$.

For linear motion of center of mass $v = v_0 - \mu g t_0$ ... (i)

Here v = velocity of center of mass at $t = t_0$

For purely rolling motion,

$$v = \omega R \quad \text{or} \quad v = \left[\frac{\mu mg R}{(mR^2/2}t_0\right]R \quad \text{or} \quad v = 2\mu g t_0 \qquad \text{... (ii)}$$

Substituting the value of $\mu g t_0$ from equation (ii) in equation (i), we get

$$v = v_0 - \frac{v}{2} \quad \text{or} \quad v = \left(\frac{2v_0}{3}\right) \qquad \text{... (iii)}$$

(ii) Substituting the value of v from equation (iii) in equation (ii), we get

$$\frac{2v_0}{3} = 2\mu g t_0 \quad \text{or} \quad t_0 = \frac{v_0}{3\mu g} \qquad \text{... (iv)}$$

Let W be the workdone by frictional force in time t, then

$$\frac{1}{2}mv_0^2 = \frac{1}{2}m(v_0 - \mu g t)^2 + \frac{1}{2}\frac{mR^2}{2}\left(\frac{2\mu g t}{R}\right) + W$$

or $$W = \frac{1}{2}mv_0^2 - \frac{1}{2}m\left[v_0^2 - 2v_0\mu g t + \mu^2 g^2 t^2\right] - m\mu^2 g^2 t^2$$

or $$W = \frac{1}{2}m\mu g t\left[2v_0 - 3\mu g t\right]$$

The workdone by frictional force for $t = t_0$

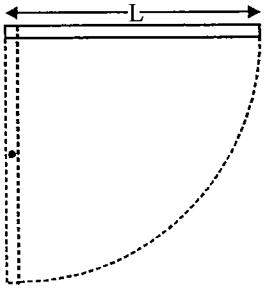
$$= \frac{1}{2}m\mu g\left(\frac{v_0}{3\mu g}\right)\left[-v_0 + 2v_0\right] = \frac{1}{6}mv_0 \times v_0 = \frac{mv_0^2}{6}.$$

After t_0, when the motion becomes pure rotational, there will be no workdone by frictional force.

So, for $t \gg t_0$, the total workdone by frictional force $= \frac{mv_0^2}{6}$.

15. A uniform beam held horizontally at two ends is set into motion by releasing it at one end, due to which it rotates about the other end. When rod becomes vertical the second end is also released. Find the trajectory of centre of rod and angular velocity of rod in subsequent motion.

Soln. Let L be length and M be mass of the rod. Let ω be the angular speed of the rod when the second end is released.



$\therefore$ From conservation of energy we get

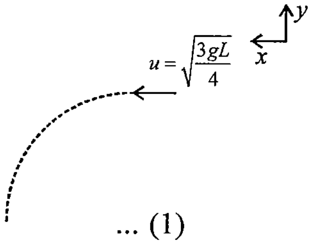
$$\frac{1}{2}\cdot\left(\frac{ML^2}{3}\right)\omega^2 = Mg\cdot\frac{L}{2}$$

$$\therefore \quad \omega = \sqrt{\frac{3g}{L}}$$

Therefore, velocity of centre of rod at this moment

$$u = \omega \cdot \frac{L}{2} = \sqrt{\frac{3gL}{4}}$$

Now centre of the rod moves like a particle falling under gravity.
Therefore its path will be parabolic.
To find equation of path let us assume that at an instant 't' the co-ordinate of the centre is (x, y).



$$\therefore \quad x = ut \qquad \text{... (1)}$$

and $\quad y = \frac{1}{2} g t^2$

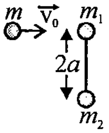
To get equation of path we eliminate t from the above two equations

$$\therefore \quad y = \frac{1}{2} g \left(\frac{x}{u}\right)^2 \text{ or } y = \frac{g}{2} \cdot \frac{\frac{x^2}{3gL}}{4}$$

$y = \frac{2}{3}\frac{x^2}{L}$ (with origin L/2 below the second support)

Angular velocity of rod in subsequent motion will be same as the angular velocity of rod at the time of the release of second support.

16. A particle of mass m strikes a dumbell and comes to rest as shown in the figure. Calculate the distance moved by the centre of mass of dumbell in the time dumbell rotates through $\pi/2$.



Soln. Distance of centre of mass of dumbel from m_1 is

$$X = \frac{m_2 a}{m_1 + m_2}$$

M.I. of dumbell about centre of mass

$$I_C = \frac{m_1 \cdot m_2}{m_1 + m_2} \cdot a^2$$

From conservation of momentum we get $mv_0 = (m_1 + m_2)v$

where v is velocity of centre of mass of dumbel after collision

$$\therefore \quad v = \frac{m_1 v_0}{m_1 + m_2}$$

Now using conservation of angular momentum we get.

$$mv_0 X = I_C \omega$$

where ω is angular velocity of dumbel about centre of mass after collision.

$$\therefore \quad \omega = \frac{m}{m_1} \cdot \frac{v_0}{a}$$

Let 't' be time for rotation by $\pi/2$

$$\therefore \quad \pi/2 = \omega t$$

$$\therefore \quad t = \frac{2}{\pi} \cdot \frac{m_1}{m_1} \cdot \frac{a}{v_0}$$

$\therefore$ Distance covered in time $t = vt = \frac{2}{\pi} \cdot \frac{m_1 a}{m_1 + m_2}$

17. A uniform rod of mass M and length L is rotating with angular velocity ω about an axis passing through its centre and $\perp$ to its length. If suddenly one end of the rod is fixed, calculate new angular velocity.

Soln. Since, torque about the end is zero, angular momentum is conserved.

Initial angular momentum of the rod.

$$L_i = \frac{ML^2}{12} \omega$$

This is also the initial angular momentum of the rod about its one end.

Moment of inertia of rod about its end is $\frac{ML^2}{3}$

Let ω' be the final angular velocity

$\therefore$ Final angular momentum $L_f = \frac{ML^2}{3} \cdot \omega'$

Since $L_i = L_f$

$$\therefore \quad \frac{ML^2}{12} \omega = \frac{ML^2}{3} \omega' \qquad \therefore \quad \omega' = \frac{\omega}{4}$$

18. A solid body rotates with deceleration about a stationary axis with an angular deceleration $\beta \propto \sqrt{\omega}$, where ω is its angular velocity. Find the mean angular velocity of the body averaged over the whole time of rotation if at the initial moment of time its angular velocity was equal to ω_0.

Soln. $-\beta = k\sqrt{\omega}$ or $-\frac{d\omega}{dt} = k\sqrt{\omega}$ $\int_{\omega_0}^{0} \frac{d\omega}{\sqrt{\omega}} = -\int_0^t k\,dt \quad \therefore \quad t = \frac{2\sqrt{\omega_0}}{k}$

Average angular velocity $\overline{\omega} = \frac{\int \omega(t)\,dt}{\int dt}$

From $2\sqrt{\omega} = -kt, \ \omega = \frac{k^2 t^2}{4}$

$$\text{Therefore, } \overline{\omega} = \frac{\frac{k^2}{4} \int_0^{2\sqrt{\omega_0}/k} t^2\,dt}{\int_0^{2\sqrt{\omega_0}/k} dt} = \frac{k^2}{12}\left[\frac{t^3}{t}\right]_0^{2\sqrt{\omega_0}/k} = \frac{k^2}{12}\left[\frac{4\omega_0}{k^2}\right] = \frac{\omega_0}{3}\Bigg].$$

19. A solid body rotates about a stationary axis according to the law $\phi = kt - bt^3$, where $k = 6.0$ rad/s and $b = 2.0$ rad/s². Find

(a) The mean values of the angular velocity and angular acceleration averaged over the time interval between $t = 0$ and the complete stop;

(b) The angular acceleration at the moment when the body stops.

Ans. (a) $\overline{\omega} = 4$ rad/sec, $\overline{\beta} = 6$ rad/sec^2

(b) $\beta = 12$ rad/sec

Soln. $\phi = kt - bt^3$

$$\omega = \frac{d\phi}{dt} = k - 3bt^2; \quad \frac{d\omega}{dt} = 0 - 6bt$$

For stopping $\omega = 0$, then $t = \sqrt{\left(\frac{k}{3b}\right)}$

$$\text{Now, } \overline{\omega} = \frac{\int_0^1 \omega(t)\,dt}{\int_0^t dt} = \frac{\int_0^{\sqrt{k/3b}} (k - 3bt^2)\,dt}{\int_0^{\sqrt{k/3b}} dt} = \frac{\left[kt - (3bt^3/3)\right]_0^{\sqrt{k/3b}}}{\sqrt{k/3b}}$$

Solving we get $\overline{\omega} = 4$ rad/sec.

$$\text{Now, } \overline{\beta} = \frac{\int_0^t \beta(t)\,dt}{\int_0^t dt} = \frac{\int_0^{\sqrt{k/3b}} 6bt\,dt}{\int_0^{\sqrt{k/3b}} dt}$$

Solving we get $\overline{\beta} = 6$ rad/sec.

Further $\quad \beta = -6bt = -6b\sqrt{k/3b}$

$$|\beta| = -6b\sqrt{(k/3b)} = 2\sqrt{(3kb)} = 12 \text{ rad/sec}^2.$$

20. A solid body rotates about a stationary axis so that its angular velocity depends on the rotation angle ϕ as $\omega = \omega_0 - k\phi$, where ω_0 and k are positive constants. At the moment $t = 0$, the angle $\phi = 0$. Find the time dependent of

(a) rotation angle

(b) angular velocity

Soln. (a) Now rotation angle at time t:

$$\phi(t) = \int_{t=0}^{t} \omega\,dt = \int_0^1 \omega_0 e^{-kt} = -\frac{\omega_0}{k} e^{-kt}\Big|_0^t = \frac{\omega_0}{k}(1 - e^{-kt})\Big].$$

(b) Angular acceleration

$$\alpha = \frac{d\omega}{dt} = -\frac{d\phi}{dt} = -k\omega \quad \text{or} \quad \frac{d\omega}{\omega} = -k\,dt$$

$$\int_{\omega_0}^{\omega} \frac{d\omega}{\omega} = -k\int_0^t dt = -kt \quad \text{or} \quad \omega = \omega_0 e^{-kt}.$$

21. Inertia tensor of a system is $\begin{pmatrix} \frac{2}{3} & -\frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{2}{3} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{2}{3} \end{pmatrix}$. Calculate moment of inertia about an axis which is oriented in the direction $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

Soln. Given $\overleftrightarrow{I} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{2}{3} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{2}{3} \end{pmatrix}$ and $\hat{n} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

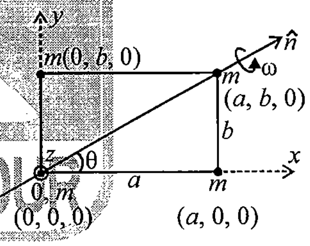
$$\therefore \quad I = \hat{n}.\overleftrightarrow{I}.\hat{n}$$

$$= \frac{1}{3}(1 \quad 1 \quad 1)\begin{pmatrix} \frac{2}{3} & -\frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{2}{3} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{2}{3} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{3}(1 \quad 1 \quad 1)\begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{pmatrix} = \frac{1}{18}(1 \quad 1 \quad 1)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{18}(1+1+1) = \frac{1}{6}$$

22. Four equal point masses m are placed the corners of a rectangle of sides 'a' and 'b'. The system is rotated about a diagonal with uniform angular velocity ω. Calculate rotational kinetic energy, angular momentum about the axis of rotation and total angular momentum of system.

Soln. Let us choose x-y-z axes as shown in figure.



$$I_{xx} = \sum m_i(y_i^2 + z_i^2) = \sum m_i y_i^2 \quad (\because z_i = 0)$$

$$= m0^2 + m0^2 + mb^2 + mb^2 = 2mb^2$$

$$I_{yy} = \sum_i m_i(x_i^2 + z_i^2) = \sum m_i x_i^2 = m0^2 + ma^2 + m0^2 + ma^2 = 2ma^2$$

$$I_{zz} = \sum_i m_i(x_i^2 + y_i^2) = 2m(a^2 + b^2)$$

$$I_{xy} = -\sum m_i x_i y_i = -mab = I_{yx} \qquad \therefore \overleftrightarrow{I} = m\begin{pmatrix} 2b^2 & -ab & 0 \\ -ab & 2a^2 & 0 \\ 0 & 0 & 2(a^2+b^2) \end{pmatrix}$$

Direction of diagonal,

$$\hat{n} = \cos\theta\hat{i} + \sin\theta\hat{j} = \frac{1}{\sqrt{a^2+b^2}}(a\hat{i} + b\hat{j}) \equiv \frac{1}{\sqrt{a^2+b^2}}\begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$$

moment of inertia about the axis of rotation

$$I = \hat{n}.\overleftrightarrow{I}.\hat{n} = \frac{m}{(a^2+b^2)}(a \quad b \quad 0)\begin{pmatrix} 2b^2 & -ab & 0 \\ -ab & 2a^2 & 0 \\ 0 & 0 & 2(a^2+b^2) \end{pmatrix}\begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$$

$$=\frac{m}{a^2+b^2}(a \quad b \quad 0)\begin{pmatrix} ab^2 \\ a^2b \\ 0 \end{pmatrix} = \frac{2ma^2b^2}{(a^2+b^2)}$$

Therefore, rotational kinetic energy $=\frac{1}{2}I\omega^2 = \frac{1}{2}\frac{2ma^2b^2}{(a^2+b^2)}\omega^2 = \frac{ma^2b^2\omega^2}{a^2+b^2}$

Angular momentum about the axis of rotation $= I\omega = \frac{2ma^2b^2\omega}{(a^2+b^2)}$

To calculate total angular momentum of system we use $\vec{L} = \overleftrightarrow{I}.\vec{\omega}$

or $\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$, Here $\omega_x = \omega\cos\theta = \frac{\omega a}{\sqrt{a^2+b^2}}$, $\omega_y = \omega\sin\theta = \frac{\omega b}{\sqrt{a^2+b^2}}$, $\omega_z = 0$

$$\therefore \begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \frac{m\omega}{\sqrt{a^2+b^2}}\begin{pmatrix} 2b^2 & -ab & 0 \\ -ab & 2a^2 & 0 \\ 0 & 0 & 2(a^2+b^2) \end{pmatrix}\begin{pmatrix} a \\ b \\ 0 \end{pmatrix} = \frac{m\omega}{\sqrt{a^2+b^2}}\begin{pmatrix} ab^2 \\ a^2b \\ 0 \end{pmatrix}$$

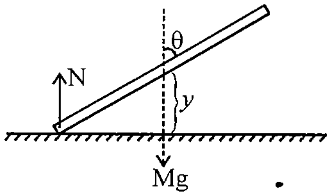
$$\therefore L_x = \frac{m\omega ab^2}{\sqrt{a^2+b^2}},\ L_y = \frac{m\omega a^2 b}{\sqrt{a^2+b^2}}, L_z = 0$$

$$\therefore L = \sqrt{L_x^2 + L_y^2 + L_z^2} = \frac{m\omega ab}{\sqrt{a^2+b^2}}\sqrt{a^2+b^2}$$

$$\therefore L = m\omega ab$$

23. A rod of mass M and length L held vertically upward with its one end touching a smooth horizontal surface. The rod is released from this position. Show that when it is about to strike the surface, normal reaction is $Mg/4$.

Soln. Let at some instant of time the rod makes angle θ with the vertical as shown in the figure.



$$\therefore \ y = \frac{L}{2}\cos\theta$$

$$\dot{y} = -\frac{L}{2}\sin\theta\dot{\theta}$$

$$\ddot{y} = -\frac{L}{2}\sin\theta\ddot{\theta} - \frac{L}{2}\cos\theta\dot{\theta}^2 \quad \text{... (i)}$$

downward acceleration of the rod $= -\ddot{y}$

$\therefore$ Equation of motion $Mg - N = -M\ddot{y}$ ... (ii)

For angular motion $\tau = \frac{ML^2}{12}\ddot{\theta}$ (about centre of rod)

$$\frac{NL}{2}\sin\theta = \frac{ML^2}{12}\ddot{\theta} \qquad \text{... (iii)}$$

using equations (i), (ii) and (iii) we get

$$Mg - N = +M\left[\frac{L}{2}\sin\theta\left(\frac{6N}{ML}\sin\theta\right) + \frac{L}{2}\cos\theta\dot{\theta}^2\right]$$

when rod is about to strike the horizontal surface

$\theta = 90°,\ \cos 90 = 0,\ \sin 90 = 1$

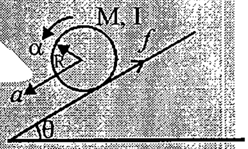
$$\therefore\ Mg - N = M\left[\frac{3N}{M}\times 1 + 0\right]$$

$$Mg = 4N$$

$$\therefore\ N = \frac{Mg}{4}$$

24. Show that least coefficient of friction of a solid sphere to roll down an inclined plane is $\frac{2}{7}\tan\theta$ and that for a hollow sphere is $\frac{2}{5}\tan\theta$.

Soln. Consider an inclined plane of inclination θ and an object of mass M, radius R, and moment of inertia I rolling down on it.



Equation of motion;

$$I\alpha = f\cdot R$$

for rolling $\alpha = \frac{a}{R}$ $\qquad \therefore\ f = \frac{Ia}{R^2}$ ... (i)

and $\ Mg\sin\theta - f = Ma$

using (i) we get $\ a = \frac{g\sin\theta}{1+\frac{I}{MR^2}}$

$$\therefore\ f = \frac{I}{R^2}\cdot\frac{g\sin\theta}{1+\frac{I}{MR^2}} \qquad N = Mg\cos\theta$$

since $f \le \mu N$

$$\therefore\ \frac{I}{R^2}\cdot\frac{g\sin\theta}{1+\frac{I}{(MR^2)}} \le \mu.Mg\cos\theta \qquad \therefore\ \mu \ge \frac{g\tan\theta}{1+\frac{MR^2}{I}}$$

For solid sphere

$$I = \frac{2}{5}MR^2 \qquad \therefore\ \mu \ge \frac{2}{7}g\tan\theta$$

For hollow sphere

$$I = \frac{2}{3}MR^2 \qquad \therefore\ \mu \ge \frac{2}{5}g\tan\theta$$

25. Calculate the radius of Gyration of a uniform solid cylinder of length L and radius R about an axis passing through its centre and perpendicular to its axis.

Soln. Moment of inertia of a solid cylinder about the given axis is

$$I=\frac{M}{4}\left(R^2+\frac{L^2}{3}\right)$$

Radius of Gyration (k) is defined as

$$k=\sqrt{\frac{I}{M}} \qquad \therefore\ k=\frac{1}{2}\sqrt{R^2+\tfrac{L^2}{3}}$$

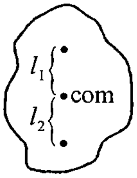
26. A physical pendulum has same period of oscillations T about two pivots which are L distance apart. Show that $T=2\pi\sqrt{\frac{L}{g}}$.

Soln. Let l_1 and l_2 be the distances of the two pivots from the centre of mass of the physical pendulum $l_1+l_2=L$

$$\therefore\ T=2\pi\sqrt{\frac{(I_c+Ml_1^2)}{Mgl_1}} \text{ and } T=2\pi\sqrt{\frac{(I_c+Ml_2^2)}{Mgl_2}}$$

where I_C is moment of inertia about centre of mass and M is mass of physical pendulum.

∴ The above two expressions can be re-written as



$$\frac{T^2Mg}{4\pi^2}l_1=I_C+Ml_1^2 \qquad \text{... (i)}$$

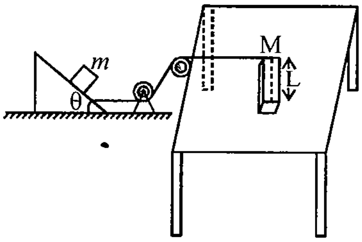
and $$\frac{T^2Mg}{4\pi^2}\cdot l_2=I_C+Ml_2^2 \qquad \text{... (ii)}$$

Subtracting (i) and (ii) we get

$$\frac{T^2Mg}{4\pi^2}\cdot(l_1-l_2)=M(l_1+l_2)(l_1-l_2)$$

or $$\frac{T^2g}{4\pi^2}=L \quad \text{or} \quad T=2\pi\sqrt{\tfrac{L}{g}}$$

27. Consider the situation shown in the figure below. Table is fixed and the rod lying on the table can rotate about one end which is hinged to the table. Coefficient of static friction between rod and table is μ.

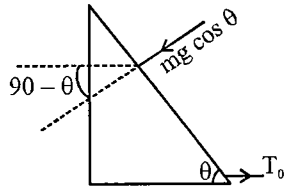


All the strings and pulleys are light and frictionless. There is no friction between the two blocks and between triangular block and ground. What should be maximum value of mass 'm' so that the triangular block stays in equilibrium.

Soln. Let T_0 be the maximum tension that can be applied in the string without making the rod rotate.

$\therefore\ T_0\cdot L$ = torque due to friction

$$= \int_0^L \mu\left(\frac{M}{L}dxg\right)\cdot x \;=\frac{\mu MgL}{2}$$



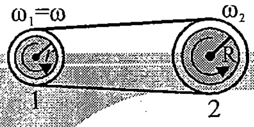
$$\therefore\quad T_0 = \frac{\mu Mg}{2}$$

Now considering the equilibrium of triangular block we get,

$$mg\cos\theta\cos(90-\theta) = T_0$$

$$\therefore\quad m=\frac{\mu M}{\sin 2\theta}$$

28. Two pulleys of radii r and R are connected with a rubber band of weight w. If moment of inertia of the two pulleys be I_1 and I_2 respectively and angular velocity of smaller pulley be ω. Calculate the kinetic energy of the system.



Soln. Rubber band has same speed everywhere

$$\therefore\quad \omega_1 r=\omega_2 R \qquad\qquad \therefore\quad \omega_2=\omega\cdot\frac{r}{R}$$

speed of rubber band $= \omega r$

$\therefore$ kinetic energy of the pulley 1 (smaller pulley) $=\frac{1}{2}I_1\omega^2$

$\therefore$ kinetic energy of the pulley 2 (bigger pulley) $=\frac{1}{2}I_2\omega_2^2$

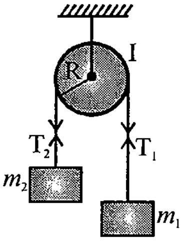
$$=\frac{1}{2}I_2\cdot\omega^2\cdot\frac{r^2}{R^2}$$

Mass of rubber band $=\frac{w}{g}$

$\therefore$ kinetic energy of rubber band $=\frac{1}{2}\left(\frac{w}{g}\right)\cdot\omega^2 r^2$

Total energy of the system $=\frac{1}{2}\left[I_1+I_2\cdot\frac{r^2}{R^2}+\frac{w}{g}r^2\right]\omega^2$

29. Calculate acceleration of two masses shown in the figure. The strings is light and pulley has moment of inertia I and radius R.



Soln. Let T_1 and T_2 be tension in strings connected to m_1 and m_2 respectively.

$\therefore$ Equation of motion, for m_1:

$$m_1 g - T_1 = m_1 a \qquad \text{... (i)}$$

where 'a' is acceleration (downward) for m_1:

$$T_2 - m_2 g = m_2 a \qquad \text{... (ii)}$$

for the pulley:

$$(T_1 - T_2)R = I\alpha$$

α is angular acceleration of pulley.

$$\therefore \quad \alpha = \frac{a}{R} \qquad \therefore \quad T_1 - T_2 = \frac{Ia}{R^2} \qquad \text{... (iii)}$$

Adding equation (i), (ii) and (iii) we get,

$$(m_1 - m_2)g = \left(m_1 + m_2 + \frac{I}{R^2}\right)a$$

$$\therefore \quad a = \frac{(m_1 - m_2)g}{m_1 + m_2 + \frac{I}{R^2}}$$

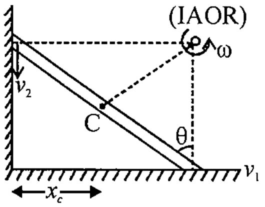
30. A thin rod of length L and mass M lying in vertical position on a horizontal surface and against a smooth vertical wall is displaced slightly due to which its one end starts sliding on the vertical wall. Find angle made with the horizontal when the rod leaves contact with the wall.

Soln. Angular velocity of rod is given by $\omega = \dot{\theta}$

where θ is angle made by rod with vertical. The rod has rotational as well as translational motion. However, motion is pure rotational about the instantaneous axis of rotation (IAOR)

OC = $l/2$

$\therefore$ M.I. about IAOR



$$I_C = \frac{ML^2}{12} + M(L/2)^2 = \frac{ML^2}{3}$$

$\therefore$ from conservation of energy we get,

$$\frac{1}{2}\cdot\frac{ML^2}{3}\cdot\omega^2 = Mg\frac{L}{2}(1-\cos\theta)$$

$$\frac{\omega^2 L}{3} = g(1-\cos\theta)$$

or $$\frac{L\dot{\theta}^2}{3g} = 1-\cos\theta \qquad \text{... (i)}$$

Distance of centre of rod from the vertical wall

$$x_C = \frac{L}{2}\sin\theta \qquad \therefore\ \dot{x}_C = \frac{L}{2}\cos\theta\dot{\theta}$$

$$\therefore\ \ddot{x}_C = \frac{L}{2}(\cos\theta\ddot{\theta} - \sin\theta\dot{\theta}^2)$$

when rod leaves the vertical wall, there is no horizontal force on the rod

$$\therefore\ \ddot{x}_C = 0$$

$$\therefore\ \cos\theta\ddot{\theta} - \sin\theta\dot{\theta}^2 = 0 \qquad \text{... (ii)}$$

from (i) $\dot{\theta}^2 = \frac{6g}{L}\sin^2\frac{\theta}{2}$

$$\therefore\ \dot{\theta} = \sqrt{\frac{6g}{L}}\cdot\sin\frac{\theta}{2} \quad \text{or} \quad \ddot{\theta} = +\sqrt{\frac{3g}{2L}}\cos\frac{\theta}{2}\cdot\dot{\theta}$$

$\therefore$ from (ii) we get

$$+\cos\theta\cdot\sqrt{\frac{3g}{2L}}\cos\frac{\theta}{2}\dot{\theta} - \sin\theta\dot{\theta}^2 = 0$$

or $$\sqrt{\frac{3g}{2L}}\cos\theta\cdot\cos\frac{\theta}{2} - \sin\theta\cdot\sqrt{\frac{8g}{L}}\cdot\sin\frac{\theta}{2} = 0$$

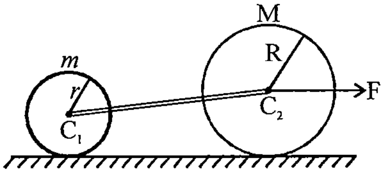
$$\cos\theta\cdot\cos\frac{\theta}{2} - 2\sin\theta\cdot\sin\frac{\theta}{2} = 0$$

$$\frac{\cos\theta\cdot\cos\frac{\theta}{2}}{\sin\theta\cdot\sin\frac{\theta}{2}} = \frac{2}{1} \quad \text{or} \quad \frac{(2\cos^2\frac{\theta}{2}-1)}{(1-\cos^2\frac{\theta}{2})} = \frac{4}{1}$$

solving we get $\theta = \cos^{-1}\left(\frac{2}{3}\right)$

Therefore, angle made with horizontal $= \frac{\pi}{2} - \cos^{-1}\left(\frac{2}{3}\right) = \sin^{-1}\left(\frac{2}{3}\right)$

31. Two wheels of masses M and m and radii R and r respectively are connected with a rod of length L. The system rolls on a rough surface under the action of an applied force F. Find tension in the connecting rod.



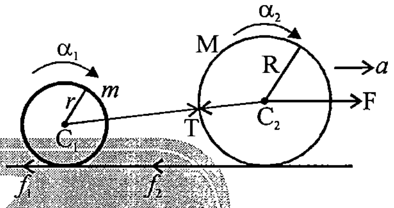
Soln. Since the two wheels are connected their linear acceleration must be same.

Let T be the tension in the connecting rod.

$\therefore$ Equation of motion for 'M'

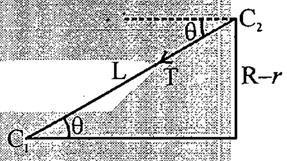
$$F - T\cos\theta - f_2 = Ma \qquad \text{... (i)}$$

and $f_2 \cdot R = \dfrac{MR^2}{2}\cdot\alpha_2$



Here $\alpha_2 = \dfrac{a}{R}$ for rolling motion

$\therefore\ f_2 = \dfrac{MR}{2}\cdot\left(\dfrac{a}{R}\right)$



or $f_2 = \dfrac{MR}{2}$ ... (ii)

Equation of motion for 'm'

$$T\cos\theta - f_1 = ma \qquad \text{... (iii)}$$

and $f_1 \cdot r = \dfrac{mr^2}{2}\cdot\alpha_1$

here $\alpha_1 = \dfrac{a}{r}$ for rolling motion

$\therefore\ f_1 = \dfrac{ma}{2}$ ... (iv)

adding (i), (ii), (iii) and (iv) we get,

$$F = \frac{3}{2}(M+m)a \qquad \therefore \quad a = \frac{2F}{3(M+m)}$$

from equation (iv)

$$f_1 = \frac{ma}{2} = \frac{mF}{3(M+m)}$$

$\therefore$ from (iii) $T\cos\theta = ma + f_1$

$$= \frac{2mF}{3(M+m)} + \frac{mF}{3(M+m)} = \frac{mF}{(M+m)}$$

$$\therefore\ T=\frac{mF}{(M+m)\cos\theta}$$

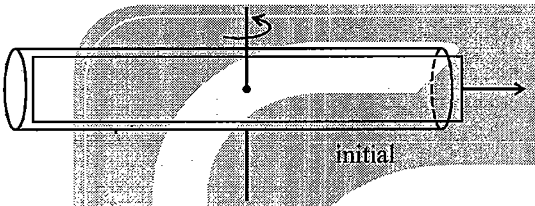
$$T=\left(\frac{m}{M+m}\right)\cdot\frac{F}{\cos\theta}$$

since $\cos\theta=\frac{\sqrt{L^2-(R-r)^2}}{L}$

$$\therefore\ T=\frac{m}{M+m}\cdot\frac{FL}{\sqrt{L^2-(R^2-r)^2}}$$

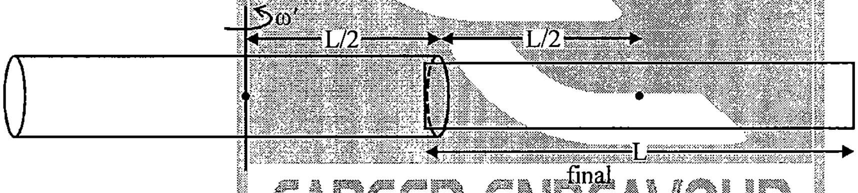
32. A rod of mass M and length L lies inside a smooth tube of same mass and length with their centers slightly off. The tube is rotated with initial angular speed ω about its center. Find the angular speed of the tube when the rod completely comes out of it.

Soln. Since, the rod gains radial momentum, angular momentum of system remains conserved.



Initial angular momentum $L_1=2\times\frac{ML^2}{12}\cdot\omega$

Final angular momentum (when rod has just come out)



$$L_2=L_2^{\text{rod}}+L_2^{\text{tube}}$$

$$=\underbrace{\left(\frac{ML^2}{12}+ML^2\right)}_{\text{M.I. of rod about the axis of tube}}\omega'+\frac{ML^2}{12}\omega'$$

ω' = final angular velocity

$$\because\ L_1=L_2$$

$$\therefore\ \frac{ML^2}{6}\cdot\omega=\frac{7ML^2}{6}\cdot\omega'$$

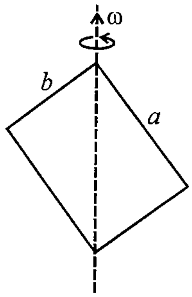
or $\omega'=\frac{\omega}{7}$

33. A thin rectangular plate of mass M and sides '*a*' and '*b*' (*a* > *b*) is rotating about a vertical axis passing through one of its diagonal. Find the torque required to sustain the rotation.

Soln. The rotation axis is not a principal axis, therefore angular momentum is not parallel to $\vec{\omega}$. Let us take *x* and *y* axes as shown in the figure.

$$\therefore\ L_x=\frac{Ma^2}{12}\omega_x=\frac{Ma^2}{12}\omega\sin\theta,\quad L_y=\frac{Mb^2}{12}\omega_y=\frac{Mb^2}{12}\cdot\omega\cos\theta$$

Here $\sin\theta = \dfrac{b}{\sqrt{a^2+b^2}}, \cos\theta = \dfrac{a}{\sqrt{a^2+b^2}}$

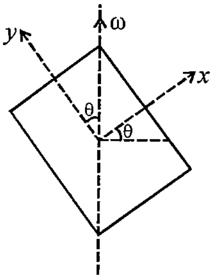


Horizontal component of angular momentum

$$L_h = L_x \cos\theta - L_y \sin\theta$$

$$= \frac{M(a^2-b^2)\omega\sin\theta\cdot\cos\theta}{12}$$

$$= \frac{M\omega}{12}\cdot\frac{(a^2-b^2)\cdot ab}{(a^2+b^2)}$$



L_h rotates in horizontal circle with angular velocity ω.

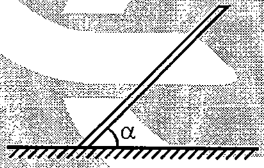
$\therefore$ Torque required to sustain the rotation.

$$\tau = L_h\,\omega$$

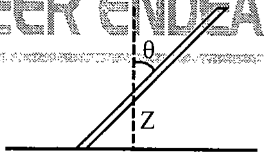
$$= \frac{M\omega^2\cdot ab}{12}\cdot\left(\frac{a^2-b^2}{a^2+b^2}\right)$$

For a square plate, diagonal is an axis of symmetry. Therefore, no torque is required to sustain the rotation as can be seen by putting $a = b$ in the expression for torque.

34. A uniform thin rod of length L and mass M is held at angle α to a smooth horizontal surface as shown in the figure. The rod is released from this position, find the angular velocity of rod when it becomes horizontal.



Soln. Since there is no horizontal force, centre of mass falls straight in vertical line and the lower end of the rod slides on the smooth surface. The rod becomes horizontal when the upper end is about to touch the surface.



Let θ = angle with the vertical line at time t

$\therefore$ angular speed $= \dot{\theta}$

Total K.E. $= \dfrac{1}{2}M\dot{Z}^2 + \dfrac{1}{2}I_c\dot{\theta}^2$

since $Z = \dfrac{L}{2}\cos\theta$

$\therefore \quad \dot{Z} = -\dfrac{L}{2}\sin\theta\cdot\dot{\theta}$

$\therefore \quad$ T.K.E. $= \dfrac{1}{2}\left(ML^2\dfrac{\sin^2\theta}{4} + I_c\right)\dot{\theta}^2 = ML^2\left(\dfrac{\sin^2\theta}{4} + \dfrac{1}{12}\right)\dot{\theta}^2$

$$\text{Loss in P.E.} = Mg\frac{L}{2}(\sin\alpha - \cos\theta)$$

$\therefore$ Gain in T.K.E. = loss in P.E.

$$\frac{1}{8}ML^2\left(\sin^2\theta + \frac{1}{3}\right)\dot{\theta}^2 = \frac{ML}{2}g\,(\sin\alpha - \cos\theta)$$

$$\therefore\ \dot{\theta} = \sqrt{4g/L\frac{(\sin\alpha - \cos\theta)}{(\sin^2\theta + 1/3)}} \leftarrow \text{Angular speed at time 't'}$$

When rod becomes horizontal $\theta = 90°$

$$\therefore\ \dot{\theta} = \sqrt{3g/L\sin\alpha} = \sqrt{\frac{3g}{L}\sin\alpha}$$

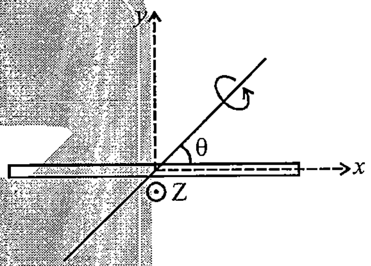
35. Calculate M.I. of a rod about an axis passing through its centre and making an angle θ with it.

Soln. This can be easily calculated by integration method. However, we will use a different approach to illustrate the use of inertia tensor.

M.I. about an axis in the direction given by unit vector $\hat{n}$ is given by

$\mathrm{I} = \hat{n}\cdot\overleftrightarrow{\mathrm{II}}\cdot\hat{n}$; $\overleftrightarrow{\mathrm{II}}$ is inertia tensor.

Let us take the co-ordinate system as shown in the figure, therefore.



$$\text{Here, } \hat{n} \equiv \begin{pmatrix}\cos\theta \\ \sin\theta \\ 0\end{pmatrix}$$

Let L = length and M = mass of the rod.

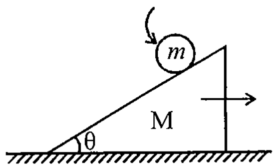
$$\therefore\ \mathrm{I}_{xx} = 0,\ \mathrm{I}_{yy} = \mathrm{I}_{zz} = \frac{ML^2}{12}$$

As the rod is lying along x-axis

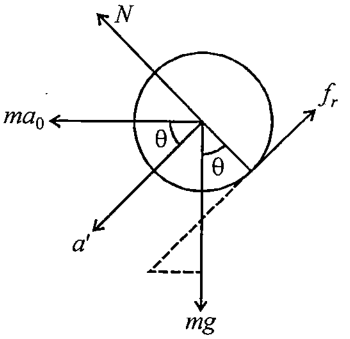
$$\therefore\ \mathrm{I}_{xy} = \mathrm{I}_{xz} = f_{yz} = 0 \qquad \therefore\ \overleftrightarrow{\mathrm{II}} = \begin{bmatrix}0 & 0 & 0 \\ 0 & \frac{ML^2}{12} & 0 \\ 0 & 0 & \frac{ML^2}{12}\end{bmatrix}$$

$$\therefore\ \mathrm{I} = \frac{ML^2}{12}[\cos\theta\ \sin\theta\ 0]\begin{bmatrix}0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos\theta \\ \sin\theta \\ 0\end{bmatrix} = \frac{ML^2}{12}(\cos\theta\ \sin\theta\ 0)\begin{pmatrix}0 \\ \sin\theta \\ 0\end{pmatrix} = \frac{ML^2}{12}\sin^2\theta$$

36. A solid sphere rolls on a wedge kept on a smooth surfaces, as shown in the figure, find the acceleration of the wedge.



Soln. Let a_0 is acceleration of wedge and 'a' is acceleration of solid sphere with respect to the wedge.
Equation of motion of sphere in the frame of wedge



$$mg\sin\theta + m a_0\cos\theta - f_r = ma \qquad ...(1)$$

and $N = m\cdot g\cos\theta - m a_0\sin\theta \qquad ...(2)$

and $f_r\cdot R = I\alpha$

since motion is pure rolling

$$\alpha = \frac{a}{R}$$

$$\therefore\ f_r = \frac{Ia}{R^2} \qquad ...(3)$$

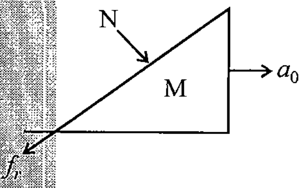
From (1) $mg\sin\theta + m a_0\cos\theta = a\left(m + \frac{I}{R^2}\right)$

$$a = \frac{g\sin\theta + a_0\cos\theta}{1 + \frac{I}{mR^2}} \qquad ...(4)$$

Equation of motion of wedge

$$-f_r\cos\theta + N\sin\theta = m a_0$$

or $-\frac{Ia}{R^2}\cdot\cos\theta + N\sin\theta = m a_0$



using (4) and (2) we get,

$$-\frac{I}{R^2}\cos\theta\left(\frac{g\sin\theta + a_0\cos\theta}{1 + \frac{I}{mR^2}}\right) + (mg\cos\theta - m a_0\sin\theta)\sin\theta = m a_0$$

$$\therefore\ -\frac{I}{mR^2}\cdot g\sin\theta\cdot\cos\theta + \left(\frac{I}{mR^2} + 1\right) g\sin\theta\cdot\cos\theta = a_0\left[\frac{M}{m} + \frac{I}{mR^2}\cos^2\theta + \left(1 + \frac{I}{mR^2}\right)\sin^2\theta\right]$$

$$\therefore\ a_0 = \frac{(g\sin\theta\cos\theta)}{\left(\frac{M}{m} + \frac{I}{mR^2} + \sin^2\theta\right)}$$

$$I = \frac{2}{5}mR^2$$

$$\therefore\ a_0 = \frac{g\sin\theta\cdot\cos\theta}{\left(\frac{M}{m} + \frac{2}{5} + \sin^2\theta\right)}$$

37. A cylinder of 1kg mass and 0.02m diameter left at the top of an inclined plane of height 1m rolls down without slipping (Take $g = 9.8$ m/s^2). **[GP-2006]**

(a) What is kinetic energy of cylinder when it reaches at the bottom of inclined plane?

(b) Find velocity of centre of mass of cylinder on reaching the bottom of inclined plane.

Soln. (a) In rolling motion total mechanical energy remains conserved. Therefore, kinetic energy at the bottom will be equal to potential energy at the top.

∴ kinetic energy at bottom = mgh

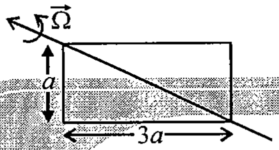
$= 1 \times 9.8 \times 1 = 9.8$ Joule

(b) In rolling motion, kinetic of a cylinder is $\frac{3}{4}mv_c^2$ where v_c is velocity of centre of mass.

$$\therefore \quad \frac{3}{4}mv_c^2 = 9.8$$

$$\frac{3}{4}\times 1\times v_c^2 = 9.8 \Rightarrow v_c = \sqrt{\frac{9.8\times 4}{3}} = 3.6\text{m/s}$$

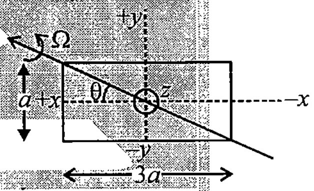
38. (a) A uniform rectangular plate of length $3a$ and width a has mass M. Ignoring gravity, determine the magnitude and direction of the torque $\vec{\tau}$ required to rotate the plate with uniform angular velocity $\vec{\Omega}$ in counter clock-wise direction about the diagonal shown below.



(b) A smaller rectangular plate of length $3a/2$ and width $a/2$ is cut from the plate shown above. Find the torque $\vec{\tau}$ to rotate the new plate along the diagonal with the same angular velocity $\vec{\Omega}$. **[GP-2012]**

Soln. Let us choose principal axes (x, y, z) as shown in figure.

Angular momentum of the plate



$$\vec{L} = I_{xx}\omega_x\hat{x} + I_{yy}\omega_y\hat{y} + I_{zz}\omega_z\hat{z}$$

$$= \frac{Ma^2}{12}\Omega\cos\theta\hat{x} + \frac{M(3a)^2}{12}\Omega\sin\theta\hat{y} + \frac{M}{12}[a^2+(3a)^2]\times 0$$

$$\vec{L} = \frac{Ma^2}{12}\Omega\cdot\frac{3}{\sqrt{10}}\hat{x} + \frac{3Ma^2}{4}\Omega\frac{1}{\sqrt{10}}\hat{y} = \frac{3\Omega}{4\sqrt{10}}Ma^2\left(\frac{1}{3}\hat{x}+\hat{y}\right)$$

$$\vec{\Omega} = \Omega(\cos\theta\hat{x} + \sin\theta\hat{y}), \quad \vec{\Omega} = \frac{\Omega}{\sqrt{10}}(3\hat{x}+\hat{y})$$

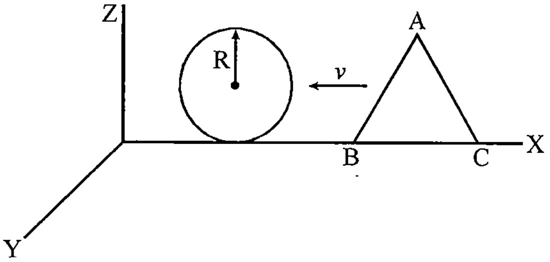
Torque required $\vec{\tau} = \vec{\Omega}\times\vec{L}$

$$= \frac{3Ma^2\Omega^2}{40}(3\hat{x}+\hat{y})\times\left(\frac{1}{3}\hat{x}+\hat{y}\right) = \frac{3Ma^2\Omega^2}{40}\left(3\hat{z}-\frac{1}{3}\hat{z}\right) = \frac{Ma^2\Omega^2\hat{z}}{5}$$

Thus a torque of magnitude $\frac{Ma^2\Omega^2}{5}$ is required in a direction perpendicular to the plane of the plate.

(b) Replacing a by $a/2$ we get $\vec{\tau} = \frac{Ma^2\Omega^2}{20}\hat{z}$.

39. A wedge of mass m and triangular cross section $(AB = BC = CA = 2R)$ is moving with a constant velocity $-v\,i$ toward sphere of radius R fixed on smooth horizontal table as shown in figure. The wedge makes an elastic collision with the fixed sphere and return along the same path without any rotation. Neglect all friction and suppose that the wedge remains in contact with the sphere for a very short time Δt, during which the sphere exerts a constant force F on the wedge.



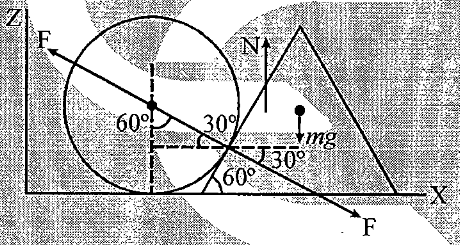
(a) Find the force F and also normal force N exerted by the table on the wedge during the time Δt.

(b) Let h denote the perpendicular distance between the center of mass of the wedge and the line of action of F. Find the magnitude of the torque due to the normal force N about the center of the wedge during the interval Δt.

Soln. (a) The collision is perfectly elastic collision. Hence the wedge will recede with the same velocity as sphere is fixed.

The change in momentum of the wedge is given by $\Delta p_x = 2mv$ ... (i)

At the time of collision, the different forces are shown in figure.



Component of force along X-axis $= F\cos 30$

Hence $F\cos 30 \times \Delta t = 2mv$ or $F = \dfrac{4mv}{\sqrt{3}\,\Delta t}$ ... (ii)

In vector form,

$$F = F\cos 30\,i + F\sin 30\,(-k)$$

or $$F = \frac{4mv}{\sqrt{3}\,\Delta t}\cdot\frac{\sqrt{3}}{2}\,i + \frac{4mv}{\sqrt{3}\,\Delta t}\cdot\frac{1}{2}\,(-k) = \frac{2mv}{\Delta t}\left[i - \frac{1}{\sqrt{3}}\cdot k\right] \quad \text{... (iii)}$$

For vertical direction

$$mg + F\sin 30 = N$$

$$mg + \frac{4mv}{\sqrt{3}\cdot\Delta t}\cdot\frac{1}{2} = N$$

$$\therefore \quad N = \frac{2mv}{\sqrt{3}\cdot\Delta t} + mg \quad \text{... (iv)}$$

In vector form, $$N = -\left[2mv/\sqrt{3}\cdot\Delta t + mg\right]k \quad \text{... (v)}$$

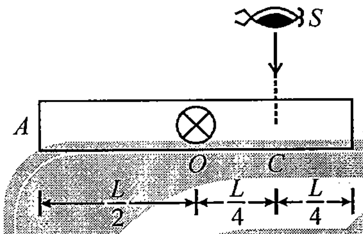
(b) Taking the moments about the center of the wedge

$$\sum \vec{\tau}=0 \text{ i.e., } \vec{\tau_F}+\vec{\tau_N}+\vec{\tau_m}=0$$

Here $\vec{\tau_{mg}}=0,\ \therefore\ \vec{\tau_N}=-\vec{\tau_F}$ or $\vec{\tau_N}=F\cdot hj=\dfrac{4mvh}{\sqrt{3}\cdot\Delta t}\cdot j$

Magnitude of $\tau_N=\dfrac{4mvh}{\sqrt{3}\,\Delta t}$ ... (vi)

40. A homogeneous rod AB of length $L=1.8$ m and mass M is pivoted at the center O in such a way that it can rotate freely in the vertical plane (see figure). The rod is initially in the horizontal position. An insect S of the same mass M falls vertically with speed v on the point C, midway between the points O and B. Immediately after falling, the insect moves towards the end B such that the rod rotates with a constant angular velocity ω.



(a) Determine the angular velocity ω in terms of v and L.

(b) If the insect reaches the end B when the rod has turned through an angle of 90°, determine v.

Soln. (a) The initial angular momentum of the insect about $O=(M\,v)\,(L/4)$.

Moment of inertia of the rod about an axis passing through O and perpendicular to the rod $=(ML^2/12)$.

The moment of inertia of the insect when at $C=M\,(L/4)^2$.

Therefore, moment of inertia of the rod plus insect when at C:

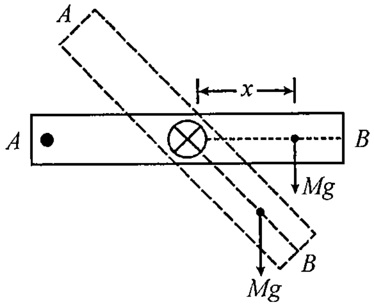
$$=\frac{ML^2}{12}+M\left(\frac{L}{4}\right)^2=\frac{ML^2}{12}+\frac{ML^2}{16}.$$

Angular momentum in this case $=\left(\dfrac{ML^2}{12}+\dfrac{ML^2}{16}\right)\omega$.

From the law of conservation of angular momentum $\dfrac{MvL}{4}=\left(\dfrac{ML^2}{12}+\dfrac{ML^2}{16}\right)\omega$

Therefore, $\omega=\dfrac{12v}{7L}$.

(b) After falling, the insect moves towards the end B. Now the moment of inertia of the system changes. Let the insect be at a distance x after time t as shown in figure.



At this instant, the moment of inertia of the system $= \left(\frac{ML^2}{12} + Mx^2 \right)$

Angular momentum $= \left(\frac{ML^2}{12} + Mx^2 \right)\omega$

Torque $\tau =$ rate of change of angular momentum $= \frac{d}{dt}\left[\frac{ML^2}{12} + Mx^2 \right]\omega = M\omega 2x\frac{dx}{dt}$ ($\because$ ω is constant).

The torque is provided by the force Mg acting downward i.e., $\tau = Mgx$

Therefore, $Mgx = M\omega 2x\frac{dx}{dt}$ or $\frac{dx}{dt} = \frac{g}{2\omega}$

Therefore, velocity of insect $v = \frac{dx}{dt} = \frac{g}{2\omega}$

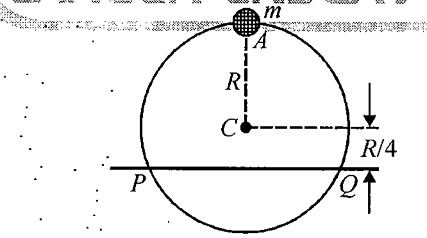
Time taken to rotate through $\frac{\pi}{2} = \frac{\pi}{2\omega}$

Time taken by insect to move to the other end of the rod $= \frac{L/4}{v} = \frac{L}{4} \times \frac{2\omega}{g} = \frac{\omega L}{2g}$

Equating the two time periods, we have $\frac{\pi}{2\omega} = \frac{\omega L}{2g}$ or $\omega^2 L = \pi g$ or $\omega = \sqrt{\left(\frac{\pi g}{L}\right)}$.

$$\frac{12v}{7L} = \sqrt{\left(\frac{\pi g}{L}\right)} \Rightarrow v = \frac{7}{12}L\sqrt{\left(\frac{\pi g}{L}\right)} = \frac{7}{12}\sqrt{(\pi g L)} = \frac{7}{12}\sqrt{(3.14 \times 10 \times 1.8)} = \mathbf{4.4\ m/s.}$$

41. A uniform circular disc has radius R and mass m. A particle also of mass m, is fixed at point A on the edge of the disc as shown in figure. The disc can rotate freely about a fixed horizontal chord PQ that is at a distance $R/4$ from the center C at the disc. The line AC is perpendicular to PQ.



Initially, the disc is held vertical with the point A at its highest position. It is then allowed to fall so that it starts rotating about PQ. Find the linear speed of the particle as it reaches the lowest position.

Soln. First of all we shall calculate the moment of inertia of the disc about the axis PQ. We know that the moment of inertia of a disc about any diameter is $(mR^2/4)$. Applying the theorem of parallel axis, the moment of inertia of the disc about PQ is given by

$$(I_{\text{disc}})_{PQ} = \left[\frac{mR^2}{4} + m\left(\frac{R}{4}\right)^2 \right] = \frac{5}{16}mR^2 \qquad \text{...(i)}$$

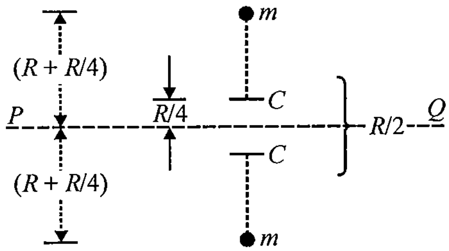
The moment of inertia of mass m about PQ is given by

$$(I_{\text{particle}})_{PQ} = m\left(R+\frac{R}{4}\right)^2 = m\left(\frac{5R}{4}\right)^2 = \frac{25\,mR^2}{16} \qquad \text{... (ii)}$$

So, the moment of inertia of the system (disc + particle) about PQ is given by

$$I = \frac{5}{16}mR^2 + \frac{25}{16}mR^2 = \frac{30\,mR^2}{16} = \frac{15\,mR^2}{8} \qquad \text{... (iii)}$$

Now we shall calculate the loss in P.E. when the particle reaches in lowest position. The lowest position of particle is shown in figure.



Loss in potential energy of disc $= mg\left(\frac{R}{4}+\frac{R}{4}\right) = \frac{mgR}{2}$.

Loss in potential energy of particle $= mg\left[\left(R+\frac{R}{4}\right)+\left(R+\frac{R}{4}\right)\right] = \frac{5\,mgR}{2}$.

Total loss in potential energy $= \frac{mgR}{2} + \frac{5\,mgR}{2} = 3\,mgR$.

Let ω be the angular velocity of the disc at the lowest position. So,

$$\frac{1}{2}I\,\omega^2 = \text{Loss in potential energy} \quad \text{or} \quad \frac{1}{2}\times\frac{15\,mR^2}{8}\times\omega^2 = 3\,mgR.$$

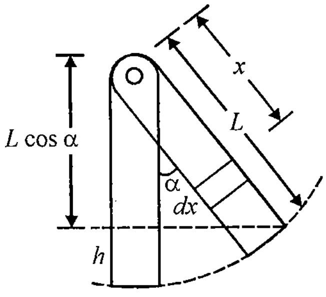
Solving, we get $\omega = \sqrt{\left(\frac{16\,g}{5R}\right)}$.

Hence, the linear speed of the particle is given by

$$v = \left(R+\frac{R}{4}\right)\omega = \frac{5R}{4}\times\sqrt{\left(\frac{16g}{5R}\right)} = \sqrt{(5gR)}.$$

42. One end of a uniform rod of mass M and length L is supported by a frictionless hinge which can withstand a tension of $1.75\,Mg$. The rod is free to rotate in a vertical plane. To what maximum angle should the rod be rotated from the vertical position so that when left, the hinge does not break.

Soln. See figure



From figure, $h = L - L\cos\alpha = L(1-\cos\alpha)$

Rotational kinetic energy $= \frac{1}{2}I\omega^2$, where $I = \frac{ML^2}{12} + M\left(\frac{L}{2}\right)^2 = \frac{ML^2}{3}$.

According to the law of conservation of energy

$$\frac{1}{2}I\omega^2 = \frac{1}{2}MgL(1-\cos\alpha)$$

or $$\frac{1}{2}\left(\frac{ML^2}{3}\right)\omega^2 = \frac{1}{2}MgL(1-\cos\alpha)$$

or $$\omega^2 = \frac{3g}{L}(1-\cos\alpha) \qquad \text{... (i)}$$

Consider an element of length dx of the rod at a distance x from hinge. Centrifugal force on this element

$$dF = \frac{M}{L}x\,dx\,\omega^2 \text{ or } F = \int_0^L dF = \frac{ML\omega^2}{2} \qquad \text{... (ii)}$$

Substituting the value of ω^2 from equation (i) in equation (ii), we get

$$F = \frac{3}{2}Mg(1-\cos\alpha) \qquad \text{... (iii)}$$

Now, total downward force will be $(F + Mg)$. Hence according to the given problem

$$F + Mg = 1.75\,Mg \text{ or } F = 0.75\,Mg \qquad \text{... (iv)}$$

From equation (iii) and (iv), we get

$$0.75\,Mg = \frac{3}{2}Mg(1-\cos\alpha)$$

or $$(1-\cos\alpha) = \frac{1}{2} \text{ or } \cos\alpha = \frac{1}{2}$$

$\therefore$ $\alpha = 60°$.

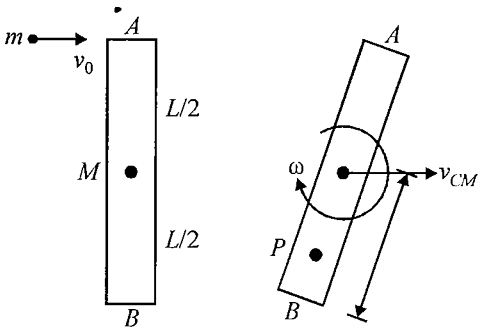
43. A rod AB of mass M and length L is lying on a horizontal frictionless surface. A particle of mass m travelling along the surface hits the end 'A' of the rod with a velocity v_0 in a direction perpendicular to AB. The collision is completely elastic. After the collision the particle comes to rest.

(a) Find the ratio m/M

(b) A point P on the rod is at rest immediately after the collision. Find the distance AP

(c) Find the linear speed of the point P at a time $\frac{\pi L}{(3v_0)}$ after the collision

Soln. (a) See figure



Applying the conservation of momentum $m v_0 = M v_{CM}$

$$\therefore \quad v_{CM} = \frac{m v_0}{M} \qquad \text{... (i)}$$

Applying the conservation of angular momentum

$$m v_0 \left(\frac{L}{2}\right) = \left(\frac{ML^2}{12}\right)\omega \quad \text{or} \quad \omega = \frac{6 m v_0}{ML} \qquad \text{... (ii)}$$

Applying the conservation of kinetic energy, we get

$$\frac{1}{2} m v_0^2 = \frac{1}{2} M v_{CM}^2 + \frac{1}{2}\left(\frac{ML^2}{12}\right)\omega^2 \qquad \text{... (iii)}$$

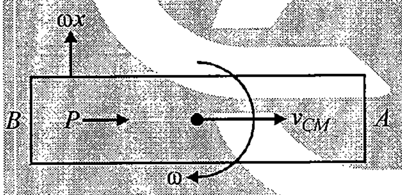
Substituting the values of v_{CM} and ω from equation (i) and (ii) in equation (iii) and solving we have

$$\frac{m}{M} = \frac{1}{4} \qquad \text{... (a)}$$

(b) Let the point P be at a distance of x from center of rod.

$$\therefore \quad v_{CM} - \omega x = 0 \quad \text{or} \quad x = \left(\frac{v_{CM}}{\omega}\right)$$

$$\therefore \quad x = \frac{(m v_0/M)}{(6 m v_0/ML)} = \frac{L}{6} \qquad \text{... (b)}$$



(c) After time $t = \left(\frac{\pi L}{3 v_0}\right)$, the angle traversed θ is given by $\theta = \omega t = \frac{6 m v_0}{ML} \times \frac{\pi L}{3 v_0} = \frac{\pi}{2}$ radian.

Net velocity of P at this instant $v_P = \sqrt{(v_{CM})^2 + \omega^2 x^2}$

Substituting the values and solving, we get $v_P = \frac{v_0\sqrt{2}}{4}$.

Chapter 10

Fluid Dynamics

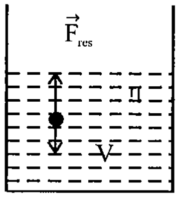
Viscosity is the property of fluid due to which fluid flow is apposed.

Frictional in nature: It is typically proportional to the velocity and non-conserved in nature.

$$F_{\text{resistance}} \propto (-V)$$

$$\vec{F} = -bV \qquad \text{where } B = \text{demping constant medium property.}$$

In case of fluids, $b = \eta \rightarrow$ coefficient of viscosity.



Stokes law

$$\vec{F}_{\text{resistance}} = k\eta\vec{V} \qquad \left(f = \eta\frac{dv}{dx}, \text{ viscous drag}\right)$$

Dimension analysis:

F depends on γ, η, v we write

$$F = k\,\gamma^x\,\eta^y\,v^z$$

$$\left(MLT^{-2}\right) = \left(ML^{-1}T^{-1}\right)^y\left(LT^{-1}\right)^z L_x = M^y L^{x-y+z} T^{-y-z}$$

Dimension of η

$$\eta = \frac{f}{\frac{dv}{dx}} = \frac{F/A}{\frac{dv}{dx}} = \frac{MLT^{-2}/L^2}{T^{-1}} = \boxed{ML^{-1}T^{-1}}$$

Here, $y = 1$

$(y+z) = 2 \qquad z = 1\ (y+z) = 2 \qquad z = 1$

$x - y + z = 1 \qquad x = 1\ x - y + z = 1 \qquad x = 1$

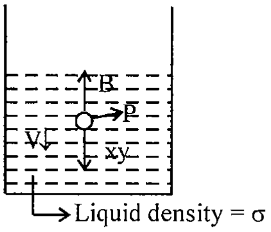
$\vec{F} = k\eta\, r\, \vec{v}$

$k = 6\pi$ experimentally.

Then, $F = 6\pi\eta r\vec{v}$ (r = radius of the particle)

Application of Stokes Law:

$m = \frac{4}{3}\pi r^3\rho$, π = density of the ball



Buoyant force $=\left(\frac{4}{3}\pi r^3\sigma\right)g$

Equation of motion, $m\frac{d\vec{v}}{dt}=\frac{4}{3}\pi r^3(\rho-\sigma)g-6\pi\eta r\vec{v},\ \frac{4}{3}\pi r^3\rho\frac{d\vec{v}}{dt}$

Or, $$\frac{d\vec{v}}{dt}=\left(1-\frac{\sigma}{\rho}\right)g-\frac{9}{2}\frac{\eta}{\rho}\frac{\vec{v}}{r^2}$$

If $F_{net}=0,\ \frac{d\vec{v}}{dt}=0,\ \vec{v}=\text{constant}$ (terminal velocity)

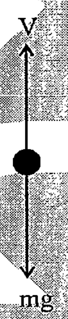
Then, $$\left|\vec{V}_T\right|=\frac{2}{9}\frac{r^2}{\eta}(\rho-\sigma)g$$

$\rho>\sigma\to$ downward motion

$\rho<\sigma\to$ upward motion

General Motion:

Resistive force $=-bV$



$$m\frac{d\vec{V}}{dt}=\frac{4}{3}\pi r^3(\rho_{air}-\rho_{ball})g-bV$$

$$\frac{4}{3}\pi r^3\rho_{ball}\left(\frac{d\vec{v}}{dt}\right)$$

Or, $$\frac{d\vec{v}}{dt}=g\left(\frac{\rho_{air}}{\rho ball}-1\right)-\frac{bV}{\frac{4}{3}\pi r^3\rho}$$

Or, $$\frac{d\vec{v}}{dt}=-g\left(1-\frac{\rho_{air}}{\rho_{ball}}\right)-\frac{b}{\frac{4}{3}\pi r^3\rho_{ball}}$$

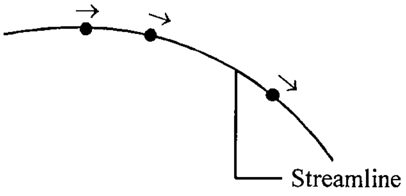
Take, $$g\left(1-\frac{\rho_{air}}{\rho_{ball}}\right)=C$$

And $$\frac{b}{\frac{4}{3}\pi r^3\tau_{ball}}=\frac{b}{m};\ \frac{dv}{dt}=-\left(c+\frac{b}{mr}\right)$$

Or, $$\int_{v_0}^{v}\frac{dv}{c+\frac{b}{mv}}=-c\int_0^t dt;\ \left(\frac{m}{b}\right)\ell n\left(c+\frac{b}{m}\right)\Bigg|_{v_0}^{v}=-ct \qquad \Rightarrow\left(\frac{m}{b}\right)\ell n\left(\frac{c+\frac{b}{m}}{c+\frac{b}{m}v_0}\right)=-ct$$

$$\Rightarrow \frac{c+\frac{b}{m}v}{c+\frac{b}{m}v_0}=\exp-\left(\frac{cb}{m}\right)t \quad\Rightarrow v=\left(\frac{m}{b}\right)\left(c+\frac{b}{m}v_0\right)e^{-\frac{cb}{m}t}-\frac{cm}{b}$$

Sreamline (Laminar) flow: Flow where are paticles (preceding or succeeding) have same velocity at a particle point.



The path followed by the particles is called steamline.

Velocity at any point of streamline is tangential to the streamline at that point.

No two streamline cross each other.

Turbulant: Which is not streamline. Velocity of particles at a given point varies with time both in magnitude and direction in an irregular manner.

The streamline flow is maintained when the velocity is below a particular limit called critical velocity.

Critical velocity V_c is related to the density of the liquid ρ, coefficient of viscosity (η) and the radius of the tube.

$$R = \frac{\text{ratio of inertial force}}{\text{viscous force}} = \left(\frac{\rho V_c r}{\eta}\right)$$

$R < 2000 =$ streamline

$2000 < R < 3000 =$ transition

$R > 3000 =$ turbulant.

Dimension: $v = k\eta^x \rho^y r^z$; $LT^{-1} = k\left(ML^{-1}T^{-1}\right)^x \left(ML^3\right)^y L^z = k\,M^{x+y} L^{-x-3y+z} T^{-x}$; $x + y = 0$

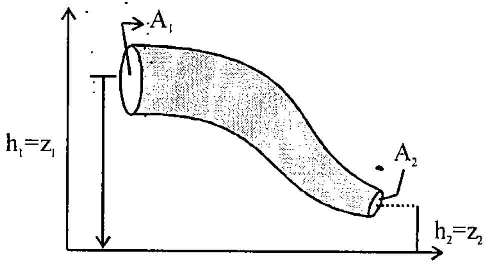
$-x = -1, x = 1, \ y = -x = -1$ $\qquad (x = 1; \ y = z = -1)$

$-x - 3y + z = 1, \ -1 + 3 + z = 1, \ z = -1$

Hence, $\quad v = \left(\frac{k\eta}{\rho r}\right)$

Bernoulli's equation: Ideal fluid flow

(i) Streamline (ii) Incompressible (iii) Non viscous



Mass entering through A_1

$$\Delta m = \rho A_1 \Delta x_1$$

$$\lim_{\Delta t \to 0} \frac{\Delta m}{\Delta t} = \frac{dm}{dt} = \rho \Delta_1 v_1 \qquad \text{Rate of mass entering.}$$

Rate of mass coming out $= \frac{dm}{dt} = \rho A_1 v_2$

As the fluid is incompressible

$$\rho A_1 v_1 = \rho A_2 v_2 \Rightarrow \boxed{A_1 v_1 = A_2 v_2}\text{; General continuity equation, } \vec{\nabla}.(\rho\vec{v}) = 0$$

Now let work done = change in mechanical energy

Work done on fluid element of mass Δm by the pressure force is

where, Δx = displacement

$$(P_1 A_1 v_1 \Delta t - P_2 A_2 v_2 \Delta t) = \Delta m\left[\frac{P_1}{\rho} - \frac{P_2}{\rho}\right] = (P_1 - P_2)\frac{\Delta m}{\rho}$$

$$\Delta m = \rho_1 v_1 \Delta t = \rho A_2 v_2 \Delta t$$

Net change in mechanical energy

$$\left(\frac{1}{2}\Delta m v_2^2 + \Delta m g z_2\right) - \left(\frac{1}{2}\Delta m v_1^2 + \Delta m g z_1\right)$$

Equating with work done

$$(P_1 - P_2)\frac{\Delta m}{\rho} = \frac{1}{2}\Delta m v_2^2 + \Delta m g z_2 - \frac{1}{2}\Delta m v_1^2 - \Delta m g z_1$$

$$\Rightarrow \quad P_2 + \frac{1}{2}\rho v_2^2 + \rho g z_2 = P_1 + \frac{1}{2}\rho v_1^2 + \rho g z_1$$

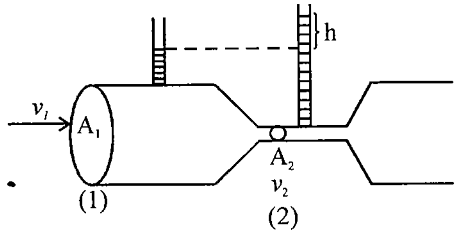
i.e. $$\boxed{\left(P + \frac{1}{2}\rho v^2 + \rho g z\right) = \text{constant}}$$

Or, $$\boxed{\frac{P}{\rho g} + \frac{v^2}{2g} + z = \text{constant}}$$

or, $$\boxed{PV + \frac{1}{2}mv^2 + mgz = \text{constant}}$$

Venturimeter: Measures the flow

Both point have same height.



$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2 \ \& \ A_1 v_1 = A_2 v_2$$

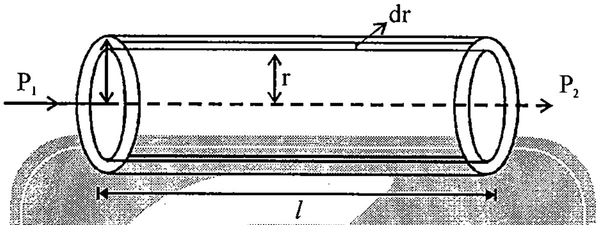
$$(P_2 - P_1) = \frac{1}{2}\rho\left(v_1^2 - v_2^2\right)$$

Or, $$\rho gh = \frac{1}{2}\rho\left(v_1^2 - \left(\frac{A_1}{A_2}v_1\right)^2\right)$$

Or, $$v_1^2 = \frac{2ghA_2^2}{A_2^2 - A_1^2}$$

Flow, $$Q = v_1A_1 = \left(\frac{A_1A_2\sqrt{2g}}{\sqrt{A_1^2 - A_2^2}}\right)\sqrt{h} = k\sqrt{h}$$

Poiseulle's formula:



Assumption:

(1) No radial flow only streamline flow due to constant pressure difference.

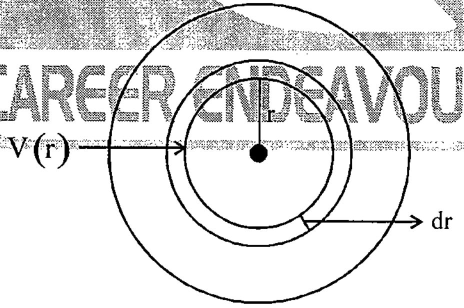
(2) Uniform cross section of the pipe.

(3) Fluid in contact with walls of the pipe is at rest.

Lets take a pipe of uniform cross section of length 'l' and radius 'a', pressure at entering and leaving point of pipe is P_1 and P_2 respectively.

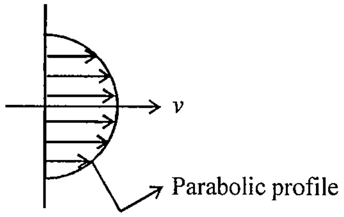
Velocity of the liquid decreases gradually from the axis to the wall where it is zero.

Take V = V(r) at some distance 'r' from the axis.



Viscous drag/unit area $= \eta\frac{dv}{dr}$. Total resisting force over the surface of cylinder of radius 'r' is $\eta\frac{dv}{dr}A$

$$= 2\pi r\ell\eta\frac{dv}{dr}$$



$$P_1 - P_2 = P$$

Now pressure difference = P

Active force due to pressure difference = $P\pi r^2$

$$\therefore \qquad P\pi r^2 = -\eta 2\pi r\ell\frac{dv}{dr} \qquad \left[\frac{dv}{dt} = -\text{ve as v decreases with increasing r}\right]$$

Or, $$Pr = -2\pi\ell\frac{dv}{dr}$$

Or, $$\int_{v(r)}^{0} dv = -\frac{P}{2\eta\ell}\int_{r}^{a} rder\,;\ -V(r) = -\frac{P}{4\eta\ell}\left(a^2-r^2\right);\ V(r) = \frac{P}{4\eta\ell}\left(a^2-r^2\right)$$

Or, $$V(0) = \frac{Pa^2}{4\eta\ell},\ V(a) = 0$$

Now if dv is the volume passed through cylinderical shell of radius 'r' and 'r+dr' then

$$dv = 2\pi rdrv \qquad (Av)$$

$$= \frac{\pi P}{2\eta\ell}\left(a^2-v^2\right)rdr$$

Total volume passed through the tube

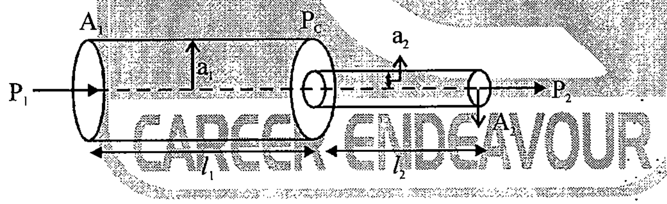
$$= \frac{\pi P}{2\eta\ell}\int_0^a \left(a^2-r^2\right) = \frac{\pi P}{2\eta\ell}\left(\frac{a^2r^2}{2}-\frac{r^4}{4}\right)_0^a = \frac{\pi P}{2\eta\ell}\left(\frac{a^4}{2}-\frac{a^4}{4}\right)$$

$$\boxed{Q = \frac{\pi Pa^4}{8\eta\ell}}$$

Applications: Joining of pipes or capilaries (series or parallel)

$$\begin{bmatrix} \text{Voltage} \Leftrightarrow \text{pressure (P)} \\ \text{Current} \Leftrightarrow \text{Flow(Q)} \end{bmatrix}$$

1. Pipes in series:

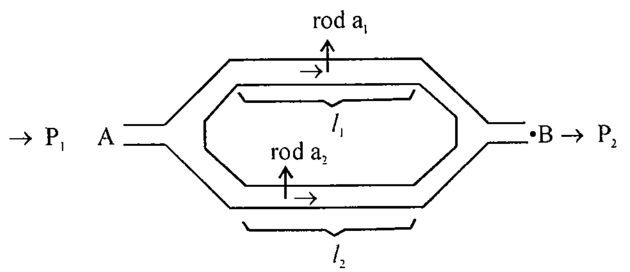


$(P_1 - P_2)$ = Net pressure difference.

$$Q_1 = \frac{\pi(P_1-P_C)a_1^4}{\rho\eta\ell_1} \qquad Q_1 = Q_2 \Rightarrow \frac{(P_1-P_C)a_1^4}{\ell_1} = \frac{(P_C-P_2)a_2^4}{\ell_2}$$

$$Q_2 = \frac{\pi(P_C-P_2)a_2^4}{\rho\eta\ell_2} \qquad \Rightarrow P_C = \left(\frac{P_1a_1^4}{\ell_1}+\frac{P_2a_2^4}{\ell_2}\right)\frac{1}{\left(\frac{a_1^4}{\ell_1}+\frac{a_2^4}{\ell_2}\right)}$$

Pipes in parallel:



ΔP same for both; $Q_1 \neq Q_2$; $Q = Q_1 + Q_2$

$$Q_1 = \frac{\pi P a_1^4}{8\eta \ell_1} \qquad Q_2 = \frac{\pi P a_2^4}{8\eta \ell_2};\ Q = Q_1 + Q_2 = \frac{\pi P}{8\eta}\left(\frac{a_1^4}{\ell_1} + \frac{a_2^4}{\ell_2}\right)$$

Put, $R = \dfrac{8\eta\ell}{\pi a^4} \quad \Rightarrow V = \dfrac{P}{R}$

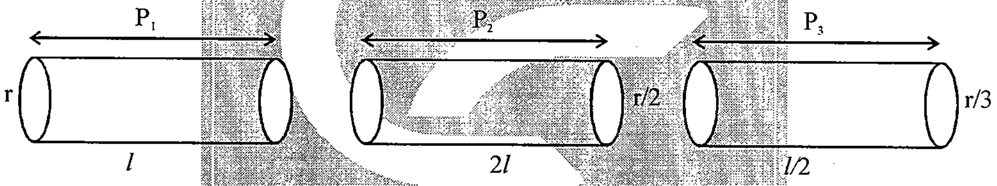
For parallel, $V = P\left(\dfrac{1}{R_1} + \dfrac{1}{R_2}\right)$

For series, $P_1 - P_C = Q8\eta \dfrac{\ell_1}{\pi a_1^4},\ P_C - P_2 = \dfrac{Q8\eta\ell_2}{\pi a_2^4}$

$$P_1 - P_2 = \frac{Q8\eta}{\pi}\left(\frac{\ell_1}{a_1^4} + \frac{\ell_2}{a_2^4}\right)$$

Or, $Q = \dfrac{\pi P}{8\eta} \Big/ \left(\dfrac{\ell_1}{a_1^4} + \dfrac{\ell_2}{a_2^4}\right) = P/(R_1 + R_2)$

Example:



Connected in series:

$$Q = Q_1 = Q_2 = Q_3 = \frac{\pi P_1 r^4}{8\eta\ell} = \frac{\pi P_2 \left(\frac{r}{2}\right)^4}{8\eta(x)} = \frac{\pi P_3 \left(\frac{r}{3}\right)^4}{8\eta \ell/2}$$

Or, $\dfrac{\pi P_1 r^4}{8\eta\ell} = \dfrac{\pi P_2 r^4}{32\eta\ell} = \dfrac{\pi P_3 r^4}{8\eta\ell}\left(\dfrac{2}{81}\right)$

Or, $P_2 = 32P_1;\ P_3 = \dfrac{81}{2}P_1$

Net pressure $= P = P_1 + P_2 + P_3 = (1 + 32 + 81)\ P_1$.

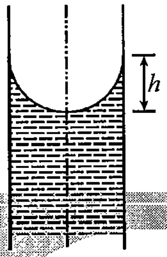
SOLVED PROBLEMS

1. A liquid is kept in a cylindrical vessel which is rotated along its axis. The liquid rises at the sides. If the radius of the vessel is 0.05 m and the speed of rotation is 2 rev. per sec., find the difference in the height of the liquid at the center of the vessel and at its sides.

Soln. When the cylindrical vessel rotates, the velocity at the sides is higher. According to Bernoulli's theorem

$$P+\frac{1}{2}\rho v^2=\text{constant.}$$

Hence, the pressure at sides is lower. Since the pressure at a given horizontal level must be equal, the liquid rises at sides to difference in the height of the liquid at the center of the vessel and at its sides as shown in figure.



Now, $\frac{1}{2}\rho v^2=\rho g h$ ... (i)

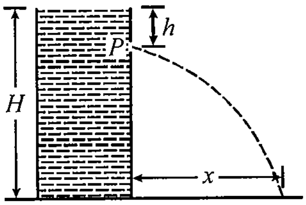
Further, $v=\omega r=2\pi n\times r=2\pi\times 2\times 0.05$ ... (ii)

From equation (i), we get $h=\frac{v^2}{2g}$

Substituting the values, we have $h=\frac{(2\pi\times 2\times 0.05)^2}{2\times 9.8}=\mathbf{0.02\ m.}$

2. Water stands at a depth H in a tank, whose side walls are vertical. A hole is made on one of the walls at a depth h below the water surface. Find at what distance from the foot of the wall does the emerging stream of water strike the floor and for what value of h this range is maximum ?

Soln. The situation is shown in figure.



Here, we have $v_A=\sqrt{(2gh)}$, ... (i)

and $(H-h)=\frac{1}{2}gt^2$ ... (ii)

The distance x is given by $x=v_A\times t$ ... (iii)

From equation (ii), $t=\sqrt{\left(\frac{2(H-h)}{g}\right)}$ ... (iv)

Substituting the value of v_A from equation (i) and the value of t from equation (iv) in equation (iii), we get

$$x=\sqrt{(2gh)}\times\sqrt{[2(H-h)/g}=2\sqrt{[h(H-h)]}.$$

The range x will be maximum when $\frac{dx}{dh}=0$

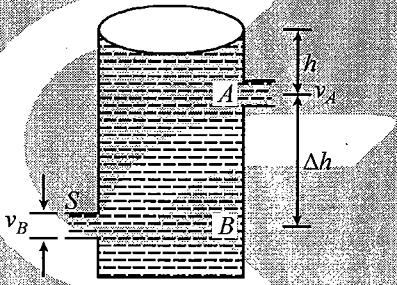
$$\therefore\qquad 2\cdot\frac{1}{2}h^{-1/2}(H-h)^{-1/2}-2h^{1/2}\cdot\frac{1}{2}(H-h)^{1/2}=0$$

Solving we get $h=\frac{H}{2}$.

$$x=2\sqrt{\frac{H}{2}\times\left(H-\frac{H}{2}\right)}=H.$$

3. On the opposite sides of a wide vertical vessel filled with water two identical holes are opened, each having cross-sectional area S. The height difference between them is equal to Δh. Find the resultant force of reaction of the water flowing out of the vessel.

Soln. The situation is shown in figure.



Discharge per sec. through $A=S\,v_A$

Discharge per sec. through $B=S\,v_B$

Force of reaction at $A=\rho Q_A v_A=\rho S v_A^2$

Similarly, force of reaction at $B=\rho S v_B^2$

Net force $F=\rho S(v_A^2-v_B^2)$ ... (i)

Applying Bernoulli's theorem at point A and B, we have

$$P_a+h\rho g+\frac{1}{2}\rho v_A^2=P_a+(h+\Delta h)\rho g+\frac{1}{2}\rho v_B^2$$

Here P_a is atmosperic pressure $\frac{1}{2}\rho\left[v_A^2-v_B^2\right]=\Delta h\,\rho\,g$ ... (ii)

From equation (i) $(v_A^2-v_B^2)=\left(\frac{F}{\rho S}\right)$

Substituting this value in equation (ii), we get $\frac{1}{2}\rho\left(\frac{F}{\rho S}\right)=\Delta h\,\rho\,g$

$$\therefore\qquad F=(2\,\Delta h\,\rho\,g\,S).$$

4. Water is flowing through two horizontal pipes of different diameters which are connected together. In the first pipe the speed of water is 4 m/sec and the pressure is 2.0×10^4 n/m^2. Calculate the speed and pressure of water in the second pipe. The diameters of the pipes are 3 cm and 6 cm respectively.

Soln. From continuity equation, $A_1 v_1 = A_2 v_2$ or $(\pi r_1^2) v_1 = (\pi r_2^2) v_2$

Therefore, $v_2 = \left(\frac{r_1}{r_2}\right)^2 v_1$. Here, $r = 1.5\text{ cm} = 1.5 \times 10^{-2}$ meter, $r_2 = 3 \times 10^{-2}$ meter and $v_1 = 4$ m/sec.

Therefore, $v_2 = \left(\frac{1.5 \times 10^{-2}}{3 \times 10^{-2}}\right) \times 4 = \mathbf{1\ m/sec.}$

Now, by Bernoulli's theorem, $P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$.

Therefore, $P_2 = P_1 + \frac{1}{2}\rho(v_1^2 - v_2^2)$

Substituting the given values, we get

$$P_2 = 2.0 \times 10^4 + \frac{1}{2} \times (1 \times 10^3) \times (4^2 - 1^2) = 2.0 \times 10^4 + 0.75 \times 10^4 = \mathbf{2.75 \times 10^4\ newton/meter^2}.$$

5. In a horizontal pipe line of uniform area of cross section, the pressure falls by 8 N-m^2 between two points separated by a distance of 1 km. What is the change in kinetic energy per kg of the oil flowing at these points? Density of oil is 806 kg-m^{-3}.

Soln. According to Bernoulli's theorem, $P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$ (pipe horizontal)

or $\quad P_1 - P_2 = \frac{1}{2}\rho(v_2^2 - v_1^2)$

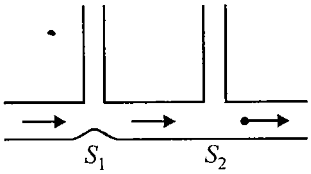
or $\quad \frac{P_1 - P_2}{\rho} = \frac{1}{2}(v_2^2 - v_1^2)$

$\quad = $ change in kinetic energy per kg mass.

Therefore, change in kinetic energy per kg mass of oil $= \frac{P_1 - P_2}{\rho}$

Substituting the given values, we have change in K.E. per kg mass $= \frac{8}{800} = \mathbf{10^{-2}\ J/kg.}$

6. Two manometric tubes are mounted on a horizontal pipe of varying cross-section at the section S_1 and S_2 as shown in figure.



Find the volume of water flowing across the pipe's section per unit time if the difference in water columns is equal to Δh.

Soln. Given that the difference in water columns is equal to Δh, hence

$$P_1 - P_2 = \Delta h \rho g \text{ or } P_2 - P_1 = \Delta h \rho g \qquad \ldots \text{(i)}$$

According to continuity equation

$$S_1 v_1 = S_2 v_2 \text{ or } \frac{v_2}{v_1} = \frac{S_1}{S_2} \qquad \text{... (ii)}$$

Applying Bernoulli's theorem

$$\frac{P_1}{\rho} + \frac{1}{2}v_1^2 = \frac{P_2}{\rho} + \frac{1}{2}v_2^2$$

or $$\frac{P_1 - P_2}{\rho} = \frac{1}{2}v_2^2 - \frac{1}{2}v_1^2 = \frac{v_1^2}{2}\left[\left(\frac{v_2}{v_1}\right)^2 - 1\right] \qquad \text{... (iii)}$$

Substituting the value of $(P_1 - P_2)$ from equation (i) and $\left(\frac{v_2}{v_1}\right)$ from equation (ii) in equation (iii), we get

$$\frac{\Delta h \rho g}{\rho} = \frac{v_1^2}{2}\left[\left(\frac{S_1}{S_2}\right)^2 - 1\right] \text{ or } \Delta h g = \frac{v_1^2}{2}\left[\frac{S_1^2 - S_2^2}{S_2^2}\right] \text{ or } v_1 = \sqrt{\left[\frac{2\Delta h g S_2^2}{(S_1^2 - S_2^2)}\right]} = S_2\sqrt{\left[\frac{2\Delta h g}{(S_1^2 - S_2^2)}\right]}$$

Now, $$Q = S_1 v_1 = S_1 S_2\sqrt{\left[\frac{2\Delta h g}{(S_1^2 - S_2^2)}\right]} = S_1 S_2\sqrt{\left[\frac{2\Delta h g}{(S_2^2 - S_1^2)}\right]}.$$

7. A vertical tube has diameter 0.016 m at its bottom end from which water flows out at the rate of 1.2 kg per minute. The pressure at the end is atmospheric pressure 0.7 m of mercury. If the diameter of the tube is 0.004 m at a height of 0.3 m from the bottom end, find the pressure there.

Soln. Mass of water flowing in one second = 1.2/60 = 0.02 kg.

Let v_1 be the velocity of flow of water at the bottom, then the mass of the water flowing per second $(\pi r^2) v_1 \rho$.

$$\therefore \qquad (\pi r^2) v_1 \rho = 0.02$$

Here, $r_1 = \frac{0.006}{2} = 0.003$ m and $\rho = 10^3$ kg/m^3

$$\therefore \qquad \{3.14 \times (0.003)^2\} v_1 10^3 = 0.02 \text{ or } v_1 = \frac{0.02}{3.14 \times (0.003)^2 \times 10^3} = 0.7074 \text{ m/sec.}$$

Let v_2 be the velocity of flow of water at the point where the radius of the tube is 0.004/2, i.e., 0.002 m. The mass of the water flowing per second is $\pi(0.002)^2 v_2 \times 10^3$. As the mass of the water flowing through any point in one second is the same, hence $\pi(0.002)^2 v_2 10^3 = 0.02$.

$$\therefore \qquad v_2 = \frac{0.02}{3.14 \times (0.002)^2 \times 10^3} = 1.592 \text{ m/sec.}$$

Applying Bernoulli's theorem, we have $P_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2$.

Here, $P_1 = 0.76$ m of mercury

$= 0.76 \times (13.6 \times 10^3) \times 9.8$ N/m^2

$P_2 = h$ m of mercury

$$= h \times (13.6 \times 10^3) \times 9.8 \text{ N/m}^3$$

$$(h_2 - h_1) = 0.3 \text{ m},$$

$$v_1 = 0.7074 \text{ m/sec and } v_2 = 1.592 \text{ m/sec.}$$

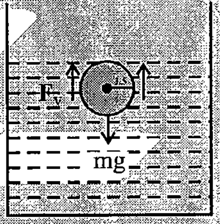
Now, $$\frac{P_1 - P_2}{\rho} = \frac{1}{2}(v_2^2 - v_1^2) + g(h_2 - h_1)$$

or $$\frac{(0.76 - h)\, 13.6 \times 10^3 \times 9.8}{10^3} = \frac{1}{2}\left[(1.592)^2 - (0.70742)^2\right] + 0.3 \times 9.8$$

$$0.76 - h = \frac{3.957}{13.6 \times 9.8} = 0.0297$$

$\therefore$ $h = 0.76 - 0.0297 =$ **0.7303 m of mercury.**

8. A steel ball of diameter d= 3.0mm sinking with zero initial velocity in olive oil whose viscosity $\eta = 0.90\text{P}$. How soon after the beging of motion will the velocity of the ball differ from the steady state velocity be $n = 1.0\,\%$?



Soln. According to the question. The equation of motion.

$$m\frac{dV}{dt} = mg - 6\pi\eta rv$$

$$\Rightarrow \quad \frac{dV}{dt} + \frac{6\pi\eta r}{m}v = g$$

$$\Rightarrow \quad \frac{dV}{dt} + kV = g \quad , k = \frac{6\pi\eta r}{m}$$

Or, $$e^{kt}\frac{dv}{dt} + ke^{kt}v = ge^{kt};\; v\,e^{kt} = \frac{g}{k}e^{kt} + C;\; v = \frac{g}{k} + C\,e^{-kt} \text{ (where C = constant)}$$

Since $v = 0$ for $t = 0$, $0 = \frac{g}{k} + C,\; C = \frac{-g}{k}$

Thus, $$v = \frac{g}{k}\left(1 - e^{-kt}\right)$$

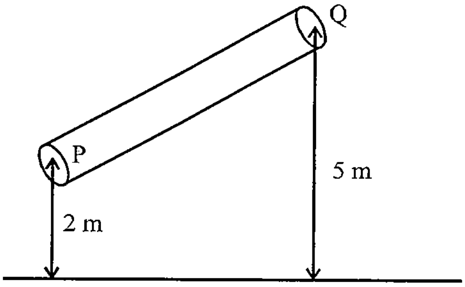
The steady stete velocity is $\frac{g}{k}$

v differs from $\frac{g}{k}$ by 'n' where $e^{-kt} = n$, $t = \frac{1}{k}\ln n$

$$\frac{1}{k} = -\frac{\frac{4\Pi}{3}r^3\rho}{6\Pi\eta r} = \frac{-4r^2\rho}{18\eta};\; \frac{1}{k} = -\frac{-d^2\rho}{18\eta}$$ (By neglecting buoyncy in olive oil).

Now one can get eavily the value of 't' by potting the values of d, ρ α η.

9. A non-viscous liquid of constant density $1000 \text{kg}/\text{m}^3$ flows in a streamline motion along a tube of variable cross section. The tube is kept inclined in a plane as shown in the fig. The area of crossection of the tube of two points at height of 2meter & 5 meter are respeetrelly $4\times10^{-3}\text{m}^2$ & $8\times10^{-3}\text{m}^2$. The velocity of liquid at point 'P' is 1m/se. Find the work done per unit volume by the pressure and the gravity forces as the fluid flows from point p to Q



Soln. As gravitational find is conserrative i.e. $W = -U$

Or, $$\left(\frac{dW}{dV}\right)_g = \frac{dU}{dV} = -\frac{mg(h_2-h_1)}{V} = -\rho g(h_2-h_1)$$

So work done by the force of gravity per unit volume

$$\left(\frac{dW}{dV}\right)_g = -\rho g(h_1-h_2) = -10^3\times9.8(5-2) = -2094\times10^4\frac{3}{m^3} \quad ...(1)$$

Now in case of aideal fluid motion by consentration of mass i.e.,

$$\left(\frac{dm}{dt}\right)_1 = \left(\frac{dm}{dt}\right)_2$$

i.e., $$(\rho AV)_1 = (\rho AV)_2$$

Or, $$A_1V_1 = A_2V_2 = (B = \text{conet})$$

So $$V_2 = \frac{A_1V_1}{A_2} = \frac{4\times10^{-3}\times1}{8\times10^{-3}} = \frac{1}{2}\text{m/se} \quad ...(2)$$

Now as work done unit volume by pressure,

$$\left(\frac{dW}{dV}\right)_P = \frac{PdV}{dV} = P = P_2 - P_1$$

But by Bernoull is theorm

$$P_1 + \rho gh_1 + \frac{1}{2}\rho v_1^2 = P_2 + \rho gh_2 + \frac{1}{2}\rho v_2^2$$

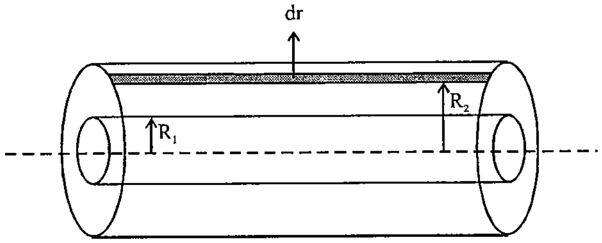
$$\Rightarrow \left(\frac{dw}{dv}\right)_P = P_1 - P_2 = \rho g(h_2-h_1) + \frac{1}{2}\rho(v_2^2 - v_1^2)$$

Which in light of equation (1) & (2)

$$\left(\frac{dW}{dV}\right)_P = 2.94\times10^4 + \frac{1}{2}10^3\left[(0.5)^2 - 1^2\right] = 29025 \text{ J}$$

10. Along Cylindrical radius R_1 is displaced along its axis with a containt velocity V_0 inside a stationary co-oxial cylinder of Radius . The space between the cylinder s is filled with the viscous liquids are function of the distance r from the axis of the cylinders.The flow is laminar

Soln.



Let's consider a co-oxial cylinder of radius 'r' & thickness dr, then force of friction or riscous force on this general Layer

$$F = 2\pi r \ell \eta \frac{dv}{dr}$$

This force must be constent form Layer to layer so that study motion may be possible

Or, $$\frac{fdr}{r} = 2\pi \ell \eta dv \qquad ...(1)$$

On Intergrating,

$$f\int_{R_2} \frac{dr}{r} = 2\pi \ell \eta \int_0^v dv \; ; \; f \ln\left(\frac{r}{R_2}\right) = 2\pi \ell \eta V \qquad ...(2)$$

Putting, $r = R_1$ we get

$$f \ln\left(\frac{R_1}{R_2}\right) = 2\pi \ell \eta v_0 \qquad ...(3)$$

form equation (2) & (3) we get

$$\boxed{v = v_0 \frac{\ell n(r/R_2)}{\ell n(R_1/R_2)}}$$

Note: The force f is supplied by the agency which tries to carry the inner cylinder with velocity V_0.

11. When a sphere of radius $r_1 = 1.2$ mm moves in glycerin the laminar flow is observed if the velocity of the sphere dues not exceed $V_1 = 23$ cm / sec . At what minimum velocity V_2 of a sphere of radius $r_2 = 5.5$ cm will the flow in water because turbulent? the viscositives of glycerin and water are equal to $\eta_1 = 13.9\rho$ $\propto \eta_2 = 0.071\rho$ respectively

Soln. We know that reynolais number for turbuient flow is greater than that of laminor flow

$$(Re)\ell = \frac{2\rho_1 V_1 r_1}{\eta_1} \qquad \left(\therefore = \frac{\rho v d}{\eta}\right)$$

And, $$(Re)_t = \frac{2\rho_2 V_2 r_2}{\eta_2} \qquad \text{But, } (Re)_t \geq (Re)\ell$$

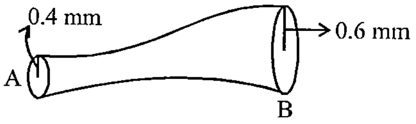
Then, $$\left(V_2\right)_{min} = \frac{S_1 V_1 r_1 \eta_2}{S_2 r_2 \eta_1}$$

By putting thye values weget

$$\boxed{\left(V_2\right)_{min} = 5\mu / se}$$

12. A tube as shown i fig. consisting of two values at its two ends. At end 'A' the the fluid having riscosity 9.1P exert a force of 1.5 N and gets out at another end 'B' by a force one the value measured as 0.6N. If the smaller and larger radii of the tube is given by 0.4 mm and 0.6 mm respectvely. Find out the flow rate at an average point inside the tube.

Soln.



Given that the forece at end 'A'

$$F_A = 1.5\text{ N}.$$

Area of $'A' = \pi r^2 = \pi(4)^2 = 16\pi\text{ cm}^2$

13. The gross-seetional radius of a pipeline decrases gradually as $r = r_0 e^{-ax}$, where $\alpha = 0.50\text{m}^{-1}$, x is the distance from the pipeline inlet. Find the ratio of Reynold number for two cross-see times seperated by $\Delta x = 3.2$ m.

Soln. we know that Reynold's number (Re) is defined as $Re = \frac{\rho V \ell}{\eta}$, where v is the velocity ℓ is characteristic length and η the coeffient of viscosity. In this case of circular cross section the charectiristic length is the diameter of cross- section 'd' and V is taken as average velocity of flow of liquied,

Now Re, (Reynold os number at x, from the pipe end) $= \frac{\rho d_1 V_1}{m\eta}$ where V_1 is the velocity at distance x,

And similarly, $Re_2 = \frac{\rho d_2 V_2}{\eta}$ so $\frac{Re_1}{Re_2} = \frac{d_1 V_1}{d_2 V_2}$

From equation of continuity, $A_1 V_1 = A_2 V_2$

Or, $$\pi r_1^2 V_1 = \pi r_2^2 V_2$$

Or, $$d_1 V_1 r_1 = d_2 V_2 r_2\;;\; \frac{d_1 V_1}{d_2 V_2} = \frac{r_2}{r_1} = \frac{r_0 e^{-ax_2}}{r_0 e^{-ax_1}} = e^{-a\Delta x}\left(\text{as } x_2 - x_1 = \Delta x\right)$$

Thus, $$\frac{Re_2}{Re_1} = e^{a\Delta x} = 5$$

14. A 3.6 m long vertical pipe resonates with a source of ferquency 212.5Hz when water level is at a certain heights is the pipe. Find the heights of water level (from the botton o the pipe) at which resonance occur. Neglet end correction. New the pipe is fvies to a heigher $H(\approx 3.6\text{m})$. A small hole is drilled very close to its bottom and water is allowed to leak. Obtain an expression for the rate of fall of water leves in the pipe as a function of H. If the radii of the pipe and hole are $2\times10^{-2}\,\text{m}$ and $1\times10^{-3}\,\text{m}$ respectivelly, calculates the time interval between the occurance of first two resonances. Speed of sound is air is 340 m/s and $g = 10\,\text{m}/\text{s}^2$.

Soln. For resonance to occur in a closed pipe,

$$\ell = \frac{(2n-1)\lambda}{4} \text{ where } n = 1, 2, 3........$$

Here, $$\lambda = \frac{V}{\eta} = \frac{340}{212.5} = 1.6\text{ m}$$

So resonce occuer for length of air column 0.4m, 1.2m, 2.0 m, 2.8m and 3.6m, and corresponding heighet of water column from bottom.

$$= 3.2\text{m}, 2.4\text{m}, 1.6\text{m}, 0.8\text{m}, \text{and } 0\text{m}$$

The velocity of eff;ux, when height of water level is h above bottom $= \sqrt{2gh}$

Applying continuity equation

$$A\left(-\frac{dh}{dt}\right) = a\sqrt{2gh}$$

where a and a are area of cross- section of hole and pipe respectively

$$dt = -\frac{A}{a\sqrt{2gh}}dh$$

So, the time in which the height of water column changes from 3.2 m to 2.4 m is time interva between first to resonances,

$$t = -\frac{A}{a}\int_{3.2}^{2.4}\frac{1}{\sqrt{2gh}}dh = -\frac{A}{a\sqrt{2g}}2\left|h^{1/2}\right|_{3.2}^{2.4}$$

$$= \frac{\pi\left(3\times10^{-2}\right)^2\times2}{\pi\left(1\times10^{-3}\right)\sqrt{2\times10}}\left[\sqrt{3.2}-\sqrt{2.4}\right] = 42.87\text{ s}$$

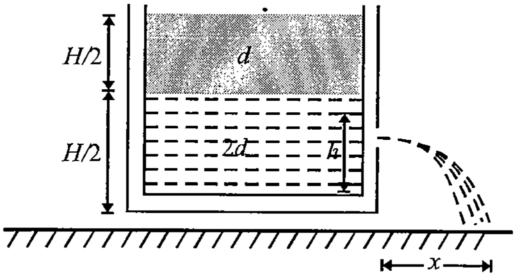
15. A container of large uniform cross- section area A resting on a horizontal surface holds two immiscible non-viscoss and incompressible liquids of densities d are 2d each of height $(H/2)$ as shown is fig. 1.The lower density liquid is ofen to the atoms phere having prssures Po (a) A homogeneous solid cylinder of length $L\left(L < H/2\right)$, cross-sectional area (A/s) is Immrsed such that it floats with uits axis vertical a at the liquid-liquid interface with length (L/4) in the densder liquid. Dtermine (i) the density of solid and (ii) the total prssure at the bottom of the contaier (b) the cylinder is removed and original arrangement is restored. A ting hole of area S(S<<A) is punched on the vertical side of the container at a hight $h\left(h < H/2\right)$.

Determine (i) the initial speed of efflux of the liquid of the hole

(ii) The horizontal distance x travelled by the liquied inttally and

(iii) The hight h_m at which the hole should be punched so that the liquid travels the maximum distance x_m ......

Also calculate x_m



Soln. (a) (i) As for floating, w = Th

$$VPg = V_1d_1g + V_2d_2g$$

Or, $$L\left(\frac{A}{5}\right)\rho = \left(\frac{3}{4}L\right)\left(\frac{A}{5}\right)d + \left(\frac{1}{4}L\right)\left(\frac{A}{5}\right)2d$$

i.e., $$\rho = \frac{3}{4}d + \frac{2}{4}d = \frac{5}{4}d$$

(ii) Total pressure = P_0 + (weight of liquid + weight of solid) |A

$$P = P_0 + \frac{H}{2}dg + \frac{H}{2}2dg + \frac{5}{4}d \times \left(\frac{A}{5} \times L\right) \times g \times \frac{1}{A}$$

$$P = P_0 + \frac{3}{2}Hdg + \frac{1}{4}Ldg = P_0 + \frac{1}{4}(6H + L)dg$$

(b) (i) By Bernoulli 's theoram for a point first inside and outside the hole

$$P_1 + \frac{1}{2}\rho V_1^2 = P_2 + \frac{1}{2}\rho V_2^2$$

i.e., $$P_0 + \frac{H}{2}dg + \left(\frac{H}{2} - h\right)2dg = P_0 \frac{1}{2}(2d)V^2$$

$$g(3H - 4h) = 2V^2 \qquad V = \sqrt{(g/2)(3H - 4h)}$$

(ii) As at the hole vertical velocity of liquid is zero to time taken by it to reach the ground,

$$t = \sqrt{(2h/g)}$$

So that $$x = vt + \sqrt{\frac{g}{2}(3H - 4h)}\sqrt{\frac{2h}{g}} = \sqrt{h(3H - 4h)}$$

(iii) For x to be maximum, x^2 must be maximum i.e.,

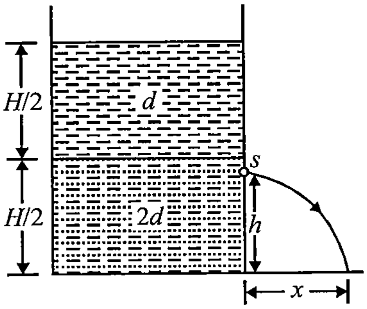
$$\frac{d}{dh}\left(x^2\right) = 0$$

Or, $$\frac{d}{dh}\left(3Hh - 4n^2\right) = 0$$

3H-8h =0, i.e., $h = (3/8)^H$

And $$x_{max} = \sqrt{\frac{3H}{8}\left(3H - \frac{3}{2}H\right)} \qquad = \frac{3}{4}H$$

16. A container of large uniform cross-sectional area A resting on a horizontal surface holds two immiscible, non-viscous and incompressible liquids of densities d and $2d$, each of height $H/2$ as shown. The lower density liquid is open to the atmosphere having pressure P_0. A tiny hole of area $s\ (s \ll A)$ is punched on the vertical side of the container at a height $h (h < H/2)$. Determine :



(i) the initial speed of efflux of the liquid at the hole.

(ii) the horizontal distance distance x travelled by the liquid initially, and

(iii) the height h_m at which the hole should be punched so that the liquid travels the maximum distance x_m initially. Also calculate x_m.

(Neglect the air resistance in these calculations.

Soln. (i) Here the effective height of orifice can not be taken as $(H-h)$ due to the pressure of two liquids. Using Bernoulli's theorem, just inside and just outside the orifice the orifice, we have

$$P_0 + \frac{H}{2}dg + \left(\frac{H}{2} - h\right)(2d)g = P_0 + \frac{1}{2}(2d)v^2$$

where v is the velocity of efflux

$$\therefore \quad \frac{H}{2}dg + \frac{H}{2}(2d)g - h(2d)g = \frac{1}{2}(2d)v^2$$

$$\text{or} \quad \frac{3}{2}Hg - 2hg = v^2$$

$$\text{or} \quad v = \sqrt{\left[\frac{g}{2}(3H-4h)\right]} \quad \text{...(i)}$$

(ii) Let t be the time taken by the liquid to hit the ground. Then

$$h = \frac{1}{2}gt^2 \quad \text{or} \quad t = \sqrt{\left(\frac{2h}{g}\right)}$$

The horizontal range $x = v \times t$

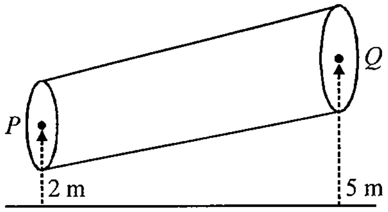
$$\therefore \quad x = \sqrt{\left[\frac{g}{2}(3H-4h)\right]} \times \sqrt{\left(\frac{2h}{g}\right)} = \sqrt{[(3H-4h)h]} \quad \text{... (ii)}$$

(iii) To find out the maximum height h_m, we differentiate x with respect to h and equate it to zero for maximum h. Hence, $\frac{dx}{dh} = \frac{d}{dh}\sqrt{[(3H-4h)h]} = 0$.

$$\therefore \quad 3H - 8h = 0 \quad \text{or} \quad h = \frac{3}{8}H \quad \text{... (iii)}$$

$$\therefore \quad x_m = \sqrt{\left[\left(3H - \frac{4 \times 3}{8}H\right)\frac{3}{8}H\right]} = \frac{3H}{4}. \quad \text{... (iv)}$$

17. A nonviscous liquid of constant density 1000 kg/m^3 flows in a streamline motion along a tube of variable cross section. The tube is kept inclined in the vertical plane as shown in figure. The area of cross-section of the tube at two points P and Q at height of 2 meters and 5 meters are respectively 4×10^{-3} m^2 and 8×10^{-3} m^2. The velocity of the liquid at point P is 1 m/s. Find the workdone per unit volume by the pressure and the gravity forces as the liquid flows from point P to Q.



Soln. The workdone per unit volume by the force of gravity,

$$W_{g/vol.} = \rho g \Delta h$$

$$= 10^3 \times 9.8 \times 3 = \mathbf{2.94 \times 10^4\ J/m^3.}$$

From the equation of continuity, $A_1 v_1 = A_2 v_2$ or $v_2 = \dfrac{A_1 v_1}{A_2}$

$$\therefore \quad v_2 = \frac{4 \times 10^{-3} \times 1}{8 \times 10^{-3}} = 0.5 \text{ m/s.}$$

Now, increment of kinetic energy of fluid per unit volume.

$$= \frac{1}{2}\rho(v_2^2 - v_1^2) = \left(\frac{1}{2} \times 10^3\right)\left[\frac{1}{4} - 1\right] = -3.75 \times 10^2 \text{ J/m}^3.$$

We know that net energy of the fluid per unit volume, in a stream line flow remains the same and hence the net workdone by pressure per unit volume of the fluid is given by

$$W_{p/vol.} = (2.94 \times 10^4 - 3.75 \times 10^2) = \mathbf{29025\ J/m^3.}$$

Oscillations, Waves & Optics

for

M.Sc. Entrance

IIT-JAM | JNU | TIFR | BHU | DU

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PREFACE

The present booklet on Oscillations, Waves & Optics section of the Physics-PH deals with the syllabi mentioned in the IIT-JAM & other M.Sc. Entrance Exams in their respective notificaions. The Booklet has been designed and organised in a manner so that the aspirants can make full use of it for their immaculate preparation.

The panel of experts and faculties in the R&D team of Career Endeavour have done lots of hardwork and that too with immense care in formulating the booklet into its present form. A brief discussion is given on every topic mentioned in syllabus alongwith ample of solved examples on the concepts discussed. The idea is not only to bring out the basic concepts clearly but also to help the students in developing the analytical approach for solving any problem.

This booklet contains ***Previous Year Questions*** *of IIT-JEE, IIT-JAM, IIT-JAM(GP), JNU, TIFR, JEST and IISc as solved examples relevant to the topic. It is expected that the booklet will form an integral part of the Classroom/ Corresspondence program which the student would join for his/her success at Career Endeavour. Even though due care has been taken while preparing the booklet, the booklet may contain errors/mistakes here and there, we sincerely apologize for the same. We would be also thankful to those readers who point out the errors.*

Finally, we from the R&D team of Career Endeavour, wish all the aspirants 'GOOD LUCK' and will always be available to help them out so that they come up with 'flying colors'.

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Chapter 1

Simple Harmonic Motion

1.1 DESCRIPTION OF SIMPLE HARMONIC MOTION (SHM):

1.1.1 Equation of motion:

Suppose a particle of mass m is undergoing SHM along a line. If x be its displacement at any instant from the position of stable equilibrium, then restoring force F under small oscillation approximation may be written as

$$F = -sx \qquad \ldots (1)$$

where s is the restoring force per unit displacement, called force constant.

The negative sign indicates that F and x are oppositely directed. For spring restoring force follows Hooke's law of elasticity. From Newton's law, the equation of motion is

$$m\frac{d^2x}{dt^2} = -sx \Rightarrow \frac{d^2x}{dt^2} + \omega^2 x = 0 \qquad \ldots (2)$$

where $\omega^2 = s/m$.

Equation (2) is the differential equation of motion of a simple harmonic oscillator.

Its general solution is given by the principle of superposition as

$$x = c_1 e^{j\omega t} + c_2 e^{-j\omega t} \qquad \ldots (3)$$

$$\Rightarrow \quad x = c_1(\cos\omega t + j\sin\omega t) + c_2(\cos\omega t - j\sin\omega t)$$

$$= A\cos\omega t + B\sin\omega t \qquad \ldots (4)$$

where, $A = (c_1 + c_2)$ and $B = j(c_1 - c_2)$.

Again, introducing two other constants a and ε by the relations $A = a\sin\varepsilon$ and $B = a\cos\varepsilon$, such that $a = \sqrt{A^2 + B^2}$ and $\tan\varepsilon = A/B$, solution (4) can be put in the form,

$$x = a\sin(\omega t + \varepsilon) \qquad \ldots (5)$$

Putting $\varepsilon = \frac{\pi}{2} + \delta$, the solution (5) can also be expressed as

$$x = a\cos(\omega t + \delta) \qquad \ldots (6)$$

1.1.2 Characteristics of SHM:

Let us choose the solution, $x = a\cos(\omega t + \delta)$

Here 'a' represents the maximum value of displacement from the position of stable equilibrium. It is called the **amplitude** of SHM. The **time period** T is the time interval in which the motion repeats itself and is given by

$$x(t) = x(t+T)$$

$$\Rightarrow \quad a\cos(\omega t + \delta) = a\cos\left[\omega(t+T) + \delta\right] = a\cos\left[\omega t + \delta + \omega T\right]$$

$$\Rightarrow \quad a\cos\left[(\omega t + \delta) + 2\pi\right] = a\cos\left[(\omega t + \delta) + \omega T\right]$$

For equality we must have, $\omega T = 2\pi$ or, $T = 2\pi/\omega = 2\pi\sqrt{\frac{m}{s}}$ ... (7)

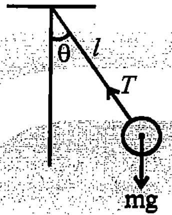
The quantity ω is called the **angular frequency**. The usual frequency ν which gives the number of oscillations per unit time is given by $\nu = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{s}{m}}$ ... (8)

The quantity $(\omega t + \delta)$ in equation (6) is called the phase of motion and δ is called the initial phase or epoch.

Effective acceleration due to gravity (g')

When simple pendulum is kept in inertial frame its time period is given by

$$T = 2\pi\sqrt{\frac{l}{g}}$$



When simple pendulum is kept in non-inertial frame its time period is given by

$$T = 2\pi\sqrt{\frac{l}{g'}},$$

where g' is called effective gravity and is given by

$$\vec{g}' = \vec{g} - \vec{a} \quad \text{... (9)}$$

where $\vec{a}$ is acceleration of the frame.

Ex. 1. A simple pendulum, suspended from the ceiling of a stationary cart, has a time period 2 seconds. When the cart accelerates in the horizontal direction with an acceleration of 10 m/s², the time period of the pendulum is ($g = 10\ \text{m/s}^{-2}$)

(a) $2^{1/2}$ seconds (b) $2^{3/2}$ seconds (c) $2^{1/4}$ seconds (d) $2^{3/4}$ seconds

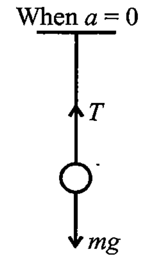
[JAM-GP: 2007]

Ans. (d)

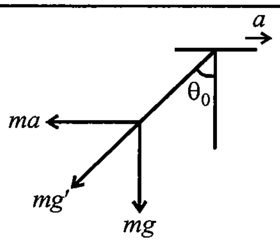
Soln. Forces on bob

1. Gravity downward : mg
2. Tension force : T
3. Psuedo force : $-ma$

Now, at equilibrium condition, sum of gravitational force and pseudo force will be balanced by tension force.



Mean position
$\theta = 0°$



Mean position
$\theta = \theta_0$

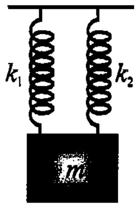
Here $g' = \vec{g} - \vec{a}$

$\Rightarrow g' = \sqrt{a^2 + g^2}$

$$\text{Time period} = 2\pi\sqrt{\frac{l}{g'}} = 2\pi\sqrt{\frac{l}{\sqrt{g^2+a^2}}} = 2\pi\left[\sqrt{\frac{l}{g\sqrt{1+\frac{a^2}{g^2}}}}\right] = 2\pi\sqrt{\frac{l}{g}}\frac{1}{\left(1+\frac{a^2}{g^2}\right)^{1/4}} = 2\times\frac{1}{2^{1/4}} = 2^{3/4}\text{ sec.}$$

Ex. 2. Two springs of force constant k_1 and k_2 are arranged in a parallel arrangement and a mass 'm' is suspended from it. The arrangement is equivalent to a single spring of constant k given by **[DU 2015]**

(a) $k_1 - k_2$ (b) $k_1 + k_2$ (c) $k_1 k_2 / (k_1 + k_2)$ (d) k_1 / k_2

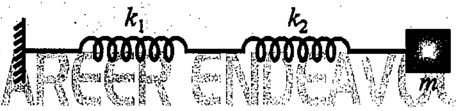


Soln. **(b)** If an elongation of x is given to the system, then restoring force

$$F = -k_1 x - k_2 x = -(k_1 + k_2)x = -kx$$

where, $k = k_1 + k_2 \equiv$ equivalent spring constant

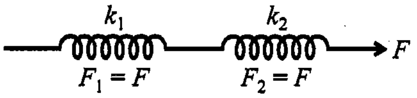
Ex. 3. Find the time period for the arrangement of the spring mass system having constant k_1 and k_2.



Soln. Let x be the displacement of mass m from its equilibrium position at an instant and x_1 and x_2 are the extension in the springs k_1 and k_2 respectively.

$x = x_1 + x_2$ ... (i)

Tension force at each points in a spring is equal to the external force applied at one of its end,



i.e., $F = F_1 = F_2$ ... (ii)

$-k'x = -k_1 x_1 = -k_2 x_2$

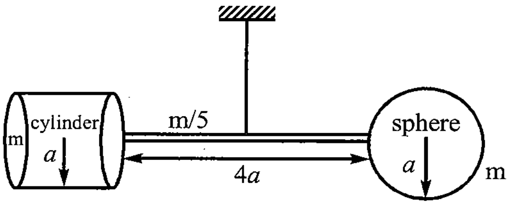
where k' is equivalent spring constant,

From equation (i)

$$\frac{F}{k'} = \frac{F}{k_1} + \frac{F}{k_2} \Rightarrow \boxed{\frac{1}{k'} = \frac{1}{k_1} + \frac{1}{k_2}} \quad \text{... (iii)}$$

i.e., if there is same tension force in all connected springs, equivalent spring constant is given by equation (iii).

Ex. 4. A solid cylinder of mass m, radius a and height also a and a solid sphere of the same mass and the same radius are connected to the two ends of a thin rod of mass m/5 and length $4a$. The line joining the centres of mass of the cylinder and the sphere coincides with the axis of the rod. The vertical cross-section of the assembly is shown in the figure below. The whole assembly is suspended vertically by a massless wire passing through its centre of mass. The torsional constant of the wire is 76 ma^2/sec^2. Calculate the frequency of the torsional oscillation about the suspension in the horizontal plane.



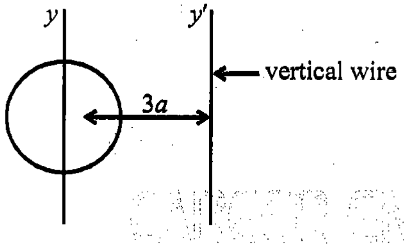
[JAM-GP: 2010]

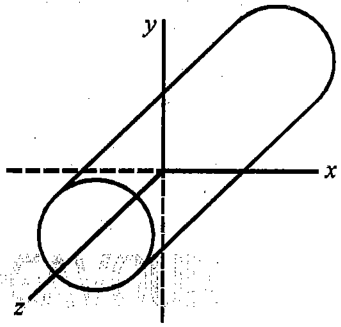
Soln. Let us calculate the moment of inertia about of cylinder and sphere, and the rod about vertical wire.

Cylinder : $I_z = I_x + I_y = \frac{1}{2}ma^2$ (Perpendicular Axes Theorem)

$I_x = I_y = \frac{1}{4}ma^2$

Applying Parallel Axes Theorem, moment of inertia about vertical wire of :





Cylinder : $I_c' = I_{cm} + m(3a)^2 = \frac{1}{4}ma^2 + 9ma^2 = \frac{37}{4}ma^2.$

Sphere : $I_s = I_{cm} + m.(3a)^2 = \frac{2}{5}ma^2 + 9ma^2 = \frac{47}{5}ma^2.$

Rod : $I_r = \frac{1}{12}ML^2 = \frac{1}{12}\cdot\frac{m}{5}\times 16a^2 = \frac{4}{15}ma^2.$

Moment of inertia of system about the vertical wire,

I = sum of moment of inertia of individual bodies about vertical wire

$$= \frac{37}{4}ma^2 + \frac{47}{5}ma^2 + \frac{4}{15}ma^2$$

$$I = \frac{227}{12}ma^2$$

Now, if the rod along with the cylinder and sphere is twisted by an angle of θ, a restoring torsional torque, $\tau = -k\theta$ acts and tends to brings the system back to its original position.

$$\tau = -k\theta$$

$$I\frac{d^2\theta}{dt^2} = -k\theta \Rightarrow I\frac{d^2\theta}{dt^2} + k\theta = 0$$

$$\frac{227}{12}ma^2\frac{d^2\theta}{dt^2} + 76ma^2\theta = 0$$

$$\frac{d^2\theta}{dt^2} + \frac{912}{227}\theta = 0$$

Angular frequency of torsional oscialltion $\omega = \sqrt{\frac{912}{227}}$ $\quad f = \frac{1}{2\pi}\omega = \frac{1}{2\pi}\sqrt{\frac{912}{227}}$.

1.1.3 Energy in SHM:

Kinetic energy at any instant is given by $E_k = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2$

$$\Rightarrow \quad E_k = \frac{1}{2}ma^2\omega^2\sin^2(\omega t + \delta) \qquad \text{... (10)}$$

Total potential energy acquired by the particle for the displacement x is

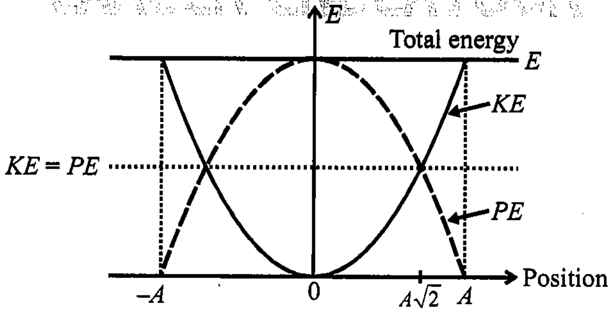
$$E_p = -\int F\cdot dx = \int_0^x sx\,dx = \frac{1}{2}sx^2$$

$$\Rightarrow E_p = \frac{1}{2}m\omega^2x^2 = \frac{1}{2}\text{m}\omega^2\text{a}^2\cos^2(\omega\text{t} + \delta) \qquad \text{... (11)}$$

Total instantaneous energy, $\text{E} = \text{E}_\text{k} + \text{E}_\text{p} = \frac{1}{2}\text{m}\omega^2\text{a}^2 = \text{constant}$ ... (12)

and it is equal to the maximum value of the kinetic energy or the potential energy. Thus in SHM energy oscillates between the kinetic and potential forms but total energy remains constant.

Energy position graph:



PE : Graph is parabola with vertex at $x = 0$.

KE : The KE curve is inverted parabola.

x. 5. Find the position in terms of amplitude when kinetic energy is equal to potential energy of a particle is SHM.

ns. If $\quad x = A\sin\omega t$ ... (i)

then $\quad v = \frac{dx}{dt} = A\omega\cos\omega t$

$$= A\omega\left[1-\left(\frac{x}{A}\right)^2\right]^{1/2}$$

$$= \omega\left[A^2 - x^2\right]^{1/2} \qquad \ldots \text{(ii)}$$

Kinetic energy, $KE = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2\left[A^2 - x^2\right] \qquad \ldots$ (iii)

Potential energy, $PE = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2 x^2$,

when, $PE = KE$

$$\frac{1}{2}m\omega^2 x^2 = \frac{1}{2}m\omega^2\left[A^2 - x^2\right]$$

$$\Rightarrow \quad 2x^2 = A^2 \quad \Rightarrow \quad x = \pm\frac{A}{\sqrt{2}}.$$

Ex. 6. Frequency of energy change ω from KE to PE in SHM is related with the frequency ω_0 of SHM by :

(a) $\omega = 2\omega_0$ (b) $\omega = \frac{\omega_0}{2}$ (c) $\omega = \omega_0^2$ (d) $\omega = \omega_0$

Soln. Potential energy $PE = \frac{1}{2}m\omega_0^2 x^2$

If $x = A\sin\omega_0 t$ then

$$PE = \frac{1}{2}m\omega^2 A^2 \sin^2\omega_0 t = \frac{1}{2}m\omega_0^2 A^2\left[\frac{1-\cos 2\omega_0 t}{2}\right]$$

$$= \frac{1}{4}m\omega_0^2 A^2 - \frac{1}{4}m\omega_0^2 A^2 \cos 2\omega_0 t = c - c\cos 2\omega_0 t, \text{ where } c = \frac{1}{4}m\omega_0^2 A^2 = \text{constant}$$

$$= c - c\cos\omega_E t$$

$\Rightarrow$ frequency of energy change $\boxed{\omega_E = 2\omega_0}$.

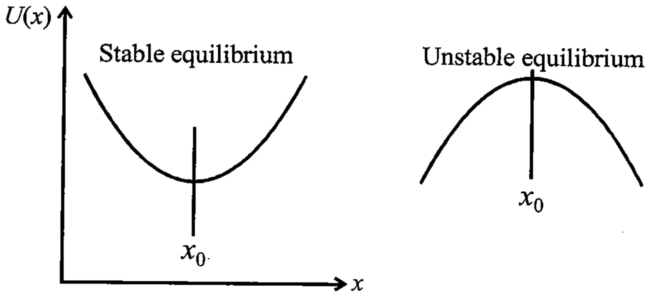
Force Constant in terms of potential energy:

If $U(x)$ is potential energy of the oscillating system in SHM, then at equilibrium point $x = x_0$,

$$F = -\left.\frac{\partial U(x)}{\partial x}\right|_{x_0} = 0.$$

Solving this, we get mean position x_0. The force constant of the oscillating system can be given by

$$K = \left.\frac{d^2U(x)}{dx^2}\right|_{x_0}$$



Ex. 7. A particle of mass '*m*' is moving in a potential

$$V(x)=\frac{1}{2}m\omega_0^2x^2+\frac{a}{2mx^2}$$

where ω_0 and 'a' are positive constants. The angular frequency of small oscillations for the simple harmonic motion of the particle about a stable minimum of the potential V(x) is:

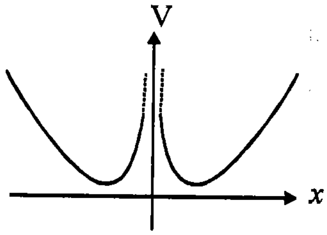
(a) $\sqrt{2}\omega_0$ (b) $2\omega_0$ (c) $4\omega_0$ (d) $4\sqrt{2}\omega_0$

[IIT-JAM : 2011]

Soln. **(b)** $V(x)=\frac{1}{2}m\omega_0^2x^2+\frac{a}{2mx^2}$; $\frac{\partial V}{\partial x}=m\omega_0^2x-\frac{a}{mx^3}$; $\frac{\partial^2 V}{\partial x^2}=m\omega_0^2+\frac{3a}{mx^4}$

At minimum potential $\frac{\partial V}{\partial x}=0\left(\text{at } x=x_0\right)$

$m\omega_0^2x_0=\frac{a}{mx_0^3}\Rightarrow x_0=\left(\frac{a}{m^2\omega_0^2}\right)^{1/4}$, equilibrium position



Force constant, $k=\left.\frac{\partial^2 V}{\partial x^2}\right|_{x=x_0}=m\omega_0^2+\frac{3a}{mx_0^4}=m\omega_0^2+\frac{3m^2\omega_0^2x_0^4}{mx_0^4}=4m\omega_0^2$

$$\therefore \quad \omega=\sqrt{\frac{k}{m}}=\sqrt{\frac{4m\omega_0^2}{m}}\Rightarrow \boxed{\omega=2\omega_0}$$

1.1.4 Equation of motion from principle of conservation of energy:

Total energy of the oscillator is $\frac{1}{2}m\left(\frac{dx}{dt}\right)^2+\frac{1}{2}sx^2=E$ (constant) ... (13)

$$\Rightarrow m\frac{dx}{dt}.\frac{d^2x}{dt^2}+sx\frac{dx}{dt}=0 \qquad \Rightarrow \left[m\frac{d^2x}{dt^2}+sx\right]\frac{dx}{dt}=0$$

Since dx/dt cannot be zero for all values of t, $m\frac{d^2x}{dt^2}+sx=0$ ... (14)

which is the equation of motion.

1. 2 REPRESENTATION OF SHM BY ROTATING VECTOR:

$\overrightarrow{OB}$ is a vector of constant magnitude a which rotates with constant angular velocity ω in anticlockwise sense. At time t = 0 the vector coincides with the line OA which makes an angle δ with x - axis. The x–component of the rotating vector is $x = a\cos(\omega t + \delta)$

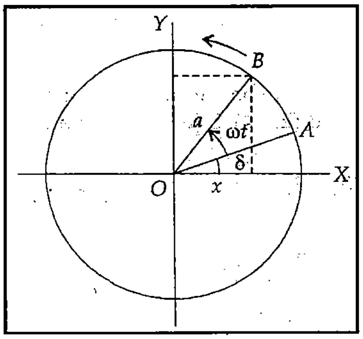


Figure 1.1

which represents a SHM. The y-component of the rotating vector is $y = a\sin(\omega t + \delta)$ which also represents a SHM. From the above treatment it is clear that SHM can be viewed as a projection of circular motion on any fixed diameter of the circle.

1.3 SUPERPOSITION OF COLLINEAR SHMs:

The principle of superposition according to which resultant displacement due to a number of sources is given by the algebraic sum of the displacements caused by the individual sources.

1.3.1 SHMs of same frequency acting along the same direction but having different amplitudes and phases:

Let two SHMs be represented by

$$x_1 = a_1\cos(\omega t + \delta_1) \qquad \dots(15)$$

$$x_2 = a_2\cos(\omega t + \delta_2) \qquad \dots(16)$$

where a_1 and a_2 are the amplitudes, δ_1 and δ_2 are the initial phase angles of the two SHMs of same angular frequency ω.

By the superposition principle the resultant displacement is given by

$$x = x_1 + x_2 = a_1\cos(\omega t + \delta_1) + a_2\cos(\omega t + \delta_2)$$

$$= (a_1\cos\delta_1 + a_2\cos\delta_2)\cos\omega t - (a_1\sin\delta_1 + a_2\sin\delta_2)\sin\omega t$$

Putting $a_1\cos\delta_1 + a_2\cos\delta_2 = A\cos\phi$

$a_1\sin\delta_1 + a_2\sin\delta_2 = A\sin\phi$

we get $x = A\cos(\omega t + \phi) \qquad \dots(16)$

It shows that the resultant motion is also simple harmonic, amplitude of which is given by

$$A^2 = a_1^2 + a_2^2 + 2a_1a_2\cos(\delta_1 - \delta_2) \qquad \dots(17)$$

And $$\tan\phi = \frac{a_1\sin\delta_1 + a_2\sin\delta_2}{a_1\cos\delta_1 + a_2\cos\delta_2} \qquad \dots(18)$$

1.3.2 Two SHMs of slightly different frequencies acting along the same direction:

Let us consider two SHMs having slightly different angular frequencies ω_1 and ω_2.

$x_1 = A_0\sin\omega_1 t$

$$x_2 = A_0 \sin \omega_2 t$$

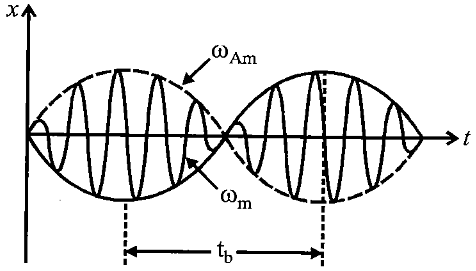
$$x = x_1 + x_2 = 2A_0\left[\sin\left(\frac{\omega_1+\omega_2}{2}\right)t \cos\left(\frac{\omega_1-\omega_2}{2}\right)t\right]$$

Let $\frac{\omega_1+\omega_2}{2} = \omega_m$ and $\frac{\omega_1-\omega_2}{2} = \omega_{Am}$

$$\boxed{x = 2A_0 \sin \omega_m t \cos \omega_{Am} t}$$

$$x = (2A_0 \cos \omega_{Am} t) \sin \omega t$$

$$x = A \cos \omega_m t$$



where $A = (2A_0 \cos \omega_{Am} t)$, if $\omega_1 \sim \omega_2$, $\omega_m \to 0$

amplitude $A = 2A_0 \to$ nearly constant.

Then the result oscillation is nearly SHM with frequency ω_{Am} and amplitude $2A_0$.

For maximum amplitude

$$A_{max} \Rightarrow \cos \omega_m t = 1$$

$$\Rightarrow \omega_m t = 2n\pi$$

$$\Rightarrow t = \frac{2n\pi}{(\omega_2-\omega_1)} = \frac{2n\pi}{2\pi(\nu_2-\nu_1)} = \frac{n}{\nu_2-\nu_1} = 0, \frac{1}{\nu_2-\nu_1}, \frac{2}{\nu_2-\nu_1}, \frac{3}{\nu_2-\nu_1}, \ldots\ldots$$

time difference between subsequent maxima, $\Delta t = \frac{1}{\nu_2-\nu_1}$.

Similarly,

time difference between subsequent minima, $\Delta t = \frac{1}{\nu_2-\nu_1}$.

Beat : One maximum amplitude followed by a minimum, is called a Beat.

Beat time : $t_b = \frac{1}{\nu_2-\nu_1}$

Beat frequency : $\frac{1}{t_b} = |\nu_2 - \nu_1|$

Ex. 8. The superposition of two harmonic oscillator oscillating in same direction result motion of a point given by $x = a\cos 2.1t \cos 50t$

Calculate (i) individual frequency, (ii) Beat frequency

Soln. (i) If $x_1 = A\cos\omega_1 t$ and $x_2 = A\cos\omega_2 t$, then

$$x = x_1 + x_2$$

$$= 2A\cos\left(\frac{\omega_2-\omega_1}{2}\right)t \cos\left(\frac{\omega_2+\omega_1}{2}\right)t$$

By comparing, $\frac{\omega_2-\omega_1}{2} = 2.1 \Rightarrow \omega_2 - \omega_1 = 4.2$ and $\frac{\omega_2+\omega_1}{2} = 50.0 \Rightarrow \omega_2+\omega_1 = 100.0$

$$\Rightarrow 2\omega_2 = 100 + 4.2 \Rightarrow \omega_2 = 52.1\,\text{Hz}$$

(ii) Since, $\omega_2 - \omega_1 = 2\pi(\nu_2 - \nu_1) = 4.2$

Beat frequency, $\nu_2 - \nu_1 = \dfrac{4.2}{2\pi} = 0.66$ Hz

Ex. 9. At a given point in space the total light wave is composed of three phasors $P_1 = a$, $P_2 = \frac{a}{2}e^{i\theta}$ and $P_3 = \frac{a}{2}e^{-i\theta}$. The intensity of light at this point is

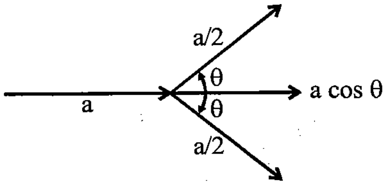
(a) $4a^2\cos^2\left(\frac{\theta}{2}\right)$ (b) $4a^2\cos^4\left(\frac{\theta}{2}\right)$ (c) $a^2\cos^2(\theta)$ (d) $4a\cos^2(2\theta)$

[IIT-JAM : 2006]

Soln. **(b)** Resultant phasors,

$$P = P_1 + P_2 + P_3 = a + \frac{a}{2}e^{i\theta} + \frac{a}{2}e^{-i\theta}$$

$$P_y = 0 + \frac{a}{2}\sin\theta - \frac{a}{2}\sin\theta = 0$$



$$P_x = a + \frac{a}{2}\cos\theta + \frac{a}{2}\cos\theta$$

$$= a + a\cos\theta = 2a\cos^2 \theta/2$$

The intensity of light at this point is $I = P^2 = 4a^2\cos^4 \theta/2$

1.4 SUPERPOSITION OF TWO SHMS ACTING AT RIGHT ANGLES TO EACH OTHER:

1.4.1 Oscillations having same frequencies:

Let two simple harmonic motions acting at right angles be represented by the equations

$$x = a\cos(\omega t + \delta_1) \text{ and} \quad \ldots (23)$$

$$y = b\cos(\omega t + \delta_2) \quad \ldots (24)$$

$$\Rightarrow \quad \frac{y}{b} = \cos(\omega t + \delta_1 + \delta_2 - \delta_1) = \cos(\omega t + \delta_1 + \delta)$$

where $\delta = \delta_2 - \delta_1$ is the phase difference between the two SHMs.

Therefore, $\dfrac{y}{b} = \cos(\omega t + \delta_1).\cos\delta - \sin(\omega t + \delta_1).\sin\delta$

$$\Rightarrow \quad \frac{y}{b} = \frac{x}{a}\cos\delta - \sqrt{1 - x^2/a^2}\,.\sin\delta$$

$$\Rightarrow \quad \left(\frac{x}{a}\cos\delta - \frac{y}{b}\right)^2 = \left(1 - \frac{x^2}{a^2}\right)\sin^2\delta$$

$$\Rightarrow \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab}\cos\delta = \sin^2\delta \quad \ldots (25)$$

It represents the general equation of an ellipse bounded within a rectangle of sides 2a and 2b. The major axis of the ellipse makes an angle θ with the x-axis, which is given by

$$\tan 2\theta = \frac{2ab}{a^2 - b^2}\cos\delta \quad \ldots (26)$$

Case-I: If $\delta = \delta_2 - \delta_1 = 0$ i.e, the two SHMs are in phase, then the Equation (25) reduces to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab} = 0 \qquad \Rightarrow \left(\frac{x}{a} - \frac{y}{b}\right)^2 = 0$$

It represents a pair of coincident straight lines $y = \frac{b}{a}x$, passing through the origin and inclined to the x-axis at an angle $\tan^{-1}(b/a)$

Case-II: When $\delta = \pi$, Equation. (25) becomes $\left(\frac{x}{a} + \frac{y}{b}\right)^2 = 0$

This also represents a pair of coincident straight lines passing through the origin inclined to the x–axis at an angle θ given by $\tan\theta = (-b/a)$.

Case-III: When $\delta = \pi/2$, Equation (25) reduces to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

It represents an ellipse with semi-axes a and b along the coordinate axes. If, in addition, a = b ellipse degenerates into a circle, $x^2 + y^2 = a^2$

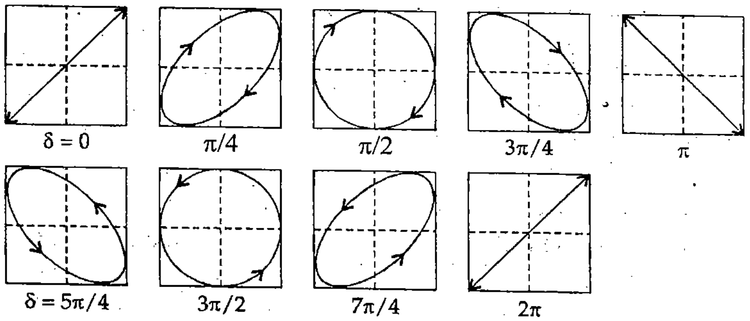


***Figure* 1.3:** Resultant pattern due to superposition of two rectangular SHMs

1.4.2 Oscillations having slightly different frequencies:

Let two rectangular SHMs having slightly different frequencies be represented by

$$x = a\cos(\omega t + \delta_1) \qquad \text{... (27)}$$

$$y = b\cos[(\omega + \Delta\omega)t + \delta_2] = b\cos(\omega t + \delta_1 + \delta) \qquad \text{... (28)}$$

where $\delta = \Delta\omega t + \delta_2 - \delta_1$ is the phase difference between the two SHMs.

As δ changes from 0 to 2π the resultant pattern goes through all the phases shown in ***Figure*1.3**. As δ goes on changing with time the whole pattern will be repeated in a cyclic fashion. The pattern goes through a cycle of changes in time T given by

$$\delta(t + T) = \delta(t) + 2\pi$$

$$\Rightarrow \quad \Delta\omega T = 2\pi$$

$$\Rightarrow \quad T = \frac{2\pi}{\Delta\omega} = \frac{1}{\Delta v} \qquad \text{... (29)}$$

1.4.3 Frequency ratio 1 : 2:

Let two rectangular SHMs of frequencies in the ratio 1 : 2 differing in phase by δ be represented by

$$x = a\cos\omega t \qquad \text{... (30)}$$

$$y = b\cos(2\omega t + \delta) \qquad \text{... (31)}$$

$$\Rightarrow \quad \frac{y}{b} = \cos 2\omega t.\cos\delta - \sin 2\omega t.\sin\delta = \left(2\cos^2\omega t - 1\right)\cos\delta - 2\sin\omega t\cos\omega t\cdot\sin\delta$$

$$\Rightarrow \quad \frac{y}{b} = \left(2.\frac{x^2}{a^2} - 1\right)\cos\delta - 2\frac{x}{a}\sqrt{1-\frac{x^2}{a^2}}\sin\delta$$

$$\Rightarrow \quad \left[\frac{y}{b} + \cos\delta - \frac{2x^2}{a^2}\cos\delta\right]^2 = \frac{4x^2}{a^2}\left(1 - \frac{x^2}{a^2}\right)\sin^2\delta \qquad \text{... (32)}$$

This is an equation of fourth degree in x and y, in general, represents closed curve having two loops. For a given value of δ the exact nature of the curve can be traced. For example, if $\delta = 0$ then Equation (32) reduces to

$$\left(\frac{y}{b} + 1 - \frac{2x^2}{a^2}\right)^2 = 0 \qquad \text{... (33)}$$

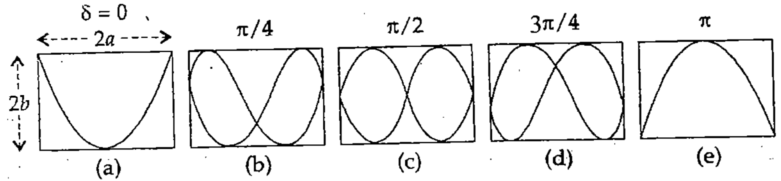


***Figure* 1.4:** Resultant pattern of two rectangular SHMs having frequency ratio 1 : 2

It represents two coincident parabolas (***Figure*** 1.4 (a)) given by

$$x^2 = \frac{a^2}{2b}(y + b) \qquad \text{... (34)}$$

If $\delta = \pi/2$ equation (32) reduces to $\frac{4x^2}{a^2}\left(\frac{x^2}{a^2} - 1\right) + \frac{y^2}{b^2} = 0$ ... (35)

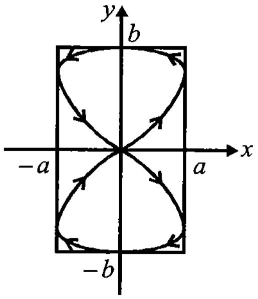
This equation represents a curve containing two loops as shown in ***Figure*** 1.4(c).

Note: If particle displacement are of following form

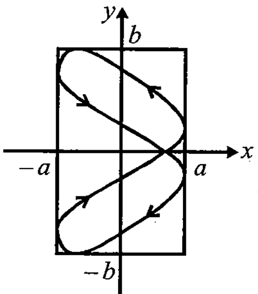
$x = a\sin(2\omega t + \delta)$, $y = b\sin(\omega t)$, the general equation of resultant curve is given by

$$\left(\frac{x}{a} - \sin\delta\right)^2 + \frac{4y^2}{b^2}\left(\frac{y^2}{b^2} - 1 + \frac{x}{a}\sin\delta\right) = 0$$

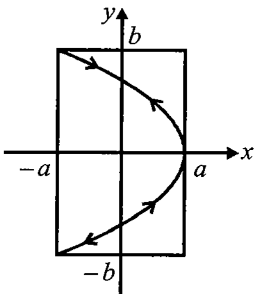
And resultant Lissajous figures loops are



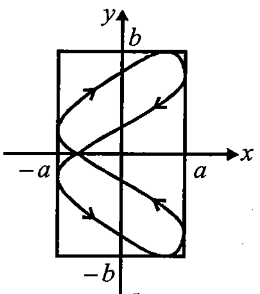
$\delta = 0$



$\delta = \frac{\pi}{4}$



$\delta = \frac{\pi}{2}$



$\delta = \frac{5\pi}{4}$

1.5 LISSAJOUS FIGURES:

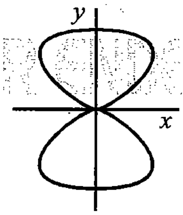
The figures or curves formed by the superposition of two simple harmonic motions at right angles to each other are known as Lissajous figures. The shape of these curves depend on the ratio of frequencies as well as on the initial phase relationship of the component simple harmonic motions.

1.5.1 Uses of Lissajous figures:

1. **Ratio of frequencies**
These figures may be used to find the ratio of two exactly commensurate frequencies of the component vibrations. For this purpose draw a horizontal and a vertical line such that it intersect curve at maximum point and find the number of intersections each line makes with the curve. Now the *ratio of the number of cut of the vertical line to that on the horizontal line will give the ratio of the horizontal to the vertical frequency* i.e.,

$$\frac{\text{Number of cuts on vertical line}}{\text{Number of cuts on horizontal line}} = \frac{N_y}{N_x} = \frac{\omega_x}{\omega_y} \quad \text{... (36)}$$

Ex. 10. Superposition of two sinusoidal waves $x(t) = A\sin(\omega_1 t)$ and $y(t) = B\sin(\omega_2 t)$ produces the following Lissajous figure. The ratio ω_1/ω_2 is



(a) 1/2 (b) 1 (c) 2 (d) 4

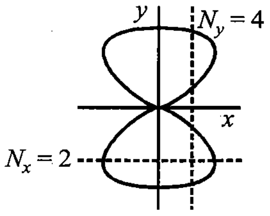
[IIT-JAM GP: 2010]

Soln. **(c)**

$$\frac{\text{Number of times figure cuts } y\text{-axis}}{\text{Number of times figure cuts } x\text{-axis}} = \frac{N_y}{N_x} = \frac{\omega_x}{\omega_y}$$

$$\Rightarrow \quad \frac{4}{2} = \frac{\omega_x}{\omega_y}$$

$$\omega_x = 2\omega_y$$



So, $\frac{\omega_1}{\omega_2} = 2$

2. **Beat frequency**

   Lissajous figure can be used to compare two nearly equal frequencies. If the frequencies differ slightly Lissajous figures change gradually and pass through a complete cycle of changes in a time T given by

$$T = \frac{1}{\Delta v} \qquad \text{... (37)}$$

3. **Phase difference**

   Lissajous figure can also be used to measure the phase difference δ between two signals of same frequency.

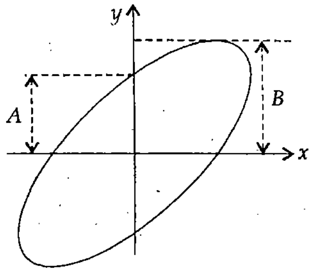


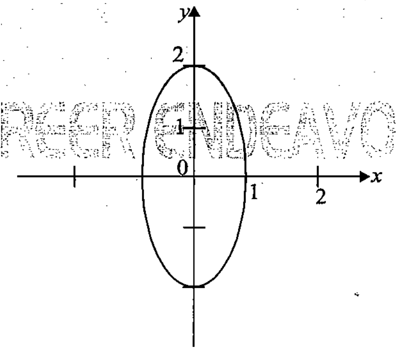
***Figure* 1.5:** Measurement of phase difference

$$\delta = \sin^{-1}(A/B) \qquad \text{... (38)}$$

Ex. 11. When two simple harmonic oscillations represented by

$$x = A_0 \cos(\omega t + \alpha) \text{ and } y = B_0 \cos(\omega t + \beta)$$

are superposed at right angle, the result is an ellipse with its major axis along the y-axis as shown in the figure. The conditions which corresponds to this are



(a) $\beta = \alpha + \frac{\pi}{2};\ A_0 = 2B_0$

(b) $\beta = \alpha - \frac{\pi}{4};\ A_0 = B_0$

(c) $\beta = \alpha + \frac{\pi}{2};\ 2A_0 = B_0$

(d) $\beta = \alpha + \frac{\pi}{4};\ A_0 = B_0$

[IIT-JAM : 2007]

Soln. **(c)** The lissajous figure is ellipse only when phase difference is $\frac{\pi}{2}$ and from given figure, amplitude of y-vibration is twice of amplitude of a x-vibration.

$$\left.\begin{array}{l} \beta = \alpha + \frac{\pi}{2} \\ B_0 = 2A_0 \end{array}\right]$$

SOLVED EXAMPLES

1. Show that a SHM can be considered as the superposition of two equal and opposite circular motions.

Soln. Circular motions can be represented by

$$\left.\begin{aligned} x_1 &= a\cos\omega t \\ y_1 &= a\sin\omega t \end{aligned}\right\}\text{Anticlockwise} \qquad \left.\begin{aligned} x_2 &= a\cos\omega t \\ y_2 &= -a\sin\omega t \end{aligned}\right\}\text{Clockwise}$$

Adding these equations we get $x = 2a\cos\omega t, y = 0$. It represents a SHM.

2. A Particle of mass m is under a unidimensional potential $V(x) = V_0(1-\cos Kx)$ where V_0 and K are constants. Find the time period of small oscillations about the equilibrium position.

Soln. For equilibrium, $\frac{dV}{dx} = 0 \quad \Rightarrow \sin Kx = 0 \quad \Rightarrow x = 0$

Now if x be the displacement from equilibrium at any instant then equation of motion is

$$m\frac{d^2x}{dt^2} = -\frac{dV}{dx} = -V_0K\sin Kx$$

For small oscillation $\sin Kx \to Kx$

$$\Rightarrow \quad \frac{d^2x}{dt^2} = -\frac{V_0K^2}{m}x = -\omega^2 x \text{ where } \omega = \sqrt{\frac{V_0k^2}{m}}$$

It represents a SHM. Its time period is $T = \frac{2\pi}{\omega} = \frac{2\pi}{K}\sqrt{\frac{m}{V_0}}$

Alternate solution :

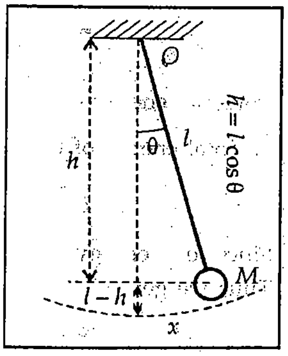
Spring constant $= \left.\frac{d^2V}{dx^2}\right|_{x_0} = V_0K^2,\ \omega = \sqrt{\frac{V_0K^2}{m}}$

3. A simple pendulum consists of a rod of mass m and length *l* which is capable of oscillation about one end and the other end carries a mass M. Determine the natural frequency of oscillation.

Soln. Let θ be the small angular displacement at any instant as shown in figure. Now the kinetic energy of the system is

$$KE = \frac{1}{2}I_{rod}\dot{\theta}^2 + \frac{1}{2}I_{mass}\dot{\theta}^2$$

$$= \frac{1}{2}\left(\frac{1}{3}ml^2\right)\dot{\theta}^2 + \frac{1}{2}Ml^2\dot{\theta}^2 = \frac{1}{2}\left(M + \frac{m}{3}\right)l^2\dot{\theta}^2$$



Potential energy of the system is

$$PE = Mgl(1-\cos\theta) + mg\frac{\ell}{2}(1-\cos\theta)$$

$$= \left(M + \frac{m}{2}\right)gl(1-\cos\theta)$$

Since total energy (KE + PE) is constant, $\frac{d}{dt}(KE + PE) = 0$

$$\Rightarrow \quad \left(M + \frac{m}{3}\right)l^2\ddot{\theta} + \left(M + \frac{m}{2}\right)gl\sin\theta = 0$$

For small oscillation $\sin\theta \approx \theta$ and the equation of motion becomes;

$$\ddot{\theta} = -\omega^2\theta, \text{ where } \omega^2 = \frac{M + m/2}{M + m/3}.\frac{g}{l}$$

Therefore, the time period of oscillation is $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{M + m/3}{M + m/2}.\frac{l}{g}}$

4. A sphere of mass M capable of producing SHM under a restoring force sx is at rest. Another sphere of mass m moving with velocity v_o along x-axis makes a head-on elastic collision with the first sphere. Find the subsequent motion of the first sphere.

Soln. Applying the principles of conservation of linear momentum and energy we can write

$$mv_0 + M.0 = mv + MV \text{ and} \qquad ...(1)$$

$$\frac{1}{2}mv_0^2 + 0 = \frac{1}{2}mv^2 + \frac{1}{2}MV^2 \qquad ...(2)$$

where v and V are the velocities of m and M respectively along x-axis

Rearranging Equation (1) and (2) we get,

$$m(v_0 - v) = MV \text{ and} \qquad ...(3)$$

$$m(v_0^2 - v^2) = MV^2 \qquad ...(4)$$

Dividing Equation (4) by Equation (3) we get

$$v_0 + v = V \qquad ...(5)$$

Now from Equation (1) we get $mv_0 = m(V - v_0) + MV$

$$\Rightarrow \quad V = \frac{2m}{m + M}v_0 \qquad ...(6)$$

The equation of motion of M is

$$M\ddot{x} = -sx \quad \Rightarrow \ddot{x} + \omega^2 x = 0 \text{ where } \omega = \sqrt{s/M}$$

The general solution is $x = A\cos\omega t + B\sin\omega t$

To find the constants A and B we apply the initial conditions, x = 0 and dx/dt = V at t = 0. Thus we get A = 0 and $B = V/\omega$.

Therefore, the subsequent motion of M will be described by the equation $x = \frac{V}{\omega}\sin\omega t$

where $\omega = \sqrt{s/M}$ and $V = \frac{2m}{m + M}v_0$.

5. Lissajous figure obtained by using two tuning two forks A and B having slightly different frequencies undergoes complete cycle of changes in 20 sec. When B is loaded with a little wax, the time reduces to 10 sec. If the frequency of A is 288, calculate the frequency of B before loading it with wax.

Soln. The frequency difference before loading is $\Delta v = 1/20$, $v_A \sim v_B = 0.05\,\text{Hz}$

After loading ν_B decreases to ν_B' and the difference $\nu_A \sim \nu_B' = 1/10 = 0.1$ Hz increases. This indicates that $\nu_A > \nu_B$. Thus, $\nu_A - \nu_B = 0.05\,\text{Hz}$ or, $\nu_B = \nu_A - 0.05\,\text{Hz} = (288 - 0.05)\,\text{Hz} = 287.95\,\text{Hz}$

5. Suppose a particle is simultaneously subjected to three collinear SHMs all of the same frequency and having amplitudes 1 cm, 0.5 cm and 0.25 cm respectively. The phase of the second relative to the first is 30° and that of the third relative to the second is 60°. Find the resultant motion.

Soln. The resultant is

$$x = \cos\omega t + 0.5\cos(\omega t + 30^\circ) + 0.25\cos(\omega t + 90^\circ)$$

$$= \cos\omega t + 0.5\left[\cos\omega t.\frac{\sqrt{3}}{2} - \sin\omega t.\frac{1}{2}\right] - 0.25\sin\omega t = \left(1 + \frac{\sqrt{3}}{4}\right)\cos\omega t - \frac{1}{2}\sin\omega t$$

$$= A\cos(\omega t + \phi)$$

where $A\cos\phi = 1 + \frac{\sqrt{3}}{4} = 1.433,\ A\sin\phi = \frac{1}{2} = 0.5$

Therefore, for the resultant motion, amplitude $A = \sqrt{(1.433)^2 + (0.5)^2} = 1.52\,\text{cm}$ and the phase relative to the first is $\phi = \tan^{-1}\frac{0.5}{1.433} = 19.23^\circ$.

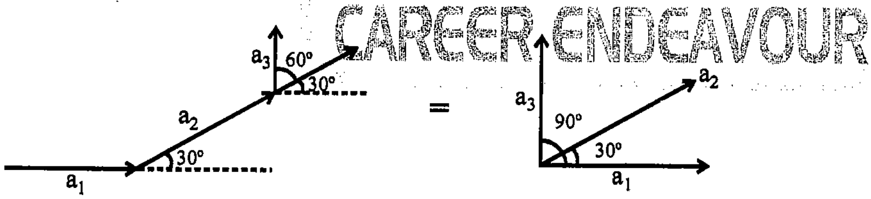
Alternate solution:

Resultant displacement is given by

$x = a_1\cos\omega t + a_2\cos(\omega t + 30^\circ) + a_3\cos(\omega t + 90^\circ)$,

where $a_1 = 1\,\text{cm},\ a_2 = 0.5\,\text{cm},\ a_3 = 0.25\,\text{cm}$.

Taking amplitude as a vector and phase difference as a angle between two vectors resultant amplitude is given by vector sum of a_1, a_2, a_3.



From above figure

x-component of amplitude $A_x = a_1 + a_2\cos 30^\circ = 1 + 0.5\frac{\sqrt{3}}{2} = 1.433$

y-component of amplitude $A_y = a_3 + a_2\sin 30^\circ = 0.25 + 0.5\frac{1}{2} = 0.5$

Therefore, resultant amplitude is given by $A = \sqrt{A_x^2 + A_y^2} = \sqrt{(0.5)^2 + (1.433)^2} = 1.52\,\text{cm}$, and

phase with the first SHM is, $\tan\phi = \frac{A_y}{A_x} = \frac{0.5}{1.433} = 0.348$.

$\Rightarrow\ \phi = \tan^{-1}(0.348) = 19.23^\circ$.

The resultant motion is $x = A\cos(\omega t + \phi)$, where $A = 1.52$.

7. A particle of mass 0.2 kg oscillates according to the law $x = 0.08\cos\left(20\pi t + \pi/4\right)$ where x is in metre and t is in second. Find expressions for the instantaneous velocity and acceleration of the particle. What is the instantaneous force acting on it? Determine the amplitudes of velocity, acceleration and force. What is the frequency of oscillation? What is the frequency of variation of its kinetic energy?

Soln. Let $x = A\cos(\omega t + \delta)$

Instantaneous velocity $v = \dfrac{dx}{dt} = -A\omega\sin(\omega t + \delta) = v_0\sin(\omega t + \delta) = -5.02\sin\left(20\pi t + \pi/4\right)\text{m/s}$

where $v_0 = -A\omega = -5.02$ m/s.

Acceleration $a = \dfrac{d^2x}{dt^2} = -\omega^2 A\cos(\omega t + \delta) = a_0\cos(\omega t + \delta) = -315.5\cos(20\pi t + \pi/4)\,\text{m/s}^2$.

Force F = mass × acceleration $= -0.2 \times (20\pi)^2 \times 0.08\cos\left(20\pi t + \pi/4\right) = 63.1\cos\left(20\pi t + \pi/4\right)\text{N}$

Therfore, Amplitudes of velocity, acceleration and force are 5.02 m/s, $315.5\,\text{m/s}^2$ and 63.1 N respectively.

Frequency of oscillation $\nu = \dfrac{\omega}{2\pi} = \dfrac{20\pi}{2\pi} = 10\,\text{Hz}$

$$\text{Kinetic energy K.E.} = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 = \frac{1}{2}\ 0.2 \times (5.02)^2\sin^2\left(20\pi t + \pi/4\right)\text{J}$$

$$= 2.52 \times \frac{1}{2}\left[1 - \cos 2\left(20\pi t + \pi/4\right)\right]\text{J} = 1.26\left[1 - \cos\left(40\pi t + \pi/2\right)\right]\text{J}$$

$$= 1.26\left[1 + \sin 40\pi t\right]\text{J}$$

Therefore, the frequency of variation of kinetic energy is $\dfrac{40\pi}{2\pi} = 20\,\text{Hz}$, which is twice the frequency of particle's motion.

8. Find the resultant of the following two SHMs:

$$x_1 = 0.03\cos 10\pi t,\ x_2 = 0.03\cos 12\pi t$$

Hence find the beat frequency and beat amplitude.

Soln. The resultant is given by $x = x_1 + x_2 = 0.03\left[\cos 10\pi t + \cos\left(10\pi t + 2\pi t\right)\right]$

$$= 0.03\left(1 + \cos 2\pi t\right)\cos 10\pi t - 0.03\sin 2\pi t\sin 10\pi t = A\cos\left(10\pi t + \phi\right)$$

where we have put $0.03\left(1 + \cos 2\pi t\right) = A\cos\phi$ and $0.03\sin 2\pi t = A\sin\phi$

Thus, $A = 0.03\sqrt{\left(1 + \cos 2\pi t\right)^2 + \sin^2 2\pi t} = 0.03\sqrt{2 + 2\cos 2\pi t}$

Therefore, the required amplitude is $A = 0.03\sqrt{2 + 2 \times 1} = 0.06$ unit.

The beat frequency = difference of the two frequencies

$$= \nu_2 - \nu_1 = \frac{1}{2\pi}\left(\omega_2 - \omega_1\right) = (6 - 5) = 1 \text{ cycle per sec.}$$

9. A particle vibrates harmonically with a frequency 0.5 Hz,. Assuming that the motion is started from its equilibrium position with a speed of 0.2 m/s, towards negative x-axis write down its equation of vibration.

Soln. Let the motion be represented by $x(t) = a\sin(\omega t + \theta_0) = a\sin(2\pi\nu t + \theta_0)$

$$= a\sin(\pi t + \theta_0) \quad [\because \nu = 0.5\text{ Hz}]$$

where a and θ_0 are two arbitrary constants to be fixed by the initial conditions. At t = 0, x = 0,

$x(t) = 0 = a\sin\theta_0 \Rightarrow \theta_0 = 0, \pi, 2\pi, ...$

Now $\frac{dx}{dt} = a\pi\cos(\pi t + \theta_0)$

At $t = 0, \frac{dx}{dt} = -0.2\text{ m/s}$. So, $-0.2 = a\omega\cos\theta_0 \Rightarrow \theta_0 = \pi$

and $a\pi = 0.2 \Rightarrow a = (0.2/\pi)\text{m}$

Therefore, the required equation of vibration is $x(t) = (0.2/\pi)\sin(\pi t + \pi)$

10. Two mutually perpendicular oscillations are represented by

$$x(t) = a\sin\omega t \text{ and } y(t) = b\sin(2\omega t + \phi)$$

Sketch the Lissajous figure resulting from these oscillations with $\phi = -\pi$. Find the value of x for which the Lissajous figure touches the line $y = \pm b$.

Soln. $x(t) = a\sin\omega t, y(t) = b\sin(2\omega t - \pi) = -b\sin 2\omega t = -2b\sin\omega t.\cos\omega t$

Using, $\sin\omega t = x/a, \cos\omega t = \sqrt{1 - x^2/a^2}$ we get,

$$y = -2b.\frac{x}{a}.\sqrt{1 - x^2/a^2}$$

$$\Rightarrow \frac{y^2}{b^2} = \frac{4x^2}{a^2}\left(1 - \frac{x^2}{a^2}\right)$$

This equation represents a curve containing two loops as shown in ***Figure*** 1.4(c).
Putting $y = \pm b$ we get

$$\frac{4x^4}{a^4} - \frac{4x^2}{a^2} + 1 = 0 \quad \Rightarrow \quad \left(\frac{2x^2}{a^2} - 1\right)^2 = 0 \quad \Rightarrow \quad x = \pm\frac{a}{\sqrt{2}}$$

These are the required values of x.

11. A mass *M* is attached to a solid support with an ideal spring constant *k*. The natural frequency of the system is ω. If the mass now carries a charge + *Q* and external electric field *E* along the axis of the spring is turned on, the frequency for small oscillations

(a) becomes zero (b) increases (c) decreases (d) remains unchanged

Ans. (d) This is because the force due to the electric field is a constant one and does not depend on the displacement of the mass from the mean position. Hence, the force due to the electric field will not be able to change the frequency of the SHM. Restoring force, which is present here is only spring force. Therefore, equation of motion of SHM do not changes. Thus frequency remain constant.

PRACTICE SET

[Multiple Choice Questions (MCQ)]

1. A uniform spring whose unstressed length is l, has a constant K. The spring is cut into two pieces of unstressed length l_1 and l_2, where $l_2 = nl_1$, n being an integer. Now a mass m is made to oscillate with first spring. The time period of its oscillation would be

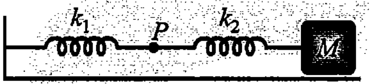
(a) $T = 2\pi\sqrt{\frac{mn}{K(n+1)}}$ (b) $T = 2\pi\sqrt{\frac{m}{nK}}$

(c) $T = 2\pi\sqrt{\frac{m}{K(n+1)}}$ (d) $T = 2\pi\sqrt{\frac{m(n+1)}{nK}}$

2. If the displacement (x) and velocity v of a particle executing simple harmonic motion are related through the expression $4v^2 = 25 - x^2$, then its time period is

(a) π (b) 2π (c) 4π (d) 6π

3. The mass M shown in the figure oscillates in simple harmonic motion with amplitude A. The amplitude of the point P is



(a) $\frac{k_1 A}{k_2}$ (b) $\frac{k_2 A}{k_1}$ (c) $\frac{k_1 A}{k_1 + k_2}$ (d) $\frac{k_2 A}{k_1 + k_2}$

4. A point mass is subjected to two simultaneous sinusoidal displacements in x-direction, $x_1(t) = A\sin\omega t$ and $x_2(t) = A\sin\left(\omega t + \frac{2\pi}{3}\right)$. Adding a third sinusoidal displacement $x_3(t) = B\sin(\omega t + \phi)$ brings the mass to a complete rest. The values of B and ϕ are

(a) $\sqrt{2}A, \frac{3\pi}{4}$ (b) $A, \frac{4\pi}{3}$ (c) $\sqrt{3}A, \frac{5\pi}{6}$ (d) $A, \frac{\pi}{3}$

5. A particle of mass m is executing oscillation about the origin on the x-axis. Its potential energy is $U(x) = k|x|^3$, where k is a positive constant. If the amplitude of oscillation is a, then its time period T is

(a) proportional to $\frac{1}{\sqrt{a}}$ (b) independent of a

(c) proportional to $\sqrt{a}$ (d) proportional to $a^{3/2}$

6. The period of oscillation of simple pendulum of length L suspended from the roof of the vehicle which moves without friction, down an inclined plane of inclination α, is given by :

(a) $2\pi\sqrt{\frac{L}{g\cos\alpha}}$ (b) $2\pi\sqrt{\frac{L}{g\sin\alpha}}$ (c) $2\pi\sqrt{\frac{L}{g}}$ (d) $2\pi\sqrt{\frac{L}{g\tan\alpha}}$

7. The function $x = A\sin^2\omega t + B\cos^2\omega t + C\sin\omega t\cos\omega t$ represents SHM

(a) for every value of A, B and C (except $C = 0$)

(b) if $A = -B$, $C = B$, amplitude $= |B\sqrt{2}|$

(c) if $A = B, C = 0$

(d) if $A = B, C = 3B$, amplitude $= |B|$

. A particle oscillates simple harmonically along x-axis. At times t, $2t$ and $3t$ it is located at a distance a, b and c respectively from the equilibrium position. The period of oscillation is

(a) $T = \dfrac{2\pi t}{\cos^{-1}\left(\dfrac{a+c}{2b}\right)}$ (b) $T = \dfrac{2\pi t}{\cos^{-1}\left(\dfrac{ac}{2b}\right)}$ (c) $T = \dfrac{2\pi t}{\cos^{-1}\dfrac{a}{2b+c}}$ (d) $T = \dfrac{\pi t}{\cos^{-1}\left(\dfrac{a+c}{2b}\right)}$

. A particle of mass m is located in a potential $u(x) = \dfrac{2}{x^2} - \dfrac{4}{x}$. The period of oscillation is

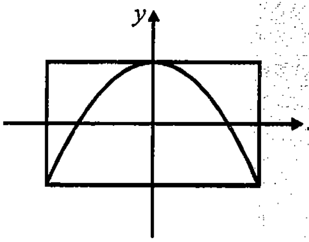
(a) $T = \dfrac{\pi}{4}\sqrt{4m}$ (b) $T = \dfrac{\pi}{2}\sqrt{2m}$ (c) $T = \dfrac{\pi}{2}\sqrt{4m}$ (d) $T = \dfrac{\pi}{2}\sqrt{m}$

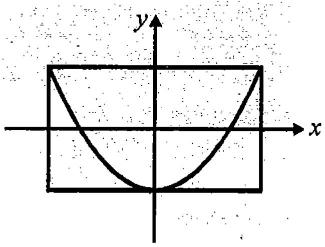
0. A system with potential energy $V(x)$ will perform SHM if

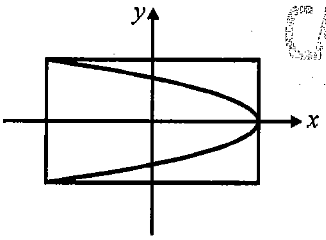
(a) $\dfrac{\partial v}{\partial x} = 0,\ \dfrac{\partial^2 v}{\partial x^2} > 0$ (b) $\dfrac{\partial v}{\partial x} = 0,\ \dfrac{\partial^2 v}{\partial x^2} < 0$ (c) only $\dfrac{\partial v}{\partial x} = 0$ (d) only $\dfrac{\partial^2 v}{\partial x^2} < 0$

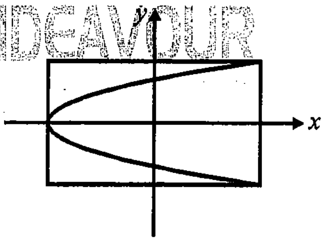
1. A particle is subjected to two SHM's represented by the following equations

$x = a\sin(\omega t),\quad y = b\sin(2\omega t + \delta)$. The Lissajous figure for $\delta = \dfrac{\pi}{2}$ is

(a)


(b)


(c)


(d)


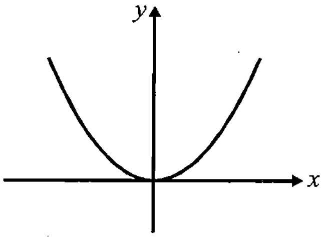
2. A simple pendulum consisting of a ball of mass m tied to a string of length l is made to swing on a circular arc of angle θ in a vertical plane. At the end of this arc, another ball of mass m is placed at rest. The momentum translated to this ball at rest by the swinging ball is

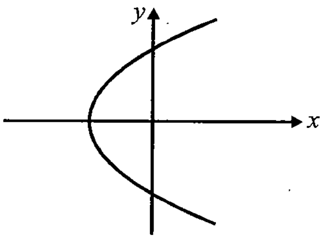
(a) zero (b) $m\theta\sqrt{l/g}$ (c) $m\theta\sqrt{2l/g}$ (d) $\dfrac{m}{2}\sqrt{(l/g)}$

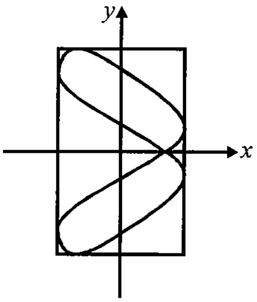
3. A particle of mass m moves in two dimensions under the following potential energy function:

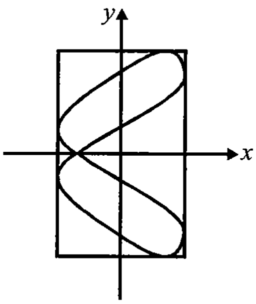
$V(\vec{r}) = \dfrac{1}{2}k(x^2 + 4y^2)$

The possible Lissajous figure formed is

(a)


(b)


(c)


(d)


14. A simple pendulum with a solid metal has a period T. The metal bob is now immersed in a liquid with a density one-tenth that of the metal of the bob. The liquid is non-viscous. Now the period of the same pendulum with its bob remaining all the time in the liquid will be :

(a) $\left(\frac{9}{10}\right)T$ (b) $T\sqrt{\left(\frac{10}{9}\right)}$ (c) unchanged (d) $T\sqrt{\left(\frac{9}{10}\right)}$

15. The coordinates of a particle moving in a plane are given by $x(t) = a\cos pt,\ y(t) = b\sin pt$ where $a, b(< a)$ and p are positive constants of appropriate dimensions. Then

(a) the path of the particle is a straight line

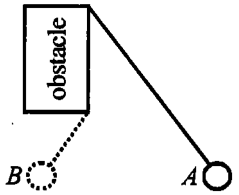
(b) the velocity and acceleration of the particle are normal to each other at $t = \frac{\pi}{(2p)}$

(c) the acceleration of the particle is always directed towards a focus

(d) the distance travelled by the particle in time interval $t = 0$ to $t = \frac{\pi}{2p}$ is a

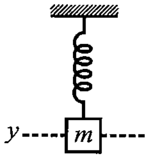
[Numerical Answer Type (NAT)]

1. A simple pendulum has a time period $T = 2\,\text{sec}$ when it swings freely. The pendulum is hung as shown in figure, so that only one-fourth of its total length is free to swing to the left of obstacle. It is displaced to position A and released. How long does it take to swing to extreme displacement B and return to A ?
Assume that displacement angle is always small.



2. An object of mass 0.2 kg executes simple harmonic oscillational along the x-axis with a frequency of $(25/\pi)$ Hz. At the position $x = 0.04$, the object has kinetic energy of 0.5 J and potential energy 0.4 J. The amplitude of oscillations is______m.

3. A mass m is undergoing SHM in the vertical direction about the mean position y_0 with amplitude A and angular frequency $\omega = 1$ rad/sec. At a distance y from the mean position, the mass detaches from the spring. Assume that the spring contracts and does not obstruct the motion of m.



The distance y______(measured from the mean position) such that the height h attained by the block is maximum. $(A\omega^2 > g)$.

4. The frequency of the combined motion of the following displacement $\sin\left(2\pi t - \frac{\pi}{4}\right) + \cos(2\pi t)$ is______Hz.

5. In an experiment to obtain Lissajous' figures, one tuning fork of higher frequency 250 Hz and a circular figure occurs after five seconds. The frequency of the other tuning fork is______.

ANSWER KEY

[Multiple Choice Questions (MCQ)]

1. (c)	**2. (c)**	**3. (d)**	**4. (b)**	**5. (a)**
6. (a)	**7. (a)**	**8. (a)**	**9. (c)**	**10. (a)**
11. (a)	**12. (a)**	**13. (a)**	**14. (b)**	**15. (b)**

[Numerical Answer Type (NAT)]

1. (1.5 sec)	**2. (0.06)**	**3. (9.8)**	**4. (1 Hz)**
5. (249.8 to 250.2 Hz)			

Chapter 2

Damped And Force Vibrations

2.1 DAMPED SIMPLE HARMONIC MOTION:

Let us consider the motion of a simple harmonic oscillator in the presence of damping force. The damping force arises due to frictional, viscous and other effects and acts in opposition to the motion. In the simplest situation the damping force may be taken to be proportional to the instantaneous velocity.

2.1.1 Mathematical Analysis of Damped SHM:

Suppose a particle of mass m is oscillating along x-axis under the action of a restoring force proportional to the displacement (x) from the position of equilibrium and a damping force proportional to the instantaneous velocity (dx/dt). The equation of motion of such an oscillator is

$$m\frac{d^2x}{dt^2} = -sx - R_m\frac{dx}{dt} \qquad \text{... (1)}$$

where s is the restoring force per unit displacement and R_m is the damping force per unit velocity.

$$\Rightarrow \qquad \frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega_0^2 x = 0 \qquad \text{... (2)}$$

where $2b = R_m / m$ and $\omega_0 = \sqrt{s/m}$. ω_0 may be identified as the angular frequency of vibration in the absence of damping.

To solve the equation let $x = ce^{\alpha t}$ be a trial solution, where c and α are arbitrary constants. Differentiating we get

$$\frac{dx}{dt} = \alpha . ce^{\alpha t} \text{ and } \frac{d^2x}{dt^2} = \alpha^2 . ce^{\alpha t}$$

$$\Rightarrow \qquad \left(\alpha^2 + 2b\alpha + \omega_0^2\right)ce^{\alpha t} = 0 \qquad \Rightarrow \alpha^2 + 2b\alpha + \omega_0^2 = 0$$

It has two roots,

$$\alpha_1 = -b + \sqrt{b^2 - \omega_0^2} \text{ and } \alpha_2 = -b - \sqrt{b^2 - \omega_0^2}$$

$$x_1 = c_1 e^{\alpha_1 t} \text{ and } x_2 = c_2 e^{\alpha_2 t},$$

So, its general solution is given by the principle of superposition as

$$x = c_1 e^{\alpha_1 t} + c_2 e^{\alpha_2 t} = e^{-bt}\left[c_1 e^{\sqrt{b^2 - \omega_0^2}\cdot t} + c_2 e^{-\sqrt{b^2 - \omega_0^2}\cdot t}\right] \qquad \text{... (3)}$$

Large damping : $b > \omega_0$

In this case $b^2 - \omega_0^2$ is positive quantity and hence the solution is *non-oscillatory*. However, the actual behaviour depends upon the initial conditions.

Initial condition:

Case-I : At t = 0,

Position $x = x_0$ and, Velocity $dx/dt = v_0$.

Putting these conditions in Equation (3) we get

$$x_0 = c_1 + c_2$$

$$v_0 = \alpha_1 c_1 + \alpha_2 c_2 \qquad \text{... (4)}$$

Solving these equation we get

$$c_1 = \frac{\alpha_2 x_0 - v_0}{\alpha_2 - \alpha_1} = \frac{x_0}{2}\left(1 + \frac{b + v_0/x_0}{\sqrt{b^2 - \omega_0^2}}\right)$$

and $$c_2 = \frac{v_0 - \alpha_1 x_0}{\alpha_2 - \alpha_1} = \frac{x_0}{2}\left(1 - \frac{b + v_0/x_0}{\sqrt{b^2 - \omega_0^2}}\right)$$

Thus from Equation (3) we get

$$x = \frac{x_0}{2}e^{-bt}\left[\left(1 + \frac{b + v_0/x_0}{\sqrt{b^2 - \omega_0^2}}\right)e^{\sqrt{b^2 - \omega_0^2}t} + \left(1 - \frac{b + v_0/x_0}{\sqrt{b^2 - \omega_0^2}}\right)e^{-\sqrt{b^2 - \omega_0^2}t}\right] \qquad \text{... (5)}$$

As $b^2 - \omega_0^2$ is a positive quantity the motion is obviously non-oscillatory, aperiodic or dead beat. If the motion is started with an initial displacement but no initial velocity the displacement gradually falls off to zero with time and the body returns to the equilibrium position without any oscillation about the equilibrium position.

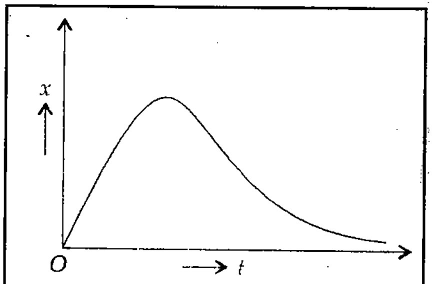


Figure 2.1: Non-oscillatory behaviour of a heavily damped oscillator

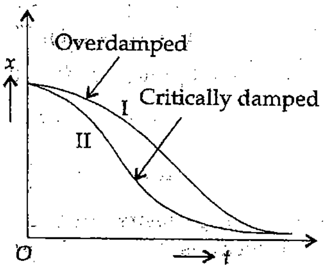


Figure 2.2: Non-oscillatory behaviour of overdamped and critically damped oscillator('t' measured from the instant of maximum displacement)

Case-II : At $t = 0, x = 0$ and $dx/dt = v_0$

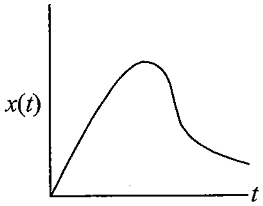
$$c_1 = -c_2 = -\frac{v_0}{\alpha_2 - \alpha_1} = \frac{v_0}{2\sqrt{b^2 - \omega_0^2}}$$

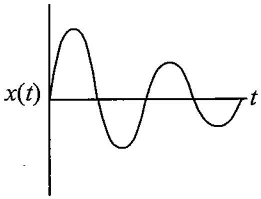
Then $$x(t) = \frac{v_0}{2\cdot\sqrt{b^2 - \omega_0^2}}e^{-bt}\left[e^{\sqrt{b^2 - \omega_0^2}t} - e^{-\sqrt{b^2 - \omega_0^2}t}\right]$$

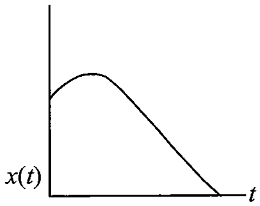
$$= \frac{v_0}{\sqrt{b^2 - \omega_0^2}}e^{-bt}\sinh\sqrt{b^2 - \omega_0^2}t \qquad \text{... (6)}$$

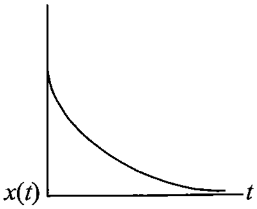
Describes the behaviour of a heavily damped system which is disturbed from equilibrium by a sudden impulse at t = 0. The motion is illustrated in figure (2.2). For small 't' the term e^{-bt} very nearly equal to unity. Displacement increases with time as $\sinh\sqrt{b^2-\omega_0^2}\,t$ increases with 't'. The displacement reaches a maximum value and thereafter the term e^{-bt} dominates and the displacement decays exponentially with time. This type of motion is found in a dead beat galvanometer or a pendulum immersed in a highly viscous liquid.

Ex. 1. Among the following displacement versus time plots, which ones may represent an overdamped oscillator?

(P)


(Q)


(R)


(S)


(a) Only (P) and (Q) (b) Only (P) and (R) (c) Only (P) and (S) (d) Only (P), (R), (S)

[IIT-JAM : 2009]

Soln. **(d)** Over damped oscillator does not have oscillatory motion, Therefore any of the (P),(R) and (S) can represent its motion.

Critical damping $b=\omega_0$**:**

It is a limiting case of heavy damping when the motion changes from non-oscillatory to oscillatory nature. Putting $b=\omega_0$ in equation (2) we find that $\alpha_1=\alpha_2=-\omega_0$. As the two roots are equal to the general solution of equation (2) in this case would be given by

$$x=(c_1+c_2t)e^{-\omega_0t} \quad \text{... (7)}$$

Initial condition:

Case-I : At t = 0, $x = x_0$ and $dx/dt = v_0$,

then $x_0 = c_1$ and

$$v_0=-\omega_0c_1+c_2 \quad \Rightarrow c_2=v_0+\omega_0x_0$$

Thus, $$\mathbf{x=e^{-\omega_0t}\left[x_0+(v_0+\omega_0x_0)t\right]} \quad \text{... (8)}$$

The motion represented by the equation (8) is also non-oscillatory. But here the rate of decay is much faster than the overdamped case. The motion is now said to be critically damped.

Case-II : At t = 0, $x = x_0$ and dx/dt = 0 then

$$\mathbf{x=x_0(1+\omega_0t)e^{-\omega_0t},\ \ v=x_0\omega^2t\,e^{-\omega_0t}} \quad \text{... (9)}$$

Maximum velocity at $t=1/\omega_0$

The displacement decreases from x_0 with time. This is illustrated by the curve II in figure (2.2). In fact in this case particle returns most quickly to the position of equilibrium.

Case-III : At t = 0, x = 0 and $dx/dt = v_0$ then we have

$$\mathbf{x=v_0t.e^{-\omega_0t}}$$

And $$\frac{dx}{dt}=v_0e^{-\omega_0t}(1-\omega_0t) \quad \text{... (10)}$$

In this case initially the displacement increases almost linearly with time and becomes maximum i.e., $dx/dt = 0$ at $t = 1/\omega_0$. Therefore, the displacement decays exponentially with time and ultimately becomes zero (figure (2.3)). Note that the time $t = 1/\omega_0$ is independent of initial velocity. The maximum displacement is

$$x_{\max} = v_0 \cdot \frac{1}{\omega_0} e^{-\omega_0 \cdot \frac{1}{\omega_0}} = \frac{v_0}{e.\omega_0} \qquad \dots (11)$$

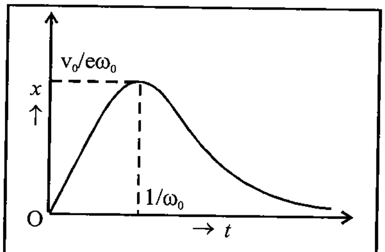


***Figure* 2.3:** Critically damped motion

The critically damped condition is useful if we desire most quick decay without oscillation. In a pointer-type galvanometer we want the pointer to move smoothly and quickly to a new equilibrium position when current is sent through it. This helps us to take the reading immediately after the meter is connected to a circuit.

. 2. A spring - mass system has undamped natural angular frequency $\omega_0 = 100$ rad s^{-1}. The solution x(t) at critial damping is given by $x(t) = x_0(1+\omega_0 t)\exp(-\omega_0 t)$, where x_0 is a constant. The system experiences the maximum damping force at time

(a) 0.01 s (b) 0.1 s (c) 0.01 πs (d) 0.1 πs

[IIT-JAM : 2006]

ln. **(a)** Damping force is maximum when speed is maximum.

i.e. $\frac{d}{dt}\left(\frac{dx}{dt}\right) = 0 \quad \therefore -x_0 w_0^3 \exp(-w_0 t) + w_0^3 xt \exp(-w_0 t) = 0$

$\therefore \quad t = \frac{1}{w_0}; \ \frac{1}{100} = 0.01\,\text{sec}.$

Alternate Solution:

Let displacement be

$x = x_0(1+\omega_0 t)\exp(-\omega_0 t)$

Damping force $F = -p\frac{dx}{dt} = -p[x_0(0+\omega_0)\exp(-\omega_0 t) + x_0(1+\omega_0 t)(-\omega_0)\exp(-\omega_0 t)]$

$$= -px_0\omega_0[\exp(-\omega_0 t) - \exp(-\omega_0 t) - \omega_0 t\exp(-\omega_0 t)]$$

$$= px_0\omega_0^2 t\exp(-\omega_0 t)$$

Let F is maximum when $t = t_0$, then $\frac{dF}{dt|t_0} = 0$

$px_0\omega_0^2[\exp(-\omega_0 t_0) - \omega_0 t\exp(-\omega_0 t_0)] = 0$

$\Rightarrow \omega_0 t_0 = 1 \Rightarrow t_0 = \frac{1}{\omega_0} = 0.01\,\text{sec}.$

Small damping $b<\omega_0$:

The most important kind of behaviour in acoustics is obtained when $b<\omega_0$. In this case $b^2-\omega_0^2$ is negative and we an write $\sqrt{b^2-\omega_0^2}=\sqrt{-\left(\omega_0^2-b^2\right)}=j\omega$ where $\omega=\sqrt{\omega_0^2-b^2}$.

The solution (3) now takes the form

$$x=e^{-bt}\left[c_1\,e^{j\omega t}+c_2e^{-j\omega t}\right]=e^{-bt}\left[B_1\cos\omega t+B_2\sin\omega t\right]$$

where $B_1=c_1+c_2$ and $B_2=j\left(c_1-c_2\right)$

Putting $B_1=A\cos\phi$ and $B_2=A\sin\phi$, we can write

$$x=Ae^{-bt}\cos\left(\omega t-\phi\right) \qquad \text{... (12)}$$

where A and ϕ are real constants to be evaluated from the given initial conditions. Equation (12) represents a damped oscillatory motion with an angular frequency $\omega=\sqrt{\omega_0^2-b^2}$ whose amplitude, Ae^{-bt}, diminishes exponentially with time. Obviously the presence of damping reduces the frequency slightly and causes the amplitude to decay exponentially with time.

To evaluate the constants A and ϕ we are to apply initial conditions. Let at $t=0$, $x=x_0$ and $dx/dt=v_0$. Therefore, from equation (12)

$$x_0=A\cos\phi \qquad \Rightarrow\ v_0=-Ab\cos\phi+A\omega\sin\phi$$

$$\Rightarrow\qquad \frac{v_0+bx_0}{\omega}=A\sin\phi$$

Combining these two equations, we get

$$A=\sqrt{x_0^2+\left(v_0+bx_0\right)^2/\omega^2}$$

$$\text{and}\qquad \tan\phi=\frac{v_0+bx_0}{\omega x_0} \qquad \text{...(13)}$$

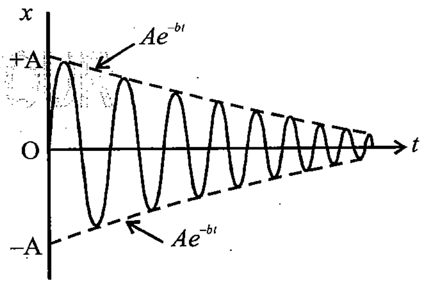


***Figure* 2.4:** Damped oscillatory motion

Case-I: At $t=0$, $x=x_0$ and $dx/dt=v_0=0$, then

$$A=x_0\sqrt{1+\frac{b^2}{\omega^2}}=x_0\sqrt{1+\frac{b^2}{\omega_0^2-b^2}}=\frac{x_0\omega_0}{\sqrt{\omega_0^2-b^2}}$$

$$\text{and}\qquad \tan\phi=\frac{b}{\omega}=\frac{b}{\sqrt{\omega_0^2-b^2}} \qquad \text{... (14)}$$

Therefore, from equation (12)

$$x=\frac{x_0\omega_0}{\sqrt{\omega_0^2-b^2}}e^{-bt}\cos\left(\sqrt{\omega_0^2-b^2}\cdot t-\tan^{-1}\frac{b}{\sqrt{\omega_0^2-b^2}}\right) \text{... (15)}$$

Case-II: At t = 0, $x = x_0 = 0$ and $dx/dt = v_0$, then $A = \frac{v_0}{\omega} = \frac{v_0}{\sqrt{\omega_0^2 - b^2}}$ and $\phi = 90°$

In this case, which is common, the motion is given by equation (12) as

$$x = \frac{v_0}{\sqrt{\omega_0^2 - b^2}} e^{-bt} \sin\sqrt{\omega_0^2 - b^2}\, t \qquad \ldots (16)$$

To discuss the motion of a weakly damped oscillator we plot the solution (16) as shown in figure (2.4). Obviously the motion is oscillatory whose amplitude decays exponentially with time.

Ex. 3. The instantaneous position $x(t)$ of a small block performing one-dimensional damped oscillation is $x(t) = Ae^{-\gamma t}\cos(\omega t + \alpha)$. Here, ω is the angular frequency, γ the damping coefficient, A the initial amplitude and α the initial phase. If $x|_{t=0} = 0$ and $\left.\frac{dx}{dt}\right|_{t=0} = -v$, the values of A and α (with $n = 0, 1, 2, \ldots\ldots\ldots.$) are

(a) $A = \frac{v}{2\omega}, \alpha = \frac{(2n+1)\pi}{2}$

(b) $A = \frac{v}{\omega}, \alpha = n\pi$

(c) $A = \frac{v}{\omega}, \alpha = \frac{(2n+1)\pi}{2}$

(d) $A = \frac{2v}{\omega}, \alpha = \frac{(2n+1)\pi}{2}$

[IIT-JAM : 2008]

oln. **(c)** $x|_{t=0} = 0 \Rightarrow A\cos(\alpha) = 0 \Rightarrow \boxed{\alpha = (2n+1)\frac{\pi}{2}}$;

$$\Rightarrow \frac{dx}{dt} = -A\,e^{-\gamma t}\omega\sin(\omega t + \alpha) - \gamma Ae^{-\gamma t}\cos(\omega t + \alpha)$$

$$\left.\frac{dx}{dt}\right|_{t=0} = v = A\omega \qquad \Rightarrow \boxed{A = \frac{v}{\omega}}$$

2.1.2 Amplitude decay constant (τ):

It is defined as the time in which the amplitude decays to 1/e of its initial value. As the amplitude at any instant is Ae^{-bt}, the amplitude decay constant τ is given by

$$\frac{Ae^{-b(t+\tau)}}{Ae^{-bt}} = \frac{1}{e} \qquad \Rightarrow \tau = \frac{1}{b} \qquad \ldots (17)$$

τ is also called **relaxation time** or modulus of decay.

2.1.3 Logarithmic decrement:

Another method of measuring damping effect on the motion is in terms of logarithmic decrement. Let at time $t = t_1$ amplitude $A_1 = Ae^{-bt_1}$. If $T = 2\pi/\omega$ is the time period of oscillation, then half time period later the amplitude A_2 is given by

$$A_2 = Ae^{-b\left(t_1+\frac{T}{2}\right)} = A_1e^{-\frac{bT}{2}}$$

Thus $\dfrac{A_1}{A_2} = e^{bT/2} = d(\text{say})$ or, $\log_e(d) = \dfrac{bT}{2} = \lambda(\text{say})$.

The quantity d which is the ratio of two successive amplitudes (on opposite sides of the mean position) is called the decrement and $\log_e d = \lambda$ *is called logarithmic decrement of the damped oscillation.* The logarithmic decrement may be defined as the natural logarithm of the ratio of two successive amplitudes of damped oscillations that are separated by half the time period. If A_1 and A_{n+1} are the amplitudes for the first and (n)th half oscillation then

$$\frac{A_1}{A_{n+1}} = \frac{A_1}{A_2}.\frac{A_2}{A_3}\ldots\ldots\frac{A_n}{A_{n+1}} = (d)^n$$

$$\Rightarrow \qquad \boldsymbol{\lambda = \log_e d = \frac{1}{n}\log_e\frac{A_1}{A_{n+1}}} \qquad \ldots(18)$$

Note : Some Text book define logarithmic decrement as, $\lambda = \dfrac{2}{n}\log\dfrac{A_1}{A_{n+1}} = bT$

2.1.4 Energy of a damped simple harmonic oscillator:

The P.E. of the particle for a displacement x is

$$E_p = \int_0^x s\cdot x\,dx = \frac{1}{2}sx^2 \qquad \ldots(19)$$

The K.E. at that instant is

$$E_k = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 \qquad \ldots(20)$$

Therefore, total energy $E = E_k + E_p$

$$= \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}sx^2 = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}m\omega_0^2x^2 \qquad \ldots(21)$$

Now from equation (12) $x = Ae^{-bt}\cos(\omega t - \phi)$

$$\Rightarrow \qquad \frac{dx}{dt} = -Abe^{-bt}\cos(\omega t-\phi) - A\omega e^{-bt}\sin(\omega t - \phi)$$

$$\Rightarrow E = \frac{1}{2}mA^2e^{-2bt}\left[b^2\cos^2(\omega t-\phi) + \omega^2\sin^2(\omega t-\phi) + b\omega\sin 2(\omega t-\phi) + \omega_0^2\cos^2(\omega t-\phi)\right] \qquad \ldots(22)$$

Therefore, time averaged energy of the oscillator is

$$< E > = \frac{1}{2}mA^2e^{-2bt}\left[b^2.\frac{1}{2} + \omega^2.\frac{1}{2} + 0 + \omega_0^2.\frac{1}{2}\right]$$

$$= \frac{1}{2}mA^2\omega_0^2e^{-2bt} = E_0e^{-2bt} \quad \left[\because\ \omega = \sqrt{\omega_0^2 - b^2}\right] \qquad \ldots(23)$$

where $E_0 = \dfrac{1}{2}mA^2\omega_0^2$ is the total energy of an undamped oscillator. So the total energy of a weakly damped oscillator decays exponentially with time.

Energy decay constant :

The initial energy falls to 1/e of its initial value after a time $\tau' = 1/2b$, known as energy decay time constant.

2.1.5 Equation of motion from energy consideration:

Total instantaneous energy

$$E = K.E. + P.E. = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}sx^2 \qquad ...(24)$$

In the presence of damping E does not remain constant because a part of it used to overcome the damping forces. The rate of loss of energy must be equal to the rate of work done against damping forces. Thus

$$-\frac{d}{dt}\left[\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}sx^2\right] = R_m\frac{dx}{dt}.\frac{dx}{dt}$$

$$\Rightarrow \qquad -\frac{dx}{dt}\left[m\frac{d^2x}{dt^2} + sx\right] = R_m\frac{dx}{dt}.\frac{dx}{dt} \qquad \Rightarrow \frac{dx}{dt}\left[m\frac{d^2x}{dt^2} + R_m\frac{dx}{dt} + sx\right] = 0$$

Since dx/dt is not zero for all values of t,

$$m\frac{d^2x}{dt^2} + R_m\frac{dx}{dt} + sx = 0$$

which is the differential equation of motion of a damped harmonic oscillator.

The rate of loss of average energy:

$$-\frac{d<E>}{dt} = 2bE_0e^{-2bt} = 2b<E> \qquad ...(25)$$

The energy loss is due to work done against damping force. So this must be equal to the average rate of work done against damping force. The rate of work done against damping force is

$$R_m\frac{dx}{dt}.\frac{dx}{dt} = R_m\left(\frac{dx}{dt}\right)^2$$

$$= R_m.Ae^{-2bt}\left[b^2\cos^2(\omega t - \phi) + \omega^2\sin^2(\omega t - \phi) + b\omega\sin 2(\omega t - \phi)\right] \qquad ...(26)$$

As before, the average rate of work done against damping force is

$$<R_m\left(\frac{dx}{dt}\right)^2> = R_mA^2e^{-2bt}\left[b^2\frac{1}{2} + \omega^2\frac{1}{2} + 0\right] = 2bm.Ae^{-2bt}\frac{1}{2}\left(b^2 + \omega_0^2 - b^2\right)$$

$$= 2b\frac{1}{2}mA^2\omega_0^2e^{-2bt} = 2b<E> \qquad ...(27)$$

Thus we prove that the rate of loss of energy is equal to the rate of work done against damping force.

2.1.6 Quality factor or Q-value of the damped oscillator:

The effect of damping on the motion of the oscillator can be expressed in terms of a quality factor defined as

$$Q = 2\pi \times \frac{\text{Average energy stored in a system}}{\text{Average energy lost per cycle}} = 2\pi \times \frac{<E>}{2b<E>.T}$$

$$= \frac{\omega}{2b} = \frac{\omega m}{R_m} = \omega\tau' \qquad ...(28)$$

Thus smaller is the damping in the system, higher is the value of the quality factor and the energy decay time constant τ'.

Ex. 4. The motion of a one-dimensional damped oscillator is described by the differential equation $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$, where ω_0 is the natural angular frequency of the oscillator and β is the damping parameter. The oscillator is given an initial velocity V_0 at time $t = 0$.

(a) Find an expression for the displacement $x(t)$ of the oscillator.

(b) If $V_0 = 0.13\ \text{m/sec}$, $\omega_0 = 13\ \text{sec}^{-1}$ and $\beta = 5\ \text{sec}^{-1}$,

(i) find the value of the 'amplitude' A_0 of the damped oscillatory motion at $t = 0$,

(ii) calculate the time t_1 at which the displacement first reaches its maximum X_1,

(iii) compute the maximum displacement X_1

[Use data: $\tan^{-1}(2.4) = 1.176$, $e^{-0.49} = 0.613$ and $\sin(1.176) = 0.923$]

[IIT-JAM GP : 2012]

Soln. Differential equation of one dimensional damped harmonic oscillation is described by the differential equation

$$\ddot{x} = 2\beta\dot{x} + \omega_0^2 x = 0$$

$$(D^2 + 2\beta D + \omega_0^2)x = 0$$

$$D = \frac{-2\beta \pm \sqrt{4\beta^2 - 4\omega^2}}{2}$$

$$= -\beta \pm i\sqrt{\omega_0^2 - \beta^2}$$

$$= -\beta \pm i\omega \text{ where } \omega' = \sqrt{\omega^2 - \beta^2}$$

So, $x = e^{-\beta t}\left[C_1 \sin\omega' t + C_2 \cos\omega' t\right]$

$$A_0 e^{-\beta t}\sin(\omega' t + \delta)$$

Velocity $V = \dfrac{dx}{dt}$

$$= A_0 e^{\beta t}\left[\omega'\cos(\omega' t + \delta) - \beta\sin(\omega' t + \delta)\right]$$

At $t = 0$, $V = V_0$

$$\Rightarrow A_0\left[\omega'\cos\delta - \beta\sin\delta\right] = V_0$$

$$A_0\left[\omega'\cos\delta - \beta\sin\delta\right] = V_0$$

At $t = 0, x = 0$

$$\Rightarrow A_0\sin\delta = 0$$

So, $\delta = 0$

So, $x = Ae^{-\beta t}\sin\omega' t$

$$V = \frac{dx}{dt} = A_0 e^{-\beta t}\left[\omega'\cos\omega' t - \beta\sin\omega' t\right]$$

At $t = 0$, $V = V_0$

$$V_0 = A_0\omega'$$

$$\omega' = \sqrt{\omega_0^2 - \beta^2} = \sqrt{169 - 25} = 12$$

$V_0 = 0.13$

$$A_0 = \frac{0.13}{12} m = 1.08\text{cm}$$

At displacement maximum, $\frac{dx}{dt} = 0$

$$\Rightarrow \tan \omega' t_1 = \frac{\omega'}{\beta} = \frac{12}{15} = 2.4$$

$$t_1 = \frac{1}{\omega}, \tan^{-1} 2.4$$

$$= \frac{1.176}{12} = 0.098 \text{ sec}$$

Maximum displacement

$$x_{\max} = A_0 e^{-\beta t} \sin \omega' t$$

$$= 1.08\, e^{-5\cdot 0.098} \sin 1.176$$

$$= 1.08\, e^{-0.49} \sin 1.176$$

$$= 0.611 \text{ cm}$$

2.2 FORCED VIBRATION AND RESONANCE:

The natural vibration of a system dies out with time due to damping force. However, the system can be maintained in a condition of vibration by applying external periodic force. When an external periodic force acts on a vibrating system the system tends to vibrate with its own natural frequency, while the external driving force tries to impress its own frequency on the vibrating system. Initially the system vibrates with both the frequencies. However, in course of time the natural vibration dies out due to the damping forces that operate on the system and finally in the steady state the system vibrates with the frequency of the driving force. Such vibrations are called forced vibration. The driven system extracts energy from the driving system.

Resonance is a special case of forced vibration. If the frequency of the driving force agrees with the natural frequency of the system then the system oscillates with maximum amplitude or velocity. This is known as resonance or sympathetic vibration. In this case the driving force communicates maximum power to the system.

2.2.1 Mathematical analysis of forced vibration:

Suppose we consider the motion of a mechanical oscillator of mass m under the action of a restoring force proportional to displacement, a damping force proportional to its instantaneous velocity and an external periodic force represented by $F = F_0 \cos \omega t$. The equation of motion of such an oscillator is

$$m\frac{d^2x}{dt^2} = F_0 \cos \omega t - sx - R_m \frac{dx}{dt} \qquad \text{... (29)}$$

where x is the displacement from equilibrium position at any instant of time t, s is the stiffness and R_m is the damping constant. The equation may be rewritten as,

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega_0^2 x = f_0 \cos \omega t = \text{Re. } f_0 e^{j(\omega t - \phi)} \cdot e^{j\phi} \qquad \text{... (30)}$$

where $2b = R_m / m$, $\omega_0 = \sqrt{s/m}$, $f_0 = F_0 / m$ and 'Re' stands for the real part of the complex quantity on its right hand side. To solve the Equation (30) we note that for small damping the complementary

function (i.e., the solution with R.H.S zero) represents a damped oscillatory motion which dies out after some time. It is called transient. The particular integral represents the forced vibration. Since ultimately in the steady state the system vibrate with the frequency of the driving force
We may try a solution of the form

$$x = \text{Re}\, Ae^{j(\omega t - \phi)}$$

Here ϕ is defined as the phase by which displacement leads force.
Substituting it in equation (30), we get

$$\text{Re}\left(-\omega^2 + j\,2b\omega + \omega_0^2\right)A\; e^{j(\omega t - \phi)} = \text{Re}\, f_0 e^{j(\omega t - \phi)} \cdot e^{j\phi}$$

This is true for all 't' and hence A will be given by

$$A = \frac{f_0 \times e^{j\phi}}{\left(\omega_0^2 - \omega^2\right) + j2b\omega} \Rightarrow Ae^{-j\phi} = \frac{f_0}{\left(\omega_0^2 - \omega^2\right) + j2b\omega} \qquad \ldots (31)$$

Simplifying, we get $A = \dfrac{f_0}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4b^2\omega^2}}$ and $\tan\phi = \dfrac{2b\omega}{\omega_0^2 - \omega^2}$ ... (32)

$$\Rightarrow \qquad x = \text{Re}\, Ae^{j(\omega t - \phi)} \qquad \Rightarrow x = A\cos\left(\omega t - \phi\right) \qquad \ldots (33)$$

For small damping the complementary function which is the solution of the equation

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega_0^2\, x = 0$$

is given by $x = c.e^{-bt}\cos\left(\sqrt{\omega_0^2 - b^2}\,t - \delta\right)$

where c and δ are two arbitrary constants. Hence the general solution of the Equation (30) is given by

$$x = c.e^{-bt}\cos\left(\sqrt{\omega_0^2 - b^2}\,t - \delta\right) + A\cos\left(\omega t - \phi\right) \qquad \ldots (34)$$

The first term on the right hand side of Equation (34) gives the transient and the second term represents the steady state motion.

2.2.2 Amplitude and velocity resonances:

For certain value of ω the amplitude becomes maximum and we say there is amplitude resonance between the driver and the driven system. Now for A to be maximum, we must have

$$\frac{dA}{d\omega} = 0 \text{ and } \frac{d^2A}{d\omega^2} < 0$$

From Equation (32) it is clear that for $dA/d\omega = 0$ we must have,

$$\frac{d}{d\omega}\left\{\left(\omega_0^2 - \omega^2\right)^2 + 4b^2\omega^2\right\} = 0 \qquad \Rightarrow \qquad 2\left(\omega_0^2 - \omega^2\right)\left(-2\omega\right) + 8b^2\omega = 0$$

$$\Rightarrow \qquad \omega = \sqrt{\omega_0^2 - 2b^2} = \omega_r\,(\text{say})$$

It is easy to show that $\dfrac{d^2A}{d\omega^2} < 0$ at $\omega = \omega_r$. Thus ω_r is the frequency at which amplitude resonance occurs.

Obviously this frequency ω_r is slightly smaller than both the natural frequency ω_0 and the frequency $\sqrt{\omega_0^2 - b^2}$ of damped oscillation. The maximum displacement amplitude at resonance is obtained by putting $\omega = \sqrt{\omega_0^2 - 2b^2}$ in Equation (32).

Thus, $$A_{\max} = \frac{f_0}{2b\sqrt{b^2+\omega^2}} = \frac{f_0}{2b\sqrt{\omega_0^2 - b^2}} \quad \ldots (35)$$

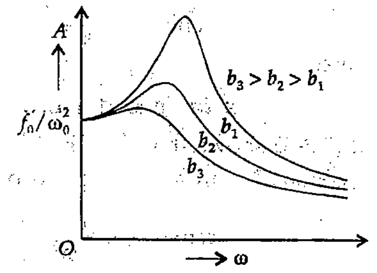


Figure 2.5: Variation of displacement amplitude with driving force frequency 'ω' for various values of b(damping factor)

Since potential energy is given by $E_p = \frac{1}{2}sx^2$, its value becomes maximum at the amplitude resonance frequency.

The instantaneous velocity of the system undergoing forced vibration may be obtained from Equation (33) as

$$v = \frac{dx}{dt} = -A\omega\sin(\omega t - \phi) = V_0\cos(\omega t - \phi + \pi/2) \quad \ldots (36)$$

where $$V_0 = A\omega = \frac{f_0\omega}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4b^2\omega^2}} = \frac{f_0}{\sqrt{\left(\frac{\omega_0^2}{\omega} - \omega\right)^2 + 4b^2}} \quad \ldots (37)$$

It will be maximum for certain value of ω and we say there is velocity or energy resonance between the driver and driven systems. Now for V to be maximum

$$\left(\frac{\omega_0^2}{\omega} - \omega\right)^2 = 0 \text{ or, } \omega = \omega_0$$

Thus the velocity resonant frequency is equal to the natural frequency of the system for free oscillation. It does not depend upon the damping in the system. The maximum velocity amplitude at resonance is obtained by putting $\omega = \omega_0$ in Equation (37). Thus

$$v_{\max} = \frac{f_0}{2b} \quad \ldots (38)$$

As the kinetic energy $E_K = \frac{1}{2}mv^2$, it will be maximum at velocity resonance.

x. 5. Consider the following forced oscillator equation:

$\frac{d^2x}{dt^2} + 2\beta\frac{dx}{dt} + 16x = 9\cos(5t)$, where the damping coefficient β is a constant. As $\beta \to 0$, the general solution $x(t)$ is given by $x(t) = \cos(5t + \varphi) + A\cos(4t + \psi)$. The phase, φ, tends to

(a) π (b) – π (c) 0 (d) π/2

[IIT-JAM GP: 2010]

Soln. **(a)**

As $\beta \to 0$

$$(D^2+16)x = 9\cos 5t$$

$$PI = \frac{9}{D^2+16}\cos 5t$$

$$= \frac{9}{-25+16}\cdot\cos 5t$$

$$= -\cos 5t$$

$$= \cos(5t+\pi)$$

Phase $\phi = \pi$

2.2.3 Phase relations in forced vibration:

Equation (33) shows that the displacement x lags behind the driving force, $F = F_0 \cos\omega t$, by an angle ϕ given by Equation (32) as

$$\tan\phi = \frac{2b\omega}{\omega_0^2-\omega^2} \qquad \text{... (39)}$$

Suppose ω is changed from 0 to ∞. Let us find how ϕ changes with ω:

(i) when $\omega = 0$, $\tan\phi = 0$ and hence $\phi = 0$.

(ii) when $\omega < \omega_0$, $\tan\phi$ is +ve. This indicates that $0 < \phi < \pi/2$.

(iii) when $\omega = \omega_0$, $\tan\phi = \infty$ and hence $\phi = \pi/2$.

(iv) when $\omega > \omega_0$, $\tan\phi$ is –ve, which indicates that $\pi/2 < \phi < \pi$.

(v) As $\omega \to \infty$, $\tan\phi \to -0$ which indicates that $\phi \to \pi$.

From equation (39), $\sec^2\phi\dfrac{d\phi}{d\omega} = \dfrac{2b}{\omega_0^2-\omega^2} - \dfrac{2b\omega}{\left(\omega_0^2-\omega^2\right)^2}(-2\omega)$

$$\Rightarrow \qquad \left(1+\tan^2\phi\right)\frac{d\phi}{d\omega} = \frac{2b\left(\omega_0^2+\omega^2\right)}{\left(\omega_0^2-\omega^2\right)^2}$$

$$\Rightarrow \qquad \frac{d\phi}{d\omega} = \frac{1}{1+\dfrac{4b^2\omega^2}{\left(\omega_0^2-\omega^2\right)^2}}\cdot\frac{2b\left(\omega_0^2+\omega^2\right)}{\left(\omega_0^2-\omega^2\right)^2} = \frac{2b\left(\omega_0^2+\omega^2\right)}{\left(\omega_0^2-\omega^2\right)^2+2b^2\omega^2}$$

At velocity resonance $\omega = \omega_0$ and then $\dfrac{d\phi}{d\omega} = \dfrac{2}{b}$ ... (40)

So smaller is the damping factor b greater is the rate of change of phase angle near resonance.

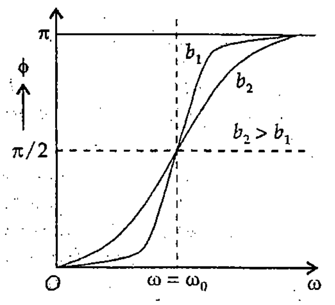


***Figure* 2.6 (a):** Phase relationship between displacement and driving force.

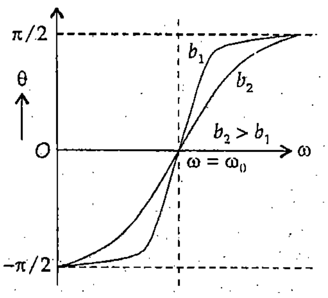


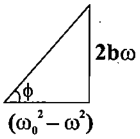
***Figure* 2.6 (b):** Phase-relationship between and driving force (θ = ϕ −π/2)

2.2.4 Energy in forced vibration:

The instantaneous rate of work done by the driving force i.e., the instantaneous power supplied to the oscillator by the driving force is $P = F_0 \cos \omega t . \frac{dx}{dt} = -F_0 A\omega \cos \omega t . \sin(\omega t - \phi)$

$$= -\frac{F_0 A\omega}{2}\left[\sin(2\omega t - \phi) - \sin\phi\right] \qquad \dots (41)$$

The average power over a complete period is



$$< P > = \frac{1}{T}\int_0^T P.dt = \frac{F_0 A\omega}{2}.\sin\phi\left[\because \int_0^T \sin(2\omega t - \phi)dt = 0\right]$$

$$= \frac{F_0 A\omega}{2}.\frac{2b\omega}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4b^2\omega^2}} = mb\omega^2 A^2 \qquad \dots (42)$$

Now the rate of work done against damping force is

$$\frac{dW}{dt} = R_m \frac{dx}{dt}.\frac{dx}{dt} = R_m \omega^2 A^2 \sin^2(\omega t - \phi) = \frac{R_m \omega^2 A^2}{2}\left[1 - \cos 2(\omega t - \phi)\right]$$

Averaging over a complete period we get

$$\left\langle \frac{dW}{dt} \right\rangle = \frac{1}{2} R_m \omega^2 A^2 = mb\omega^2 A^2 \qquad \dots (43)$$

Thus, we prove that in the steady state the average power supplied to the oscillator by the driving force is just equal to that being dissipated by the damping forces.

2.2.5 Sharpness of resonance and quality factor:

The time average power, from Equation (43), is

$$\langle P \rangle = mb\omega^2 A^2 = \frac{mb\omega^2 f_0^2}{\left(\omega_0^2 - \omega^2\right)^2 + 4b^2\omega^2}$$

$$= \frac{mbf_0^2}{\left(\frac{\omega_0^2}{\omega} - \omega\right) + 4b^2} \qquad \dots (44)$$

< P > is maximum when $\left(\frac{\omega_0^2}{\omega} - \omega\right)^2 = 0$ or, $\omega = \omega_0$.

Putting $\omega = \omega_0$ in Equation (44) we get the maximum average power at resonance as

$$< P >_{max} = \frac{mbf_0^2}{4b^2} \quad \text{... (45)}$$

Therefore, $$< P > = < P >_{max} \frac{4b^2\omega^2}{\left(\omega_0^2 - \omega^2\right)^2 + 4b^2\omega^2} \quad \text{... (46)}$$

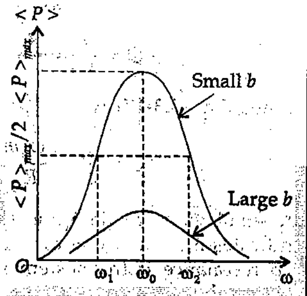


Figure **2.7: Response curve**

At half power points, $< P > = \frac{< P >_{max}}{2}$

$$\Rightarrow \frac{mb\omega^2 f_0^2}{(\omega_0^2 - \omega^2)^2 + 4b^2\omega^2} = \frac{1}{2}\frac{mbf_0^2}{4b^2}$$

$$\Rightarrow (\omega_0^2 - \omega^2)^2 + 4b^2\omega^2 = 8\omega^2 b^2$$

$$\Rightarrow \omega_0^2 - \omega^2 = \pm 2b\omega$$

$$\Rightarrow \omega^2 - \omega_0^2 \pm 2b\omega$$

$$\Rightarrow \omega = \mp b \pm \sqrt{b^2 + \omega_0^2} \quad \text{... (47)}$$

These are the two quadratic equation $\omega^2 - \omega_0^2 + 2b\omega = 0$ and $\omega^2 + \omega_0^2 + 2b\omega = 0$
Since negative frequency are not allowed, we take the position roots only

$$\omega_1 = -b + \sqrt{b^2 + \omega_0^2} \text{ and } \omega_2 = b + \sqrt{b^2 + \omega_0^2} \quad \text{... (48)}$$

Thus bandwidth, $\Delta\omega = \omega_2 - \omega_1 = 2b$

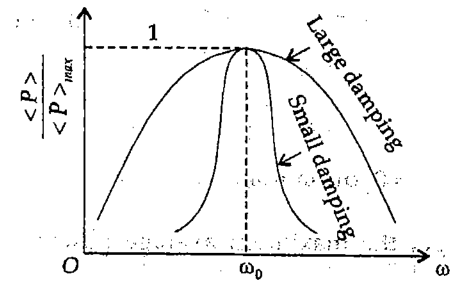


***Figure* 2.8: Dependence of sharpness of resonance on damping**

Quantitatively the sharpness of resonance is measured in terms of quality factor Q which is defined as

$$Q = \frac{\omega_0}{\Delta\omega} = \frac{\text{Resonant frequency}}{\text{Bandwidth}} \Rightarrow Q = \frac{\omega_0}{2b} = \frac{\omega_0 m}{R_m} \qquad \ldots (49)$$

A small value of damping constant R_m makes Q large. Again a large Q indicates a small value of bandwidth i.e., a sharp resonance. In case of sharp resonance if the system is driven by a force consisting of a large number of harmonic components then it will give significant response only to the components having frequencies close to the resonant frequency. The system thus possesses selective properties. This is known as selectivity.

SOLVED EXAMPLES

1. A mass of 0.01 kg is acted upon by a restoring force $0.01\,\text{N.m}^{-1}$ and a resisting force of 2×10^{-3} N.s.n^{-1}. Find out whether the motion is oscillatory or non-oscillatory. Also, find the value of resistive force for the motion to be critically damped.

Soln. Here $m = 0.01$ kg, $s = 0.01\,\text{N.m}^{-1}$ and $R_m = 2 \times 10^{-3}\,\text{N.s.m}^{-1}$

$$\text{So, } b = \frac{R_m}{2m}\ \frac{2\times 10^{-3}}{2 \times 0.01} = 0.1\,\text{N.s.m}^{-1}\,\text{kg}^{-1};\ \omega_0 = \sqrt{\frac{s}{m}} = \sqrt{\frac{0.01}{0.01}} = 1$$

As $b < \omega_0$, $\sqrt{b^2 - \omega^2} < 1$ or $i\sqrt{\omega^2 - b^2}$ the motion is oscillatory.

For the motion to be critically damped. $b = \omega_0$

$$\Rightarrow \qquad \frac{R_m}{2m} = \sqrt{s/m}$$

$$\Rightarrow \qquad R_m = 2\sqrt{sm} = 2\sqrt{0.01 \times 0.01} = 0.02\,\text{N.s.m}^{-1}$$

2. A mass m = 0.2 kg is suspended by a light spring of stiffness constant $20\,\text{N.m}^{-1}$. The energy associated with the vertical oscillation of the system is observed to decay to 1/e of initial value in 50 s. Assuming the damping force $(R_m v)$ proportional to the velocity (v), find the damping force constant R_m. Also, find the Q-value of the oscillator and the percentage change in frequency due to damping.

Soln. The energy decays according to the law, $E = E_0 e^{-2bt}$

$$\text{At } t = 50 \text{ sec, } E = \frac{E_0}{e} = E_0 e^{-2bt} \Rightarrow e = e^{2bt}$$

$$\Rightarrow 1 = 2bt \Rightarrow b = \frac{1}{2t}$$

Thus, $R_m = 2bm = \frac{m}{t} = \frac{1}{50} \times 0.2 = 4 \times 10^{-3}\ \mathrm{N.s.m^{-1}}$

The undamped natural frequency

$$\omega_0 = \sqrt{\frac{s}{m}} = \sqrt{\frac{20}{0.2}} = 10\ \mathrm{rad\,s^{-1}}$$

$$\Rightarrow \quad Q = \frac{\omega}{2b} = \frac{1}{2b}\sqrt{\omega_0^2 - b^2} = 50\sqrt{10^2 - \frac{1}{10^4}} = 500$$

Now, $\omega = \sqrt{\omega_0^2 - b^2} = \omega_0\left(1 - \frac{b^2}{\omega_0^2}\right)^{1/2} = \omega_0\left(1 - \frac{b^2}{2\omega_0^2}\right) = \omega_0\left(1 - \frac{1}{8Q^2}\right)\left[\because Q = \frac{\omega_0}{2b}\right]$

$$\Rightarrow \quad \frac{\omega_0 - \omega}{\omega_0} = \frac{1}{8Q^2}$$

Therefore, the percentage change in frequency due to damping is

$$\frac{\omega_0 - \omega}{\omega_0} \times 100 = \frac{100}{8Q^2} = \frac{100}{8 \times (500)^2} = 5 \times 10^{-4}\%$$

3. A body of mass $m = 10$ kg is subjected to a restoring force of stiffness constant $s = 10\ \mathrm{N.m^{-1}}$. The system is subjected to a damping force proportional to its velocity with the constant of proportionality as $R_m = 10\ \mathrm{N.s.m^{-1}}$. The mass is given an initial impulse of $0.68\ \mathrm{kg.m.s^{-1}}$ when at rest. Find the subsequent displacement of the mass as a function of time.

Soln. The equation of motion of the mass is $m\ddot{x} = -R_m\dot{x} - sx$

and its solution is of the form

$$x = e^{-bt}(A\cos\omega t + B\sin\omega t)$$

where $b = \frac{R_m}{2m}$, $\omega = \sqrt{\omega_0^2 - s/m}$; A and B are constants to be determined from given initial conditions.

At $t = 0, x = 0$. This indicates that $A = 0$.

$$\Rightarrow \quad x = Be^{-bt}\sin\omega t$$

Now the impulse is equal to change of momentum. Therefore

$$m\dot{x} - 0 = 0.68$$

Therefore, Initial velocity $\dot{x}_0 = \frac{0.68}{10} = 0.068\ \mathrm{ms^{-1}}$

$$b = \frac{R_m}{2m} = \frac{10}{2 \times 10} = 0.5\,\mathrm{s^{-1}},\ \omega_0$$

$$= \sqrt{s/m} = 1\ \mathrm{rad.s^{-1}},\ \omega = \sqrt{\omega_0^2 - b^2} = 0.866$$

$$\dot{x} = -bBe^{-bt}\sin\omega t + \omega Be^{-bt}\cos\omega t$$

At $t = 0, 0.068 = \omega B$.

$$\Rightarrow \qquad B = \frac{0.068}{0.866} = 0.078\,\text{m}$$

Thus $x = 0.078e^{-0.5t}\sin 0.866t$

This equation describes the subsequent motion.

4. In a damped harmonic oscillator the amplitude of vibration decreases from 10 cm to 1 cm after 100 complete cycles of oscillation. The time period of oscillation is 2.3 secs. Estimate the first undamped amplitude.

Soln. Log decrement $\lambda = \frac{1}{n}\log_e \frac{A_1'}{A_{n+1}'} = \frac{1}{2\times 100}\log_e \frac{10}{1} = 0.0115$, where n is the number of the half cycles.

Therefore, Undamped amplitude $A = A_1(1 + \lambda/2) = 10\left(1 + \frac{0.0115}{2}\right) = 10.06\,\text{cm}$

5. Show that in case of a forced vibration under a simple harmonic force the half-power frequencies correspond to the phase angle range given by $\tan\phi = \pm 1$ where ϕ is the phase difference between the displacement and the driving force.

Soln. Average power $<P> = \dfrac{mbf_0^2}{\left(\dfrac{\omega_0^2}{\omega} - \omega\right) + 4b^2}$

At half power points, $\left(\dfrac{\omega_0^2}{\omega} - \omega\right)^2 = 4b^2$ or, $\omega_0^2 - \omega^2 = \pm 2b\omega$

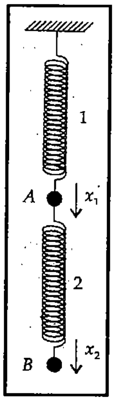
Therefore the phase angle ϕ is given by $\tan\phi = \dfrac{2b\omega}{\omega_0^2 - \omega^2} = \pm\dfrac{(\omega_0^2 - \omega^2)}{\omega_0^2 - \omega^2} = \pm 1.$

. Two equal masses m are attached with two identical massless springs of stiffness a as indicated in Fig. below. Find the normal frequencies and the ratios of amplitudes of the normal modes.

oln. Let x_1 and x_2 be the instantaneous displacements of the masses from their equilibrium positions. If $x_2 > x_1$ then the spring 2 is extended and the tension $s(x_2 - x_1)$ in the spring will act in favour of acceleration $\ddot{x}_1$ but against the acceleration $\ddot{x}_2$. Thus the equations of motion are

$$m\ddot{x}_1 = -sx_1 + s(x_2 - x_1)$$

$$m\ddot{x}_2 = -s(x_2 - x_1)$$



Putting $s/m = \omega_0^2$ and rearranging we get

$$\ddot{x}_1 + 2\omega_0^2 x_1 - \omega_0^2 x_2 = 0 \qquad \text{.... (1)}$$

and

$$\ddot{x}_2 + \omega_0^2 x_2 - \omega_0^2 x_1 = 0 \qquad \text{.... (2)}$$

In a normal mode both the masses oscillate with the same frequency. To find the normal frequencies we let

$$x_1 = A_1 e^{j\omega t} \text{ and } x_2 = A_2 e^{j\omega t}$$

Substituting these solution in Equation (1) and (2) we get

$$\left(2\omega_0^2 - \omega^2\right)A_1\,\omega_0^2 A_2 = 0 \qquad \text{.... (3)}$$

$$-\omega_0^2 A_1 + \left(\omega_0^2 - \omega^2\right) A_2 = 0 \qquad \text{.... (4)}$$

For non-trivial solution, $\begin{vmatrix} 2\omega_0^2 - \omega^2 & -\omega_0^2 \\ -\omega_0^2 & \omega_0^2 - \omega^2 \end{vmatrix} = 0 \quad \Rightarrow \quad \omega^4 - 3\omega_0^2\omega^2 + \omega_0^4 = 0$

Therefore the normal frequencies are $\omega_1 = \left(\frac{3-\sqrt{5}}{2}\right)^{1/2} \omega_0$ and $\omega_2 = \left(\frac{3+\sqrt{5}}{2}\right)^{1/2} \omega_0$

Putting $\omega = \omega_1$ in Equation (3) and (4) we get $\frac{A_1}{A_2} = -\frac{\sqrt{5}-1}{2}$

It is the ratio of amplitudes for the normal mode of frequency $\omega = \omega_1$.

Putting $\omega = \omega_2$ in Equations (3) and (4) we get $\frac{A_1}{A_2} = -\frac{\sqrt{5}+1}{2}$

It is the ratio of amplitudes for the normal mode of frequency $\omega = \omega_2$.

Chapter 3

Acoustic Wave Motion

3.1 WAVES:

A wave is a disturbance which propagate in space, transport energy from one place to another without transport of matter.

Example: Ripples on a pond, sound we hear etc.

3.1.1 Classification of wave:

- (a) Based on medium
    - Mechanical waves
    - Electro magnetic wave.
- (b) Based on motion of particles of medium
    - Longitudinal wave
    - Transverse wave
- (c) Based on energy propagation
    - Progressive/Travelling wave
    - Stationary wave
- (d) Based on direction of propagation
    - 1D wave
    - 2D wave
    - 3D wave

3.1.2 Wave Equation:

Let us consider the equation representing a plane progressive harmonic wave

$$\psi(x,t) = a\sin(\omega t - kx) \qquad \text{... (1)}$$

$$\frac{\partial^2\psi}{\partial t^2} = -\omega^2 a\sin(\omega t - kx) \qquad \text{... (2)}$$

and, $$\frac{\partial^2\psi}{\partial x^2} = -k^2 a\sin(\omega t - kx) \qquad \text{... (3)}$$

$$\frac{\partial^2\psi}{\partial t^2} = c^2\frac{\partial^2\psi}{\partial x^2} \qquad \text{... (4)}$$

where $c = \omega / k$ is the velocity of the wave. Equation (4) is the well-known one dimensional differential wave equation.

Ex. 1. Which of the following CANNOT be a solution of the one dimensional wave equation:

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u(x,t)}{\partial t^2}$$

(a) $A\cos k(x+ct)$ (b) $A\exp[ik(x-ct)]$

(c) $A\exp\left[-\alpha(x-ct)^2\right]$ (d) $\frac{A}{(x^2-c^2t^2)}$

[IIT-JAM GP : 2010]

Soln. **(d)** The solution must satisfy the given partial differential equation and only (d) does not satisfy. Also, only those expression which contain $(ax \pm bt)$ term only satisfy wave equation.

Ex. 2. Two waves described by $y_1 = A\sin(\omega t + kx)$ and $y_2 = A\sin(\omega t - kx)$ are travelling along a string. Let $A = 0.001 m$, $k = 3.142\ m^{-1}$ and $\omega = 157.1\ s^{-1}$ (Take $\pi = 3.142$)

(a) Find the magnitude and direction of velocity of these waves.

(b) What shall be the amplitude of resultant wave on the string at $x = 0.5 m$.

[IIT-JAM GP : 2006]

Soln. Waves is described by following equations

(a) $y_1 = A\sin(\omega t + kx)$ and $y_2 = A\sin(\omega t - kx)$

Velocity of each waves are $v = \frac{\omega}{k} = \frac{157.1}{3.142} = 50$ m/s

First wave is going towards negative axis and second towards positive x-axis.

Thus, $\vec{v}_1 = -50\hat{i}$ m/s, and $\vec{v}_2 = \frac{\omega}{k} = 50\hat{i}$ m/s.

(b) The resultant waves is described by

$$\begin{aligned} y &= y_1 + y_2 \\ &= A\sin(\omega t + kx) + A\sin(\omega t - kx) \\ &= 2A\sin\omega t\cos kx \\ &= 2(A\cos kx)\sin\omega t \end{aligned}$$

The amplitude of resultant wave = $2A\cos kx$

At $x = 0.5\ m$

Amplitude $= 2\times 0.001\cos(3.142\times 0.5) = 0.002\cos\frac{\pi}{2} = 0.$

Ex. 3. A plane wave of frequency ω propagates so that planes of constant phase move with speeds v_1, v_2 and v_3 along the x, y and z axes, respectively. The wave vector, K of the plane wave (in terms of the Cartesian unit vectors $\hat{i}, \hat{j}$ and $\hat{k}$) is:

(a) $K = \omega\left(\frac{1}{v_1}\hat{i} + \frac{1}{v_2}\hat{j} + \frac{1}{v_3}\hat{k}\right)$ (b) $K = \omega\left(\hat{i}+\hat{j}+\hat{k}\right)/\sqrt{v_1^2+v_2^2+v_3^2}$

(c) $K = \omega\left(v_1\hat{i}+v_2\hat{j}+v_3\hat{k}\right)/\left(v_1^2+v_2^2+v_3^2\right)$ (d) $K = \left(v_1\hat{i}+v_2\hat{j}+v_3\hat{k}\right)/\omega$

[JNU : 2010]

Soln. **(c)** $\bar{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$

Wave vector has magnitude equal to ω/v and direction is along the direction of the propagation.

$$\vec{k}=\frac{\omega}{|\vec{v}|}(\hat{v})=\frac{\omega}{\sqrt{v_1^2+v_2^2+v_3^2}}\frac{\left(v_1\hat{i}+v_2\hat{j}+v_3\hat{k}\right)}{\sqrt{v_1^2+v_2^2+v_3^2}}$$

$$\boxed{\vec{k}=\frac{\omega\left(v_1\hat{i}+v_2\hat{j}+v_3\hat{k}\right)}{v_1^2+v_2^2+v_3^2}}$$

3.2 TRAVELLING WAVES:

It progresses with fixed velocity in a medium.
Example: Waves on liquid surface.

3.2.1 Equation of travelling waves:

(i) In + X direction: $y=a\sin\omega\left(t-\frac{x}{v}\right)$ ... (5)

(ii) In –X direction: $y=a\sin\omega\left(t+\frac{x}{v}\right)$ ... (6)

x. 4. A plane harmonic light wave travelling through a medium is represented (in S1 system) by

$$E_x(y,t)=E_0\sin 2\pi\left(\frac{y}{3\times10^{-7}}-5\times10^{14}t\right)$$

The refractive index of the medium at this frequency will be

(a) 1.3 (b) 1.5 (c) 2.0 (d) 2.5

[IIT-JAM GP : 2009]

oln. **(c)**

A plane wave travelling along y-axis is given by

$E_x(y,t)=E_0\sin(ky-\omega t)$

Comparing with given equation in problem, we get

Wavevector $k=\frac{2\pi}{3\times10^{-7}}$, and

angular frequency $\omega=2\pi\times5\times10^{14}$

Velocity $v=\frac{\omega}{k}=5\times10^{14}\times3\times10^{-7}$

$=1.5\times10^{8}$

Refractive index

$$\mu=\frac{c}{v}=2$$

x. 5. If a function of distance x and time t is given by $f(x-vt)$, then it represents

(a) a wave travelling along $+x$-direction and satisfies the wave equation $\frac{\partial f}{\partial x}+\frac{1}{v}\frac{\partial f}{\partial t}=0$

(b) a wave travelling along $+x$-direction and satisfies the wave equation $\frac{\partial f}{\partial x}-\frac{1}{v}\frac{\partial f}{\partial t}=0$

(c) a wave travelling along $-x$-direction and satisfies the wave equation $\frac{\partial f}{\partial x}+\frac{1}{v}\frac{\partial f}{\partial t}=0$

(d) a wave travelling along $-x$-direction and satisfies the wave equation $\frac{\partial f}{\partial x}-\frac{1}{v}\frac{\partial f}{\partial t}=0$

[IIT-JAM GP : 2008]

Ans. **(a)**

Given wave function $f(x-vt)$

$$\text{Now, } \frac{\partial f}{\partial x}(x-vt)=\dot{f}(x-vt) \text{ and } \frac{\partial f}{\partial t}(x-vt)=-v\dot{f}(x-vt)$$

$$\text{So, } \frac{\partial f}{\partial x}(x-vt)+\frac{1}{v}\frac{\partial f}{\partial t}(x-vt)=\dot{f}(x-vt)-\frac{v}{v}\dot{f}(x-vt)=0$$

Also, for travelling wave, phase $= x-vt=$ constant

$$\Rightarrow \frac{dx}{dt}-v=0 \Rightarrow \frac{dx}{dt}=+v$$

i.e., wave travel along positive x-axis.

So, option (a) is correct.

Ex. 6. A traveling pulse is given by $f(x,t)=A\exp\left(\frac{2abxt-a^2x^2-b^2t^2}{c^2}\right)$, where A, a, b and c are positive constants of appropriate dimensions. The speed of the pulse is

(a) $\frac{b}{a}$ (b) $\frac{2b}{a}$ (c) $\frac{cb}{a}$ (d) $\frac{b}{2a}$

[IIT-JAM : 2013]

Soln. **(a)** A travelling pulse is given by

$$f(x)=A\exp\left(\frac{2abxt-a^2x^2-b^2t^2}{c^2}\right)=A\exp\left[-\frac{(a^2x^2+b^2t^2-2abxt)}{c^2}\right]=A\exp\left[-\frac{(ax-bt)^2}{c^2}\right]$$

$$=A\exp\left[-\left(\frac{a}{c}x-\frac{b}{c}t\right)^2\right]=Ae^{-\left(\frac{a}{c}x-\frac{b}{c}t\right)^2}$$

$$\boxed{v=\frac{\omega}{k}=\frac{\text{coeff of } t}{\text{coeff of } x}}=\frac{b/c}{a/c}=\frac{b}{a}$$

Ex. 7. Consider the solution to the one-dimensional wave equation: $\Psi(x,t)=\Psi_0\exp i(3x+147t)$, where x is in meters and t in seconds. The velocity of the wave is

(a) 7 m/s (b) –1/7 m/s (c) –7 m/s (d) 1/7 m/s

[IISc : 2009]

Soln. None

$\psi(x,t)=\psi_0\exp[i(3x+147t)]$

Given wave travel in negative x-axis became coefficient of x and t, both are positive.

$$\text{Velocity of wave } v=\frac{\omega}{k}=-\frac{147}{3}=-49 \text{ m/s}$$

Ex. 8. Consider the following four functions each representing a plane wave :

(I) $f_1(x,y,z,t)=\sin\left(\sqrt{222}t-2x-7y+13z\right)$

(II) $f_2(x,y,z,t)=\sin\left(\sqrt{14}t-2x-3y-z\right)$

(III) $f_3(x,y,z,t)=\sin\left(\sqrt{89}\,t-4x-3y+8z\right)$

(IV) $f_4(x,y,z,t)=\sin\left(\sqrt{125}\,t+5x-6y+8z\right)$

Which of the following pairs represents waves moving in mutually perpendicular directions ?

(a) I and III (b) 1 and IV (c) II and III (d) II and IV

[JNU : 2011]

ln. (d) Comparing with standard equation

$$f=A\sin\left(\omega t-\vec{k}.\vec{r}\right)\text{, we get}$$

$$\vec{k}_1=2\hat{i}+7\hat{j}-13\hat{k}\,;\ \vec{k}_2=2\hat{i}+3\hat{j}+\hat{k}\,;\ \vec{k}_3=4\hat{i}+3\hat{j}-8\hat{k}\,;\ \vec{k}_4=-5\hat{i}+6\hat{j}-8\hat{k}$$

Clearly, $\vec{k}_2.\vec{k}_4=0$

Therefore, wave II and IV are perpendicular to each other.

c. 9. Two vibrating strings of lengths L and 3L and radii 3R and R, respectively, are stretched under the same tension. Both the strings vibrate in their fundamental modes, the one of length L with frequency ν_1 and the other with frequency ν_2. If the two strings are made of same material then the ratio ν_1/ν_2 is given by

(a) 1 (b) $3^{1/2}$ (c) 3 (d) 9 **[DU 2015]**

ln. (a) Since both the strings are made of same material, the density of the material is same for both.

Let ρ be the density.

Now, μ_1 = mass unit length of string

$$=\frac{\rho\pi(3R)^2L}{L}=9\pi\rho R^2$$

(Since, length_1 = L and radius_1 = 3R)

Similarly, $\mu_2=\frac{\rho\pi R^2(3L)}{3L}=\pi\rho R^2$

(Since, length_2 = 3L and radius_2 = R)

Now, frequency $\nu=\frac{1}{2L}\sqrt{\frac{T}{\mu}}$

Where, ν = frequency of wave

L = length of string

T = Tension in string

μ = mass per unit length of string)

$$\therefore\quad \nu_1=\frac{1}{2L}\sqrt{\frac{T}{9\pi\rho R^2}}\quad \&\quad \nu_2=\frac{1}{6L}\sqrt{\frac{T}{\pi\rho R^2}}$$

$$\Rightarrow\quad \frac{\nu_1}{\nu_2}=\frac{1}{2L}\frac{1}{3}\sqrt{\frac{T}{\pi\rho R^2}}\times 6L\sqrt{\frac{\pi\rho R^2}{T}}=1$$

Ex. 10. A transverse wave is described by the equation $y = y_0 \sin 2\pi (ft - \frac{x}{\lambda})$. The maximum particle velocity is equal to four times the wave velocity if

(a) $\lambda = \pi$ (b) $\lambda = \pi y_0/2$ (c) $\lambda = \pi$ (d) $\lambda = 2\pi y_0$

[IIT-JEE : 1984]

Soln. (b)

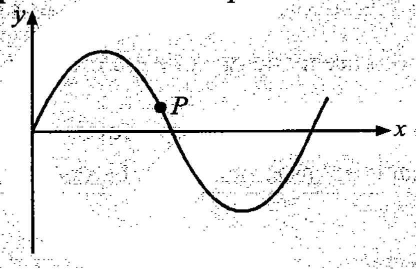
Wave velocity $v = \dfrac{\text{coefficient of } t}{\text{coefficient of } x} = \dfrac{2\pi f}{2\pi/\lambda} = \lambda f$

Maximum particle velocity $v_{pm} = \omega A = 2\pi f y_0$, $[\omega = 2\pi f]$

Given, $v_{pm} = 4v$ or $2\pi f y_0 = 4\lambda f$

$\therefore \ \lambda = \dfrac{\pi y_0}{2}$

Ex. 11. A transverse sinusoidal wave moves along a string in the positive x-direction at a speed of 10 cm/s. The wavelength of the wave is 0.5 m and its amplitude is 10 cm. At a particular time t, the snap-shot of the wave is shown in figure. The velocity of point P when its displacement is 5 cm is



(a) $\dfrac{\sqrt{3}\pi}{50}\hat{\mathbf{j}}$ m/s (b) $-\dfrac{\sqrt{3}\pi}{50}\hat{\mathbf{j}}$ m/s (c) $\dfrac{\sqrt{3}\pi}{50}\hat{\mathbf{i}}$ m/s (d) $-\dfrac{\sqrt{3}\pi}{50}\hat{\mathbf{i}}$ m/s

[IIT-JEE : 2008]

Soln. (a)

Consider a wave $y(x,t) = A\sin(kx - \omega t)$

Slope $\dfrac{dx(x,t)}{dx,t} = Ak\cos(kx - \omega t)$

$$= Ak\left[1 - \sin^2(kx - \omega t)\right]^{1/2}$$

$$= Ak\left[1 - \left(\frac{x(x,t)}{A}\right)^2\right]^{1/2}$$

At $y = 5\,\text{cm}$

$$\text{Slope} = 10\,\text{cm}\left(\frac{2\pi}{0.5\times10^2\,\text{cm}}\right)\left[1 - \left(\frac{5\,\text{cm}}{10\,\text{cm}}\right)^2\right]^{1/2} \quad \left[\because\ k = \frac{2\pi}{\lambda}\right]$$

$$= 0.2\cdot 2\pi\cdot\frac{\sqrt{3}}{2} = \frac{\pi\sqrt{3}}{5}.$$

Velocity of the particle at any position is given by

$$|v_p| = |v \cdot \text{slope}| = 10\,\text{cm/s} \times \frac{\pi\sqrt{3}}{5}$$

$$= \frac{\pi\sqrt{3}}{5} \times 10 \times 10^{-2}\ \text{m/s}$$

Particle velocity $v_p = -v$ (slope of y-x graph)

Here, v = positive, as the wave is travelling in positive x-direction.

Slope at P is negative. Therefore, velocity of particle is in positive y(or $\hat{\mathbf{j}}$) direction.

$$= \frac{\pi\sqrt{3}}{50}\ \text{m/s along } \hat{j}.$$

x. 12. The speed of sound propagation in air as a function of temperature T is given by $v = \alpha T$, where α is a constant of appropriate dimensions. Calculate the time taken for a sound wave to travel a distance L between two points A and B, if the air temperature between the points varies linearly from T_1 to T_2.

ln. It is given that air temperature varies linearly. Therefore, we can assume that at a distance 'x' from A temperature is given as

$$T = a + \beta x$$

At A, $x = 0$, $T = T_1$

$\therefore$ $T_1 = a + \beta \times 0 \Rightarrow a = T_1$

At B, $x = L$, $T = T_2$

$\therefore$ $T_2 = T_1 + \beta L \Rightarrow \beta = \dfrac{T_2 - T_1}{L}$

$$\therefore \quad T = T_1 + \frac{T_2 - T_1}{L}x \qquad \ldots \text{(i)}$$

Given, $v = \alpha T$

$$\therefore \quad \frac{dx}{dt} = \alpha\left[T_1 + \frac{T_2 - T_1}{L}x\right]$$

$$\therefore \quad \int_0^L \frac{dx}{T_1 + \frac{T_2 - T_1}{L}x} = \int_0 \alpha dt \quad \Rightarrow \frac{L}{T_2 - T_1}\left[\log\left(T_1 + \frac{T_2 - T_1}{L}x\right)\right]_0^L = \alpha t$$

$$\frac{L}{T_2 - T_1}.\log\frac{T_2}{T_1} = \alpha t \qquad \therefore \qquad t = \frac{L}{\alpha(T_2 - T_1)}\log\frac{T_2}{T_1}$$

3.2.2 Energy Distribution in Plane Progressive (travelling) Sound Waves:

Suppose a plane, harmonic longitudinal wave is moving through a medium along +ve x-direction. The displacement of a particle at the position x at any time t may be represented by

$$\xi(x, t) = a\cos(\omega t - kx) \qquad \ldots (7)$$

where 'ω' is the angular frequency, k is the propagating constant and a is the amplitude of vibration. Let us now consider an infinitesimally thickness δx and of unit cross-section. The layer is considered so thin that all the particles within the layer may be assumed to possess the same velocity $\partial\xi/\partial t$. The mass of this layer is $\rho_0 \delta x$ where ρ_0 is the equilibrium density.

Therefore, kinetic energy of the layer is

$$\Delta E_k = \frac{1}{2}(\rho_0\,\delta x)\left(\frac{\partial\xi}{\partial t}\right)^2 \qquad \text{... (8)}$$

Using equation(7) we get the kinetic energy per unit volume as

$$E_k = \frac{1}{2}\rho_0 a^2\omega^2 \sin^2(\omega t - kx) \qquad \text{... (9)}$$

Now the time average value of $\sin^2(\omega t - kx)$ over a complete time period (or its space average value over a complete wavelength λ) is 1/2. Thus average kinetic energy per unit volume becomes

$$\bar{E}_k = \frac{1}{4}\rho_0 a^2\omega^2 \qquad \text{... (10)}$$

During the transmission of the acoustic wave the volume element is periodically compressed and rarefied. So there is change in potential energy associated with this volume change. Suppose an increase of pressure from P_0 to $P_0 + p$ compresses unit volume of the medium to a volume (1-s) where $s = \partial\xi/\partial x$ is the condensation. The potential energy associated with this volume is equal to the work done in the process of compression. Thus potential energy per unit volume is given by

$$E_p = \int_0^s p\,ds \qquad \text{... (11)}$$

Now by definition bulk modulus

$$B = \frac{p}{s} \text{ or, } p = Bs \qquad \text{... (12)}$$

Therefore, $$E_p = B\int_0^s s\,ds = \frac{1}{2}Bs^2 = \frac{1}{2}B\left(-\frac{\partial\xi}{\partial x}\right)^2 \qquad \text{... (13)}$$

Using equation (7) we get

$$E_p = \frac{1}{2}Ba^2k^2\sin^2(\omega t - kx) \qquad \text{... (14)}$$

Now the time or space average value of $\sin^2(\omega t - kx)$ is 1 / 2. Therefore the average value of potential energy per unit volume becomes

$$\bar{E}_p = \frac{1}{4}Ba^2k^2 = \frac{1}{4}\rho_0 a^2\omega^2 \quad \left[\because c = \sqrt{B/\rho_0} = \omega/k\right] \qquad \text{...(15)}$$

Therefore, energy density is $\bar{E} = \bar{E}_k + \bar{E}_p = \frac{1}{2}\rho_0 a^2\omega^2$ ... (16)

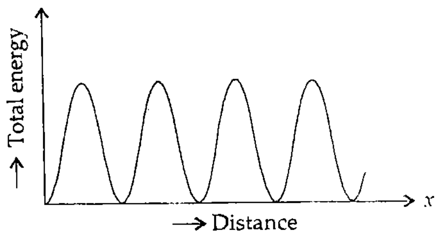


***Figure*3.1:** Energy distribution in space for a sound wave in a fluid

Ex. 13. The ends of a stretched wire of length L are fixed at $x = 0$ and $x = L$. In one experiment the displacement of the wire is $y_1 = A\sin\left(\frac{\pi x}{L}\right)\sin \omega t$ and energy is E_1 and in other experiment its displacement is $y_2 = A\sin\left(\frac{2\pi x}{L}\right)\sin 2\omega t$ and energy is E_2. Then **[IIT-JEE : 2001]**

(a) $E_2 = E_1$ (b) $E_2 = 2E_1$ (c) $E_2 = 4E_1$ (d) $E_2 = 16E_1$

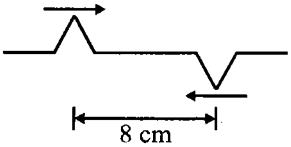
Soln. **(c)**

Energy $E \propto$ (amplitude)^2 (frequency)^2

Amplitude (A) is same in both the cases, but frequency 2ω in the second case is two times the frequency (ω) in the first case.

Therefore, $E_2 = 4E_1$

Ex. 14. Two pulses in a stretched string, whose centres are initially 8 cm apart, are moving towards each other as shown in the figure. The speed of each pulse is 2 cm/s. After 2 s the total energy of the pulses will be



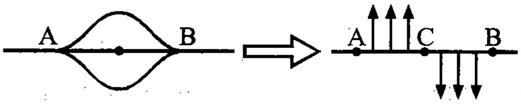
(a) zero (b) purely kinetic

(c) purely potential (d) partly kinetic and partly potential

[IIT-JEE : 2001]

Soln. **(b)**

After two seconds both the pulses will move 4 cm towards each other. So, by their superposition, the resultant displacement at every point will be zero. Therefore, total energy will be purely in the form of kinetic. Half of the particles will be moving upwards and half downwards.



3.3 STATIONARY WAVES OR STANDING WAVES:

3.3.1 Introduction:

When two progressive waves of the same frequency travelling in opposite directions superpose on each other, the resulting wave pattern does not propagate in either direction but remains confined within the region of overlap. Such waves are called stationary waves or standing waves. Suppose we consider two progressive waves of the same amplitude and frequency but propagating along the x-axis in opposite directions. Let these waves be represented by

$$\xi_1 = a\cos(\omega t - kx) \text{ and } \xi_2 = a\cos(\omega t + kx) \qquad \ldots(17)$$

where ξ_1 and ξ_2 are the particle displacements associated with the two waves. The resultant displacement is given by the principle of superposition as

$$\xi = \xi_1 + \xi_2 = 2a\cos\omega t \,.\, \cos kx = A\cos\omega t \qquad \ldots(18)$$

where $A = 2a\cos kx$.

3.3.2 Nodes and Antinodes:

Equation (18) represents a simple harmonic motion of the same frequency as the component waves but of amplitude A which is a periodic function of the position x of the particle. Equation (18) does not represent a progressive wave because the phase angle does not contain both x and t. At places where x is such that $\cos kx = \pm 1$ the amplitude becomes maximum and has a value 2a. These points of maximum amplitude are called displacement antinodes. These positions are given by

$$kx = n\pi \qquad \text{where } n = 0, 1, 2, 3, \text{ etc.}$$

$$\Rightarrow \quad \frac{2\pi}{\lambda}x = n\pi \quad \Rightarrow \quad x = n\frac{\lambda}{2} \qquad \text{...(19)}$$

The distance between two successive antinodes is $(n+1)\frac{\lambda}{2} - n\frac{\lambda}{2} = \frac{\lambda}{2}$. The amplitude A is zero at points where $\cos kx = 0$. Such points are given by

$$kx = (2n+1)\frac{\pi}{2} \qquad \text{where } n = 0, 1, 2, 3, \text{ etc.}$$

$$\Rightarrow \quad x = (2n+1)\frac{\lambda}{4} \qquad \text{...(20)}$$

These points where displacement amplitude is zero, are called displacement nodes. The distance between two successive nodes is also $\lambda / 2$.

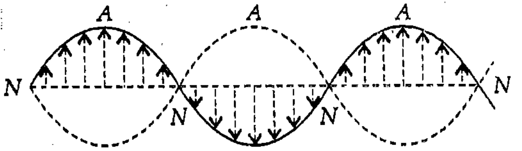


***Figure* 3.2:** Stationary wave pattern with nodes (N) and antinodes (A)

The distance between a node and its nearest antinode is $\lambda / 4$. The portion of the stationary wave pattern between two consecutive nodes is called a loop. All the particles within a loop vibrate with same phase but with different amplitudes (Fig. 3.2). The particles in two consecutive loops vibrate in opposite phase.

Ex. 15. Consider a pipe of length 65 cm with both ends open. How many modes of oscillations of the air column are possible up to a frequency of 1250 Hz? (Take the speed of sound to be 330 m/s)

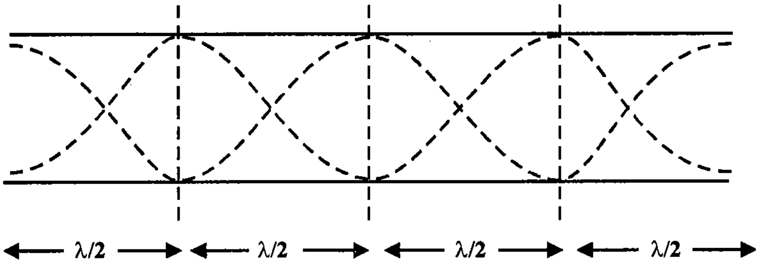
(a) 2 (b) 3 (c) 5 (d) 6

[JNU : 2010]

Soln. **(c)**

$$\therefore\ p\frac{\lambda}{2} = \ell \quad \Rightarrow p\frac{v}{2n} = \ell \quad \Rightarrow p = \frac{2n\ell}{v} = \frac{(1250)\times 2(0.65)}{330} = 4.92 \simeq 5$$

where 'p' is the number of modes of vibration 'v' is speed of sound, n is the frequency.



x. 16. A wave represented by the equation $y = a\cos(kx - \omega t)$ is superimposed with another wave to form a stationary wave such that point $x = 0$ is a node. The equation for the other wave is

(a) $a\sin(kx + \omega t)$ (b) $-a\cos(kx - \omega t)$ (c) $-a\cos(kx + \omega t)$ (d) $-a\sin(kx - \omega t)$

[IIT-JEE : 1988]

oln. **(a)**

For a stationary wave to form, two identical waves should travel in opposite direction. Further at $x = 0$, resultant y (from both the waves) should be zero at all instants.

x. 17. A vibrating string of certain length l under a tension T resonates with a mode corresponding to the first overtone (third harmonic) of an air column of length 75 cm inside a tube closed at one end. The string also generates 4 beats/s when excited along with a tuning fork of frequency n. Now when the tension of the string is slightly increased the number of beats reduces to 2 per second. Assuming the velocity of sound in air to be 340 m/s, the frequency n of the tuning fork in Hz is

(a) 344 (b) 336 (c) 117.3 (d) 109.3

[IIT-JEE : 2008]

oln. **(a)**

With increase in tension, frequency of vibrating string will increase. Since number of beats are decreasing. Therefore, frequency of vibrating string or third harmonic frequency of closed pipe should be less than the frequency of tuning fork by 4.

∴ Frequency of tuning fork

= third harmonic frequency of closed pipe + 4

$$= 3\left(\frac{v}{4l}\right) + 4 = 3\left(\frac{340}{4\times 0.75}\right) + 4 = 344\text{ Hz}$$

3.3.3 Energy density in stationary waves:

In a stationary wave formed by two identical waves propagating in two opposite directions the displacement at any point x at any time t is given by

$$\xi(x,t) = 2a\cos kx\cos\omega t \quad \text{.... (21)}$$

Let us consider the stationary waves formed within a tube of the medium of length δx and area of cross-section unity. The total energy of an element of volume δx is given by the sum of K.E. and P.E.

Thus, $$\delta E = \delta E_k + \delta E_p = \frac{1}{2}\rho_0\delta x\left(\frac{\partial\xi}{dt}\right)^2 + \frac{1}{2}B\left(\frac{\partial\xi}{\partial x}\right)^2\delta x$$

$$= \frac{1}{2}\rho_0.4a^2\omega^2\cos^2 kx\sin^2\omega t.\delta x + \frac{1}{2}B.4a^2k^2\sin^2 kx\cos^2\omega t.\delta x$$

$$= 2\rho_0 a^2\omega^2\left[\cos^2 kx.\sin^2\omega t + \sin^2 kx.\cos^2\omega t\right]\delta x \quad \text{.... (22)}$$

$$\overline{\delta E} = \rho_0 a^2\omega^2\delta x \quad \text{... (23)}$$

Therefore, Average energy per unit volume, $\bar{E} = \rho_0 a^2\omega^2$ ... (24)

Thus energy density in a stationary wave is twice that of a progressive wave.

3.3.4 Energy flow in stationary waves:

Particle velocity, $u = \partial\xi/\partial t = -2a\omega\cos kx\sin\omega t$ ... (25)

and acoustic pressure, $p = -B(\partial\xi/\partial x) = 2aBk\sin kx\cos\omega t$ ... (26)

Now the rate of work done per unit area of the wavefront i.e., the rate of energy flow through unit area is given by force per unit area × displacement in unit time = $p \times u$

$$= -4a^2 B\omega k \sin kx \cos wt . \cos kx \sin \omega t$$

$$= -a^2 B\omega k \sin 2kx . \sin 2\omega t \qquad \text{... (27)}$$

As the time average value of $\sin 2\omega t$ is zero, there is no transmission of energy through any cross-section in an ideal stationary wave.

Ex. 18. A string under tension T can support transverse waves with speed c. A displacement pattern of the form $y(x,t) = A\cos(2\pi x/\lambda)\cos(2\pi ct/\lambda)$ is set up in the string. The energy transported in a time interval $\Delta t = \lambda/c$ across $x = 0$ is

(a) 0 (b) TA (c) $T\lambda$ (d) $\frac{TA^2}{2\lambda}$ **[IISc : 2012]**

Ans. (a) $y(x,t) = A\cos\left(\frac{2\pi}{\lambda}x\right)\cos\left(2\pi\frac{c}{\lambda}t\right)$ or $y(x,t) = A\cos kx \cos \omega t$, represents standing wave. Energy transmission in time interval $\Delta t = \frac{\lambda}{c} = T$ (Time period) through only cross sectional area is zero.

Ex. 19. In a standing wave, the phase difference between any two points confined between adjacent node and antinode **[DU 2015]**

(a) is always zero (b) is between $\pi/4$ and $\pi/2$

(c) is between $\pi/2$ and π (d) changes periodically

Soln. **(a)** is always zero

3.4 DIFFERENCE BETWEEN PROGRESSIVE AND STATIONARY WAVES:

(1) A progressive wave originates from the regular vibration of a source. A stationary wave is formed by the superposition of two oppositely propagating waves.

(2) In progressive wave each particle oscillates with the same frequency and the same amplitude. In a stationary wave particles vibrate with the same frequency but with different amplitudes; at nodes amplitude is zero and at antinodes it is maximum.

(3) In progressive wave phase difference between two particles is proportional to the separation. In stationary waves all the particles within a loop vibrate in same phase and in the next loop the phase is opposite.

(4) In progressive wave variation of acoustic pressure is same at all points, but in a stationary wave variation of acoustic pressure is maximum at displacement nodes and zero at displacement antinodes.

(5) Progressive waves can carry energy from one place to another but in stationary waves there is no transfer of energy.

SOLVED EXAMPLES

1. Show that the function $\psi(x,t) = f_1(ct - x) + f_2(ct + x)$ satisfies the one dimensional differential wave equation.

Soln. $\frac{\partial \psi}{\partial x} = -f_1'(ct - x) + f_2'(ct + x)$

$$\frac{\partial^2 \psi}{\partial x^2} = f_1''(ct - x) + f_2''(ct + x) \qquad \ldots(1)$$

where $f'(ct \pm x)$ indicate differentiation with respect the argument.

Again $\frac{\partial \psi}{\partial t} = cf_1'(ct - x) + cf_2'(ct + x)$

$$\frac{\partial^2 \psi}{\partial t^2} = c^2\left[f_1''(ct - x) + f_2''(ct + x)\right] \qquad \ldots(2)$$

combining (1) and (2) we get $\frac{\partial^2 \psi}{\partial t^2} = c^2 \frac{\partial^2 \psi}{\partial x^2}$

Thus ψ satisfies the differential wave equation.

2. Compare the intensities of sound in air and water at N.T.P for the same acoustic pressure. Given the density of air $\rho_a = 1.21\,\text{kg/m}^3$, density of water $\rho_\omega = 998\,\text{kg/m}^3$, the speed of sound in air $c_a = 343$ m/s and the speed of sound in water $c_\omega = 1480$ m/s.

Soln. Intensity $I = p_{r.m.s}^2/\rho c$

Therefore, $$\frac{I_{air}}{I_{water}} = \frac{p_{r.m.s}^2/(\rho c)_{air}}{p_{r.m.s}^2/(\rho c)_{water}} = \frac{\rho_\omega}{\rho_a} \times \frac{c_\omega}{c_a} = \frac{998}{1.21} \times \frac{1480}{343} = 3559$$

3. Find the sound energy density in air of a free plane progressive acoustic wave having an IL of 80 dB. Given the velocity of sound in air = 343 m/s and reference intensity $I_0 = 10^{-12}\,\text{W/m}^2$.

Soln. $IL = 10\log_{10}\frac{I}{I_0} = 80\,\text{dB}$ or, $I = 10^8\,I_0 = 10^8 \times 10^{-12}\,\text{W/m}^2 = 10^{-4}\,\text{W/m}^2$

Now energy density $= \frac{E}{V} = \frac{E}{tAc} = \frac{I}{c} = \frac{10^{-4}}{343}\,\text{J/m}^3 = 2.9\times10^{-7}\,\text{J/m}^3$

4. An intense plane harmonic wave in air STP has an r.m.s excess pressure of $20\,\text{N/m}^2$. The frequency is 1 kHz and the phase velocity 350 m/s. Calculate the maximum value of particle displacement and particle velocity. Given $\gamma_{air} = 1.40$.

Soln. Let displacement $\xi = a\cos(\omega t - kx)$

Excess pressure $p = -B\frac{\partial \xi}{\partial x} = Bak\sin(\omega t - kx)$

where bulk mudulus $B = \gamma.P_0 = 1.40 \times 1.013 \times 10^5\,\text{N/m}^2$.

$p_{max} = Bak$ or, Maximum particle displacement

$$a = \frac{p_{max}}{Bk} = \frac{\sqrt{2}p_{r.m.s} \times c}{\gamma P_0 \times \omega} \quad [\because c = \omega/k] = \frac{\sqrt{2} \times 20 \times 350}{1.40 \times 1.013 \times 10^5 \times 2\pi \times 10^3}\,\text{m} = 1.11 \times 10^{-5}\,\text{m}$$

Particle velocity $u = \partial\xi/\partial t = -a\omega\sin(\omega t - kx) = a\omega\cos(\omega t - kx + 90^\circ)$

Therefore, maximum particle velocity $= a\omega = 1.11 \times 10^{-5} \times 2\pi \times 10^3\,\text{m/s} = 6.97 \times 10^{-2}\,\text{m/s}$.

5. Write the wave function for a plane harmonic progressive wave which is progressing with velocity 340 m/s along a direction having direction cosines $1/\sqrt{2},\ 1/\sqrt{3}$ and $1/\sqrt{6}$

Soln. $\psi(x,y,z,t) = a\cos\frac{2\pi}{\lambda}(ct - lx - my - nz) = a\cos\frac{2\pi}{\lambda}\left(340t - \frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{3}}y - \frac{1}{\sqrt{6}}z\right)$

6. Two plane progressive harmonic waves propagating along the same line but moving in opposite direction overlap. They are represented by

$$y_1 = a_1\sin(\omega t - kx) \text{ and } y_2 = a_2\sin(\omega t + kx), \qquad a_1 > a_2$$

Show that the resultant wave consists of a stationary wave together with a progressive wave moving along the x-axis.

Soln. The resultant wave is given by

$$y = y_1 + y_2 = a_1\sin(\omega t - kx) + a_2\sin(\omega t + kx)$$

Let $a_1 = a_2 + b$, where b is a positive quantity.

Therefore, $y = 2a_2\cos kx\sin\omega t + b\sin(\omega t - kx)$

Obviously the first term represents a stationary wave and the second term represents a progressive wave moving along positive x-direction.

7. A long string having a cross-sectional area 0.80 mm² and density 12.5 g/cm³, is subjected to a tension of 64 N along the X-axis. One end of this string is attached to a vibrator moving in transverse direction at a frequency of 20 Hz. At $t = 0$, the source is at a maximum displacement $y = 1.0$ cm

(a) Find the speed of the wave travelling on the string

(b) Write the equation for the wave

(c) What is the displacement of the particle of the string at $x = 50$ cm at time $t = 0.05$ s ?

(d) What is the velocity of this particle at this instant ?

Soln. (a) The mass of 1 m long part of the string is

$$m = (0.80\ \text{mm}^2)\times(1\text{m})\times(12.5\ \text{g/cm}^3)$$
$$= (0.80\times10^{-6}\ \text{m}^3)\times(12.5\times10^3\ \text{kg/m}^3)$$
$$= 0.01\ \text{kg}.$$

The linear mass density is $\mu = 0.01$ kg/m. The wave speed is

$$v = \sqrt{F/\mu}$$
$$= \sqrt{\frac{64\ \text{N}}{0.01\ \text{kg/m}}} = 80\ \text{m/s}$$

(b) The amplitude of the source is $A = 1.0$ cm and the frequency is $\nu = 20$ Hz. The angular frequency is $\omega = 2\pi\nu = 40\pi s^{-1}$. Also at $t = 0$, the displacement is equal to its amplitude i.e., at $t = 0$, $x = A$. The equation of motion of the source is, therefore,

$$y = (1.0\,\text{cm})\cos[(40\pi s^{-1})t] \qquad \text{... (i)}$$

The equation of the wave travelling on the string along the positive X-axis is obtained by replacing t with $t - x/v$ in equation (i). It is, therefore,

$$y = (1.0\ \text{cm})\cos\left[(40\pi s^{-1})\left(t - \frac{x}{v}\right)\right]$$

$$= (1.0\,\text{cm})\cos\left[(40\pi s^{-1})t - \left(\frac{\pi}{2}m^{-1}\right)x\right], \qquad \ldots\text{(ii)}$$

where the value of v has been put from part (a).

(c) The displacement of the particle at $x = 50$ cm at time $t = 0.05\,s$ is by equation (ii),

$$y = (1.0\,\text{cm})\cos\left[(40\pi s^{-1})(0.05\,s) - \left(\frac{\pi}{2}\text{m}^{-1}\right)(0.5\,\text{m})\right]$$

$$= (1.0\,\text{cm})\cos\left[2\pi - \frac{\pi}{4}\right] = \frac{1.0\,\text{cm}}{\sqrt{2}} = 0.71\,\text{cm}$$

(d) The velocity of the particle at position x at time t is, by equation (ii),

$$v = \frac{\partial y}{\partial t} = -(1.0\,\text{cm})(40\pi s^{-1})\sin\left[(40\pi s^{-1})t - \left(\frac{\pi}{2}\text{m}^{-1}\right)x\right].$$

Putting the values of x and t,

$$v = -(40\pi\,\text{cm/s})\sin\left(2\pi - \frac{\pi}{4}\right) = \frac{40\pi}{\sqrt{2}}\,\text{cm/s} \approx 89\,\text{cm/s}.$$

3. The vibrations of a string fixed at both ends are described by the equation

$$y = (5.00\,\text{mm})\sin[(1.57\,\text{cm}^{-1})x]\sin[(314\,s^{-1})t]$$

(a) What is the maximum displacement of the particle at $x = 5.66$ cm ?

(b) What are the wavelengths and the wave speeds of the two transverse waves that combine to give the above vibration ?

(c) What is the velocity of the particle at $x = 5.66$ cm at time $t = 2.00\,s$?

(d) If the length of the string is 10.0 cm, locate the nodes and the antinodes. How many loops are formed in the vibration ?

oln. (a) The amplitude of the vibration of the particle at position x is $A = \left|(5.00\,\text{mm})\sin[(1.57\,\text{cm}^{-1})x]\right|$.

For $x = 5.66$ cm,

$$A = \left|(5.00\,\text{mm})\sin\left[\frac{\pi}{2}\times 5.66\right]\right|$$

$$= \left|(5.00\,\text{mm})\sin\left(2.5\pi + \frac{\pi}{3}\right)\right|$$

$$= \left|(5.00\,\text{mm})\cos\frac{\pi}{3}\right| = 2.50\,\text{mm}.$$

(b) From the given equation, the wave number $k = 1.57\,\text{cm}^{-1}$ and the angular frequency $\omega = 314\,s^{-1}$. Thus, the wavelength is

$$\lambda = \frac{2\pi}{k} = \frac{2\times 3.14}{1.57\,\text{cm}^{-1}} = 4.00\,\text{cm}$$

and the frequency is $\nu = \dfrac{\omega}{2\pi} = \dfrac{314\,s^{-1}}{2\times 3.14} = 50\,s^{-1}$.

The wave speed is $v = \nu\lambda = (50\,s^{-1})(4.00\,\text{cm}) = 2.00$ m/s.

(c) The velocity of the particle at position x at time t is given by

$$v=\frac{\partial y}{\partial t}=(5.00\text{ mm})\sin[(1.57\text{ cm}^{-1})x],[314s^{-1}\cos(314s^{-1})t]$$

$$=(157\text{ cm/s})\sin(1.57\text{ cm}^{-1})\,x\cos(314s^{-1})t.$$

Putting $x=5.66$ cm and $t=2.00\,s$, the velocity of this particle at the given instant is

$$(157\text{ cm/s})\sin\left(\frac{5\pi}{2}+\frac{\pi}{3}\right)\cos(200\pi)=(157\text{ cm/s})\times\cos\frac{\pi}{3}\times 1=78.5\text{ cm/s}.$$

(d) The nodes occur where the amplitude is zero i.e., $\sin(1.57\text{ cm}^{-1})x=0$.

or $\left(\frac{\pi}{2}\text{cm}^{-1}\right)x=n\pi$, where n is an integer.

Thus, $x=2n\,\text{cm}$.

The nodes, therefore, occur at $x=0$, 2 cm, 4 cm, 6 cm, 8 cm and 10 cm. Antinodes occur in between them i.e., at $x=1$ cm, 3 cm, 5 cm, 7 cm and 9 cm. The string vibrates in 5 loops.

Chapter 4

hase And Group Velocity

4.1 WAVE TRAIN AND THE CONCEPT OF WAVE GROUP:

A wave train of finite length may be considered to be formed by the superposition of a theoretically infinite number of plane harmonic waves having continuously differing frequencies. For practical purpose the frequencies are limited within a finite range depending on the length of the wave train. The shorter is the length of the wave train the wider is the effective frequency range. In this sense a short enough wave train may be considered as a group of harmonic waves. It may be called a wave group or a wave packet.

4.2 PHASE VELOCITY, GROUP VELOCITY AND THEIR RELATIONSHIP:

When a monochromatic harmonic wave moves through a medium the velocity with which planes of constant phase move is called the phase velocity or wave velocity. A plane harmonic wave may be represented by $\psi(x,t) = \text{Re } Ae^{j(\omega t - kx)}$ ... (1)

For planes of constant phase, $\omega t - kx = \text{constant}$ ... (2)

Differentiating, $\frac{d}{dt}(\omega t - kx) = 0$ or, $\frac{dx}{dt} = \frac{\omega}{k} = c_p$ ... (3)

where c_p is called the phase velocity.

Practical waves are of finite duration and are not truly monochromatic. A short enough wave train may be considered as a wave group formed by the superposition of an infinite number of plane harmonic wave having slightly differing frequencies and phases. The wave group has a maximum of amplitude which falls off to zero not far from the maximum. If the medium is dispersive i.e., the phase velocity depends on frequency then the shape of the wave group changes as it travels through the medium. The maximum of the wave group travels with a velocity that is different from the velocities of the component waves. This velocity is known as group velocity (c_g).

4.3 RELATION BETWEEN PHASE VELOCITY AND GROUP VELOCITY:

The wave group is not situated at a fixed point but moves in space with a velocity, called group velocity, given by $c_g = \frac{dx}{dt} = \frac{d\omega}{dk}$ ... (4)

Since phase velocity $c_p = \omega / k \Rightarrow \omega = c_p k$ we can write

$$c_g = \frac{d}{dk}(c_p k) = c_p + k\frac{dc_p}{dk} = c_p + k\frac{dc_p}{d\lambda}\cdot\frac{d\lambda}{dk}$$

Now, $k = 2\pi/\lambda$ or, $d\lambda/dk = -2\pi/k^2$.

$$\Rightarrow \qquad c_g = c_p - \frac{2\pi}{k}.\frac{dc_p}{d\lambda} = c - \lambda\frac{dc_p}{d\lambda}$$

This is the required relation between phase velocity and group velocity for a dispersive medium. For a non-dispersive medium $dc_p/d\lambda = 0$. Then $v_g = c_p$ i.e, group velocity equals phase velocity.

SOLVED EXAMPLES

1. For gravity waves in a liquid the phase velocity c depends on the wavelength λ according to the formula $c_p = A\sqrt{\lambda}$, A being a constant. Show that the group velocity is half the phase velocity.

Soln. $c_p = A\sqrt{\lambda} \quad \Rightarrow \frac{dc_p}{d\lambda} = \frac{1}{2}A\lambda^{-1/2}$

Therefore, $\quad c_g = c_p - \lambda\frac{dc_p}{d\lambda} = c_p - \frac{1}{2}A\sqrt{\lambda} = c_p - \frac{c_p}{2} = \frac{1}{2}c_p$

2. The dispersion relation for microwaves in ionosphere is given by

$$\omega^2 = \omega_p^2 + c^2k^2$$

where c is the velocity of light in free space and ω_p is a constant depending on the electron density of the ionosphere. Show that the phase velocity c_p is greater than c. Does it violate the principle of relativity?

Soln. $\omega = \sqrt{\omega_p^2 + c^2k^2}$

Therefore, $\quad$ Phase velocity $c_p = \frac{\omega}{k} = \sqrt{c^2 + \frac{\omega_p^2}{k^2}}$ which is greater than c.

Differentiating the dispersion relation with respect to k we get

$$2\omega\frac{d\omega}{dk} = c^2.2k \Rightarrow \frac{\omega}{k}.\frac{d\omega}{dk} = c^2$$

$$\Rightarrow \qquad c_pc_g = c^2 \qquad \left[\text{since } c_p > c,\ c_g < c\right]$$

It does not violate the principle of relativity because energy or signal is transmitted not with the phase velocity but with group velocity which is less than c.

3. Two harmonic waves represented by

$$\xi_1 = 3\cos(7t - 10x)\,\text{m} \text{ and } \xi_2 = 3\cos(5t - 8x)\,\text{m}$$

are superposed to form a wave group. Find the group velocity.

Soln. Angular frequencies are $\omega_1 = 7$ rad/s and 5 rad/s. Wave numbers are $k_1 = 10\,\text{m}^{-1}$ and $k_2 = 8\,\text{m}^{-1}$.

Therefore, $\quad$ Group velocity $c_g = \frac{\Delta\omega}{\Delta k} = \frac{7-5}{10-8} = 1\,\text{m/s}$.

4. The phase velocity c_p of surface waves on a liquid of density ρ and surface tension S is given by

$$c_p^2 = \frac{g\lambda}{2\lambda} + \frac{2\pi S}{\rho\lambda}$$

where 'g' is the accleration due to gravity and λ is the wavelength of the wave.

(i) Obtain an expression for the group velocity of the wave.

(ii) Calculate the wavelength (λ_0) for which c is the minimum

(iii) Show that for $\lambda = \lambda_0$, group velocity $c_g = c_p$.

oln. (i) Group velocity $c_g = \dfrac{d\omega}{dk} = \dfrac{d}{dk}\left(c_p k\right) = c_p + k\dfrac{dc_p}{dk}$

Since $k = \dfrac{2\pi}{\lambda}$, $c_g = c_p - \lambda\dfrac{dc_p}{d\lambda}$

Now differentiating the given expression with respect to λ,

$$2c_p\frac{dc_p}{d\lambda} = \frac{g}{2\pi} - \frac{2\pi S}{\rho\lambda^2}$$

$$\Rightarrow \qquad c_g = c_p - \frac{\lambda}{2c_p}\left(\frac{g}{2\pi} - \frac{2\pi S}{\rho\lambda^2}\right)$$

(ii) For c to be minimum, $\dfrac{dc}{d\lambda} = 0 \quad \Rightarrow \dfrac{g}{2\pi} - \dfrac{2\pi S}{\rho\lambda_0^2} = 0 \quad \Rightarrow \lambda_0 = 2\pi\sqrt{S/\rho g}$

(iii) For $\lambda = \lambda_0 \quad \Rightarrow c_g = c_p - \dfrac{\lambda_0}{2c_p}\left(\dfrac{g}{2\pi} - \dfrac{2\pi S}{\rho.4\pi^2(S/\rho g)}\right) = c_p - 0 = c_p$

A wave packet in a certain medium is constructed by superposing waves of frequency ω around $\omega_0 = 100$ and the corresponding wave-number k with $k_0 = 10$ as given in the table below

ω	k
81.00	9.0
90.25	9.5
100.00	10.0
110.25	10.5
121.00	11.0

Find the ratio v_g / g_p of the group velocity v_g and the phase velocity v_p.

(a) $\dfrac{1}{2}$ (b) 1 (c) $\dfrac{3}{2}$ (d) 2

[IIT-JAM : 2009]

ln. **(d)** $v_p = \dfrac{\omega}{k} = \dfrac{100}{10} = 10\ ;\ v_g^{(1)} = \dfrac{100-90.25}{10-9.5} = \dfrac{9.75}{0.5} = 19.5$

$v_g^{(2)} = \dfrac{110.25-100}{10.5-10} = \dfrac{10.25}{0.5} = 20.5\ ;\ v_g = \dfrac{v_g^{(1)} + v_g^{(2)}}{2} = \dfrac{19.5+20.5}{2} = 20$

$\therefore \dfrac{v_g}{v_p} = \dfrac{20}{10} = 2$

6. For a wave in a medium the angular frequency 'ω' and the wave vector $\vec{k}$ are related by, $\omega^2 = \left(\omega_0^2 + c^2k^2\right)$, where ω_0 and c are constants. The product of group and phase velocities, i.e. $v_g v_p$ is:

(a) $0.25c^2$ (b) $0.4c^2$ (c) $0.5c^2$ (d) c^2

[IIT-JAM : 2010]

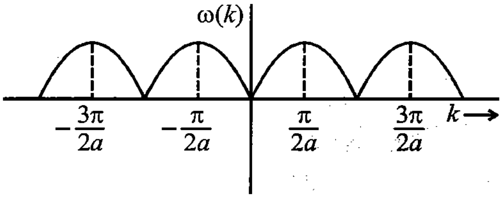
Soln. **(d)** $v_p = \frac{\omega}{k} = \frac{\sqrt{\omega_0^2 + c^2k^2}}{k}$; $v_g = \frac{d\omega}{dk} = \frac{c^2k}{\sqrt{\omega_0^2 + c^2k^2}}$ $\Rightarrow \boxed{v_g.v_p = c^2}$

7. The dispersion relation for a wave is given as $\omega(k) = C|\sin ka|$, where C is a constant. The group velocity of the wave vanishes at

(a) $k = 0$ (b) $k = \pm\,\pi/2a$ (c) $k = \pm\,\pi/2$ (d) $k = \pm\,2\pi/a$

[IIT-JAM GP : 2008]

Soln. **(b)**



$$\omega(k) = C|\sin ka|$$

$$v_g = \frac{d\omega}{dk} = 0 \text{ at } k = \pm\frac{\pi}{2a}$$

8. In a certain medium, the wave number k and the frequency ω are related by the dispersion relation $\omega^2 = c^2k^2(1 + \alpha k^2)$, where c and α are constants. If v_g is the group velocity and v_p is the phase velocity, then

(a) $v_g = v_p$ (b) $\frac{v_g}{v_p} = \frac{(1+\alpha k^2)}{(1+2\alpha k^2)}$ (c) $v_g \cdot v_p = c^2$ (d) $v_g \cdot v_p = c^2(1+2\alpha k^2)$

[IIT-JAM GP : 2010]

Soln. **(d)**

$$\omega = ck\,(1+\alpha k^2)^{1/2}$$

$$v_g = \frac{d\omega}{dk} = c(1+\alpha k^2)^{1/2} + ck\frac{1}{2}(1+\alpha k^2)^{1/2}(2\alpha k)$$

$$= c(1+\alpha k^2)^{1/2} + \frac{\alpha ck^2}{(1+\alpha k^2)^{1/2}} = \frac{c\,(1+\alpha k^2) + \alpha ck^2}{(1+\alpha k^2)^{1/2}}$$

$$= \frac{c\,(1+2\alpha k^2)}{(1+\alpha k^2)^{1/2}}$$

Since $v_p = \frac{\omega}{k} = c\,(1+\alpha k^2)^{1/2}$

Therefore, $v_g \cdot v_p = c^2(1+2\alpha k^2)$

9. The dispersion relation for surface waves propagating in a fluid is given as: $\omega^2 = \alpha k + \beta k^3$, where α and β are constants with appropriate units. The phase velocity βV_p becomes equal to the group velocity V_g at $k = k_0$. The value of k_0 is

(a) $\sqrt{\frac{\alpha}{2\beta}}$ (b) $\sqrt{\frac{2\alpha}{\beta}}$ (c) $\sqrt{\frac{\alpha}{\beta}}$ (d) $\sqrt{\frac{3\alpha}{\beta}}$

[IIT-JAM GP : 2012]

Soln. **(c)** $\omega = \sqrt{\alpha k + \beta k^3}$

Phase velocity $V_p = \frac{\omega}{k} = \frac{\sqrt{\alpha k + \beta k^3}}{k}$

Group velocity $V_y = \frac{d\omega}{dk} = \frac{(\alpha + 3\beta k^2)}{2\sqrt{\alpha k + \beta k^3}}$

$V_p = V_g$

$$\Rightarrow \frac{\sqrt{\alpha k + \beta k^3}}{k} = \frac{\alpha + 3\beta k^2}{2\sqrt{\alpha k + \beta k^3}} \Rightarrow 2(\alpha k + \beta k^3) = \alpha k + 3\beta k^3$$

$$\Rightarrow \beta k^3 - \alpha k = 0$$

$$k = \sqrt{\frac{\alpha}{\beta}}$$

0. The angular frequency ω of deep water waves varies as the inverse square root of the wavelength λ, *i.e.*, $\omega(\lambda) \propto 1/\sqrt{\lambda}$. Which of the following is the relation between its group velocity v_g and phase velocity v_p?

(a) $v_g = v_p/2$ (b) $v_g = v_p$ (c) $v_g = 2v_p$ (d) $v_g = \lambda v_p$

[JNU : 2009]

oln. **(a)** $\omega \propto \frac{1}{\sqrt{\lambda}}$; $\omega = \frac{a}{\sqrt{\lambda}}$, '$a$' is constant

$$\therefore \quad \omega = \frac{a}{\sqrt{2\pi}}.\sqrt{k} \qquad \because \lambda = \frac{2\pi}{k}$$

Or, $\omega^2 = \frac{a^2}{2\pi}k$

$$\therefore \quad 2\omega\frac{d\omega}{dk} = \frac{a^2}{2\pi}; \qquad \frac{d\omega}{dk} = \frac{a^2}{4\pi}.\frac{1}{\omega} = \frac{a^2}{4\pi}.\frac{\sqrt{2\pi}}{a\sqrt{k}}$$

$$\Rightarrow \frac{d\omega}{dk} = \frac{a}{2\sqrt{2\pi}}.\frac{1}{\sqrt{k}} = v_g \text{ and } \frac{\omega}{k} = \frac{a}{\sqrt{2\pi}}.\frac{\sqrt{k}}{k} = \frac{a}{\sqrt{2\pi}}.\frac{1}{\sqrt{k}} = v_p$$

$$\Rightarrow v_g = \frac{1}{2}\left(\frac{a}{\sqrt{2\pi}}.\frac{1}{k}\right) = \frac{1}{2}v_p \Rightarrow \boxed{v_g = \frac{v_p}{2}}$$

Chapter 5

Doppler Effect

5.1 INTRODUCTION:

The pitch of the note emitted by a source of sound appears to change when there is a relative motion between the source and the listener with respect to each other or to the medium. This apparent change in the pitch of a note is known as Doppler effect in sound. It is our common experience that the pitch or frequency of whistling railway engine appears to increase as it approaches a listener and the pitch or frequency suddenly drops as it passes and recedes from the listener. In case of a moving source the change in frequency can be explained as due to the fact that, though the source is emitting waves at constant rate, the waves at constant rate, the waves emitted in a given time will now occupy a shorter or longer distance than when the source is stationary. Consequently the apparent wavelength and hence the frequency appears to change. In case a listener is moving towards or away from a source, the wavelength remain unchanged but the listener encounters a larger or smaller number of waves in a given time than if he or she remained stationary. The change in frequency due to the motion of the medium is associated with the change in the velocity of sound. The fundamental fact underlying the Doppler effect is that Doppler shift in frequency occurs when the transit time from the source to the observer is different for different wavefronts.

5.2 CALCULATION OF DOPPLER SHIFT:

5.2.1 Source moving towards a stationary observer:

Let in Figure-5.1 S be source moving with a velocity υ_s towards the stationary observer O. The medium is assumed to be at rest. If ν is the frequency of the source, it will emit ν complete waves in 1 sec. If the source is at rest these ν waves will occupy a length SP ($=\lambda$) = c, where c is speed of wave. However, if the source

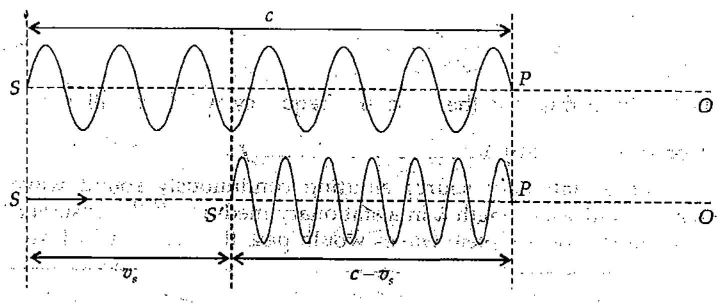


***Figure* 5.1:** Source moving towards observer

moves with velocity υ_s, in one second, it goes from S to S' where SS' = υ_s. As the velocity of sound is independent of the motion of the source, the ν waves emitted during this one second will now occupy a smaller length $c - \upsilon_s = S'P$.

The original wavelength $\lambda = c/\nu$ will now be changed to $\lambda' = \dfrac{c - \upsilon_s}{\nu}$. The observer will interpret them as waves of frequency $\nu' = \frac{c}{\lambda'} = \frac{c}{c-\upsilon_s}\nu$ ...(1)

Obviously $\nu' > \nu$. As ν_s is increased from zero, ν' increases. At $\upsilon_s = c$, ν' becomes infinite. In this case waves travel with the source and reach the observer at the same time.

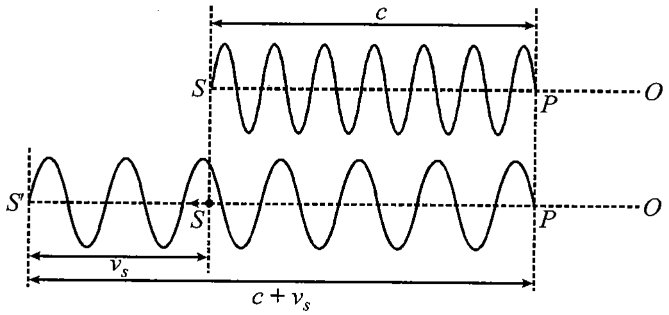


***Figure* 5.2 :** Source moving away from the observer

If $\upsilon_s > c$ then ν' is negative which means that the waves emitted by the source are received in the reversed order. In this case the source moves ahead and waves follow it.

Source moving away from station are observer:

If the source moves away from the observer the ν waves emitted in 1 sec. will now occupy a greater length $c + \upsilon_s$ as shown in Figure-5.2. So the new wavelength in this case would be $\lambda' = \frac{c+\upsilon_s}{\nu}$ and hence the apparent frequency would appear to be

$$\nu' = \frac{c}{\lambda'} = \frac{c}{c+\upsilon_s}.\nu \qquad ...(2)$$

Obviously in this case the apparent frequency will be smaller.

x. 1. The engine of a train, emitting the sound of frequency ν_0 approaches an observer with constant speed. If the observer measures the frequencies as ν_1 when it is approaching and ν_2 while it is going away, the relation between the frequencies is given by

(a) $\nu_1 = \nu_2 = \nu_0$ (b) $\nu_1 > \nu_0 > \nu_2$ (c) $\nu_1 < \nu_0 < \nu_2$ (d) $\nu_1 = \nu_2 \neq \nu_0$

[IIT-JAM GP : 2008]

oln. **(b)** When source is moving the towards the observer, the apparent frequency is more then the actual frequency and vice-versa.

x. 2. A siren of natural frequency 600 Hz is fitted on an ambulance. When it approaches a pedestrian, she heard a frequency of 640 Hz. At what speed was the ambulance approaching her? (Note that the speed of sound under those conditions was 360 m/s)

(a) 24.0 km/hr (b) 30.2 km/hr (c) 60.8 km/hr (d) 86.4 km/hr

[IISc : 2007]

oln. **(a)**

When source (ambulance) is moving towards stationary observer, aparent frequency is given by

$$\nu' = \nu_0\left(\frac{c}{c-v_s}\right)$$

$$\Rightarrow\ 640 = 600\left(\frac{360}{360 - v_s}\right)$$

$v_s = 20.7$ km/hr

Ex. 3. A train moves towards a stationary observer with speed 34 m/s. The train sounds a whistle and its frequency registered by the observer is f_1. If the train's speed is reduced to 17 m/s, the frequency registered is f_2. If the speed of sound is 340 m/s then the ratio f_1/f_2 is
(a) 18/19 (b) 1/2 (c) 2 (d) 19/18

[IIT-JEE : 2000]

Soln. (d)

$$f_1 = \left(\frac{c}{c - v_s}\right) f_0$$

$$f_1 = f\left(\frac{340}{340 - 34}\right) = f\left(\frac{340}{306}\right) \text{ and } f_2 = f\left(\frac{340}{340 - 17}\right) = f\left(\frac{340}{323}\right)$$

$$\therefore\ \frac{f_1}{f_2} = \frac{323}{306} = \frac{19}{18}$$

5.2.2 Observer moving towards a stationary source:

Let S be a stationary source emitting continuously sound waves of frequency ν and wavelength λ in a stationary medium. If the observer O is at rest, ν number of complete waves would pass the observer in 1 sec. and occupy the length OP = c, where c is speed of sound. However, if the observer moves with velocity υ_0, in 1 sec. it moves from O to O' where $OO' = \upsilon_0$. So all the waves contained in the length $O'P = c + \upsilon_0$ will be received by the observer in 1 sec. Here the wavelength remain unchanged hence the number of waves in the length O'P i.e., the apparent frequency

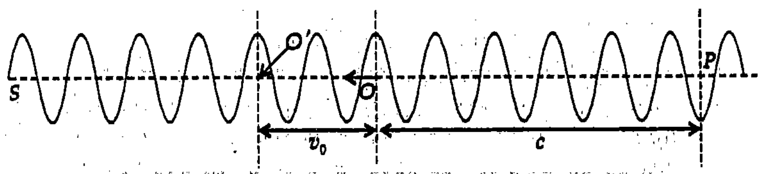


Figure **5.3: Observer moving towards the source**

$$\nu' = \frac{c + \upsilon_0}{\lambda} = \frac{c + \upsilon_0}{c}.\nu \qquad \text{... (3)}$$

Obviously $\nu' > \nu$ i.e., the frequency appears to increase when the observer moves towards the source S. Here wavelength remains unchanged.

The Doppler effect is due to the fact that the observer encounters a larger number of waves in a given time than when he or she was at rest relative to the source.

Observer moving away from the source:

If the observer moves away from the source (Figure-5.4) then only the waves contained in the length $c - v_0$ will be received by the listener in 1 sec. Therefore, apparent frequency in this case would be

$$\nu' = \frac{c - \upsilon_0}{c}\nu \qquad \ldots(4)$$

Obviously the frequency appears to decrease when the observer is moving away from the source. If $\upsilon_0 = c$ then $\nu' = 0$. If $\upsilon_0 = 2c$ then $\nu' = -\nu$, which indicates that the listener will hear the sound in correct time but backward. If $\upsilon_0 >> c, \nu' \to -\infty$; this means that sound waves lying ahead of the observer will be heard in reverse of the natural order but the waves lying behind the observer will never reach the observer i.e., he or she will hear nothing. Therefore, the above formulas are applicable when the speed of the source and observer is smaller than c.

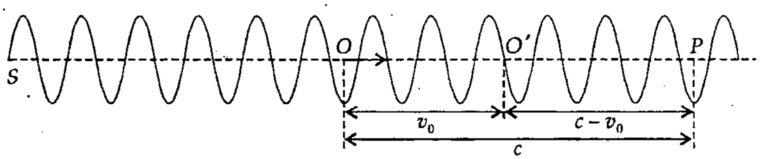


***Figure* 5.4:** Observer moving away from the source

x. 4. An object of specific gravity ρ is hung from a thin steel wire. The fundamental frequency for transverse standing waves in the wire is 300 Hz. The object is immersed in water, so that one half of its volume is submerged. The new fundamental frequency (in Hz) is

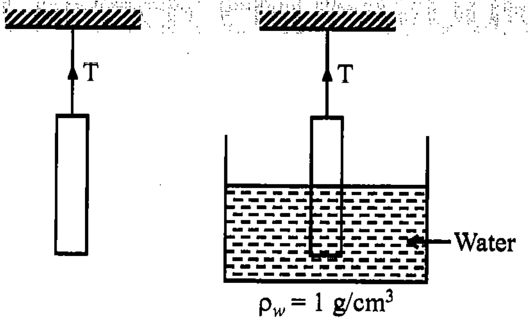
(a) $300\left(\frac{2\rho-1}{2\rho}\right)^{1/2}$ (b) $300\left(\frac{2\rho}{2\rho-1}\right)^{1/2}$ (c) $300\left(\frac{2\rho}{2\rho-1}\right)$ (d) $300\left(\frac{2\rho-1}{2\rho}\right)$

[IIT-JEE : 1995]

ln. (a)

The diagramatic representation of the given problem is shown in figure. The expression of fundamental frequency is

$$\nu = \frac{1}{2l}\sqrt{\frac{T}{\mu}}$$



In air $T = mg = (V\rho)g$

$$\therefore \quad \nu = \frac{1}{2l}\sqrt{\frac{V\rho g}{\mu}} \qquad \ldots(i)$$

When the object is half immersed in water

$$T' = mg - \text{upthrust} = V\rho g - \left(\frac{V}{2}\right)\rho_w g$$

$$= \left(\frac{V}{2}\right)g\,(2\rho - \rho_w)$$

The new fundamental frequency is

$$\nu' = \frac{1}{2l} \times \sqrt{\frac{T'}{\mu}} = \frac{1}{2l}\sqrt{\frac{(V/2)(2\rho - \rho_w)g}{\mu}} \qquad \text{...(ii)}$$

$$\therefore \frac{\nu'}{\nu} = \left(\sqrt{\frac{2\rho - \rho_w}{2\rho}}\right) \text{ or } \nu' = \nu\left(\frac{2\rho - \rho_w}{2\rho}\right)^{1/2} = 300\left(\frac{2\rho - 1}{2\rho}\right)^{1/2} \text{ Hz}$$

5.2.3 Both the source and observer are moving:

Suppose the source S and observer O both are moving along the same direction SO with speeds c and υ_o respectively. Let the source S be emitting continuously sound waves of frequency ν and wavelength λ in a stationary medium. If the source were stationary the ν waves emitted in 1 sec. would occupy the length SP = c, where c is the speed of sound. Now as the source moves through in 1 sec., the ν waves will now occupy the length $S'P = c - \upsilon_s$ and hence the original wavelength SS' (= λ) = υ_s will be changed to $\lambda' = \frac{c - \upsilon_s}{v}$

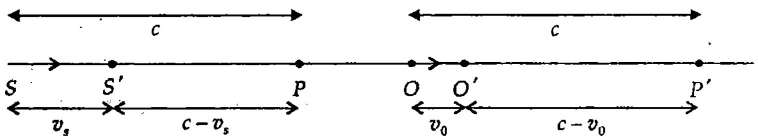


***Figure* 5.5:** Both the source and observer moving in the same direction

Now this wave of wavelength λ' is approaching the observer O. The number of waves received by a stationary observer is equal to the number of waves contained in $OP' = c$. As the observer moves in 1 sec. through $OO' = \upsilon_0$, he or she will receive only those waves contained in the length $O'P' = c - \upsilon_0$. Therefore, the apparent frequency would be

$$v' = \frac{c - \upsilon_0}{\lambda'} = \frac{c - \upsilon_0}{c - \upsilon_s} \cdot \upsilon \qquad \text{... (5)}$$

This formula can also be obtained easily by combining the Equations (1) and (4).

If the observer moves towards the source then υ_0 is to be taken with a negative sign. Thus if the source and observer move towards each other then

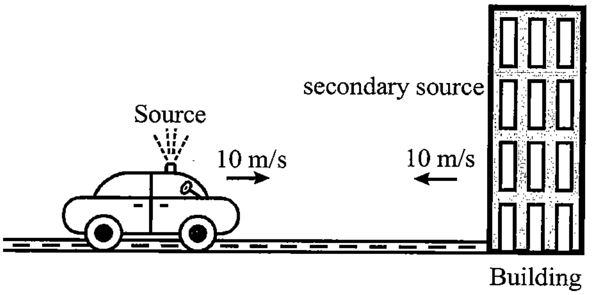
$$v' = \frac{c + \upsilon_0}{c - \upsilon_s} v \qquad \text{... (6)}$$

Ex. 5. A police car with a siren of frequency 8 kHz is moving with uniform velocity 36 km/h towards a tall building which reflects the sound waves. The speed of sound in air is 320 m/s. The frequency of the siren heard by the car driver is

(a) 8.50 kHz (b) 8.25 kHz (c) 7.75 kHz (d) 7.50 kHz

[IIT-JEE : 2011]

Soln. (a)



In this case, wall will act like secondary source of sound moving towards car driver with speed equal to car.

Speed of observer (driver) $v_0 = -10\,\text{m/s}$

[minus sign implies that observer is moving towards secondary source (wall)]

Speed of the source $v_s = 10\,\text{m/s}$

Apparent frequency of sound heard by the driver will be

$$v' = \left(\frac{c-\upsilon_0}{c-v_s}\right)v = 8\left[\frac{320+10}{320-10}\right] = 8.5\text{ kHz}$$

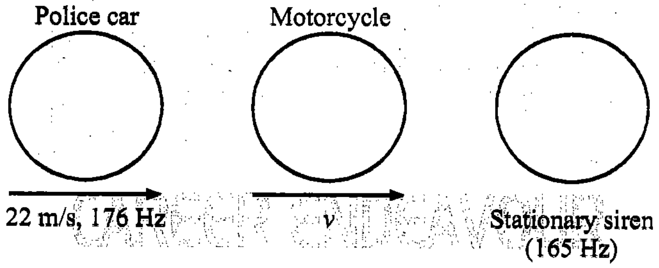
Ex. 6. A police car moving at 22 m/s chases a motorcyclist. The police man sounds his horn at 176 Hz, while both of them move towards a stationary siren of frequency 165 Hz. Calculate the speed of the motorcycle. If it is given that the motorcyclist does not observe any beats (speed of sound = 330 m/s)

(a) 33 m/s (b) 22 m/s (c) zero (d) 11 m/s

[IIT-JEE : 2003]

Soln. **(b)**

The motorcyclist observes no beats. So, the apparent frequency observed by him from the two sources must be equal.



$$f_1 = f_2$$

$$\therefore\ 176\left(\frac{330-v_o}{330-22}\right) = 165\left(\frac{330+v_o}{330}\right)$$

Solving this equation, we get $v_o = 22$ m/s.

5.2.4 Source, observer and medium all are moving

If the medium carrying the waves moves with a velocity υ_m with respect to some inertial frame of reference in the direction of sound, the apparent sound speed increase to $c + \upsilon_m$. In this in all the above formulas (1) –(6), c is to be replaced by $c + \upsilon_m$. On the other hand, if the medium moves against the direction of sound then the apparent sound speed becomes $c - \upsilon_m$ and we are to replace c by $c - \upsilon_m$ in each of the above formulas (1) – (6). Thus if the observer, source and the medium all are moving in the same direction then apparent frequency

$$\nu' = \frac{c + \upsilon_m - \upsilon_0}{c + \upsilon_m - \upsilon_s} . \nu \qquad \text{... (7)}$$

Here ν is the frequency of the source, υ_s its speed towards the observer, υ_0 the speed of observer away from the source, υ_m the speed of the medium towards the observer and c the speed of sound in stationary medium.

If $\upsilon_0 = \upsilon_s$, then $\nu' = \nu$ i.e. when the velocities of the source and observer are the same in magnitude and direction, the velocity of the medium will have no effect on the frequency.

5.2.5 Source moving not in line with stationary observer:

In most practical cases the stationary observer O is situated away from the line of motion AB of the source (Figure-5.6).

When the source is at S_1 the component of the source velocity (υ_s) along S_1O is $\upsilon_s \cos\theta_1$ i.e., the source appears to move towards the observer with a velocity $\upsilon_s \cos\theta_1$. At this moment the observer receives a note frequency ν' given by the equation (1) as

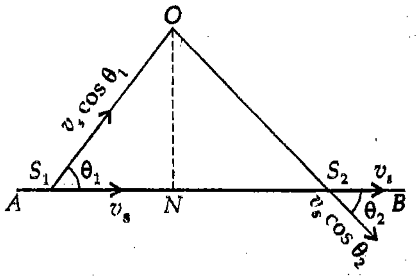


***Figure* 5.6:** Stationary observer not in line with the moving source

$$\nu' = \frac{c}{c - \upsilon_s \cos\theta_1} \nu \qquad \text{... (8)}$$

where ν is the original frequency and c is the velocity of sound in the medium which is assumed to be stationary.

As the source approaches N, the foot of the perpendicular from O on AB, θ_1 increases continuously. Consequently, $\cos\theta_1$ decreases and the frequency decreases continuously. When the source is at N, $\theta_1 = \pi/2$ and $\nu' = \nu$.

When the source is at S_2: the velocity component $\upsilon_s \cos\theta_2$ is away from the observer and hence from Equation(2) the apparent frequency becomes smaller and is given by

$$\nu'' = \frac{c}{c + \upsilon_s \cos\theta_2} \nu \qquad \text{... (9)}$$

When the source is act and, θ_2 decreases continuously and ultimately at large distance θ_2 tends to zero.

5.2.6 Observer moving not in line with the stationary source:

Suppose the stationary source S is situated away from the line of motion AB of the observer (Figure-5.7). When the observer is at O_1 the component of his velocity (υ_0) along O_1S is $\upsilon_0 \cos\theta_1$. At this point we may consider that an observer is moving towards the source with a velocity $\upsilon_0 \cos\theta_1$. Hence the apparent frequency as perceived by the observer would be

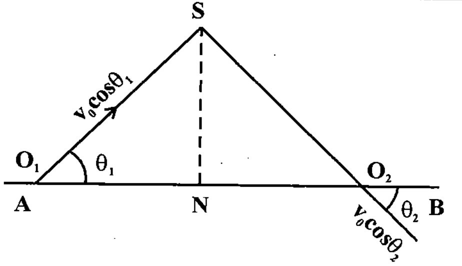


Figure **5.7:** Stationary source not in line with the moving observer

$$\nu' = \frac{c + \upsilon_0 \cos\theta_1}{\lambda} = \frac{c + \upsilon_0 \cos\theta_1}{c}\nu \qquad .... (10)$$

where ν is the original frequency of the source and c is the velocity of sound in the medium which is assumed to be stationary.

As the observer approaches N, the foot of the perpendicular from S on AB, θ_1 increases continuously. Consequently $\cos\theta_1$ and hence ν' decreases continuously. At , $\theta_1 = \pi/2$ and $\nu = \nu'$ i.e., the observer now hears a note of the same frequency as emitted by the source. On the other side of N when the observer is at O_2 the velocity component $\upsilon_0 \cos\theta_2$ is along is along SO_2 i.e., away from the source. So the frequency now appears to be

$$\nu' = \frac{c - \upsilon_0 \cos\theta_2}{c}\nu \qquad ... (11)$$

In this case $\nu' < \nu$. When the observer moves to a large distance from N then $\theta_2 \to 0$ and the formula (11) reduces to (4).

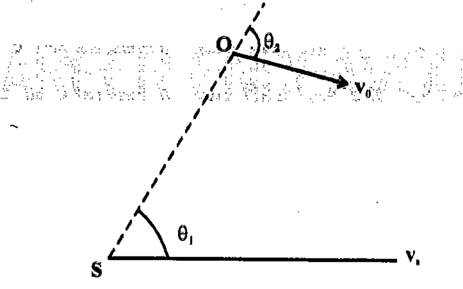


Figure **5.8:** Source and observer moving in arbitary directions in a plane.

5.2.7 Source and observer moving in any direction in a plane:

Suppose the source is moving with velocity υ_s in a direction making an angle θ_1 with the line SO joining the source S and observer O at any instant. Let at the same instant the direction of the velocity v_0 of the observer makes an angle θ_2 with the same line SO as shown in Figure-5.8. We may consider the situation to be equivalent to the case where the source is moving towards the observer with a velocity $\upsilon_s \cos\theta_1$ and the observer is moving away from the source with a velocity $\upsilon_0 \cos\theta_2$ and hence the apparent frequency is given by

$$\nu' = \frac{c - \upsilon_0 \cos\theta_2}{c - \upsilon_s \cos\theta_1}\nu \qquad ...(12)$$

5.3 INTERCONNECTION BETWEEN DOPPLER EFFECT AND STATIONARY WAVES

Suppose we have two identical stationary sources S_1 and S_2 emitting sound of same frequency ν and a listener O is moving along the line S_1S_2 as shown in Figure 5.9. As the listener is moving away from the source S_1 with a velocity υ_0 the frequency of the sound from S_1 as perceived by the listener would be

$$\nu_1 = \nu.\frac{c - \upsilon_0}{c} \qquad ... (13)$$

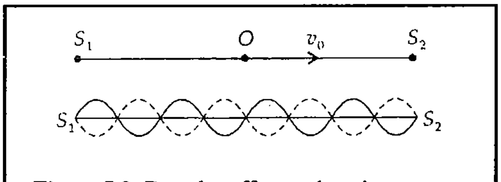


***Figure* 5.9:** Doppler effect and stationary waves

Again as the observer is moving towards S_2, the apparent frequency of the sound from S_2 will be

$$\nu_2 = \nu.\frac{c + \upsilon_0}{c} \qquad ... (14)$$

These notes of frequencies ν_1 and ν_2 will produce beats of frequency

$$\nu_2 - \nu_1 = \nu.\frac{2\upsilon_0}{c} = \frac{2\upsilon_0}{\lambda} \qquad ... (15)$$

where $\lambda = c/\nu$ is the wavelength of the emitted note.

Alternatively, we may think that the listener is moving through a system of stationary waves. The distance between successive nodes or antinodes is $\lambda/2$. Therefore, the number of minimum or maximum sound the listener hears $\upsilon_0/(\lambda/2) = 2\upsilon_0/\lambda$ which agrees with the result obtained from Doppler's principle. Thus we find an interesting connection between Doppler effect, stationary waves and the phenomenon of beats.

SOLVED EXAMPLES

1. A plane wave of frequency ω propagates so that planes of constant phase move with speeds v_1, v_2 and v_3 along the x, y and z axes, respectively. The wave vector, K of the plane wave (in terms of the Cartesian unit vectors $\hat{i}, \hat{j}$ and $\hat{k}$) is:

(a) $K = \omega\left(\frac{1}{v_1}\hat{i} + \frac{1}{v_2}\hat{j} + \frac{1}{v_3}\hat{k}\right)$

(b) $K = \omega\left(\hat{i} + \hat{j} + \hat{k}\right)/\sqrt{v_1^2 + v_2^2 + v_3^2}$

(c) $K = \omega\left(v_1\hat{i} + v_2\hat{j} + v_3\hat{k}\right)/\left(v_1^2 + v_2^2 + v_3^2\right)$

(d) $K = \left(v_1\hat{i} + v_2\hat{j} + v_3\hat{k}\right)/\omega$

[JNU : 2010]

Soln. **(c)**

$\bar{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$

$$\vec{k} = \frac{\omega}{|\vec{v}|}(\hat{v}) = \frac{\omega}{\sqrt{v_1^2+v_2^2+v_3^2}}\frac{\left(v_1\hat{i}+v_2\hat{j}+v_3\hat{k}\right)}{\sqrt{v_1^2+v_2^2+v_3^2}}$$

$$\boxed{\vec{k} = \frac{\omega\left(v_1\hat{i}+v_2\hat{j}+v_3\hat{k}\right)}{v_1^2+v_2^2+v_3^2}}.$$

Consider two cases (i) a source is approaching a stationary observer with a relative velocity υ and (ii) an observer is approaching a stationary source with the same relative velocity υ. Will the Doppler shift be the same in the two cases?

ln. For the case (i) the apparent frequency is

$$\nu_1 = \nu\frac{c}{c-\upsilon}$$

where ν is the actual frequency and c is the speed of sound.
For the case (ii) the apparent frequency is

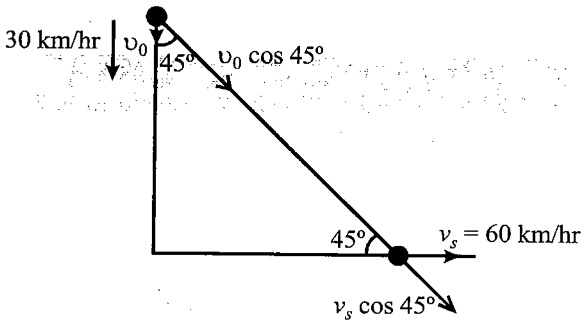
$$\nu_2 = \nu.\frac{c+\upsilon}{c}, \quad \therefore \frac{\nu_2}{\nu_1} = 1-\frac{\upsilon^2}{c^2}$$

Thus unless υ approaches c, the two frequency will be different.

Two roads meet at right angles to each other. A motor cyclist is moving away from the junction along one road with a speed 60 km/hr and blowing a horn of frequency 200 Hz. Along the other road a bi-cyclist is approaching the first road at speed of 30 km/hr. When they are at equal distances from the junction, what would be the frequency as heard by the bi-cyclist? Given the velocity of sound = 340 m/s.

ln. The required apparent frequency

$$\nu' = \nu.\frac{c-\upsilon_0\cos\theta_2}{c-\upsilon_s\cos\theta_1}$$



Here $\nu = 200\,\text{Hz}, \theta_1 = \theta_2 = 45°, \upsilon_0 = 30\,\text{km/hr}, \upsilon_s = 60\,\text{km/hr}$

and $\quad c = 340\,\text{m/s} = \frac{340}{1000}\times 60\times 60\ \text{km/hr} = 1224\,\text{km/hr}$

$$\therefore \quad \nu' = 200\frac{1224-30/\sqrt{2}}{1224-660/\sqrt{2}} = 203.6\ \text{Hz}$$

A traffic police is blowing a whistle of frequency 200 Hz. If the sound is reflected from an oncoming double decker bus with relative (speed 30 km/hr) what would be the number of beats heard by a stationary observer? Assume the velocity of sound = 340 m/s.

Soln.. The apparent frequency of the reflected sound is given by $\nu' = \nu\dfrac{c+\upsilon_r}{c-\upsilon_r}$

where v_r is relative speed between source and reflecting object

Here $\nu = 200\,\text{Hz},\ \upsilon_r = 30\,\text{km/hr}$ and $c = \dfrac{340}{1000}\times 60\times 60 = 1224\ \text{km/hr}$

$$\therefore \quad \nu' = 200\frac{1224+30}{1224-30} = 210\ \text{Hz}$$

Therefore, number of beats heard per second = 210 – 200 = 10.

5. A train is blowing a whistle while passing by a stationary observer. The frequency of sound appears to decrease from 220 Hz to 190 Hz as the train crosses the observer. Find the actual frequency of the whistle.

Soln. If ν is the actual frequency and v_s is the speed of the train then observed frequency would be

$$\nu' = \nu\frac{c}{c-\upsilon_s} \text{ and } \nu'' = \nu\frac{c}{c+\upsilon_s} \qquad \text{where c is the speed of sound}$$

$$\therefore \quad \frac{\nu'}{\nu''} = \frac{c+\upsilon_s}{c-\upsilon_s} = \text{or, } \frac{\nu'+\nu''}{\nu'-\nu''} = \frac{2c}{2\upsilon_s} = \frac{c}{\upsilon_s}$$

Using this we can write

$$\nu = \nu'\frac{c-\upsilon_s}{c} = \nu'\left(1-\frac{\upsilon_s}{c}\right) = \nu'\left(1-\frac{\nu'-\nu''}{\nu'+\nu''}\right) = \frac{2\nu'\nu''}{\nu'+\nu''} = \frac{2\times 220\times 190}{220+190}\ \text{Hz} = 203.9\ \text{Hz}$$

6. A prominent spectral line of a certain substance has a wavelength 5000 A in the laboratory. The same spectral line is identified in the light coming from a distant star but with a different wavelength 5001 A. Is the star receding from or moving towards to earth? What is speed with respect to the earth?

Soln. As the wavelength increases, the frequency decreases which indicates that the star is receding from the earth Now, frequency of the light coming from the star moving away from the earth with a speed υ will be given by

$$\nu' = \nu.\frac{c}{c+\nu} \text{ where } \nu \text{ is the actual frequency.} \qquad \Rightarrow \frac{c}{\lambda'} = \frac{c}{\lambda}.\frac{c}{c+\upsilon}$$

$$\text{or,} \quad 1+\frac{\upsilon}{c} = \frac{\lambda'}{\lambda} = \frac{5001}{5000} = 1+\frac{1}{5000} \qquad \Rightarrow \upsilon = \frac{1}{5000}c = \frac{1}{5000}\times 3\times 10^8\,\text{m/s} = 6\times 10^4\,\text{km/s}\cdot$$

Chapter 6

Fermat s Principle

6.1 OPTICAL PATH:

When light travels a distance l in a medium of refractive index n the product nl is termed the **optical path** corresponding to the distance l. By definition,

$$n = \frac{\text{velocity of light in vacuum}\,(c)}{\text{velocity of light in medium}\,(\upsilon)} \qquad \dots (1)$$

Therefore optical path,

$$nl = \frac{c}{\upsilon}.l = c.\frac{l}{\upsilon} = c \times \text{time taken by the light to travel the distance } l \text{ in the medium.}$$

$$\dots (2)$$

6.2 FERMAT'S PRINCIPLE:

The path followed by a ray of light in moving from one point to another point after any number of reflections or refractions would always be stationary (i.e., either maximum or minimum) with respect to variations of that path.

If the light from one point reaches another point directly or after reflection or refraction at a plane surface the path is minimum. Hence time taken by light in this case must also be minimum. Thus the principle of least path may also be called the principal of least time. If the reflection or refraction occurs at spherical surfaces the path may be either maximum or minimum, depending on the curvature of the surfaces. Hence, in general, the path would be stationary.

The optical path between two points may be expressed as $\sum nl$ or $\int ndl$ (if the refractive index n varies continuously over the path). From Fermat's principle,

$$\delta\sum nl = \delta\int ndl = 0 \qquad \dots (3)$$

6.2.1 Rectilinear propagation of light:

If light moves from one point to another in a homogeneous medium, Fermat's principle implies that path of the ray between the two points should be least. Obviously the path is a straight line. Thus the rectilinear propagation of light results as direct consequence of Fermat's principle.

6.2.2 Principle of reversibility of light:

Fermat's principle stated in the form of the Equation (3), involves only the optical path and not the direction of propagation of light. So a ray in going from the point P to Q will trace the same route as one from Q to P. Thus the principle of reversibility is also included in Fermat's principle.

6.3 TO ESTABLISH LAWS OF REFLECTION AT A SPHERICAL SURFACE FROM FERMAT'S PRINCIPLE:

Suppose a ray of light from a fixed point P is arriving at another fixed point Q via the point A of a spherical surface as shown in Figure 6.1. Let ϕ_1 and ϕ_2 be the angles which the rays PA and AQ make with the normal CA at A drawn from the centre of curvature C of the spherical surface. Let OC = R = radius of the spherical surface. Now from ΔAPC and ΔAQC we can write

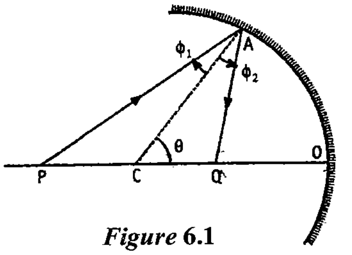


Figure **6.1**

$$PA = \left(R^2 + CP^2 + 2R.CP.\cos\theta\right)^{\frac{1}{2}} \text{ and } \quad AQ = \left(R^2 + CQ^2 - 2R.CQ.\cos\theta\right)^{\frac{1}{2}} \quad \text{... (4)}$$

According to Fermat's principle the total optical path PA + AQ between P and Q should be stationary and hence the variation of PA + AQ with the angle θ must be zero. Thus,

$$\frac{d}{d\theta}(PA + AQ) = 0$$

Since R, CP and CQ are constants we get from above Equation, using Equations(4)

$$-\frac{CP}{PA} + \frac{CQ}{AQ} = 0 \quad \Rightarrow -\frac{\sin\phi_1}{\sin(180^\circ - \theta)} + \frac{\sin\phi_2}{\sin\theta} = 0$$

or, $\phi_1 = \phi_2$ i.e., angle of incidence (ϕ_1) = angle of reflection (ϕ_2) ... (5)

6.3.1 Reflection formula:

Under paraxial approximation the angle θ as indicated in Figure 6.1 is small so that we may use

$$\cos\theta = 1 - \frac{\theta^2}{2} \quad \text{... (6)}$$

Now assuming OP = x and OQ = y as the magnitude of distances we can write from Equation (4),

$$PA = \left[R^2 + (x - R)^2 + 2R(x - R)\left(1 - \frac{\theta^2}{2}\right)\right]^{\frac{1}{2}}$$

$$= \left[x^2 - R(x - R)\theta^2\right]^{\frac{1}{2}} \simeq x\left[1 - \frac{R(R-x)^2\theta^2}{2x^2}\right]$$

$$= x - \frac{1}{2}R^2\left(\frac{1}{R} - \frac{1}{x}\right)\theta^2 \quad \text{... (7)}$$

Similarly we get from Equation (4),

$$AQ \simeq y - \frac{1}{2}R^2\left(\frac{1}{R}-\frac{1}{y}\right)\theta^2 \qquad \text{... (8)}$$

Now for the optical path $L_{op} = PA + AQ$ to an extremum we must have $\frac{dL_{op}}{d\theta} = \frac{d}{d\theta}(PA + AQ) = 0$

Using Equation (7) and (8) we get, $\left[\frac{2}{R}-\frac{1}{x}-\frac{1}{y}\right]\theta = 0$ ... (9)

If the quantity within the bracket is not zero then we must have $\theta = 0$. In this case light falls normally on the surface along PO and gets reflected normally back to Q along OQ. Laws of reflection is obeyed here. On the other hand, if $\frac{1}{y}+\frac{1}{x}=\frac{2}{R}$ ... (10)

$\frac{dL_{op}}{d\theta}$ would vanish for all values of θ. This indicates that all paths like PAQ are allowed ray paths i.e., all paraxial rays emanating from P will, after reflection, pass through Q. Therefore, in this case Q will be the image point of the object point P. Thus Equation (10) is reflection formula at the concave spherical surface. Using the sign convention that all the distances measured to the left of O are negative we may write object distance OP = u = - x, image distance OQ = υ = - y and radius of curvature OC = r = – R. Then formula (10) may be rewritten for a concave spherical reflector as

$$\frac{1}{\upsilon}+\frac{1}{u}=\frac{2}{r} \qquad \text{... (11)}$$

6.4 TO ESTABLISH LAWS OF REFRACTION AT A SPHERICAL SURFACE FROM FERMAT'S PRINCIPLE:

Suppose a ray of light from a fixed point Q via the point A on a spherical surface separating two media of refractive indices n_1 and n_2. Let ϕ_1 and ϕ_2 be the angles which the rays PA and AQ make with the normal CA at A drawn from the centre of curvature C of the spherical surface. From ΔAPC and ΔAQC we can write

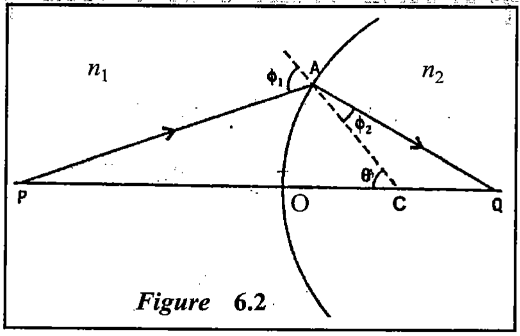


Figure **6.2**

$$PA = \left(AC^2 + CP^2 - 2.AC.CP.\cos\theta\right)^{\frac{1}{2}} \text{ and } AQ = \left(AC^2 + CQ^2 + 2.AC.CQ.\cos\theta\right)^{\frac{1}{2}} \qquad \text{... (12)}$$

According to Fermat's principle the total optical path $n_1PA + n_2AQ$ between P and Q should be stationary and hence

$$\frac{d}{d\theta}(n_1PA + n_2AQ) = 0 \qquad \text{... (13)}$$

Since AC, CP and CQ are constants we get from Equation (13). Using Equations (12).

$$\frac{n_1 CP}{PA}-\frac{n_2 CQ}{AQ}=0 \quad \Rightarrow \quad \frac{n_1\sin\left(180^\circ-\phi_1\right)}{\sin\theta}-\frac{n_2\sin\phi_2}{\sin\left(180^\circ-\theta\right)}=0$$

$\Rightarrow$ $n_1\sin\phi_1 = n_2\sin\phi_2$ which is the snell's law of refraction. ... (14)

6.4.1 Refraction formula:

Under paraxial approximation the angle θ as indicated in (Figure 6.2) is small so that we can use

$$\cos\theta = 1-\frac{\theta^2}{2} \qquad \text{... (15)}$$

Now putting OP = x, OQ = y and AC = OC = R as the magnitude of the distances we can write from Equation (12),

$$PA=\left[R^2+\left(x+R\right)^2-2R.\left(x+R\right)\left(1-\frac{\theta^2}{2}\right)\right]^{\frac{1}{2}}$$

$$=\left[x^2+R\left(x+R\right)\theta^2\right]^{\frac{1}{2}}\simeq x\left[1+\frac{R\left(x+R\right)\theta^2}{2x^2}\right]=x+\frac{1}{2}R^2\left(\frac{1}{x}+\frac{1}{R}\right)\theta^2 \qquad \text{... (16)}$$

Similarly from Equation (12) we get,

$$AQ=y+\frac{1}{2}R^2\left(\frac{1}{y}-\frac{1}{R}\right)\theta^2 \qquad \text{... (17)}$$

Now for the optical path $L_{op}=n_1PA+n_2AQ$ be an extremum we must have

$$\frac{dL_{op}}{d\theta}=\frac{d}{d\theta}\left[n_1PA+n_2AQ\right]=0$$

$$\Rightarrow \quad R^2\left[n_1\left(\frac{1}{x}+\frac{1}{R}\right)+n_2\left(\frac{1}{y}-\frac{1}{R}\right)\right]\theta=0$$

$$\Rightarrow \quad \left[\frac{n_2}{y}+\frac{n_1}{x}-\frac{n_2-n_1}{R}\right]\theta=0 \qquad \text{... (18)}$$

If the quantity within the bracket is not zero then must have $\theta=0$. In this case light from P falls on the surface normally along PO and proceeds undeviated to the point Q, obeying the laws of refraction. On the other hand, if

$$\frac{n_2}{y}+\frac{n_1}{x}-\frac{n_2-n_1}{R}=0 \qquad \text{... (19)}$$

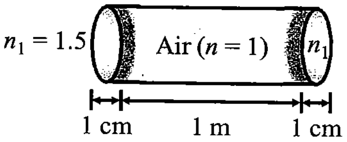
$\frac{dL_{op}}{d\theta}=0$ for all values of θ. This indicates that all paths like PAQ are allowed ray paths i.e., all paraxial rays emanating from P will, after refraction, pass through Q. Therefore, Q in that case will be the image point of the object point P. Equation (19) then represents the refraction formula at the curved surface.

Using the sign convention that all distances measured to the left of O are negative and those to its right positive we may write object distance OP = u = – x, image distance OQ = v = y and radius of curvature OC = r = R. Then formula 19 may be rewritten for a convex spherical surface in form

$$\frac{n_2}{v}-\frac{n_1}{u}=\frac{n_2-n_1}{r} \qquad \text{... (20)}$$

SOLVED EXAMPLES

A straight hollow pipe 1m long is closed at both ends with glass plates of 1 cm thick. If the pipe is evacuated what would be the optical path-length between the two outer glass surfaces. Take the refractive index of glass = 1.5.

oln. 

Total optical path

$$S = (\text{actual path}) \cdot (\text{refractive index})$$
$$= (1.5 \times 0.01 + 1 \times 1 + 1.5 \times 0.01)\,\text{m} = 1.03\ \text{m}.$$

If the pipe in the above problem is filled with a liquid of refractive index 1.33, find the change in optical path.

ıln. In this case total optical path $S_2 = (1.5 \times 0.01 + 1.33 \times 1 + 1.5 \times 0.01)\,\text{m} = 1.36\,\text{m}$

Therefore, increase in optical path is $S_2 - S_1 = 0.33\,\text{m}$.

Consider a refracting surface of radius of curvature r with 'O' as the pole. P and Q are two points such that PO = OQ = x and the centre of curvature lies on the line PQ. Show that the straight optical path POQ is an extremum and this extremum is a minimum when (i) $x < r.\dfrac{n+1}{n-1}$, (ii) a maximum when $x > r.\dfrac{n+1}{n-1}$ and (iii) stationary when $x = r.\dfrac{n+1}{n-1}$.

Given $n = n_2 / n_1 > 1$, where n_1 and n_2 are refractive indices of media on the left and on the right of the refracting surface.

ln.. From Equations (16) and (17) total optical path

$$L_{op} = n_1 PA + n_2 AQ = n_1 x + \frac{1}{2} n_1 r^2 \left(\frac{1}{x} + \frac{1}{r}\right)\theta^2 + n_2 y + \frac{1}{2} n_2 r^2 \left(\frac{1}{y} - \frac{1}{r}\right)\theta^2$$

where we put PO = OQ = x, R = r

Now $$\frac{dL_{op}}{d\theta} = r^2 \left[n_1 \left(\frac{1}{x} + \frac{1}{r}\right) + n_2 \left(\frac{1}{x} - \frac{1}{r}\right) \right] \theta$$

For $\theta = 0$, L_{op} becomes the straight optical path POQ and $\dfrac{dL_{op}}{d\theta} = 0$

Thus the straight path POQ is an extremum.

Now $$\frac{d^2L_{op}}{d\theta^2} = r^2 \left[n_1 \left(\frac{1}{x} + \frac{1}{r}\right) + n_2 \left(\frac{1}{x} - \frac{1}{r}\right) \right] \quad \Rightarrow \frac{d^2L_{op}}{d\theta^2} = r^2.n_1 \left[(n+1).\frac{1}{x} - (n-1)\frac{1}{r} \right]$$

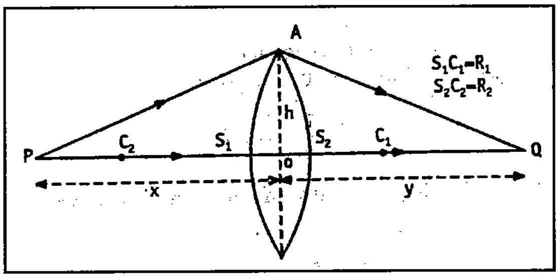
where we put $n = n_2 / n_1$.

For L_{op} to be maximum, $\dfrac{d^2L_{op}}{d\theta^2} < 0 \quad \Rightarrow (n-1).\dfrac{1}{r} > (n+1).\dfrac{1}{x} \quad \Rightarrow x > \dfrac{n+1}{n-1}.r$

For L_{op} to be minimum, $\frac{d^2L_{op}}{d\theta^2} > 0 \quad \Rightarrow \quad x < r.\frac{n+1}{n-1}$

For L_{op} to be stationary, $\frac{d^2L_{op}}{d\theta^2} = 0 \quad \Rightarrow \quad x = r.\frac{n+1}{n-1}$

4. Establish thin lens formula by using Fermat's principle.



Soln.. Let in figure above, Q be the image point of an axial object at point P. According to Fermat's principle the two optical paths from P to Q as shown in figure must be same i.e.,

$$PA + AQ = PS_1 + n.S_1S_2 + S_2Q \qquad \dots (1)$$

Now from above figure

$$PA = \left(x^2 + h^2\right)^{\frac{1}{2}} \simeq x + \frac{1}{2}\frac{h^2}{x}$$

$$AQ = \left(y^2 + h^2\right)^{\frac{1}{2}} \simeq y + \frac{1}{2}\frac{h^2}{y}$$

where we assume *h* small (i.e., paraxial approximation).

Therefore, from Equation (1)

$$x + y + \frac{1}{2}h^2\left(\frac{1}{x} + \frac{1}{y}\right) = PO - OS_1 + n\left(OS_1 + OS_2\right) + OQ - OS_2$$

$$= x + y + (n-1)\left(OS_1 + OS_2\right)$$

Therefore, $\frac{1}{x} + \frac{1}{y} = (n-1).\frac{2}{h^2}.\left(OS_1 + OS_2\right) \qquad \dots (2)$

Now for the spherical surface S_1 with the centre of curvature at C_1, we can write from geometry

$$h^2 = OS_1\left(2S_1C_1 - OS_1\right)$$

For a thin lens $OS_1 << S_1C_1$ and we can write

$$h^2 = OS_1.2R_1$$

Similarly for the surface S_2,

$$h^2 = OS_2.2R_2$$

substituting these values of h^2 in Equation (2) we get

$$\frac{1}{x} + \frac{1}{y} = (n-1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \qquad \dots (3)$$

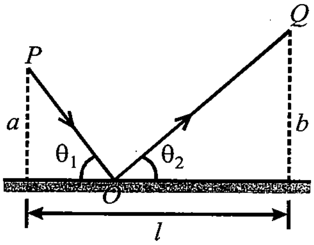
Now according to our sign convention, object distance OP = $u = -x$, image distance OQ = $\upsilon = y$, radius of curvature of the first surface S_1, $S_1C_1 = r_1 = +R_1$, radius of curvature of the second surface S_2,$S_2C_2 = r_2 = -R_2$. Therefore, formula (2) takes the usual form

$$\frac{1}{\upsilon}-\frac{1}{u}=\frac{1}{f} \qquad \ldots (4)$$

where f, known as the focal length of the lens, is given by

$$\frac{1}{f}=(n-1)\left(\frac{1}{r_1}-\frac{1}{r_2}\right) \qquad \ldots (5)$$

Consider a ray of light starting from point P in front of mirror and going to another point Q after getting reflected from the mirror, as shown in the figure.



Show that the incident and reflected light make same angle from the mirror using Fermat principal.

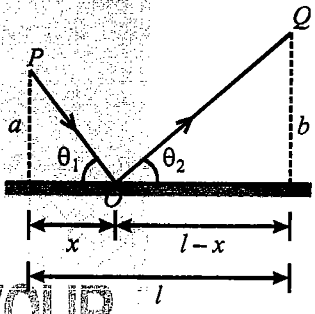
ln. Let O is at x distance from point M. Now, total path travel by the light ray

$$S=PO+OQ=\sqrt{a^2+x^2}+\sqrt{b^2+(l-x)^2}$$

Now, according to Fermat principal, path followed by the light will be minimum, so that time taken is the least. Thus, time taken will be least, if path S is minimum.

$$\frac{dS}{dx}=\frac{d(a^2+x^2)^{1/2}}{dx}+\frac{d[b^2+(l-x)^2]^{1/2}}{dx}=0$$

$$=\frac{x}{(a^2+x^2)^{1/2}}-\frac{(l-x)}{\sqrt{b^2+(l-x)^2}}=0$$



from given figure, we can write $\cos\theta_1-\cos\theta_2=0$

$\Rightarrow \cos\theta_1=\cos\theta_2$

$\Rightarrow \theta_1=\theta_2$

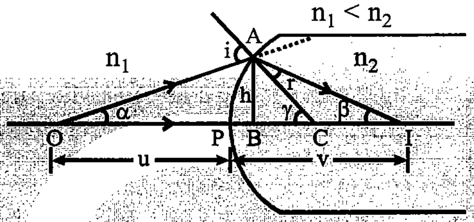
i.e., incident angle is equal to the reflected angle.

Chapter 7

Refraction at Spherical Surfaces

7.1 REFRACTION AT SPHERICAL SURFACES:

Consider a convex spherical surface separating two media of refractive index n_1 and n_2 $(n_2 > n_1)$.



Ray diagram for the real image is shown in above figure where O is the point object and I is the image formed by the spherical surface.
Object distance u, image distance v and radius of curvature r is related by the formula

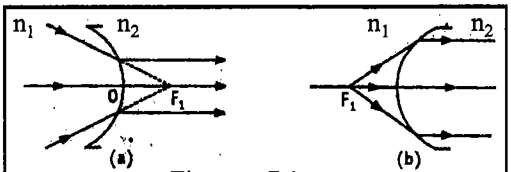
$$\frac{n_2}{v} - \frac{n_1}{u} = \frac{n_2 - n_1}{r} \qquad \dots (1)$$

This equation is known as **Gaussian formula** for the single refracting surface.

Here, $\frac{n_2 - n_1}{r}$ is called power of the refracting surface.

If r is expressed in meter, unit of power is dioptre (D). Above formula is applicable for refraction at concave surface also.

7.1.1 Principal foci of a spherical surface:



(a) (b)

Figure 7.1

First principal focus : The first focal point (F_1) is a point on the principal axis such that rays moving towards that point [Figure 7.1(a)] or diverging from that point [Figure 7.1(b)] become parallel to the principal axis after refraction. *The distance of F_1 from the pole of the refracting surface is called the first principal focal length* (f′) *of the surface.* When f′ = u, then v = ∞. From the general formula (1)

$$f' = -\frac{n_1 r}{n_2 - n_1} \qquad \dots (2)$$

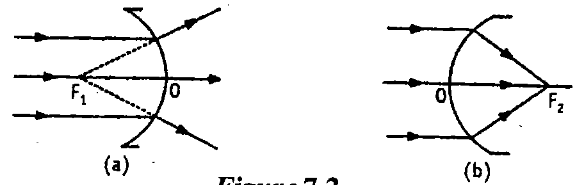


Figure 7.2

Second principal focus : The second focal point (F_2) is a point on the principal axis such that the incident rays moving parallel to the principal axis, after refraction, either appear to diverge from this point [Figure 7.2(a)] or actually converge to this point [Figure 7.2(b)].

This distance of F_2 from the pole of the refracting surface is called the second principal focal length (f) of the surface. When $f = v$, then $u = \infty$. From the general formula (1),

$$f = \frac{n_2 r}{n_2 - n_1} \quad \text{... (3)}$$

From Equation (2) and (3) we get, $\frac{f}{f'} = -\frac{n_2}{n_1}$ ... (4)

This indicates that the two principal focal lengths are always of opposite signs.

7.2 MAGNIFICATION FOR A SPHERICAL SURFACE:

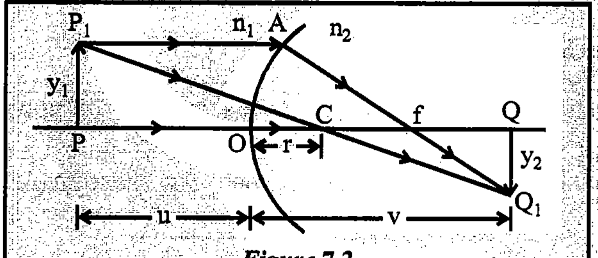


Figure 7.3

In Figure 7.3 let PP_1 be an object and QQ_1 be its image formed by refraction at the concave spherical surface separating two media of refractive indices n_1 and n_2.

7.2.1 Lateral or linear magnification:

Linear magnification (m) is defined as the ratio of the image height (y_2) to the object height (y_1) both measured perpendicular to the axis. Thus,

$$m = \frac{y_2}{y_1} = \frac{QQ_1}{PP_1} = \frac{CQ}{CP} = \frac{OQ - OC}{OP - OC} \quad \text{... (5)}$$

$[\because \; \Delta PP_1C$ and ΔQQ_1C are similar$]$

Now according to our sign convention all distances measured to the left of O are negative. Let the object distance u = – OP, image distance v = –OQ and the radius of curvature r = –OC.

$$m = \frac{v - r}{u - r} \quad \text{... (6)}$$

From the relation, $\frac{n_2}{v} - \frac{n_1}{u} = \frac{n_2 - n_1}{r}$ we get,

$$n_1\left(\frac{1}{r}-\frac{1}{u}\right)=n_2\left(\frac{1}{r}-\frac{1}{v}\right) \Rightarrow n_1\frac{u-r}{u}=n_2\frac{v-r}{v} \qquad \ldots(7)$$

Hence from Equation (6) and (7), $m=\frac{n_1}{n_2}\cdot\frac{v}{u}$ ... (8)

7.2.2 Longitudinal magnification:

If a small object of length δu, placed along the axis produces an images of length δv, the longitudinal magnification (l) is defined as

$$l = \lim\frac{\delta v}{\delta u}=\frac{dv}{du}\,.$$

Differentiating the equation, $\frac{n_2}{v}-\frac{n_1}{u}=\frac{n_2-n_1}{r}$ w.r.t. u, we get

$$-\frac{n_2}{v^2}\cdot\frac{dv}{du}+\frac{n_1}{u^2}=0 \Rightarrow \frac{dv}{du}=\frac{n_1}{n_2}\cdot\frac{v^2}{u^2}$$

From equation (8)

$$\Rightarrow \quad l=\frac{dv}{du}=\left(\frac{n_1 v}{n_2 u}\right)^2\cdot\frac{n_2}{n_1}=m^2\cdot\frac{n_2}{n_1} \qquad \ldots(9)$$

7.2.3 Angular magnification:

If the incident ray PA and the corresponding refracted ray AB make angles α_1 and α_2 respectively with the axis (Figure 7.3) the ratio $\frac{\tan\alpha_2}{\tan\alpha_1}$ is called the angular magnification (M).

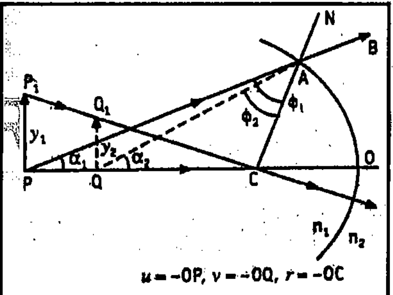


Figure 7.4

$$\Rightarrow \quad M=\frac{\tan\alpha_2}{\tan\alpha_1}=\frac{AO/OQ}{AO/OP}=\frac{OP}{OQ}=\frac{u}{v} \qquad \ldots(10)$$

Using Equation (8) we can write, $M=\frac{n_1}{n_2}\cdot\frac{1}{m}$ ... (11)

7.2.4 Inter-relation:

Combining Equation (9) and (11) we get $l=m\times\frac{1}{M} \Rightarrow \frac{l\times M}{m}=1$

$$\Rightarrow \quad \frac{\text{Longitudinal magnification}\times\text{Angular magnification}}{\text{Linear magnification}}=1 \qquad \ldots(12)$$

7.3 LAGRANGE'S LAW RELATION:

In the paraxial region there exists an important relation between object size, image size and the angles which the conjugate rays make with the principal axis. Let in Figure 7.4, PP_1 be a small object of size y_1 placed transversely to the axis in front of a certain refracting surface. C is the centre of curvature and QQ_1 is the image of size y_2. The conjugate rays PA and AB make angles α_1 and α_2 respectively with the axis. From Figure 7.4 we can write,

$$\frac{y_2}{y_1}\cdot\frac{\tan\alpha_2}{\tan\alpha_1}=\frac{QQ_1}{PP_1}\times\frac{AO/OQ}{AO/OP}$$ [in the paraxial region the point A is very close to pole O]

Using equation (5)

$$=\frac{CQ}{CP}\times\frac{OP}{OQ}=\frac{OQ-OC}{OP-OC}\times\frac{OP}{OQ} \qquad \text{... (13)}$$

According to our sign convention all distances measured to the left of O are negative. Let the object distance u = – OP, image distance $\upsilon = -OQ$ and radius of curvature r = - OC. Therefore,

$$\frac{y_2}{y_1}\cdot\frac{\tan\alpha_2}{\tan\alpha_1}=\frac{v-r}{u-r}\times\frac{u}{v} \qquad \text{... (14)}$$

Now we have the relation

$$\frac{n_2}{v}-\frac{n_1}{u}=\frac{n_2-n_1}{r} \quad\Rightarrow\quad n_1\left(\frac{1}{r}-\frac{1}{u}\right)=n_2\left(\frac{1}{r}-\frac{1}{v}\right)$$

$$\Rightarrow \qquad \frac{n_1}{n_2}=\frac{v-r}{u-r}\cdot\frac{u}{v} \qquad \text{... (15)}$$

Combining Equation (14) and (15) we get

$$\boldsymbol{n_1 y_1 \tan\alpha_1 = n_2 y_2 \tan\alpha_2}$$

This relation is also called Lagrange's law

Helmholtz's relation : If there is m number of co-axial refractive surface of indices $n_1, n_2, n_3, ..., n_m$, then final n_m, y_m and $\tan\alpha_m$ is related by

$$n_1 y_1 \tan\alpha_1 = n_m y_m \tan\alpha_m \qquad \text{... (16)}$$

Its importance lies in the fact that it expresses linear magnification in terms of angular magnification. *Note that this relationship between the magnifications is independent of the refractive indices of the intervening media.*

A system is known as telescopic system if all the cardinal (focal, principal and nodal) points lie at infinity. In this case, usually $n_1 = n_m$. Hence we get

$$\frac{y_1}{y_m}=\frac{\tan\alpha_m}{\tan\alpha_1} \qquad \text{... (17)}$$

Thus in a telescopic system angular magnification $(\tan\alpha_m/\tan\alpha_1)$ is equal to the reciprocal of linear magnification (y_m/y_1).

7.4 ABBE'S SINE CONDITION:

$$n_1 y_1 \sin\alpha_1 = n_2 y_2 \sin\alpha_2$$

This is known as Abbe's sine condition. It holds good for all rays whether paraxial or not.

When the rays are paraxial i.e., α_1 and α_2 are small we can write $\sin\alpha_1 \approx \tan\alpha_1$ and $\sin\alpha_2 \approx \tan\alpha_2$. In this case Abbe's sine condition reduces to Helmholtz's relation.

7.5 REFRACTION THROUGH THIN LENS:

7.5.1 Some important terms:

(i) Principal axis: Principal axis of a lens is the straight line joining the centres of curvature of its two bounding surfaces.

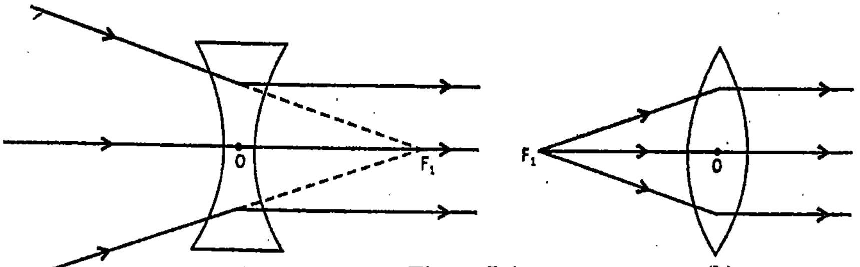
(ii) Poles: The points of intersection of the two surfaces with the principal axis are called the poles of the lens. The distance between the two poles is called the thickness of the lens (which might be considered zero for thin lens).

(iii) Principal Section: The section of a lens by a plane through its principal axis is called principal section.

(iv) Aperture: When the bounding line of the lens is circular, the diameter of this circular boundary is called aperture of the lens.

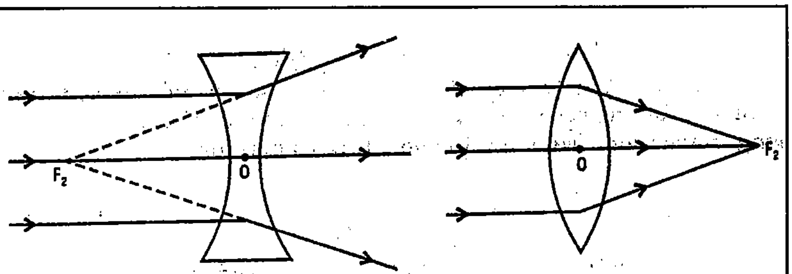
(v) Optical centre: Optical centre of a lens is a point on the principal axis such that any ray through it emerges parallel to its direction of incidence. For equi-convex lens the optical centre is situated at the mid point.

(vi) First focal point: First focal point (F_1) of a lens is a point on the principal axis such that rays moving towards that point [Figure 7.4(a)] or diverging from that point [Figure 7.4(b)] become parallel to the principal axis after refraction. The distance of F_1 from the optical centre (O) of the lens is called the first focal length (f'), when $v = \infty$.



(a) *Figure* 7.4 (b)

(vii) Second focal length: Second focal length (F_2) is a point on the principal axis such that the incident rays moving parallel to the principal axis will, after refraction, either appear to diverge from this point [Figure 7.5(a)] or actually converge to point [Figure 7.5(b)]. The distance of F_2 from the optical centre (O) of the lens is called the second focal length (f). Hence $f = \upsilon$ when $u = \infty$. According to our sign convention f is negative for a diverging lens and positive for a converging lens.



(a) *Figure* 7.5 (b)

(viii) Power: Power of a lens is defined as $P = \dfrac{1}{\text{focal length}} = \dfrac{1}{f}$ ... (19)

The unit of power is **dioptre** which is the power of a lens of one metre focal length. The power of a convex lens is considered to be positive and that of a concave lens negative.

Ex. 1. A convex lens of focal length 40 cm is in contact with a concave lens of focal length 25 cm. The power of the combination is

(a) – 1.5 D (b) – 6.5 D (c) + 6.5 D (d) + 6.67 D

[IIT-JEE : 1982]

Soln. **(a)**

$$P = P_1 + P_2 = \frac{1}{f_1(m)} = \frac{1}{0.4} - \frac{1}{0.25} = -1.5\ D$$

7.5.2 Lens formula with optical centre as origin:

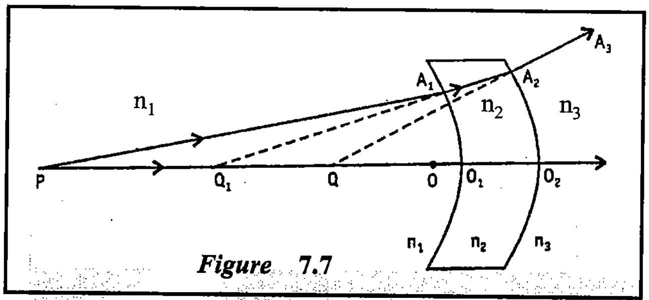


Figure 7.7

$$\frac{n_3}{v} - \frac{n_1}{u} = \frac{n_2 - n_1}{r_1} - \frac{n_2 - n_3}{r_2} \qquad \text{... (20)}$$

$$\frac{n_3}{v} - \frac{n_1}{u} = \frac{1}{F}$$

where, u = object distance, v = image distance ,

$$\frac{1}{F} = \frac{n_2 - n_1}{r_1} - \frac{n_2 - n_3}{r_2} \qquad \text{... (21)}$$

According to the definition of focal lengths,

$u = f'\left(1^{st}\text{ focal length}\right)$ when $v = \infty$;

$v = f'\left(2^{nd}\text{ focal length}\right)$ when $u = \infty$;

Putting these conditions in equation (20), we get

$$-\frac{n_1}{f'} = \frac{n_3}{f} = \frac{1}{F} \qquad \text{... (22)}$$

$$\Rightarrow \qquad \frac{f'}{f} = -\frac{n_1}{n_3} = -\frac{\text{R.I. of incident medium}}{\text{R.I. of emergent medium}} \qquad \text{... (23)}$$

Combining (20) and (22) we get, $\frac{n_3}{v} - \frac{n_1}{u} = \frac{n_3}{f}$ ... (24)

Case-I: When $n_1 = n_3$ $\Rightarrow \frac{1}{f} = -\frac{1}{f'} = \left(\frac{n_2}{n_1} - 1\right)\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$ ... (25)

And $\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$ ... (26)

Case-II: When the surrounding medium is air, $n_1 = n_3 = 1$ and if $n_2 = n$ then

$$\frac{1}{f} = \frac{1}{f'} = (n-1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \qquad \text{... (27)}$$

The power of the lens is $P = \frac{1}{F} = \frac{n_2 - n_1}{r_1} + \frac{n_3 - n_2}{r}$ ... (28)

= Sum of the powers of the refracting surfaces.

7.5.2 Conjugate foci relation with the two foci as origin – Newton's equation:

The conjugate foci relation for a thin lens with optical centre (O) as origin is given by

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f} = -\frac{1}{f'} \qquad \text{... (29)}$$

where we assume that the media on both sides of the lens are same. Therefore, $f' = -f$ and we can write

$$\frac{f}{v} - \frac{f}{u} = 1 \quad \Rightarrow \quad \frac{f}{v} + \frac{f'}{u} = 1 \quad \Rightarrow uv - uf - vf' + ff' = ff'$$

$$\Rightarrow \quad (u - f')(v - f) = ff' \qquad \text{... (30)}$$

$$UV = ff' \qquad \text{... (31)}$$

Here U and V represent the object distance and its image distance measured from the first and second foci respectively.

This equation is known as Newton's equation.

Ex. 2. When a point object is placed at a distance $u = 36$ cm from a biconvex lens, the image forms at a distance $v = 18$ cm on the other side of the lens. If the distance of the object is reduced to 12 cm, the image will be formed at

(a) 12 cm (b) ∞ (c) 6 cm (d) $-\infty$

[IIT-JAM GP : 2008]

Soln. **(b)**

$$\frac{1}{f} = \frac{1}{v} - \frac{1}{u} = \frac{1}{18} - \frac{1}{-36} = \frac{1}{12}$$

$f = 12\ cm$

Now, $u = 12\ cm$

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

$$\frac{1}{v} - \frac{1}{-12} = \frac{1}{12}$$

$v = \infty$

Ex. 3. A concave lens of glass, refractive index 1.5 has both surfaces of same radius of curvature R. On immersion in a medium of refractive index 1.75, it will behave as a

(a) convergent lens of focal length 3.5 R (b) convergent lens of focal length 3.0 R

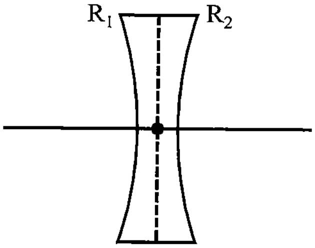
(c) divergent lens of focal length 3.5 R (d) divergent lens of focal length 3.0 R

[IIT-JEE : 1999]

Soln. **(a)**

$R_1 = -R, R_2 = +R, \mu_g = 1.5$ and $\mu_m = 1.75$

$$\therefore\ \frac{1}{f}=\left(\frac{\mu_g}{\mu_m}-1\right)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$

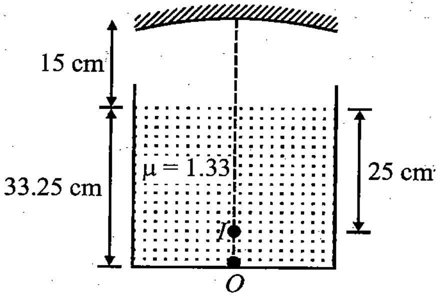


$$\frac{1}{f}=\left(\frac{1.5}{1.75}-1\right)\left(\frac{1}{-R}-\frac{1}{R}\right)=\frac{1}{3.5R}$$

$\therefore\ f=+3.5R$

Therefore, in the medium it will behave like a convergent lens of focal length 3.5R. It can be understood as, $\mu_m > \mu_g$, the lens will change its behaviour.

Ex. 4. A container is filled with water ($\mu = 1.33$) upto a height of 33.25 cm. A concave mirror is placed 15 cm above the water level and the image of an object placed at the bottom is formed 25 cm below the water level. The focal length of the mirror is



(a) 10 cm (b) 15 cm (c) 20 cm (d) 25 cm

[IIT-JEE : 2005]

oln. **(c)**

Apparent depth of the object majored from the top in a medium of refractive index n is given by $h'=\frac{h}{n}$.

Distance of object from mirror $=15+\frac{33.25}{1.33}=40$ cm.

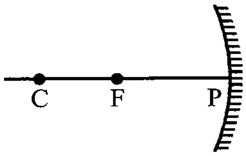
Distance of image from mirror $=15+\frac{25}{1.33}=33.8$ cm.

For the mirror, $\frac{1}{v}+\frac{1}{u}=\frac{1}{f}$

$$\therefore\ \frac{1}{-33.8}+\frac{1}{-40}=\frac{1}{f}$$

$\therefore\ f=-18.3$ cm

x. 5. In the concave mirror arrangement below C and F stand for the centre of curvature and the focal point respectively. Where on the axis would one place an object for the image to have the same dimension as the object itself?



(a) Midway between F and P (b) At F
(c) Midway between C and F (d) At C

[IISc : 2006]

Ans. (d)

7.6 EQUIVALENT LENS AND EQUIVALENT FOCAL LENGTH:

A single lens is said to be equivalent to a system of lenses when that single lens produces an image of a given object at the same place and of the same size as that formed by the system of lenses. The focal length of such a single lens is called equivalent focal length of the system of lenses.

The equivalent focal length F of two thin lenses having second focal lengths f_1 and f_2 and kept separated by a distance t in air is given by

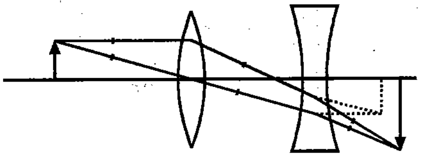
$$\frac{1}{F}=\frac{1}{f_1}+\frac{1}{f_2}-\frac{t}{f_1f_2} \qquad \text{...(32)}$$

Ex. 6. Two lenses of power +5D and –2D are in contact and are coaxially situated. The power of the combination is

(a) –3D (b) +3D (c) –7D (d) +7D **[DU 2015]**

Soln. **(b)** $P=P_1+P_2=+5D-2D=+3D$

Ex. 7. A convex lens and a concave lens, each with a focal length of 4 cm are separated by a distance of 6 cm along their axis as shown below :



An object is placed 8 cm before the first lens. Its image is

(a) real and erect (b) virtual and erect (c) real and inverted (d) virtual and inverted

[IISc : 2008]

Soln. **(c)** $u=-8\,cm,\quad f=4\,cm$

$$\frac{1}{v}-\frac{1}{u}=\frac{1}{f}$$

$$\frac{1}{v}+\frac{1}{8}=\frac{1}{4}$$

$v=8\,cm$

For concave lens

The image formed by convex lens forms a virtual object

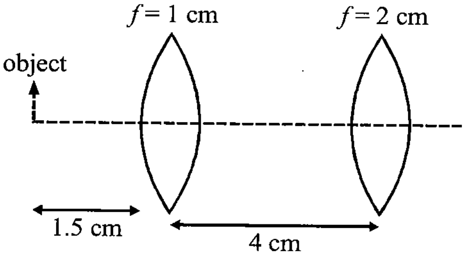
$u=2\,cm,\quad f=-4\,cm$

$$\frac{1}{v}-\frac{1}{2}=\frac{1}{-4}$$

$v=4\,cm$

Image is real and inverted.

Ex. 8. Two thin convex lenses L_1 and L_2 with focal lengths 1 cm and 2 cm respectively are separated by a distance of 4 cm along their as shown below:



An object is placed at a distance of 1.5 cm before the first lens. The ratio of the final image size to the object size is

(a) 1 (b) 2 (c) 3 (d) 4

[IISc : 2009]

Soln. **(d)**

For first lens, $u = -1.5, f = 1$ cm

From Len's maker formula

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

$$\Rightarrow \frac{1}{v} - \frac{1}{-1.5} = \frac{1}{1}$$

$$\Rightarrow \frac{1}{v} = 1 - \frac{2}{3} = \frac{1}{3}$$

So, $v = 3\,cm$

Magnification $m_1 = \frac{v}{u} = \frac{3}{-1.5} = -2$

For second lens

$u = -1\,cm, f = 2\,cm$

From Len's maker formula

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

$$\Rightarrow \frac{1}{v} - \frac{1}{-1} = \frac{1}{2}$$

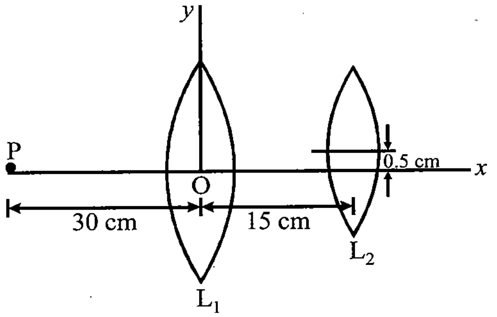
$$\frac{1}{v} = \frac{1}{2} - 1 = -\frac{1}{2}$$

Magnification, $m_2 = \frac{v}{u} = \frac{-2}{-1} = 2$

Therefore, final magnification, $m = m_1 m_2 = -2 \times 2 = -4$

So, Ratio of final image size to the object size = 4.

x. 9. Two thin lenses L_1 and L_2 of focal length 15 cm and 10 cm respectively, are kept 15 cm apart from each other. Their axes are separated by 0.5 cm as shown in figure (not to scale). If a point object P is placed on the axis of L_1 to its left at a distance of 30 cm, find the x and y coordinates (origin O) of the image formed by the combination,



[IIT-JAM : 2007]

Soln. For L_1 $\quad u_1 = -30\text{ cm},\ f_1 = 15\text{ cm};\quad$ since $\dfrac{1}{v_1} - \dfrac{1}{u_1} = \dfrac{1}{f_1} \quad \Rightarrow v_1 = 30\text{cm}$

For L_2, the image of L_1 will act as a virtual object.

$u_2 = 30 - 15 = 15$ m, $\quad f_2 = 10$ cm; $\quad$ since $\dfrac{1}{v_2} - \dfrac{1}{u_2} = \dfrac{1}{f_2} \quad \Rightarrow v_2 = 6\text{ cm}$

Magnification, $m = \dfrac{v_2}{u_2} = \dfrac{6}{15}$.

The object for L_2 lies on the principal axis of L_1, so, height of object $h = -0.5$ cm,

Magnification, $\quad m = \dfrac{h_i}{h_0} = \dfrac{6}{15} \qquad \Rightarrow h_i = \dfrac{6}{15} \times h_0 = \dfrac{6}{15} \times (-0.5) = -0.2\text{ cm}$

Negative sign means image is inverted i.e., below the principal axis of L_2.

x coordinate of image $= 15 + 6 = 21$ cm
y coordinate of image $= 0.5 - 0.2 = 0.3$ cm

So, co-ordinate of the image $(21, 0.3)$

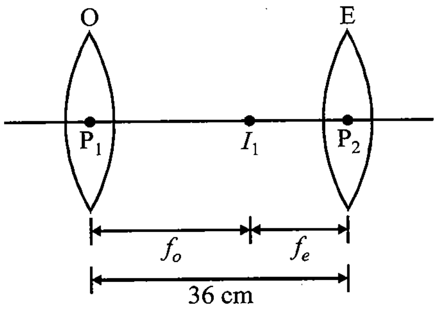
Ex. 10. An astronomical telescope has an angular magnification of magnitude 5 for far objects. The separation between the objective and the eyepiece is 36 cm and the final image is formed at infinity. The focal length f_o of the objective and the focal lenght f_e of the eyepiece are

(a) $f_o = 45$ cm and $f_e = -9$ cm $\qquad$ (b) $f_o = 50$ cm and $f_e = 10$ cm
(c) $f_o = 7.2$ cm and $f_e = 5$ cm $\qquad$ (d) $f_o = 30$ cm and $f_e = 6$ cm

[IIT-JEE : 1989]

Soln. **(d)**

Image formed by objective (I_1) is at second focus of it because objective is focussed for distant objects. Therefore, $P_1I_1 = f_o$



Further I_1 should lie at first focus of eyepiece because final image is formed at infinity.

$\therefore\ P_2I_1 = f_e$

Given $\quad P_1P_2 = 36\ cm$

$\therefore \; f_o + f_e = 36$ ...(i)

Further angular magnification is given as 5. Therefore, $\frac{f_o}{f_e} = 5$ ...(ii)

Solving equations (i) and (ii), we get $f_o = 30\,cm$ and $f_e = 6\,cm$

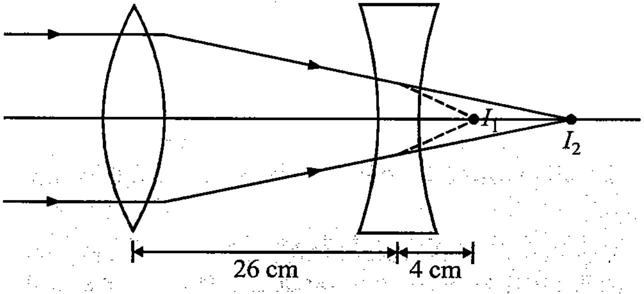
. **11.** The size of the image of an object, which is at infinity, as formed by a convex lens of focal length 30 cm is 2 cm. If a concave lens of focal length 20 cm is placed between the convex lens and the image at a distance of 26 cm from the convex lens, calculate the new size of the image.

(a) 1.25 cm (b) 2.5 cm (c) 1.05 cm (d) 2 cm

[IIT-JEE : 2003]

n. (b)

Image formed by convex lens at I_1 will act as a virtual object for concave lens. For concave lens



$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$ or $\frac{1}{v} - \frac{1}{4} = \frac{1}{-20}$ or $v = 5\,cm$

Magnification for concave lens $m = \frac{v}{u} = \frac{5}{4} = 1.25$. As size of the image at I_1 is 2 cm. Therefore, size of image at I_2 will be 2 × 1.25 = 2.5 cm.

7.7 SOME IMPORTANT PROBLEMS WITH LENSES:

7.7.1 Minimum distance between an object and its real image produced by a convex lens:

Suppose we consider a convex lens in air forming real image on its right hand side for an object placed on the left. Here object distance u is –ve, image distance v and focal length f are +ve. So by putting numerical values of u, v and f in the formula $\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$ $\Rightarrow \frac{1}{v} + \frac{1}{u} = \frac{1}{f}$ ... (33)

For the distance $D = u + v$ to be minimum, $\frac{d}{du}(u + v) = 0$ $\Rightarrow \frac{dv}{du} = -1$

Now from Equation (33), on differentiation, $-\frac{1}{v^2} - \frac{1}{u^2}\frac{du}{dv} = 0$

Therefore, for D to be minimum, $-\frac{1}{v^2} - \frac{1}{u^2}(-1) = 0$ $\Rightarrow u = v$

Therefore, from Equation (33) $u = v = 2f$

Thus $D_{\min} = u + v = 2f + 2f = 4f$

7.7.2 Condition for the formation of two real images of a given object, on the same screen, for two different positions of a convex lens and expression for focal length in terms of lens displacement:

Here the distance between the object and the screen i.e., $D = u + v$ is constant. Therefore, putting $v = D - u$ in Equation (33) we get, $\frac{1}{D-u} + \frac{1}{u} = \frac{1}{f}$

$$\Rightarrow \quad u^2 - Du + Df = 0 \qquad \text{... (34)}$$

For real distinct roots of this quadratic equation we must have $D^2 > 4Df \Rightarrow D > 4f$

So, for getting two real images on the screen the distance between the object and the screen must be greater than four times the focal length of the convex lens.

Solving Equation (34) we get for the two positions of the lens

$$u_1 = \frac{D + \sqrt{D^2 - 4Df}}{2} \text{ and } u_2 = \frac{D - \sqrt{D^2 - 4Df}}{2} \qquad \text{... (35)}$$

Therefore, displacement of the lens between the two positions is

$$x = u_1 - u_2 = \sqrt{D^2 - 4Df} \quad \Rightarrow f = \frac{D^2 - x^2}{4D} \qquad \text{... (36)}$$

7.7.3 Size of the object is the geometric mean of the sizes of the images produced by a convex lens at two positions:

Let u, v and I_1 be respectively the object distance, image distance and length of the image for the first position L_1 of the lens; while the corresponding quantities for the second position L_2 of the lens would be v, u and I_2 ($\because$ the positions of object and image are interchangeable) [See Figure 7.7]. If O be the length of the object then magnifications for the two positions are given by

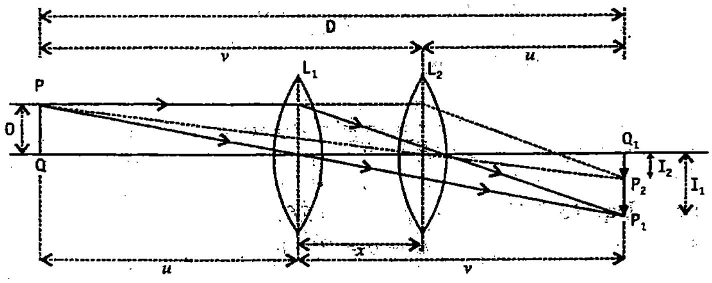


Figure 7.7

$$m_1 = \frac{I_1}{O} = \frac{v}{u} \text{ and } m_2 = \frac{I_2}{O} = \frac{u}{v} \qquad \text{... (37)}$$

$$\Rightarrow \quad \frac{I_1 I_2}{O^2} = 1 \Rightarrow O = \sqrt{I_1 I_2} \qquad \text{... (38)}$$

7.7.4 Focal length of a convex lens in terms of the linear magnifications of its images:

A convex lens produces a real image on a fixed screen at its two positions when the distance D between the object and screen is greater than $4f$. Let u_1 and v_1 be the numerical values of the object and image distances when the convex lens is nearer to the object while u_2 and υ_2 are the corresponding quantities when the lens is nearer to the screen. From the relation (33) we get,

$$\frac{1}{v_1}+\frac{1}{u_1}=\frac{1}{f} \Rightarrow 1+\frac{v_1}{u_1}=\frac{v_1}{f}$$

$$\Rightarrow \quad 1+m_1=\frac{v_1}{f} \qquad \ldots (38)$$

where $m_1=\frac{v_1}{u_1}$ = magnification for the first position.

Similarly for the second position of the lens we may write,

$$1+m_2=\frac{v_2}{f} \qquad \ldots (39)$$

From Equation (38) and (39) we get, $m_1-m_2=\frac{v_1-v_2}{f}$

$$\Rightarrow \quad f=\frac{d}{m_1-m_2} \qquad \ldots (40)$$

where $d=v_1-v_2$ is the displacement of the lens.

SOLVED EXAMPLES

A small air bubble in a sphere of glass 3 cm in diameter appears to be 1 cm from the surface when looked along a diameter. Find the true position of the bubble. Given refractive index for glass = 1.5.

ln. We have the general formula for refraction at a curved surface,

$$\frac{n_2}{v}-\frac{n_1}{u}=\frac{n_2-n_1}{r}$$

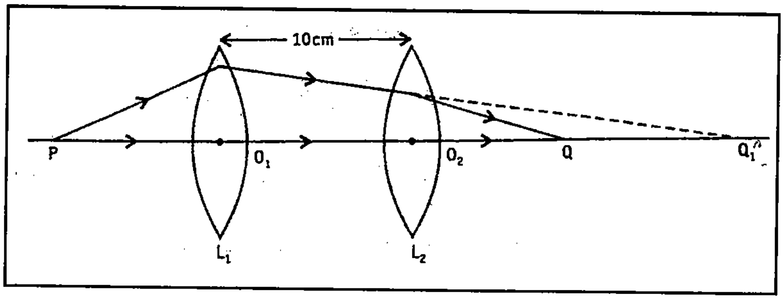
Here refraction takes place through a concave surface and $v=-1$ cm, $r=-1.5$ cm.

$$\Rightarrow \quad \frac{1}{-1}-\frac{1.5}{u}=\frac{1-1.5}{-1.5}$$

or, $u=-1.12$ cm.

Two convex lenses, each of focal length 10 cm, are situated 10 cm apart and have a common axis. An object 2 cm in height is placed on the axis at a distance of 15 cm in front of the first lens. Find the position and size of the final image.

n.



For the first lens $(L_1), u=-O_1P=-15$ cm, $f=10$ cm.

Therefore, image distance $v_1 = O_1Q_1$ is given by $\frac{1}{v_1} - \frac{1}{-15} = \frac{1}{10}$

or, $v_1 = 30$ cm

For the second lens (L_2), object distance, $u_1 = O_2Q_1 = O_1Q_1 - O_1O_2 = 30 \text{ cm} - 10 \text{ cm} = 20$ cm.

Therefore, image distance v for the second lens is given by

$$\frac{1}{\upsilon} - \frac{1}{20} = \frac{1}{10} \quad \Rightarrow \quad v = \frac{20}{3} \text{ cm} = 6\frac{2}{3} \text{ cm}$$

Now $\frac{\text{image height}}{\text{object height}} = \text{total magnification} = \frac{v_1}{u} \times \frac{v}{u_1}$

Therefore, $\frac{\text{image height}}{2} = \frac{30}{-15} \times \frac{20/3}{20} \Rightarrow \text{image height} = -1\frac{1}{3}$ cm

Thus the final image is inverted and of length $1\frac{1}{3}$ cm. It is formed at a distance of $6\frac{2}{3}$ cm from the second lens.

3. Two thin convex lenses of focal lengths 20 cm and 25 cm are placed coaxially 10 cm apart. Find the equivalent focal length.

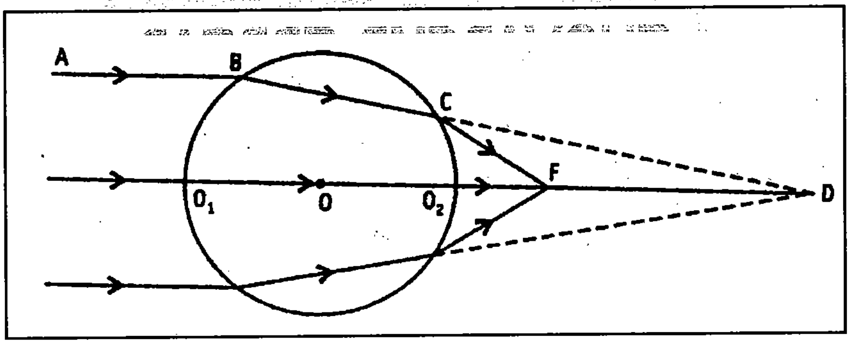
Soln. Equivalent focal length is given by, $\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2}$

Here $f_1 = 20$ cm, $f_2 = 25$ cm, d = 10 cm

Therefore, $f = \frac{f_1 f_2}{f_1 + f_2 - d} = \frac{20 \times 25}{20 + 25 - 10} \text{ cm} = 14\frac{2}{7}$ cm.

4. Consider a transparent sphere of radius 15 cm and of refractive index $n = 1.5$. Find the position of the paraxial focal point.

Soln.



For refraction at the first surface, $\frac{n}{v'} - \frac{1}{u} = \frac{n-1}{r}$

Here $u = -\infty$, r = + 15 cm, $n = 1.5$; therefore, image distance $v' = O_1 D = 45$ cm

For refraction at the second surface D is the virtual object. So object distance $= O_2D$

$= O_1D - O_1O_2 = 45 \text{ cm} - 30 \text{ cm} = 15$ cm

Radius of curvature of the second surface = – 15 cm

Therefore, in the relation, $\frac{n_2}{v} - \frac{n_1}{u} = \frac{n_2 - n_1}{r}$

we are to put $n_2 = 1$, $n_1 = 1.5$, $u = 15$ cm, $r = -15$ cm.

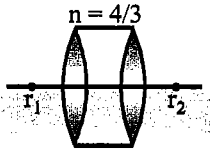
Thus we get $v = +7.5$ cm. So the distance of the paraxial focal point F from the point O_2 is $O_2F = v = +7.5$ cm.

Two thin double convex lenses of refractive index 3/2 are placed in contanct and the space between them is filled with water of refractive index 4/3. What is the focal length of the combination? Assume radius of curvature of each surface = 24 cm.

oln. Focal length of each convex lens is given by $\frac{1}{f_1} = (n-1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$

Here $n = 3/2$, $r_1 = +24$ cm, $r_2 = -24$ cm. $\Rightarrow \frac{1}{f_1} = \left(\frac{3}{2} - 1\right)\left(\frac{1}{24} + \frac{1}{24}\right)$

or, $f_1 = 24$ cm



For the concave lens formed by the water, trapped in double convex lens $n = \frac{4}{3}$, $r_1 = -24\,\text{cm}$, $r_2 = +24\,\text{cm}$.

Focal length of the double concave water lens is given by $\frac{1}{f_2} = \left(\frac{4}{3} - 1\right)\left(-\frac{1}{24} - \frac{1}{24}\right)$

or, $f_2 = -36$ cm

Therefore, equivalent focal length F is given by $\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_3} = \frac{1}{24} - \frac{1}{36} + \frac{1}{24}$

Therefore, $f = +18$ cm.

When a light ray enters a glass slab from air

(a) its wavelength decreases

(b) its wavelength increases

(c) its frequency increases

(d) neither its wavelenght nor its frequency changes

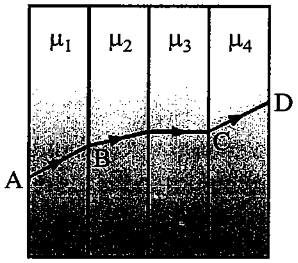
[IIT-JEE : 1980]

ln. (a)

$$\lambda = \frac{v}{f}$$

In moving from air to glass, f remains unchanged while v decreases. Hence, λ should decrease.

A ray of light passes through four transparent indices μ_1, μ_2, μ_3 and μ_4 as shown in the figure. The surfaces of all media are parallel. If the emergent ray CD is parallel to the incident ray AB, we must have



(a) $\mu_1 = \mu_2$ (b) $\mu_2 = \mu_3$ (c) $\mu_3 = \mu_4$ (d) $\mu_4 = \mu_1$

[IIT-JEE : 2001]

Soln. **(d)**

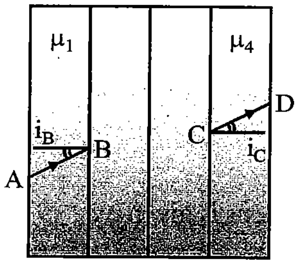
Applying Snell's law at B and C.

Product of μ and i = constant

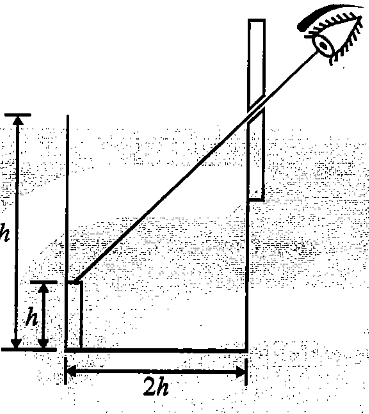
$$\text{or } \mu_1 \sin i_B = \mu_4 \sin i_C$$

But $AB \parallel CD$

$\therefore\ i_B = i_C$ or $\mu_1 = \mu_4$



8. An observer can see through a pin-hole the top end of a thin rod of height h, placed as shown in the figure. The beaker height is $3h$ and its radius h. When the beaker is filled with a liquid up to a height $2h$, he can see the lower end of the rod. Then the refractive index of the liquid is



(a) $\frac{5}{2}$ (b) $\sqrt{\frac{5}{2}}$ (c) $\sqrt{\frac{3}{2}}$ (d) $\frac{3}{2}$

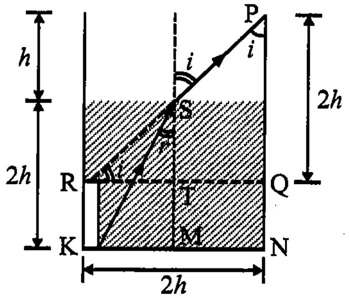
[IIT-JEE : 2002]

Soln. **(b)**

$PQ = QR = 2h \ \therefore \angle i = 45°$

$\therefore\ ST = RT = h = KM = MN$ So, $KS = \sqrt{h^2 + (2h)^2} = h\sqrt{5}$

$$\therefore\ \sin r = \frac{h}{h\sqrt{5}} = \frac{1}{\sqrt{5}}$$



$$\therefore\ \mu = \frac{\sin i}{\sin r} = \frac{\sin 45°}{1/\sqrt{5}} = \sqrt{\frac{5}{2}}$$

9. A combination of two thin convex lenses of equal focal lengths, is kept separated along the optic axes by a distance of 20 cm between them. The combination behaves as a lens system of infinite focal length. If an object is kept at 10 cm from the first lens, its image will be formed on the other side at a distance x from the second

lens. The value of x is:

(a) 10 cm (b) 20 cm (c) 6.67 cm (d) Infinite.

[IIT-JAM : 2005]

ln. (a) $\frac{1}{F}=\frac{1}{f_1}+\frac{1}{f_2}-\frac{d}{f_1 f_2}$

where d = the separation of two lenses.

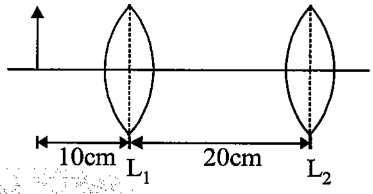
f_1 and f_2 are the focal length of lenses.

F = the focal length of combination system.

Given : $F=\infty,\ f_1=f_2=f$

$$\Rightarrow \quad \frac{1}{\infty}=\frac{1}{f}+\frac{1}{f}-\frac{d}{f^2}$$

$$\Rightarrow \quad 0=\frac{2}{f}-\frac{d}{f^2} \quad \Rightarrow f=\frac{d}{2} \quad \Rightarrow f=\frac{20}{2}=10\ cm$$



For first lens,

$v=?$

$u=-10\ cm \Rightarrow \frac{1}{v}-\frac{1}{u}=\frac{1}{f} \Rightarrow \frac{1}{v}+\frac{1}{10}=\frac{1}{10} \Rightarrow \frac{1}{v}=0 \Rightarrow v=\infty$

$f=10\ cm$

Image formed by the first lens will acts as a virtual object for second lens. Therefore,

$u=\infty$ for second lens

$$\Rightarrow \frac{1}{v}-\frac{1}{\infty}=\frac{1}{f'} \Rightarrow v=f' \Rightarrow v=10\text{ cm}$$

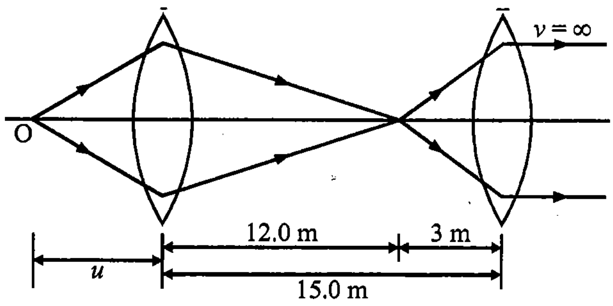
The focal lengths of the objective and the eyepiece of a compound microscope are 2.0 cm and 3.0 cm respectively. The distance between the objective and the eyepiece is 15.0 cm. The final image formed by the eyepiece is at infinity. The two lenses are thin. The distance in cm of the object and the image produced by the objective, measured from the objective lens, are respectively

(a) 2.4 and 12.0 (b) 2.4 and 15.0 (c) 2.0 and 12.0 (d) 2.0 and 3.0

[IIT-JEE : 1995]

n. (a)

Since, the final image is formed at infinity, the image formed by the objective will be at the focalpoint of the eyepiece, which is 3.0 cm. The image formed by the objective will be at a distance of 12.0 cm (= 15.0 cm − 3.0 cm) from the objective

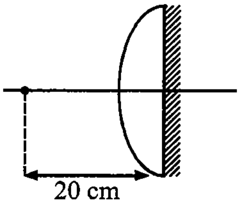


If u is the distance of the object from the objective, we have $\frac{1}{u}+\frac{1}{v}=\frac{1}{f}$

$$\Rightarrow \quad \frac{1}{u}+\frac{1}{12.0}=\frac{1}{2.0}$$

$$\Rightarrow \quad u=\frac{(12.0)\,(2.0)}{12.0-2.0}=\frac{24.0}{10.0}=2.4\text{ cm}$$

11. A point object is placed at a distance of 20 cm from a thin planoconvex lens of focal length 15 cm. The plane surface of the lens is now silvered. The image created by the system is at



(a) 60 cm to the left of the system
(b) 60 cm to the right of the system
(c) 12 cm to the left of the system
(d) 12 cm to the right of the system

[IIT-JEE : 2006]

Soln. **(c)**

Refraction from lens : $\frac{1}{v_1}-\frac{1}{-20}=\frac{1}{15} \Rightarrow v_1=\frac{15\times 20}{20-15}=+60$

i.e., first image is formed at 60 cm to the right of lens system.

Reflection from mirror

After reflection from the mirror, the second image will be formed at equal distance but in opposite side i.e., at distance of 60 cm to the left of lens system.

Refraction from lens

Now, image form by the mirror will act as a object for the lens, thus $u=+60\text{ cm}$.

$\frac{1}{v_3}-\frac{1}{60}=\frac{1}{15}$ positive direction **or** $v_3=12\text{ cm}$

Therefore, the final image is formed at 12 cm to the left of the lens system.

Chapter 8

Thick Lens And Lens Combinations

8.1 INTRODUCTION:

There are three pairs of points in an optical system which greatly simplify the description of image formation by the system. These are called cardinal points or Gauss points. For a thick lens (or a system of coaxial surfaces) we have six cardinal points: (a) two principal or unit points, (b) two nodal points and (c) two focal points.

8.1.1 Principal points:

Principal points are a pair of conjugate points on the principal axis of a thick lens, so that if an object be placed at any one of these two points an image of the same size as the object, will be formed at the other point and the image will be on the same side of the principal axis as the object. Thus lateral magnification of the image is positive unity. If planes be drawn through the two principal points perpendicular to the principal axis of the lens then these two planes will be called principal planes or unit planes. The principal planes are, therefore, conjugate planes of unit positive lateral magnification.

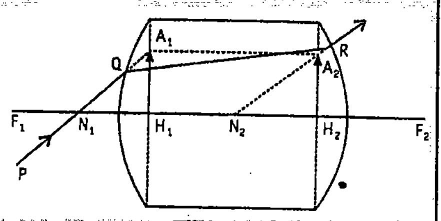


Figure 8.1

8.1.2 Nodal points:

Nodal points of a thick lens are also a pair of conjugate points on its principal axis so that if a ray be allowed to pass through or travel towards the first point (N_1) then its corresponding emergent ray from the lens will be parallel to the incident ray and will appear to diverge from or pass through the second point (N_2). If planes be drawn through the two nodal points perpendicular to the principal axis of the lens then these two planes will be called nodal planes. As the incident and emergent rays are parallel, they will make equal angles with the principal axis of the lens and hence we may say that nodal planes are conjugate planes having unit angular magnification.

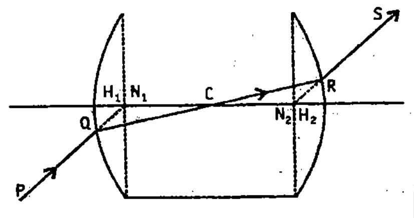


Figure 8.2

8.1.3 Focal Points:

(i) First focal point (F_1): first focal point (F_1) is the position of the object (real or virtual) whose image would be formed at infinity. The distance of first focal point (F_1) from the first principal point H_1 is called first focal length f'.

(ii) Second focal point (F_2) : the second focal point (F_2) is the position of the image (real or virtual) whose object is at infinity. The distance of the second focal point (F_2) from the second principal point H_2 is called second focal length $f\,(= H_2F_2)$.

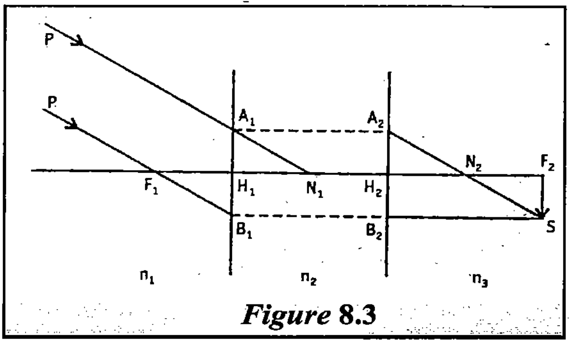


Figure 8.3

8.1.4 Relation between the distances of cardinal points:

The distance (H_1N_1) between the first principal and nodal points = distance (H_2N_2) between the second principal and nodal points.
First focal length $f'(= H_1F_1)$ = distance (F_2N_2) between the second focal and nodal points.
Second focal length $f(= H_2F_2)$ = distance (F_1N_1) between the first focal and nodal points.
The distance between two nodal points (N_1N_2) = distance between two principal points (H_1H_2).

8.2 CARDINAL POINTS OF A THICK LENS:

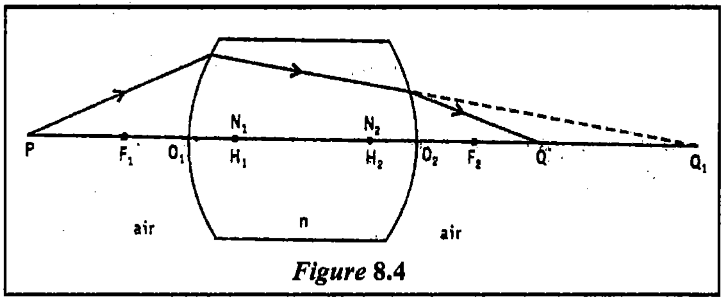


Figure 8.4

Consider a thick lens of thickness $t = O_1O_2$ placed in air. Let P be a point object and Q_1 be its image formed by refraction at the first surface of the lens whose radius of curvature is r_1. This image Q_1 serves as an object to the second surface of radius of curvature r_2 and forms the final image at Q. Let

$$u = -O_1P,\ \upsilon_1 = O_1Q_1,$$

$$O_2Q_1 = O_1Q_1 - O_1O_2 = \upsilon_1 - t \text{ and } \upsilon = O_2Q$$

Now by using the formula for refraction at a single spherical surface we get,

$$\frac{n}{\upsilon_1} - \frac{1}{u} = \frac{n-1}{r_1} \qquad \text{...(1)}$$

$$\frac{1}{\upsilon}-\frac{n}{\upsilon_1-t}=\frac{1-n}{r_2} \qquad \dots(2)$$

where n is the refractive index of the material of the lens.

Putting $y_1 = \upsilon_1/n$ and $t' = t/n$ we can rewrite the Equation (1) and (2) as

$$\frac{1}{y_1}-\frac{1}{u}=P_1;\ \frac{1}{\upsilon}-\frac{1}{y_1-t'}=P_2 \qquad \dots(3)$$

where $P_1=\frac{n-1}{r_1}$ and $P_2=\frac{1-n}{r_2}$ are respectively the refracting powers of the 1st and 2nd surfaces. Eliminating y_1 from Equation (3), we get, $\upsilon=\frac{cu+d}{au+b}$ ... (4)

where $a=P_1+P_2-P_1P_2t', b=1-P_2t', c=1-P_1t'$ and $d=-t'$

The constants a, b, c and d are interrelated by the relation bc – ad = 1

... (5)

Note that the Equation (5) is a relation between conjugate distances u and υ measured from the poles O_1 and O_2 of the 1st and 2nd surfaces of the lens respectively.

8.2.1 Principal foci (F_1 and F_2) with reference to poles:

We know that when $\upsilon=\infty, u=-O_1F_1$ = 1st focal length of the lens from O_1; but when $u=\infty, \upsilon=O_2F_2$ = 2nd focal length of the lens from O_2.

Applying these two conditions in Equation (4) we get, $au+b=0$

$$\Rightarrow \qquad u=-O_1F_1=-b/a \text{ and } \upsilon=O_2F_2=\frac{c+d/\infty}{a+b/\infty}=\frac{c}{a}$$

Thus, when distances are measured from the poles,

First focal length $(\beta')=-O_1F_1=-\frac{b}{a}$ and second focal length $(\beta)=O_2F_2=\frac{c}{a}$

8.2.2 Linear magnification:

Linear magnification at the 1st and 2nd surfaces are given by

$$m_1=\frac{\upsilon_1/n}{u} \text{ and } m_2=\frac{\upsilon}{(\upsilon_1-t)/n}=\frac{\upsilon}{y_1-t'}$$

Hence total magnification, $m=m_1m_2=\frac{y_1}{u}\cdot\frac{\upsilon}{y_1-t'}$ ... (6)

Using Equation (3) we get,

$$\frac{1}{m}=\frac{u}{y_1}\cdot\frac{y_1-t'}{\upsilon}=\frac{u}{y_1}\left[1+P_2(y_1-t')\right]=\frac{u}{y_1}(1-P_2t')+P_2u$$

$$=(1+P_1u)(1-P_2t')+P_2u=au+b$$

Now from Equation(4),

$$\upsilon(au+b)-cu-d=0 \quad \Rightarrow \quad a\upsilon(au+b)-acu-ad=0$$

$$\Rightarrow \quad a\upsilon(au+b)-acu+1-bc=0 \quad \Rightarrow \quad a\upsilon(au+b)-c(au+b)=-1$$

$$\Rightarrow \qquad (au + b)(a\upsilon - c) = -1$$

$$\Rightarrow \qquad \frac{1}{m} = au + b = \frac{1}{c - a\upsilon} \qquad \ldots (7)$$

8.2.3 Principal or unit points (H_1 and H_2):

By definition principal points or unit points are conjugate points of unit positive linear magnification i.e., $m = 1$. Therefore, the distance (p') of 1st unit H_1 from O_1 i.e., O_1H_1 will u for which $m = 1$

From Equation (7) $au + b = 1$

$$\Rightarrow \qquad p' = u = O_1H_1 = \frac{1-b}{a} \qquad \ldots (8)$$

The distance (p) of 2nd principal point H_2 from O_2 is given by υ for which $m = 1$. Therefore, from Equation(7)

$$c - a\upsilon = 1; \text{ or } p = \upsilon = -O_2H_2 = \frac{c-1}{a} \qquad \ldots (9)$$

8.2.4 Nodal points (N_1 and N_2):

The nodal points N_1 and N_2 coincide with the principal points H_1 and H_2 as the media on two sides of the lens are same (air).

8.2.5 Focal lengths (measured from principal points):

The distance of 1st focal point F_1 from the 1st principal point H_1 is the 1st focal length (f') of the thick lens. Thus

$$f' = -H_1f_1 = -(O_1H_1 + O_1F_1) = -\left[\frac{1-b}{a} + \frac{b}{a}\right] \qquad \ldots (10)$$

The distance of 2nd focal point F_2 from the 2nd principal point H_2 is the 2nd focal length (f) of the thick lens. Thus,

$$f = H_2F_2 = O_2H_2 + O_2F_2 = -\frac{c-1}{a} + \frac{c}{a} = \frac{1}{a} \qquad \ldots (11)$$

$$\Rightarrow \qquad \frac{1}{f} = a = P_1 + P_2 - P_1P_2t' = \frac{n-1}{r_1} + \frac{1-n}{r_2} + \frac{(n-1)^2}{r_1r_2}.\frac{t}{n}$$

$$= (n-1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right) + \frac{(n-1)^2.t}{r_1r_2n} \qquad \ldots (12)$$

In terms of second focal length $f = \frac{1}{a}$,

Equations (8) and (9) takes the following simple forms: $p' = fP_2t'$ and $p = fP_1t'$

8.2.6 Conjugate foci relation with distances measured from H_1 and H_2:

Let x = object distance from 1st principal point $H_1 = -H_1P = -[O_1H_1 + O_1P]$

$$= u - \frac{1-b}{a} \qquad \ldots (13)$$

and y = image distance from 2nd principal point H_2

$$= H_2Q = H_2O_2 + O_2Q = \upsilon - \frac{c-1}{a} \qquad \text{... (14)}$$

$$\Rightarrow \qquad \frac{1}{y} - \frac{1}{x} = a\left[\frac{1}{a\upsilon - c + 1} - \frac{1}{au + b - 1}\right] \qquad \text{... (15)}$$

Using Equation (4), $\dfrac{1}{a\upsilon - c + 1} = \dfrac{1}{a.\dfrac{cu+d}{au+b} - c + 1} = \dfrac{au+b}{au+b-1}$ ... (16)

Substituting this in the equation (15) we get,

$$\frac{1}{y} - \frac{1}{x} = a = \frac{1}{f} \qquad \text{... (17)}$$

This equation is identical with the conjugate foci relation of thin lens. Thus we conclude that when the distances of conjugate points of a thick lens are measured from principal points, its formula becomes identical to that of a thin lens.

8.2.7 Conjugate foci relation with distance measured from F_1 and F_2:

Let the object distance measured from 1st focus F_1

$$U = -PF_1 = -(O_1P - O_1F_1) = u + \frac{b}{a}$$

Image distance measured from 2nd focus F_2,

$$V = F_2Q = O_2Q - O_2F_2 = \upsilon - \frac{c}{a}$$

$$\therefore \quad UV = \left(u + \frac{b}{a}\right)\left(\upsilon - \frac{c}{a}\right) = \frac{1}{a^2}(au+b)(a\upsilon - c) = -\frac{1}{a^2} = ff' \qquad \text{... (18)}$$

This is known as Newton's equation.

8.3 MAGNIFICATIONS PRODUCED BY A THICK LENS:

8.3.1 Linear magnification (m):

When the distances of conjugate points are measured from principal points of a thick lens, the thick lens formula becomes identical to thin lens formula. Hence linear magnification produced by an thick lens air is,

$$m = \frac{y}{x} \qquad \text{... (19)}$$

The Equation (17) gives $\dfrac{1}{x} = \dfrac{f-y}{yf}$; and $y = -\dfrac{xf}{x+f}$

$$m = y \times \frac{f-y}{yf} = -\frac{y-f}{f} = -\frac{V}{f} \qquad \text{... (20)}$$

Also, $$m = \frac{xf}{a+f} \times \frac{1}{x} = \frac{f}{x+f} = \frac{f}{x-f'}$$

$$= \frac{f}{-PH_1 + F_1H_1} = \frac{f}{-PF_1} = \frac{f}{U} \qquad \text{... (21)}$$

8.3.2 Longitudinal magnification (*l*):

If δV be the length of the image of an object of length δU, both measured along the axis, then longitudinal magnification of the image is given by,

$$l = \lim \frac{\delta V}{\delta U} = \frac{dV}{dU} \qquad \text{... (22)}$$

Differentiating the relation $UV = ff'$ we get,

$$V + U.\frac{dV}{dU} = 0 \qquad \Rightarrow l = \frac{dV}{dU} = -\frac{V}{U}$$

Now $$m^2 = \frac{V}{f} \times \frac{f}{U} = -\frac{V}{U} = l \qquad \text{... (23)}$$

Thus longitudinal magnification (*l*) is equal to the square of the linear magnification.

8.3.3 Angular magnification (*M*):

If the incident ray and its corresponding emergent ray from the lens make angles α_1 and α_2 with the axis of the lens then angular magnification of the image is given by

$$M = \frac{\tan \alpha_2}{\tan \alpha_1} \qquad \text{... (24)}$$

By Helmholtz's law we get, $n_1 y_1 \tan \alpha_1 = n_2 y_2 \tan \alpha_2$.

Here $n_1 = n_2 = 1$ and hence angular magnification

$$M = \frac{\tan \alpha_2}{\tan \alpha_1} = \frac{y_1}{y_2} = \frac{1}{m} \qquad \text{... (25)}$$

Thus angular magnification is equal to the reciprocal of its linear magnification (*m*).

8.3.4 Relation between *m*, *l* and *M*:

From the relations (22) and (24) we get

$$Ml = \frac{1}{m} \times m^2 = m \quad \Rightarrow \frac{Ml}{m} = 1 \qquad \text{... (26)}$$

SOLVED EXAMPLES

1. Find the positions of cardinal points of a thick double convex lens in air whose surfaces have radii of curvature 5 cm and 4 cm, the thickness of the lens being 3 cm and refractive index $n = 1.5$

Soln. Power of the 1st surface $P_1 = \frac{n-1}{r_1} = \frac{1.5-1}{5} = \frac{1}{10}\text{cm}^{-1}$

Power of the 2nd surface $P_2 = \frac{n-1}{r_2} = \frac{1-1.5}{-4} = \frac{1}{8}\text{cm}^{-1}$

$$a = P_1 + P_2 - p_1 P_2 t' = \frac{1}{10} + \frac{1}{8} - \frac{1}{10} \times \frac{1}{8} \times \frac{3}{1.5} = 0.2 \text{ cm}^{-1}$$

$$b = 1 - P_2 t' = 1 - \frac{1}{8} \times \frac{3}{1.5} = 0.75$$

$$c = 1 - P_2 t' = 1 - \frac{1}{10} \times \frac{3}{1.5} = 0.8$$

Locations of the principal points (H_1, H_2) and focal points (F_1, F_2) are given as,

$$p' = O_1H_1 = \frac{1-b}{a} = \frac{1-0.75}{0.2} = +1.25 \text{ cm}$$

$$p = -O_2H_2 = \frac{c-1}{a} = \frac{0.8-1}{0.2} = -1 \text{ cm}$$

$$\beta' = -O_1F_1 = -\frac{b}{a} = -\frac{0.75}{0.2} = -3.75 \text{ cm}$$

$$\beta = O_2F_2 = \frac{c}{a} = \frac{0.8}{0.2} = +4 \text{ cm}$$

$$f' = -H_1F_1 = -\frac{1}{a} = -\frac{1}{0.2} = -5 \text{ cm}$$

$$f = H_2F_2 = \frac{1}{a} = +5 \text{ cm}$$

As the lens is surrounded by air nodal points coincide with principal points.

2. Find the cardinal points of a transparent glass sphere ($n = 1.5$) of radius 1 cm placed in air. Where will sunlight be focussed by such a sphere?

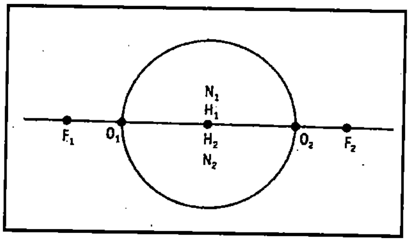
Soln. Power of 1st surface $P_1 = \frac{n-1}{r} = \frac{1.5-1}{1} = \frac{1}{2}$ cm^{-1}

Power of surface $P_2 = \frac{1-n}{-r} = \frac{1-1.5}{-1} = \frac{1}{2}$ cm^{-1}

Thickness $t = 2$ cm, reduced thickness $t' = 2/1.5$ cm

$$\Rightarrow \quad a = P_1 + P_2 - P_1P_2t' = \frac{1}{2} + \frac{1}{2} - \frac{1}{2} \times \frac{1}{2} \times \frac{2}{1.5} = \frac{2}{3} \text{ cm}^{-1}$$

$$b = 1 - P_2t' = 1 - \frac{1}{2} \times \frac{2}{1.5} = \frac{1}{3}, \; c = \frac{1}{3}$$



Therefore, positions of principal points (H_1, H_2) and focal points (F_1, F_2) are given as,

$$p' = O_1H_1 = \frac{1-b}{a} = \frac{1-\frac{1}{3}}{\frac{2}{3}} = +1 \text{ cm} \quad \text{and} \quad p = -O_2H_2 = \frac{c-1}{a} = \frac{\frac{1}{3}-1}{\frac{2}{3}} = -1 \text{ cm}$$

$$\beta' = -O_1F_1 = -\frac{b}{a} = -\frac{\frac{1}{3}}{\frac{2}{3}} = -0.5 \text{ cm and } \beta = O_2F_2 = \frac{c}{a} = \frac{\frac{1}{3}}{\frac{2}{3}} = +0.5 \text{ cm}$$

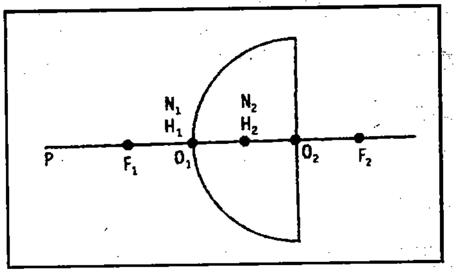
$$f' = -H_1F_1 = -\frac{1}{a} = -\frac{3}{2} = -1.5 \text{ cm and } f = H_2F_2 = \frac{1}{a} = \frac{3}{2} = +1.5 \text{ cm}$$

3. Consider a hemispherical lens of radius of curvature 12 cm and refractive index 2. A small object is placed 36 cm from its curved surface. Find the cardinal points and the position and nature of the image.

Soln. Here $r_1 = r = 12$ cm, $r_2 = \infty$, $P_1 = \frac{n-1}{r} = \frac{2-1}{12} = \frac{1}{12}$ cm^{-1}, $P_2 = 0$

$$t' = \frac{t}{n} = \frac{r}{n} = \frac{12}{2} = 6 \text{ cm.}$$

$$\Rightarrow \quad a = P_1 + P_2 - P_1P_2t' = \frac{1}{12} \text{ cm}^{-1}, \text{b} = 1, \text{c} = 1 - P_1t' = 1 - \frac{1}{12} \times 6 = \frac{1}{2}$$



Now, $p' = O_1H_1 = \frac{1-b}{a} = 0$, $p = -O_2H_2 = \frac{c-1}{a} = -6$ cm

$$\beta' = -O_1F_1 = -\frac{b}{a} = -12 \text{ cm, } \beta = O_2\,F_2 = \frac{c}{a} = +6 \text{cm}$$

$$f' = H_1F_1 = -\frac{1}{a} = -12 \text{ cm, } f = H_2F_2 = \frac{1}{a} = +12 \text{ cm}$$

Object distance measured from H_1, $x = -PH_1 = -PO_1 = -36$ cm

Therefore, the image is formed at 18 cm to the right of H_2 in Figure above. The magnification is $\frac{y}{x} = \frac{18}{-36} = -\frac{1}{2}$

4. An object P is placed at a distance of 40 cm in front of a convex lens (O_1) of focal length 20 cm. Behind the convex lens a cancave lens (O_2) of focal length 10 cm is placed at a distance of 15 cm from the convex lens. Find the cardinal points of the lens system and the position of final image from the centre (O_2) of second lens. What is the magnification of final image?

Soln. $P_1 = \frac{1}{f_1} = \frac{1}{20}$ cm^{-1}, $P_2 = \frac{1}{f_2} = -\frac{1}{10}$ cm^{-1}, $t = 15$ cm,

$$a = P_1 + P_2 - P_1P_2t = \frac{1}{20} - \frac{1}{10} + \frac{1}{20} \times \frac{1}{10} \times 15 = \frac{1}{40}$$

$$b = 1 - P_2t = 1 + \frac{15}{10} = \frac{5}{2},\ c = 1 - P_1t = 1 - \frac{1}{20} \times 15 = \frac{1}{4}$$

$$\beta' = -O_1F_1 = -\frac{b}{a} = -100 \text{ cm}, \ \beta = O_2F_2 = \frac{c}{a} = +10 \text{ cm}$$

$$f' = -H_1F_1 = -\frac{1}{a} = -40 \text{ cm}, \ f = H_2F_2 = \frac{1}{a} = +40 \text{ cm},$$

$$p' = O_1H_1 = \frac{1-b}{a} = -60 \text{ cm}, \ p = -O_2H_2 = \frac{c-1}{a} = -30 \text{ cm}$$

When the distances are measured from principal points we get,

$$\frac{1}{y} - \frac{1}{x} = \frac{1}{f} \text{ and } m = \frac{y}{x}$$

Here $x = H_1P = -[H_1O_1 + O_1P] = -[-60 + 40] = +20$ cm

$f = H_2F_2 = +40$ cm. Hence y is given by

$$\frac{1}{y} - \frac{1}{20} = \frac{1}{40}; \text{ or, } y = +\frac{40}{3} \text{ cm}$$

If Q is the image position, then $y = H_2Q = \frac{40}{3}$ cm

$$\Rightarrow \quad O_2Q = H_2Q - O_2H_2 = \frac{40}{3} - 30 = -\frac{50}{3} \text{ cm}$$

Mabnification $m = \frac{y}{x} = \frac{2}{3}$.

Consider a system of two lenses separated by a distance in air. If p' represents the distance of 1st principal plane from 1st lens and p represents the distance of 2nd principal plane from 2nd lens, show that $p'f_2 + pf_1 = 0$.

oln. $p = O_1H_1 = \frac{1-b}{a}$. Now $\frac{1}{a} = f$ and $b = 1 - P_2t = 1 - \frac{t}{f_2}$

$$\Rightarrow \quad p' = \frac{f.t}{f_2} \qquad \text{... (1)}$$

$$\Rightarrow \quad p = -O_2H_2 = \frac{c-1}{a}.$$

$$\text{Since } c = 1 - P_1t = 1 - \frac{t}{f_1}, \ p = -\frac{ft}{f_1} \qquad \text{... (2)}$$

Combining Equations (1) and (2) we get,

$$p'f_2 + pf_1 = 0$$

Chapter 9

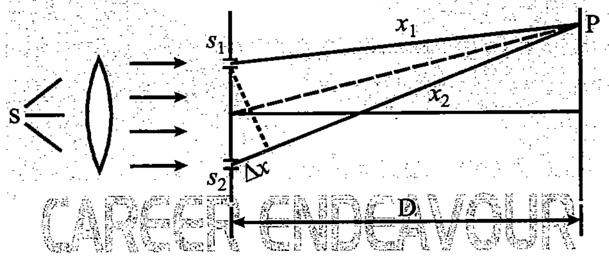
Interference of Light

9.1 INTRODUCTION:

When monochromatic waves of light from two sources proceed almost in the same direction and superpose at a point either in same or in opposite phase. Then the intensity of light at that point will be maximum or minimum according to as the waves meet the point in the same or in opposite phase. This phenomenon is known as interference of light. This phenomenon requires for its explanation that light must have a wave nature.

9.2 THEORY OF INTERFERENCE:

Suppose a narrow slit S is illuminated with monochromatic light of wavelength λ. There are two other slits S_1 and S_2 equidistant from S. According to Huygens' principle cylindrical wavelets spread out from the slit S. As $SS_1 = SS_2$, the wavelets will reach S_1 and S_2 at the same time instant. Hence new secondary wavelets will start from S_1 and S_2 and diverge towards the screen. These wavelets start with equal phase and hence S_1 and S_2 may be considered as coherent sources.



Let us represent the complex disturbances constituting light waves at S_1 by

$$y = a_1 e^{i\omega t} \qquad \text{... (1)}$$

In travelling from S_1 to any point P on the screen the phase of the wave change by $\frac{2\pi}{\lambda} S_1P = \frac{2\pi}{\lambda} x_1 = kx_1$.

Hence at the instant t the complex disturbance at P due to light from S_1 will be

$$y_1 = a_1 e^{i(\omega t - kx_1)} \qquad \text{... (2)}$$

Similarly, the disturbance at P due to light from S_2 may be represented by,

$$y_2 = a_2 e^{i(\omega t - kx_2)} \qquad \text{... (3)}$$

The resultant disturbance at P, according to the principal of superposition, is given by

$$y = y_1 + y_2 = e^{i\omega t}\left[a_1 e^{-ikx_1} + a_2 e^{-ikx_2}\right]$$

$$= e^{i\omega t}\left[a_1\left(\cos kx_1 - i\sin kx_1\right) + a_2\left(\cos kx_2 - i\sin kx_2\right)\right]$$

$$= e^{i\omega t}\left[a - ib\right] = e^{i\omega t}\sqrt{a^2 + b^2}.e^{-i\phi}; \quad \phi = \tan^{-1}\frac{b}{a}$$

where $a = a_1 \cos kx_1 + a_2 \cos kx_2$, $b = a_1 \sin kx_1 + a_2 \sin kx_2$

Therefore, $y = Ae^{i(\omega t - \phi)}$ ... (4)

where $A^2 = a^2 + b^2 = a_1^2 + a_2^2 + 2a_1 a_2 \cos k(x_2 - x_1)$ ... (5)

Here A is the amplitude of the resultant disturbance. Hence the intensity of light at P is given as

$$I \propto A^2 \quad \Rightarrow \quad I = a_1^2 + a_2^2 + 2a_1 a_2 \cos\delta$$

$$= I_1 + I_2 + 2\sqrt{I_1}\sqrt{I_2}\cos\delta \quad \text{... (6)}$$

where $\delta = k.(x_2 - x_1) = \frac{2\pi}{\lambda}(x_2 - x_1)$ is the phase difference between the interfering waves. This phase difference depends on the path difference $x_2 - x_1$ and hence on the position of the point P.

Case-I : Maximum intensity (constructive interference)

when $\delta = 2m\pi$, or, $x_2 - x_1 = 2m\frac{\lambda}{2}$; $m = 0, 1, 2, 3, \ldots,$

$$I_{\max} = I_1 + I_2 + 2\sqrt{I_1}\sqrt{I_2} \qquad [\because \cos 2m\pi = 1]$$

$$= \left(\sqrt{I_1} + \sqrt{I_2}\right)^2$$

Thus when the path difference of the point P from the two sources is even multiple of $\lambda / 2$ the intensity of light at that point becomes maximum and we get a bright band there. This is known as constructive interference.

Case-II : Minimum intensity (destructive interference)

when $\delta = (2m+1)\pi$ or $x_2 - x_1 = (2m+1)\lambda / 2$; $m = 0, 1, 2, 3, \ldots$

$$I_{\min} = I_1 + I_2 - 2\sqrt{I_1}\sqrt{I_2} \qquad [\because \cos(2m+1)\pi = -1]$$

$$= \left(\sqrt{I_1} - \sqrt{I_2}\right)^2$$

when the path difference of the point P from the two sources is odd multiple of $\lambda / 2$ the intensity of light at the point is minimum and we get a nearly dark band there. This is known as destructive interference.

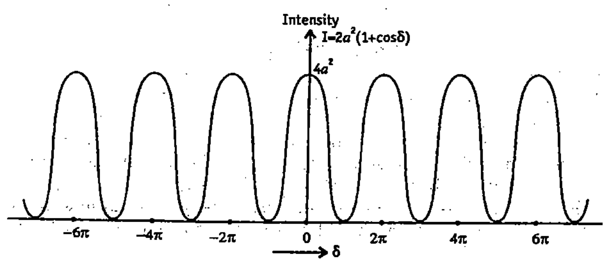


Figure 9.1

(i) No destruction of light energy occurs in interference of light. What happens is merely a redistribution of energy. The energy which disappears from the dark regions actually goes to the bright regions. The average value of energy over a number of fringes can be shown to be the same as if the interference effects were absent. Taking the constant of proportionality to be one, the intensity I can be expressed from Equation .... as

$$I = a_1^2 + a_2^2 + 2a_1 a_2 \cos\delta \qquad \text{... (7)}$$

where a_1 and a_2 are the amplitudes of the waves and δ is the phase difference.

$$\Rightarrow \quad I_{average} = \frac{\int_0^{2\pi} I d\delta}{\int_0^{2\pi} d\delta} = a_1^2 + a_2^2 = \text{sum of intensities of individual waves.} \qquad \dots (8)$$

Thus we can say that there is no destruction of energy or the principle of conservation of energy is not violated in the phenomenon of interference.

(ii) When light waves from two separated source meet at a point, the point will be bright or dark according to as the waves meet in the same or opposite phases. This phase difference depends on the initial phase difference between the sources and the optical path difference of the point from the two sources. The path difference of the point in question remains constant but as two different source cannot be coherent, the initial phase difference goes on changing very rapidly with time. As a result at every point there is rapid alternation of brightness and darkness and hence we get general illumination.

Ex. 1. Two coherent monochromatic light beams of intensities I and $4I$ are superposed. The maximum and minimum possible intensities in the resulting beam are

(a) $5I$ and I (b) $5I$ and $3I$ (c) $9I$ and I (d) $9I$ and $3I$

[IIT-JEE : 1988]

Soln. **(c)**

$$I_{\max} = \left(\sqrt{I_1} + \sqrt{I_2}\right)^2 = \left(\sqrt{4I} + \sqrt{I}\right)^2 = 9I$$

$$I_{\min} = \left(\sqrt{I_1} - \sqrt{I_2}\right)^2 = \left(\sqrt{4I} - \sqrt{I}\right)^2 = I$$

Ex. 2. Two beams of light having intensities I and $4I$ interfere to produce a fringe pattern on a screen. The phase difference between the beams is $\pi/2$ at point A and π at point B. Then the difference between resultant intensities at A and B is

(a) $2I$ (b) $4I$ (c) $5I$ (d) $7I$

[IIT-JEE : 2001]

Soln. **(b)**

$$I(\phi) = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\phi \qquad \dots\text{(i)}$$

Here, $I_1 = I$ and $I_2 = 4I$

At point $A, \phi = \frac{\pi}{2}$ $\quad \therefore\ I_A = I + 4I = 5I$

At point $B, \phi = \pi$ $\quad \therefore\ I_B = I + 4I - 4I = I$

$\therefore\ I_A - I_B = 4I$

Ex. 3. Two beams of monochromatic light with intensities I_1 and I_2 interfere constructively to produce an intensity of 100 mW. If one of the beams is shifted in phase by 60 degrees, the intensity reduces to 84 mW. Then I_1 and I_2 are

(a) 64 mW and 4 mW (b) 36 mW and 16 mW

(c) 10 mW and 9 mW (d) 92 mW and 8 mW

[IISc : 2011]

Soln. **(a)**

After interference of two beams, the intensity of beams are given by

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\delta$$

For constructive interference, $\delta = 2n\pi$

$$100 = I_1 + I_2 + 2\sqrt{I_1 I_2} \quad \text{...(i)}$$

If one of the beam is shifted in phase by 60°, i.e. $\delta = 60°$

$$84 = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos \pi/3 = I_1 + I_2 + \sqrt{I_1 I_2} \quad \text{...(ii)}$$

By (i) divide (ii),

$$\Rightarrow \quad \sqrt{I_1 I_2} = 16$$

Putting this in (ii)

$$I_1 + I_2 = 84 - \sqrt{I_1 I_2} = 68$$

So, $I_1 + I_2 = 68$ and $\sqrt{I_1 I_2} = 16$

Which are satisfied by $I_1 = 64\ mW$ and $I_2 = 4\ mW$

9.3 WIDTH AND SHAPE OF INTERFERENCE FRINGES:

9.3.1 Fringe width:

Let us consider two coherent sources S_1 and S_2 which are sending monochromatic light of wavelength λ (Figure (9.2) The point C on the screen is equidistant from S_1 and S_2 and hence the waves from S_1 and S_2 arrive at C at the same time. If the phase difference between the two coherent sources is zero the waves from S_1 and S_2 will meet at C in same phase and will produce a bright band known as central bright band. Let the mth bright band be formed at P whose distance from C is x_m (say). Now in Figure -9.2, D is the separation between the sources and the screen; d is the separation between the coherent sources.

$$(S_2P)^2 = D^2 + \left(x_m + \frac{d}{2}\right)^2 \qquad \Rightarrow S_2P = D\left[1 + \left(\frac{x_m + \frac{d}{2}}{D}\right)^2\right]^{\frac{1}{2}} \quad \text{... (9)}$$

Since usually, $D >> d$, $S_2P \approx D\left[1 + \frac{1}{2}\left(\frac{x_m + \frac{d}{2}}{D}\right)\right]$ ... (10)

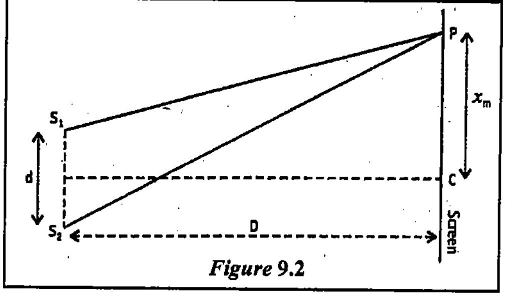


Figure 9.2

Similarly, $$S_1P \approx D\left[1+\frac{1}{2}\left(\frac{x_m-\frac{d}{2}}{D}\right)^2\right] \qquad \text{... (11)}$$

Now for mth order bright fringe at P,

Path difference $S_2P - S_1P = m\lambda$

Using Equation (10) and (11) we get, $\frac{x_m.d}{D} = m\lambda$

$$x_m = m\frac{\lambda D}{d} \qquad \text{... (15)}$$

Similarly, the distance of (m + 1)th bright fringe from C will be

$$x_m = (m+1)\lambda\frac{D}{d} \qquad \text{... (16)}$$

Thus the distance between two consecutive bright bands would be

$$\beta = x_{m+1} - x_m = \frac{\lambda D}{d} \qquad \text{... (17)}$$

where β is known as fringe width.

If P be the position of mth order dark band then

$$x_m = (2m+1)\frac{\lambda}{2}.\frac{D}{d} \text{ and } x_{m+1} = (2m+2+1)\frac{\lambda}{2}.\frac{D}{d}$$

Hence $$\beta = x_{m+1} - x_m = \frac{\lambda D}{d} \qquad \text{... (18)}$$

Equations (17) and (18) show that the distances (β) between two consecutive bright or dark bands are equal.

9.3.2 Shape of interference fringes:

An idea regarding the shape of interference fringes can be obtained by finding the locus of points having a constant path difference from the sources (slits) S_1 and S_2. Referring to Figure -9.3 we choose the mid point O between the slits as the origin of a coordinate system. Let the x-axis be along OX and y-axis be perpendicular to the plane containing the slits. For any point $P(y, x)$ we can write

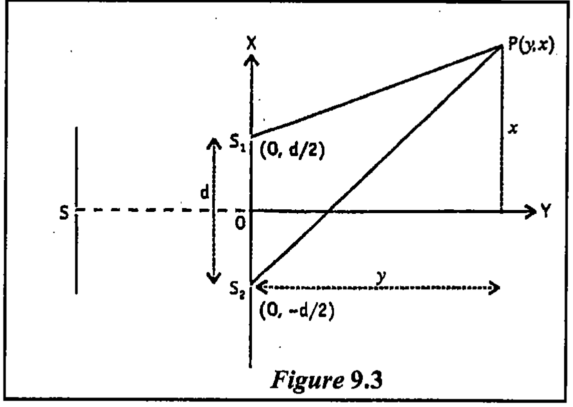


Figure 9.3

$$(S_1p)^2 = y^2 + \left(x - \frac{d}{2}\right)^2$$

and $$(S_2p)^2 = y^2 + \left(x + \frac{d}{2}\right)^2 \qquad \text{... (19)}$$

where 'd' is the separation between the the slits.

Let $\Delta = S_2P - S_1P$

$$\Rightarrow \quad S_2P = \Delta + S_1P$$

$$\Rightarrow \quad S_2P = \left[y^2 + \left(x + \frac{d}{2}\right)^2\right]^{\frac{1}{2}} = \Delta + \left[y^2 + \left(x - \frac{d}{2}\right)^2\right]^{\frac{1}{2}}$$

On squaring and re-arranging

$$\Rightarrow \quad 2\Delta\left[y^2 + \left(x - \frac{d}{2}\right)^2\right]^{\frac{1}{2}} = 2xd - \Delta^2$$

Again, squaring and re-arranging, we get

$$\Rightarrow \quad \frac{x^2}{\Delta^2/4} - \frac{y^2}{\left(d^2 - \Delta^2\right)/4} = 1 \qquad \text{...(20)}$$

Thus the loci of points of constant path difference Δ in xy plane are hyperbolae with S_1 and S_2 as foci on x-axis. The eccentricities of the hyperbolae are given by

$$e = \frac{d}{\Delta} \qquad \text{.....(21)}$$

In optical experiments the path difference $\Delta \sim 10^{-8}$ cm and $d \sim 10^{-2}$ cm. Therefore, e is very high and as a consequence the hyperbolae become practically straight lines given by

$$y = \pm\left[\frac{d^2 - \Delta^2}{\Delta^2}\right]^{\frac{1}{2}} .x \qquad \text{...(22)}$$

If two coherent point source are used the loci of maxima of different orderes will be a system of confrontal hyperboloids.

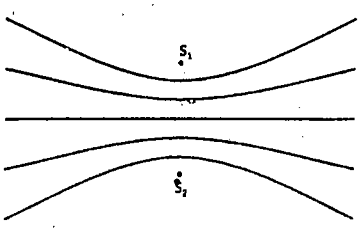
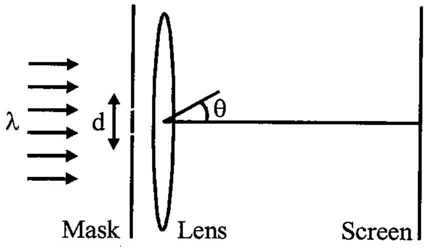


Figure 9.4

4. In an optical telescope arrangement shown below the opaque mask covers the lens completely except for two very small holes, separated by a distance d. The screen is kept at the focal plane of the lens, and a collimated monochromatic radiation with wavelength λ is incident on the mask from left. The angular dependence of the intensity distribution $f(\theta)$ at the screen is given by,



(a) $f(\theta) = 2\pi d \sin\theta/\lambda$

(b) $f(\theta) = 1 + \cos(2\pi d \sin\theta/\lambda)$

(c) $f(\theta) = \text{constant}$

(d) $f(\theta) = 1 - \cos(2\pi d \sin\theta/\lambda)$

[IISc : 2006]

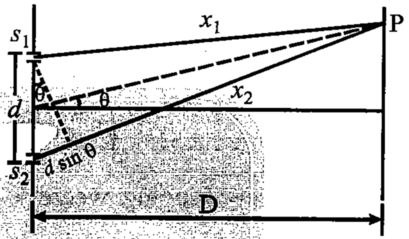
Soln. **(a)** The intensity at any point P on the screen will depend on the plase difference between two beams which interfere at P.

The phase difference

$$\delta = \frac{2\pi}{\lambda} \times \text{path difference}$$

$$= \frac{2\pi}{\lambda} \cdot [S_2P - S_1P]$$

$$= \frac{2\pi}{\lambda} \times d \sin\theta = \frac{2\pi d \sin\theta}{\lambda}$$



So, $f(\theta) = \dfrac{2\pi d \sin\theta}{\lambda}$.

9.4 INTERFERENCE WITH WHITE LIGHT AND COLOUR EFFECT:

The distance of the *m*th bright fringe from the central one is given by

$$d \sin\theta = d \cdot \frac{x_m}{D} = n\lambda \Rightarrow x_m = m \cdot \frac{\lambda D}{d} \qquad \dots (23)$$

where d is the separation between the two coherent sources and D is the distance of the screen from the source. As x_m is a function of wavelength λ, So the fringes of different colours will be in step only for the central band ($m = 0$). In this case $x_m = 0$ and bright fringe of all wavelengths will coincide and the central band will be white with white light. For higher order bright fringes ($m > 0$) x_m will be greater for a light of longer wavelength and less for a light of shorter wavelength. As the wavelength λ_r for red light is longer than the wavelength λ_υ for violet we infer that all bright bands, excepting the central bright band, will be coloured in which red will be in the outermost position while violet will be in the innermost position.

Ex. 5. If the monochromatic source of light in Young's double-slit experiment is replaced by a white-light source, then:

(a) No fringes will be formed **[DU 2014]**

(b) There will be white fringes only

(c) There will be a coloured central fringe only

(d) There will be a central white fringe flanked on either side by a few coloured fringes

Soln. **(d)** This is because at centre there will be maxima and at the either sides, the diffracted rays of different wavelength will interfere.

x. 6. Interference and diffraction of light supports its: **[DU 2014]**

(a) Wave nature
(b) Quantum nature
(c) Transverse nature
(d) Electromagnetic character

oln. (a)

9.5 CONDITIONS FOR OBSERVABLE INTERFERENCE PATTERN:

(i) The two beams of light which interfere must be coherent.

(ii) The interfering waves must have the same frequency. Also, their amplitudes must be equal or very nearly equal.

(iii) The original source must be monochromatic or very nearly monochromatic. If the light source is heterogeneous the optical path difference between the interfering beams must be kept very small.

(iv) The two interfering beams must propagate along the same direction or must intersect at a very small angle.

(v) For interference with polarised light the waves must be in the same state of polarisation.

x. 7. In a Young's double slit experiment, 12 fringes are observed to be formed in a certain segment of the screen when light of wavelength 600 nm is used. If the wavelength of light is changed to 400 nm, number of fringes observed in the same segment of the screen is given by

(a) 12 (b) 18 (c) 24 (d) 30

[IIT-JEE : 2001]

ln. (b)

Fringe width, $\beta_1 = \frac{\lambda_1 D}{d}$, $\lambda_1 = 600$ nm and $\beta_2 = \frac{\lambda_2 D}{d}$, $\lambda_2 = 400$ nm

Let n be the number of the fringe then,

$$12\beta_1 = n\beta_2$$

$$\Rightarrow n = 12\frac{\beta_1}{\beta_2} = 12\frac{\lambda_1}{\lambda_2} = 12\frac{600}{400} = 18.$$

x. 8. In Young's double slit experiment intensity at a point is (1/4) of the maximum intensity. Angular position of this point is

(a) $\sin^{-1}\left(\frac{\lambda}{d}\right)$ (b) $\sin^{-1}\left(\frac{\lambda}{2d}\right)$ (c) $\sin^{-1}\left(\frac{\lambda}{3d}\right)$ (d) $\sin^{-1}\left(\frac{\lambda}{4d}\right)$

[IIT-JEE : 2005]

ln. (c)

$$I = I_{\max}\cos^2\left(\frac{\phi}{2}\right)$$

$$\therefore \frac{I_{\max}}{4} = I_{\max}\cos^2\frac{\phi}{2} \Rightarrow \cos\frac{\phi}{2} = \frac{1}{2} \text{ or } \frac{\phi}{2} = \frac{\pi}{3} \Rightarrow \phi = \frac{2\pi}{3} \qquad \text{...(i)}$$

Also, $\phi = \left(\frac{2\pi}{\lambda}\right)\Delta x$ where $\Delta x = d\sin\theta$

Substituting in equation (i), we get

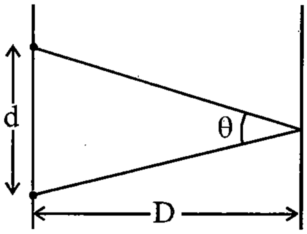
$\sin\theta = \frac{\lambda}{3d}$ or $\theta = \sin^{-1}\left(\frac{\lambda}{3d}\right)$ **Ans.**

Ex. 9. Two coherent plane waves of light of equal amplitude, and each of wavelength $20\pi\times10^{-8}\,\text{m}$, propagating at an angle of $\frac{\pi}{1080}$ radian with respect to each other, fall almost normally on a screen. The fringe-width (i mm) on the screen is:

(a) 0.108 (b) 0.216 (c) 1.080 (d) 2.160

[IIT-JAM : 2008]

Soln.



From figure, $\tan\theta \simeq \theta = \frac{d}{D}$

Fringe width, $\Delta x = \frac{D\lambda}{d} = \frac{\lambda}{\theta} = \frac{20\,\pi\times10^{-8}}{\frac{\pi}{1080}} = 21600\times10^{-8}\text{m} = 0.216\text{ mm}$

Ex. 10. A Young's double slit apparatus has a slit separation of 0.5 mm and a screen placed at a distance of 100 cm from the slits. When illuminated by a 632 nm Helium-Neon laser, the position of the first intensity minimum with respect to the central maximum will be **[IISc : 2009, 2012]**

(a) 31.6 μm (b) 63.2 μm (c) 316 μm (d) 632 μm

Soln. **(d)**

Slit separation $d = 0.5\text{ mm} = 5\times10^{-4}\text{ m}$

Distance between slit and screen $D = 100\text{ cm} = 1\text{ m}$

Wavelength $\lambda = 632\text{ nm} = 632\times10^{-9}\text{ m}$

Position of minima is given by

$$x_n = \left(n+\frac{1}{2}\right)\frac{\lambda}{d} \qquad n = 0,1,2,\ldots$$

First intensity minima will correspond to $n = 0$

$$x_0 = \frac{\lambda}{2d} = \frac{632\times10^{-9}\times1}{2\times5\times10^{-4}} = 632\ \mu m$$

Ex. 11. In a Young;s double-slit experiment, the two slits are such that the dark fringes are perfectly dark (i.e., the intensity vanishes at these points). Now, we double the width of one of the two slits. What will be the effect on the interference pattern?

(a) The fringe pattern as well as contrast between the dark and bright fringes will change

(b) The fringe pattern will not change but the contrast between the dark and bright fringes will change.

(c) The fringe pattern will change but the contrast between the dark and bright fringes will not change.

(d) There will be no change in either the interference pattern or the contrast between the dark and bright fringes.

[JNU : 2009]

Soln. Width of the slit are proportional to the amplitude of interfering waves when both slits are same, amplitudes are same, i.e.,

$a_1 = a_2 = a$ (say)

$\Rightarrow I_1 = I_2 = I$ (say)

$\Rightarrow I_{\text{dark}} = \sqrt{I_1} - \sqrt{I_2} = 0$

when one slits are double,

$a_1 = a \quad \Rightarrow I_1 = I$ and $a_2 = 2a \Rightarrow I_2 = 4I \quad \left[I \propto a^2\right]$ then,

$I_{\text{dark}} = \sqrt{I_1} - \sqrt{I_2} = \sqrt{I} - 2\sqrt{I} \neq 0$

i.e., when width of slits are double, dark bands now are not completely dark. Also white bands will have more contrast. As fringe pattern depends upon wavelength only, therefore, it will remain same.
Correct option is (b).

9.6 TWO CLASSES OF INTERFERENCE:

(i) Division of wavefront:

Optical devices which divide the incident wavefront into two parts by reflection, refraction or diffraction and thereby give rise to two coherent interfering beams come under the division of wavefront class. In order to maintain spatial coherence it is essential to use narrow sources in these cases. The formation of fringes by **Biprism, Lloyd's single mirror, Billet's divided lens etc.** belong to this category.

(ii) Division of amplitude:

Optical devices which divide the amplitude of incident light wave into two or more parts by partial reflection and refraction and there by give rise to two or more coherent interfering beams of light come under the division of amplitude class. Here we require to use broad source of light. As the interference effects corresponding to different points of the source are superposed here we get brighter bands. Since a large section of the wavefront is used diffraction effects are minimised. The formation of fringes **by thin films, Newton's ring, Michelson's interferometer, Fabry-Perot interferometer etc. belong to this category.**

9.7 FRESNEL'S BIPRISM:

A biprism is essentially two prisms each of very small refracting angle (~ 30′) placed based to base. The incident wavefront is divided into two parts and suffer separate refractions through the two halves of the biprism. The two refracted wavefronts appear to diverge from two virtual sources S_1 and S_2.

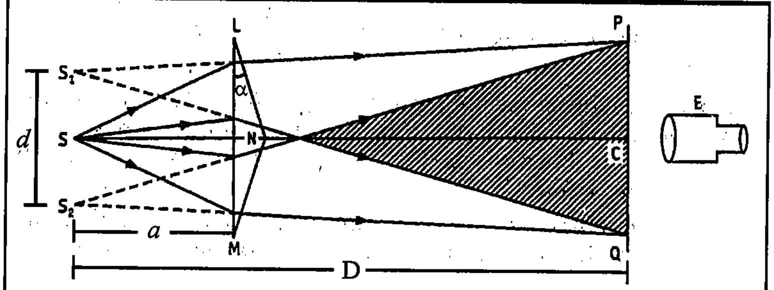


Figure 9.4

As in young double slit experiment,

The position of bright fringe $x_n = \dfrac{n\lambda D}{d}$, $n = 0, 1, 2, 3, \ldots$

The position of dark fringe $x_n = \dfrac{(2n+1)}{2}\,\dfrac{\lambda D}{d}$, $n = 1, 2, 3, \ldots$

9.7.1 Determination of unknown wavelength:

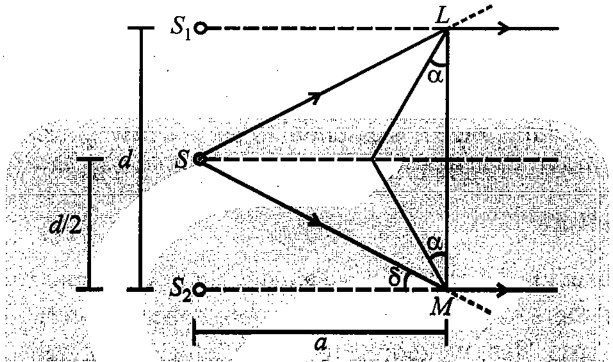
Fringe width β is given by $\beta = \dfrac{\lambda D}{d}$

where, D = distance between the slit and the screen.
d = distance between the two virtual coherent sources S_1 and S_2.

Unknown wavelength, $\lambda = \dfrac{d\beta}{D}$ ... (24)

9.7.2 Measurement of the acute angle of biprism:

It is evident that the deviation of each of the rays *SL* and *SM* after their passage through the two thin prisms having acute angles at *L* and *M* respectively, is given by $\delta = \angle SMS_2 = \angle SLS_1$.

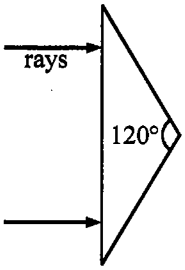


Since biprism is thin, deviation produced in a ray is given by
$\delta = (n-1)\alpha$
where *n* is refractive index of the prism.

From figure, $\tan\delta \simeq \delta = \dfrac{d/2}{a}$

$\Rightarrow d = 2a(n-1)\alpha$ ... (25)

Ex. 12. An isosceles prism of angle 120° has a refractive index 1.44. Two parallel rays of monochromatic light enter the prism parallel to each other in air as shown. The rays emerging from the opposite face



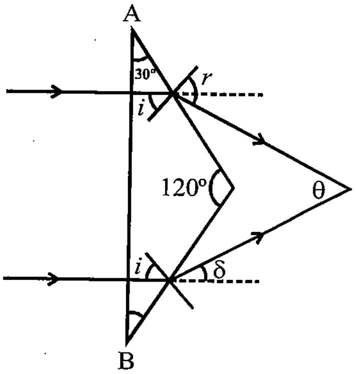
(a) are parallel to each other
(b) are diverging
(c) make an angle $2[\sin^{-1}(0.72) - 30°]$ with each other
(d) make an angle $2\sin^{-1}(0.72)$ with each other

[IIT-JEE : 1995]

Soln. **(c)**

The diagramatic representation of the given problems is shown in figure.

From figure it follows that $\angle i = \angle A = 30°$



From Snell's law, $n_1 \sin i = n_2 \sin r$ **or** $\sin r = \dfrac{1.44 \sin 30°}{1} = 0.72$

Now, $\angle\delta = \angle r - \angle i = \sin^{-1}(0.72) - 30°$

$\therefore\ \theta = 2(\angle\delta) = 2\{\sin^{-1}(0.72) - 30°\}$

9.7.3 Measurement of the thickness of a thin film:

Let S_1 and S_2 be the two virtual coherent sources which are producing interference fringes on the screen so that C is the position of the central bright band of zero optical path difference i.e. $S_1C = S_2C$.

If a thin film of thickness t be introduced into one of the path (say, S_1P) of the interfering rays, then the position of the central fringe will be shifted from C to P (say), so that the optical path S_1P is again equal to the optical path S_2P. The time taken by light in going from S_1 to P and from S_2 to P will be equal. Thus,

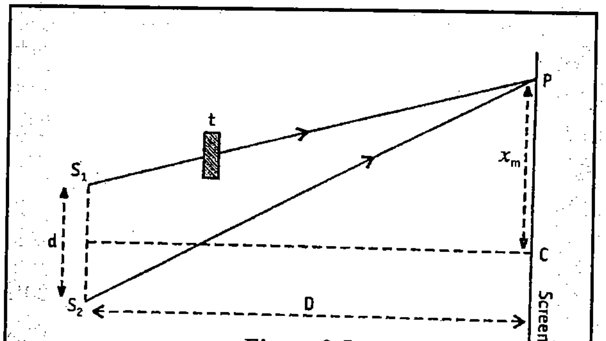


Figure 9.5

$$\frac{S_2P}{c} = \frac{S_1P - t}{c} + \frac{t}{v} \qquad \text{... (26)}$$

$$\Rightarrow \qquad S_2P - S_1P = (n-1)t \qquad \text{... (27)}$$

where c and v are the velocity of light in air and in the film; and $n = c/v$ is the refractive index of the material of the film, if P is the position originally occupied by the mth order fringe then

$$S_2P - S_1P = m.\lambda \qquad \text{... (28)}$$

Therefore, from Equation (27) and (28) we get,

$$(n-1)t = m\lambda \qquad \text{... (29)}$$

The lateral shift of the central fringe of zero optical path difference is given as

$$x_m = CP = m\beta \qquad \text{... (30)}$$

where $\beta = \dfrac{\lambda D}{d}$ is the fringe width

From Equation (29) and (30) we get

$$(n-1)t = \frac{x_m}{\beta}.\lambda \qquad \Rightarrow \quad t = \frac{x_m.\lambda}{\beta(n-1)}$$

$$\Rightarrow \qquad t = \frac{x_m}{(n-1)} \cdot \frac{d}{D} \qquad \ldots (31)$$

Ex. 13. In the ideal double-slit experiment, when a glass-plate (refractive index 1.5) of thickness t is introduced in the path of one of the interfering beams (wavelength λ), the intensity at the position where the central maximum occurred previously remains unchanged. The minimum thickness of the glass-plate is

(a) 2λ (b) $\frac{2\lambda}{3}$ (c) $\frac{\lambda}{3}$ (d) λ

[IIT-JEE : 2002]

Soln. **(a)**

Path difference due to slab should be integral multiple of λ i.e. $\Delta x = n\lambda \Rightarrow (\mu - 1)t = n\lambda \quad n = 1, 2, 3, \ldots$

or $t = \frac{n\lambda}{\mu - 1}$.

For minimum value of t, $n = 1$

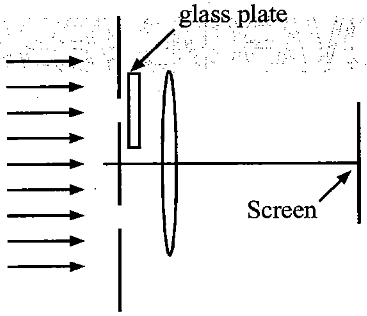
$$\therefore \quad t = \frac{\lambda}{\mu - 1} = \frac{\lambda}{1.5 - 1} = 2\lambda$$

Ex. 14. Young's double slit experimental measures
(a) temporal coherence of light
(b) spatial coherence of light
(c) polarization of light source
(d) beat frequency of light source

[IIT-JAM GP : 2011]

Ans. (b)

Ex. 15. In a Young's double-slit experiment, the central maximum occurs at zero path difference between the interfering beams. A glass plate of thickness 0.005 mm is inserted to cover the upper slit completely (see figure). If the slit separation is 0.1 mm and the diffraction pattern is observed at the focal plane of the converging lens (focal length 200 mm), which of the following statements on the position of the central maximum is true ? [refractive index of glass is 1.5]



(a) The central maximum shifts up by 15 mm
(b) The position of the central maximum remains unchanged
(c) The central maximum shifts down by 15 mm
(d) The central maximum shifts down by 1.5 mm

[IISc : 2007]

Soln. From equation 31, central maxima will shift by distance

$$x_m = \frac{(\mu - 1)\, t}{d} \cdot D = \frac{0.5 \times 0.005 \times 200}{0.1} = 5 \text{ mm}$$

9.8.LLOYD'S SINGLE MIRROR:

9.8.1 Introduction:

In Lloyd's single mirror arrangement (Figure 9.6) light from a narrow slit S_1 illuminated with a monochromatic light is partly incident at a grazing angle on a metallic mirror M_1M_2 while the rest reaches the screen AB directly. The reflected light appears to diverge from a virtual source S_2. Hence S_1 and S_2 act as coherent sources and interference fringes are formed in the region of overlapping EF.

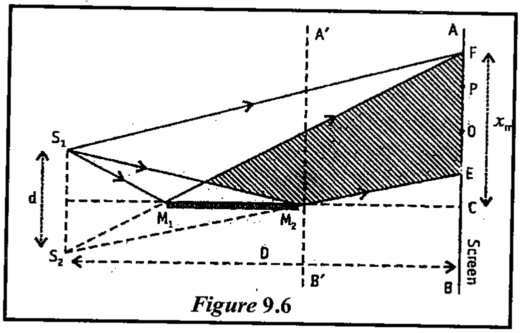


Figure 9.6

Ray reflected by the mirror suffer a phase change of π, therefore, effective path changes by $\lambda/2$.

Condition of maxima

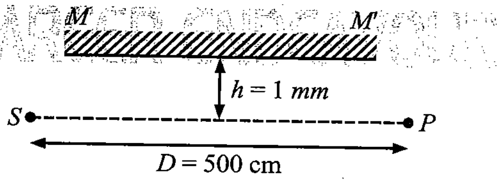
$$\Delta x = d\sin\theta + \frac{\lambda}{2} = n\lambda \Rightarrow d\sin\theta = (2n-1)\frac{\lambda}{2},\ n = 1,2,3,....$$

Condition of minima

$$\Delta x = d\sin\theta + \frac{\lambda}{2} = (2n+1)\frac{\lambda}{2} \Rightarrow d\sin\theta = n\lambda,\ n = 0,1,2,3,...$$

Fringe width is same as in YDSE, i.e., $\beta = \dfrac{\lambda D}{d}$

k. 16. A simple setup to study interference of light (Lloyd's Mirror) consists of a monochromatic point source S of wavelength $\lambda = 5000$ Å and a detector P separated from it by a distance of D = 500 cm. A plane mirror MM′ is placed above at height h which is initially adjusted to be $h = 1$mm so that bright fringe is observed at P.



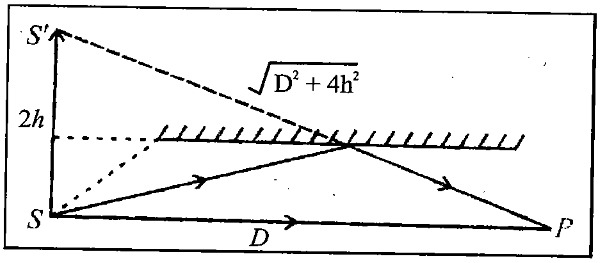
By what distance must the mirror be moved vertically so that the next bright fringe is observed at P ?

(a) 25.0 μm (b) 12.5 μm (c) 6.25 μm (d) 3.12 μm

[IISc : 2008]

ln. (None)

Since, ray reflected from the mirror suffer a additional path change $\dfrac{\lambda}{2}$.



Path difference $\Delta x = S'P - SP = \sqrt{D^2 + 4h^2} - D + \frac{\lambda}{2} = n\lambda$.

If this path difference corresponds to nth bright fringe. Let h' be the new height which corresponds to $(n+1)^{\text{th}}$ bright fringe

$$\Delta x' = \sqrt{D^2 + 4h'^2} - D + \frac{\lambda}{2} = (n+1)\lambda$$

$$\Delta x' - \Delta x = \sqrt{D^2 + 4h'^2} - \sqrt{D^2 + 4h^2} = \lambda$$

$$\Rightarrow \; D\left[1 + \frac{4h'^2}{D^2}\right]^{1/2} - D\left[1 + \frac{4h^2}{D^2}\right]^{1/2} = \lambda$$

$$\Rightarrow \; D\left[1 + \frac{1}{2}\cdot\frac{4h'^2}{D^2}\right] - D\left[1 + \frac{1}{2}\cdot\frac{4h^2}{D^2}\right] = \lambda \quad \Rightarrow \quad \frac{2(h'^2 - h^2)}{D} = \lambda$$

$$h'^2 = \frac{\lambda D}{2} + h^2 = \frac{5\times10^{-7}\times5}{2} + 10^{-6} = 2.25\times10^{-6}$$

$$h' = 1.5\times10^{-3} = 1.5\, mm$$

So, mirror should be moved vertically be a distance of $1.5 - 1 = 0.5\, mm$.

9.8.2 Difference between fresnel's biprism and Lloyed's mirror fringes:

(i) In biprism experiment fringes are formed on both sides of the central fringe, whereas in Lloyd's mirror arrangement less than half of the fringes are obtained on one side of the central line.

(ii) In biprism the central fringe is bright whereas in Lloyd's mirror it is dark.

(iii) In Fresnel's biprism the separation (d) between every pair of corresponding points of the coherent sources is same and hence fringe width is same for all parts of the source. But in Lloyd's mirror due to lateral inversion in the mirror d is different for different pair of coherent point sources. Consequently in Lloyd's mirror fringe width is not same for all parts of the source slit.

9.9 PHASE CHANGE ON REFLECTION: STOKE'S TREATMENT:

Suppose a light wave of amplitude 'a' is incident on an interface of two media as shown in Figure-9.7. Let r and t represent the fractions of the amplitude of the incident wave which are respectively reflected above and refracted below the surface. Hence the amplitudes of the reflected and refracted waves for an incident wave of amplitude a, will be 'ar' and 'at' respectively.

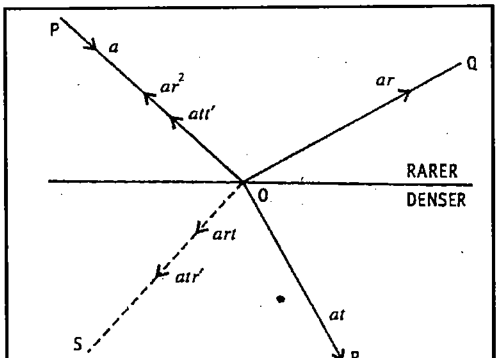


Figure 9.7

If there is no absorption of energy, the wave motion is a strictly reversible phenomenon. Now if the reflected wave along *OQ* having the amplitude atr' be reversed, we get one reflected wave of amplitude $ar \cdot r$ anlong *OP* and another refracted wave of amplitude *ar't* along *OS*. Similarly if the wave '*at*' along *OR* having the amplitude at be reversed, we get a reflected wave of amplitude *atr'* along *OS* and a refracted wave of amplitude *att'* along *OP* where *r* and *t* represent respectively the amplitude reflection and transmission coefficients when the wave is incident from lower medium. Since the resultant effect must consist of a wave along *OP* of amplitude a we have,

$$att' + ar^2 = a \qquad \dots (32)$$

$$\Rightarrow r^2 + tt' = 1$$

The total amplitude along *OS* is zero, i.e.,

$$art + atr' = 0$$

$$\Rightarrow r' = -r \qquad \dots (33)$$

The negative sign indicates that one of the rays has a positive displacement and the other has a negative displacement.

When reflection occurs from an interface beyond which the medium has a higher refractive index, the reflected wave undergoes a phase change of π. But there is no phase change occurs when reflection takes place from denser to rarer interface.

9.10 INTERFERENCE PHENOMENA IN THIN FILM, BASED ON DIVISION OF AMPLITUDE:

Beautiful colours are often exhibited when a beam of white light from an extended source is reflected from a thin film of oil floating on water or from a soap bubble. These phenomena can be explained on the basis of interference of light waves reflected from the upper and the lower surfaces of the film.

9.10.1 Interference in reflected light:

Let *AB* and *CD* be the bounding surfaces of a thin wedge-shaped film enclosing an angle α (Figure 9.8) Suppose a ray *PQ* of monochromatic light of wavelength λ is incident on the film. This ray will be partly refelcted along *QO* from the front surface and partly refracted along *QR*. The ray along *QR* after being partly reflected from the back surface *CD*, emerges along *SO* and meets the reflected ray *QO* at *O*.

The rays *QO* and *SO* are derived from the same incident ray *PQ* and hence are coherent. They combine to produce interference phenomena. Let us now calculate the phase difference between these reflected beams.

Draw $SN \perp^r QO$, $SN_1 \perp^r QR$ and $SLM \perp^r CD$. Produce *QR* and *SL* to meet at *M*. The paths of the two reflected beams (*QO* and *SO*) which are going to meet at *O*, will be equal from the dotted line *SN* upto *O* (for they are very close to each other). Hence the path difference of the two reflected interfering beams would be,

$$l = n\left(QN_1 + N_1R + RS\right) - QN \qquad \dots (34)$$

where n = refractive index of the film.

From the Figure 9.8,

RS = *RM* and *SL* = *LM* = *d* = thickness of the film at S. By Snell's law we get,

$$n = \frac{\sin i}{\sin r} = \frac{QN/QS}{QN_1/QS} \qquad \Rightarrow QN = nQN_1 \qquad \dots (35)$$

Thus Equation(35) now reduces to,

$$l = n\left(N_1R + RS\right) = n\left(N_1R + RM\right) = nN_1M$$

or, $$l = 2nd\cos\left(r - \alpha\right) \qquad \dots (36)$$

In addition to this path difference, there is an extra phase difference of π equivalent to a path difference

$\pm\lambda/2$, caused by reflection at *Q*,

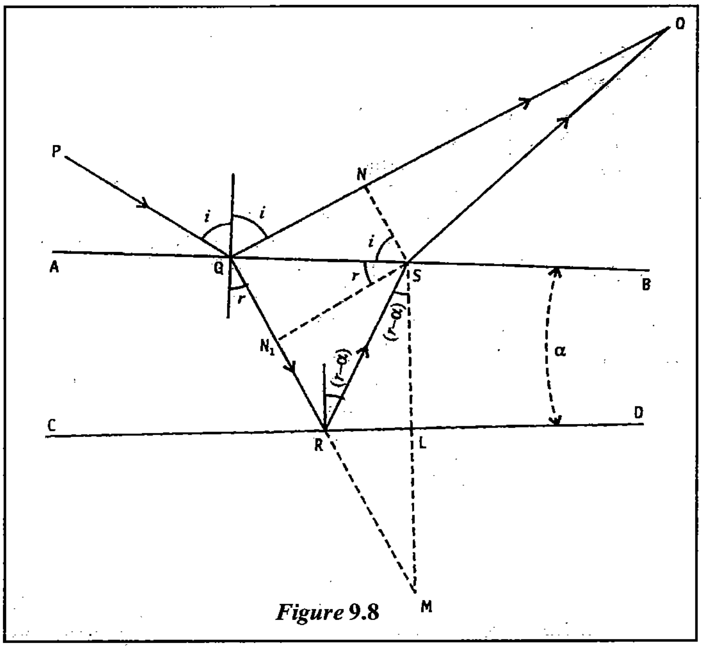


Figure **9.8**

from the surface backed by denser medium. The reflection at *R*, from the surface backed by rarer medium, will not cause any change of phase.

Hence the total path difference between the two reflected rays is,

$$l = 2nd\cos(r-\alpha) \pm \frac{\lambda}{2} \qquad \text{...(37)}$$

- **Conditions for maxima and minima:**

For maxima of brightness at O,

$$2nd\cos(r-\alpha) \pm \frac{\lambda}{2} = n\lambda$$

$$\Rightarrow \qquad 2nd\cos(r-\alpha) = (2m+1)\frac{\lambda}{2} \text{ i.e. odd multiples of } \frac{\lambda}{2} \qquad \text{... (38)}$$

where m = 0, 1, 2, 3, ...

Similarly, for minima of brightness at O,

$$2nd\cos(r-\alpha) = m\lambda \qquad \text{... (39)}$$

- **Fringes with monochromatic light:**

(i) when a parallel beam of monochromatic light is incident on a film of varying thickness, n, λ and r are all constants. Hence different order fringes will be obtained for different thickness (d) of the film. At the thin edge of the film, where d is practically zero, the path difference (= 2nd cos *r*) of the two reflected rays inside the film is zero. **The only path difference between these two reflected rays is $\lambda/2$, caused by reflection from the surface AB of the film. Hence this thin edge would be dark not only for a light of one colour but for lights of all colours.**

As the thickness, d of the film increases, the order number m of the fringe also increases. Thus a fringe of a given order number will be the locus of the points where film thickness d is remaining same. If the film surfaces are perfectly plane, we get straight fringes parallel to the line of intersection of the surfaces of the film.

(ii) **When the film is extremely thin, so that d is practically zero for all parts of the film,** then the only path difference between the two reflected rays is $\lambda/2$ and **the film surface will appear perfectly dark even with white light.**

(iii) If the film surfaces are plane-parallel (i.e., $\alpha = 0$) and a parallel beam of monochromatic light is incident on its surface obliquely, then n, λ and d are all constants but r will very with the inclination of the parallel beam with the surface. Hence the film surface will be uniformly bright or dark according as the value of 'r' satisfies the conditions of brightness and darkness given in the relations (38) and (39) respectively. If the parallel beam is incident on such a film normally then n, d and λ are all constants and $\cos r = 1$. Hence the film surface will be uniformly bright or dark according as the thickness d satisfies the conditions of brightness or darkness.

- **Fringes with white light-colours of thin film:**

When a parallel beam of white light is incident on a thin wedge-shaped film, the values of n, λ and r will be different for lights of different colours. At the thinner edge of the film, thickness (d) is practically zero and this edge will be perfectly dark for light of all colours **following to the introduction of path difference of $\lambda/2$ by reflection from the surface *AB* of the film.** As $\lambda_v < \lambda_r$, the first order bright fringe of violet light will be formed at a smaller thickness of the film while the corresponding bright fringe of red light will be formed at the greater thickness. Thus we get differently coloured fringes at different thickness and these fringes are called fringes of equal chromatic order.

Beyond this point, we get another thickness where the condition of brightness will be simultaneously satisfied by two or more colours and here we get a coloured band due to the overlapping of these coloured bright fringes. When the thickness of the film is sufficient, the overlapping of differently coloured bright fringes goes to such an extent that we get uniform illumination. Thus we explain the colouration of thin films.

9.10.2 Interference in transmitted light:

A similar interference phenomenon is seen to occur with the rays transmitted through the film and emerging on the side *CD* of it (Figure 9.9). One part of the ray *QR* incident on the boundary *CD*, will emerge out of *CD* in the direction RO_1. Another part of it will also emerge out of CD along TO_1 after two successive internal reflections at *R* and *S*. These two internal reflections will cause no change of phase and hence the phase difference of these two emergent rays will be purely determined by their path difference.

In figure $NO_1 \approx TO_1$ and hence the path difference of the two transmitted interfering beams would be,

$$l = n(RN_1 + N_1S + ST) - RN \quad \text{... (40)}$$

Now, TN_1, TN and TLM are drawn perpendiculars on RS, RO_1 and AB respectively. RS and TL are produced to meet at M. **From the geometry of the Figure 9.9 we get, $ST = SM$ and $LT = LM = d =$ thickness of the film at L.**

By Snell's law we get, $n = \dfrac{\sin i_1}{\sin(r-\alpha)} = \dfrac{RN/RT}{RN_1/RT}$

or, $\quad RN = nRN_1 \quad$ ... (41)

Thus Equation (40) reduces to,

$$l = n(N_1S + SM) = nN_1M = 2nd\cos(r - 2\alpha) \quad \text{... (42)}$$

Therefore, for maxima of brightness,

$$2nd\cos(r - 2\alpha) = 2m\frac{\lambda}{2} = m\lambda \quad \text{... (43)}$$

For minima of brightness,

$$2nd\cos(r - 2\alpha) = (2m+1)\frac{\lambda}{2} \quad \text{... (44)}$$

For a film of very small wedge angle α or for a parallel film $(\alpha = 0)$, then we note from the Equation (38), (39), (43) and (44) that the condition for maxima in the transmitted pattern corresponds to the condition for minima in the reflected pattern and vice-versa. Thus we may conclude that the fringes observed with the reflected and transmitted light are complementary to each other.

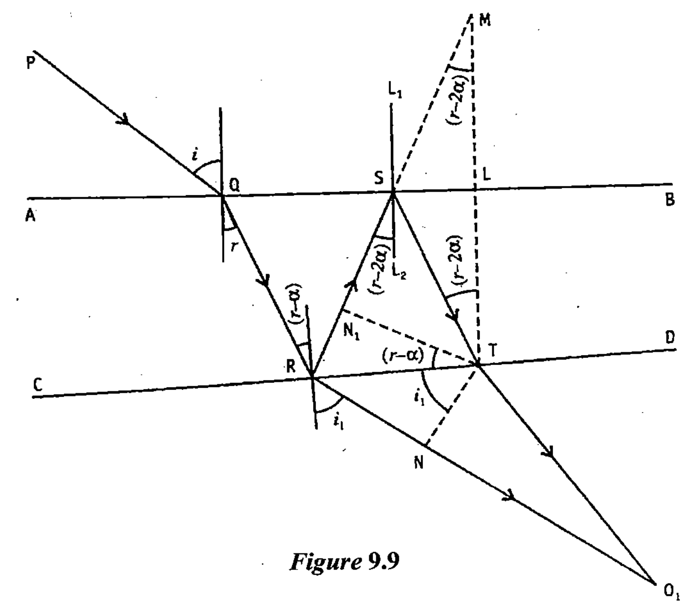


Figure 9.9

9.10.3 Necessity of using a broad source and thin film:

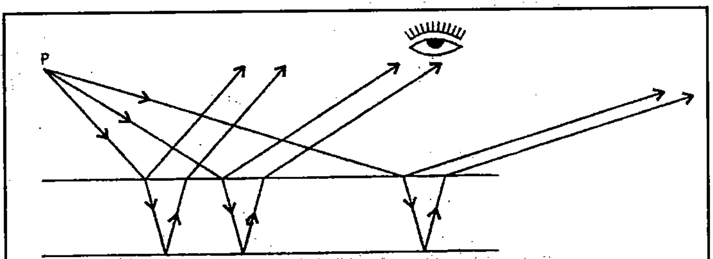


***Figure* 9.10 (a)**

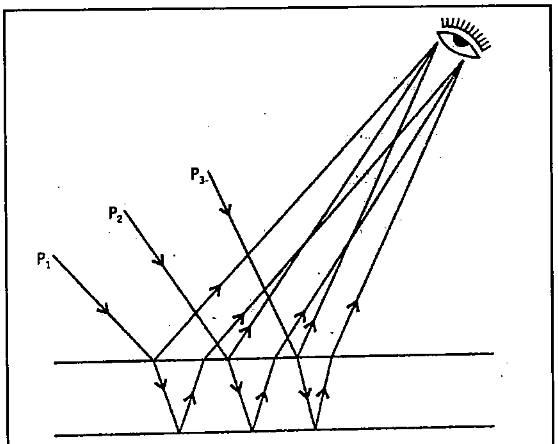


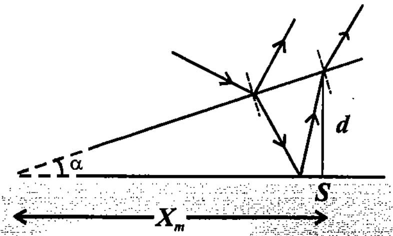
***Figure* 9.10 (b)**

If rays from a point source P [Figure 9.10(a)] be made incident on the film then the reflected or transmitted rays that can enter the eye (whose aperture is small) are confined to a small range of directions. Hence field of view will be very small. On the other had, if a broad source of light is used, light from every point of the source gives rise to a pair of coherent waves which can reach the eye after being reflected or transmitted from different points of the film. Thus the field of view become wide and fringes can be seen over the entire film [Figure 9.10(b).

For a thick film, the two adjacted reflected or transmitted rays will be wide apart and will not be able to enter the eye whose pupil is small. If the film be thin, then the two adjacent reflected or transmitted rays will be able to enter the eye where interference will occur by overlapping.

9.10.4 Fringe Width:

When a parallel beam of monochromatic light is incident normally on the film, the condition of brightness with reflected light is given by, $2nd = (2m+1)\lambda/2$ ... (45)



But the thickness d of the film at S is αx_m, where x_m is distance of the mth bright band from the apex of the film. Thus

$$2n\alpha x_m = (2m+1)\lambda/2. \quad \Rightarrow x_m = \frac{(2m+1)\lambda}{4n\alpha} \quad \text{... (46)}$$

Similarly, $$x_{m+1} = \frac{(2m+2+1)\lambda}{4n\alpha} \quad \text{... (47)}$$

The distance (β) between two consecutive bright bands is given by,

$$\beta = x_{m+1} - x_m = \frac{\lambda}{2n\alpha} \quad \text{... (48)}$$

9. 10.5 Fringes of equal width and equal inclination:

The conditions for maxima and minima of brightness of the fringes formed by the light reflected from a thin wedge-shaped film are respectively given by,

$$2nd\cos(r-\alpha) = (2m+1)\frac{\lambda}{2} \text{ (maxima)} \quad \text{... (49)}$$

$$2nd\cos(r-\alpha) = m\lambda \text{ (minima)} \quad \text{... (50)}$$

The above conditions suggest the existence of two distinct types of fringes. Suppose the incident rays are parallel and monochromatic and hence n, r and λ are all constants. Under this condition different order numbers (m) of the fringes will be controlled by the thickness d of the film. Hence **a fringe of a particular order number will lie on the locus of all the points of film having a constant thickness. These fringes are called fringes of equal width or thickness.**

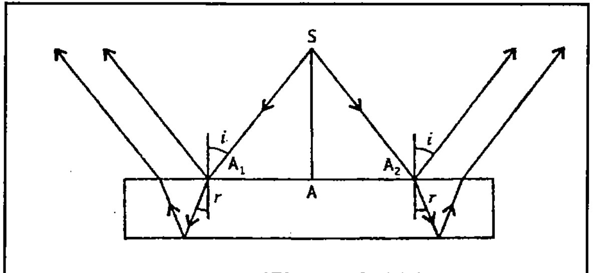


Figure 9.11

Another type of fringes will be obtained when monochromatic divergent rays from an extended source are incident on a film of uniform thickness. Here different order numbers (m) of the fringes will be controlled by the different values of r. If we draw a circle, with the cutting point A of the normal ray SA with the surface as centre, and AA_1 as radius, then all rays incident on the circumference of this circle will have the same angle of incidence i (Figure 9.11). Thus the fringe of a particular order will be a circle with the point A as centre. Fringe system will consist of concentric bright and dark rings. Fringe of a particular order is characterised by a particular angle of incidence and hence these fringes are called fringes of equal inclination. As the angle of incidence of the rays increases, the value of cos r decerases causing a decrease in the value of m. Thus the fringes of bigger radii will have smaller order number.

9.11 NON-REFLECTING FILM:

A part of the incident light is lost due to reflection at lens surfaces. Although the loss is small for a single surface, in a multi-lens system the loss is significant. Thus it is important to find ways of reducing or eliminating reflection is to use a thin layer of transparent material on the lens surface. This is known as anti-reflection coating.

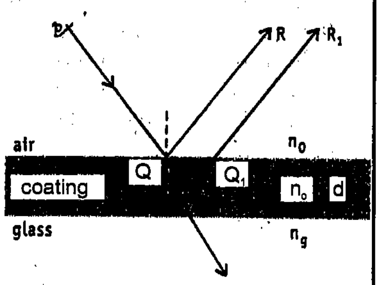


Figure 9.12

According to the electromagnetic theory of light the reflectivity r at the boundary between two media of refractive indices n_1 and n_2 and at normal incidence is given by

$$r = \left(\frac{n_2 - n_1}{n_2 + n_1}\right)^2 \qquad \text{... (51)}$$

So for equality of the amplitudes of the waves along QR and Q_1R_1 we have

$$\left(\frac{n_c - n_0}{n_c + n_0}\right)^2 = \left(\frac{n_g - n_c}{n_g + n_c}\right)^2 \qquad \text{... (52)}$$

Which, on simplification, shows that the refractive index of the coating must be

$$n_c = \sqrt{n_0\, n_g} \qquad \text{... (53)}$$

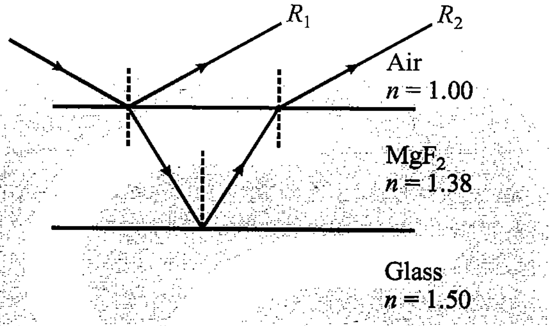
Here both the rays QR and Q_1R_1 suffer sudden phase change of π and hence for destructive interference the geometrical path difference between them must be equal to $\lambda/2$. Thus for normal incidence the minimum thickness *'d'* of the coating is given by

$$2n_c d = \lambda/2 \quad \Rightarrow d = \frac{\lambda}{4n_c} \qquad \text{... (54)}$$

Most frequently used materials are **magnesium fluoride and cryolite**. The technique of reducing the reflectivity of a surface is known as blooming.

x. 17. A lens is coated with a thin film of transparent substance magnesium fluoride (MgF_2) with $n = 1.38$ to reduce the reflection from the glass surface ($n = 1.50$). How thick a coating is needed to produce a minimum reflection at the centre of the visible spectrum ($\lambda = 550\,\text{nm}$) ?

ln. We assume that the light strikes the lens at near-normal incidence. We would like to find the thickness of the film that will bring about destructive interference between rays R_1 and R_2 as shown in the figure below.



Phase change of π is now associated with each ray and the condition of minimum intensity is now

$$2nd = \left(m + \frac{1}{2}\right)\lambda, m = 0, 1, 2, ...$$

Thus, the thickness of the thin nest possible film is (for $m = 0$)

$$d = \frac{\lambda}{4n} = \frac{550}{4 \times 1.38}\,\text{nm} = 99.64\,\text{nm}.$$

. 18. A tanker leaks kerosene (n = 1.2) into the Persian Gulf, creating a large slick on top of the water ($n = 1.3$).
(a) If you are looking straight down from an airplane onto a region of the slick where its thickness is 460 nm, for which wavelength(s) of visible light is the reflection the greatest?
(b) If you are scuba-diving directly under this same region of the slick, for which wavelength(s) of visible light is the transmitted intensity the strongest.

ln. (a) For the reflected rays, the condition for maxima is

$$2nd = m\lambda, \text{ or } \lambda = \frac{2nd}{m} = \frac{2 \times 1.2 \times 460}{m}\,\text{nm}$$

The wavelength of visible light (for $m = 2$) is 552 nm.

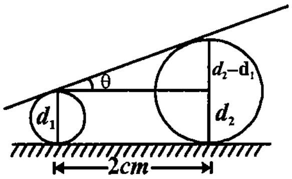
(b) For the transmitted rays, the condition for maxima is

$$2nd = \left(m + \frac{1}{2}\right)\lambda \text{ or } \lambda = \frac{2nd}{m + \frac{1}{2}} = \frac{2 \times 1.2 \times 460}{m + \frac{1}{2}}\,\text{nm}.$$

The wavelengths of visible light are 736 nm (when $m = 1$) and 441.6 nm (when $m = 2$).

Ex. 19. Two wires with diameters 0.01 cm and 0.03 cm are laid parallel to each other and 2 cm apart on a piece of plane glass. Another piece of plane glass is laid on top. Monochromatic light of wavelength 546 nm falls normally on the glass plates. Calculate the distance between consecutive bright fringes.

Soln.



Angle of the wedge,

$$\theta = \frac{d_2 - d_1}{\ell} = \frac{0.03 - 0.01}{2} = 0.01 \text{ rad.}$$

Fringe width,

$$\beta = \frac{\lambda}{2\theta} = \frac{546}{2 \times 0.01} \text{nm} = 27.3\ \mu\text{m}.$$

9.12 NEWTON'S RINGS:

9.12.1 Introudction:

Newton's rings are a particular example of interference fringes formed by thin films. By placing a plano-convex lens on a plane glass plate a thin air film of progressively increasing thickness from the point of contact *O* can be formed as shown in Figure 9.13. If it is illuminated by a monochromatic light interference fringes in the form of concentric circular rings are found. These rings were first observed by Newton and are known as Newton's rings. These fringes are the loci of points of equal film thickness. These rings are localized in the air film.

9.12.2 Mathematical analysis:

The optical path difference between two successive reflected waves QS_1R_1 and NS_2R_2 as shown in Figure 9.13 will be $2nd \pm \lambda/2$ ... (55)

where d = thickness of the film at N. Here we assume the film to be optically rarer than the glass media, above and below and consequently we are to consider the abrupt phase change of π due to refection at G. This is equivalent to a path difference of $\pm\lambda/2$.

Case-I : For constructive interference

$$2nd \pm \frac{\lambda}{2} = m\frac{\lambda}{2}$$

$$\Rightarrow \quad 2nd = (2m+1)\lambda/2 \qquad \text{... (56)}$$

where m = 0, 1, 2, 3, ...

Case-II : For destructive interference,

$$2nd \pm \lambda/2 = (2m+1)\frac{\lambda}{2}$$

$$\Rightarrow \quad 2nd = m\lambda \qquad \text{... (57)}$$

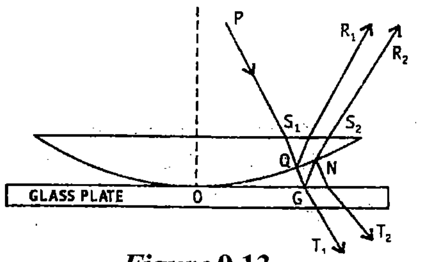


Figure 9.13

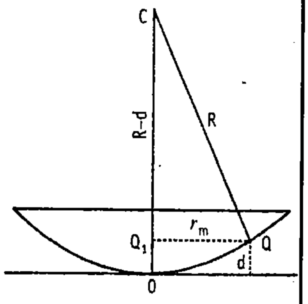


Figure 9.14

A fringe of a given order (m) will be along the loci of points of equal film thickness (d) and hence the fringes will be circular. The radii of the rings can be found out from Figure 9.14. If the point Q fulfils the condition of brightness (or darkness) then all points on the circumference of a circle of radius Q_1Q will be bright (or dark). Thus we shall get a bright (or dark) ring, of say m^{th} order, whose radius is $Q_1Q = r_m$. By geometry,

$$R^2 = r_m^2 + (R-d)^2 \qquad \ldots (58)$$

where R = radius of curvature of the convex surface.

Since R > > d, we can write $r_m^2 \approx 2Rd \Rightarrow 2d = \frac{r_m^2}{R}$ ... (59)

Using the condition (56) we can write for the m^{th} bright ring.

$$2nd = \frac{nr_m^2}{R} = (2m+1)\frac{\lambda}{2} \Rightarrow r_m^2 = \frac{(2m+1)\lambda R}{2n} \qquad \ldots (60)$$

Similarly for the m^{th} dark ring,

$$r_m^2 = \frac{2m.\lambda R}{2n} \qquad \ldots (61)$$

Thus we find that the radii of the bright rings are proportional to the square root of odd natural numbers and that of the dark rings proportional to square root of natural numbers. Corresponding diameters are given by

$$D_m^2 = \frac{2(2m+1)\lambda R}{n} \quad \text{[Bright ring]} \qquad \ldots (62)$$

$$D_m^2 = \frac{4m\lambda R}{n} \quad \text{[Dark ring]} \qquad \ldots (63)$$

The difference in diameters of the m^{th} and $(m+1)^{\text{th}}$ order dark rings is

$$D_{m+1} - D_m = \sqrt{\frac{4\lambda R}{n}}\left[\sqrt{m+1} - \sqrt{m}\right] \qquad \ldots (64)$$

Thus as the order number 'm' increases this difference decreases meaning that the rings gradually become narrower as their radii increases. There will be more crowding of the rings as we move outward from the centre.

9.12.3 Fringe width:

If D_m and D_{m+1} are the diameters of two successive bright rings then we can write from Equation (63),

$$D_{m+1}^2 - D_m^2 = \frac{4\lambda R}{n} \Rightarrow D_{m+1} - D_m = \frac{4\lambda R}{n(D_{m+1} + D_m)}$$

Writing $D_{m+1} + D_m \approx 2D_m$, we get fringe width β as, Fringe width, $$\beta = \frac{D_{m+1} - D_m}{2} = \frac{\lambda D}{n.D_m} \qquad \dots (65)$$

Thus fringe width β decreases as the diameter D_m of the ring increases.

9.12.4 Central fringe:

At the point of contact of the lens and the glass plate thickness of the film d = 0 Consequently, the condition (57) for destructive interference will be satisfied with m = 0. This indicates that the central fringe is dark and is situated at the point of contact of the lens and the glass plate. It appears as a dark spot.

9.12.5 Newton's rings with white light:

With a white light the central spot will be black and it will be surrounded by a few coloured rings and beyond this there will be general illumination due to overlapping of different coloured rings.

9.12.6 Determination of wavelength:

Newton's rings can be used for the measurement of the wavelength of monochromatic light. The diameters of m^{th} Newton's ring for an air film (n = 1) are given from Equations (62) and (63) as

$$D_m^2 = 2(2m+1)\lambda R \text{ [Bright ring]} \qquad \dots (66)$$

$$D_m^2 = 4m\lambda R \text{ [Dark ring]} \qquad \dots (67)$$

Thus if D_m and D_{m+p} are respectively be the diameters of the m^{th} and $(m+p)^{th}$ rings (bright or dark) then the wavelength λ is given by

$$\lambda = \frac{D_{m+p}^2 - D_m^2}{4pR} \qquad \dots (68)$$

9.12.7 Determination of refractive index of a liquid:

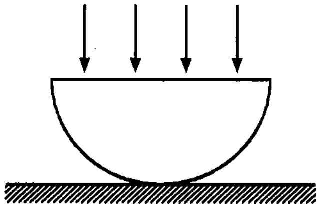
Newton's rings can also be used to measure the refractive index (n) of a liquid. At first the diameters of m^{th} and $(m+p)^{th}$ bright or dark rings are measured with air film. Then the diameters of these rings are measured again by forming a liquid film between the lens and the glass plate. Introduction of the liquid decreases the diameters of the rings.

For air film, $$\left(D_{m+p}^2 - D_m^2\right)_{air} = 4p\lambda R$$

For liquid film, $$\left(D_{m+p}^2 - D_m^2\right)_{liquid} = \frac{4p\lambda R}{n}$$

Hence, $$n = \frac{\left(D_{m+p}^2 - D_m^2\right)_{air}}{\left(D_{m+p}^2 - D_m^2\right)_{liquid}} \qquad \dots (69)$$

Ex. 20. A thin slice is cut out of a glass cylinder along a plane parallel to its axis. The slice is placed on a flat plate as shown. The observed interference fringes from this combination shall be



(a) staright (b) circular (c) equally spaced
(d) having fringe spacing which increases as we go outwards

[IIT-JEE : 1999]

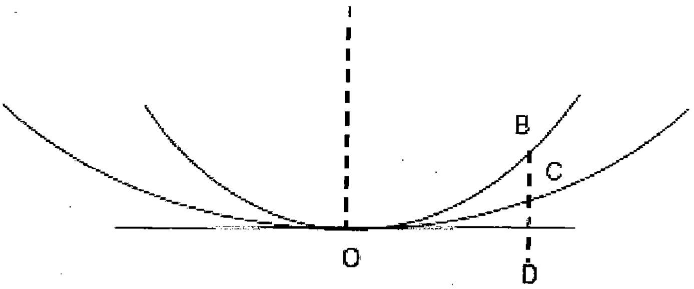
In. (a)

Locus of equal path difference are the lines running parallel to the axis of the cylinder. Hence, straight fringes are obtained.

Note: Circular rings (also called Newton's rings) are observed in interference pattern when a plano-convex lens of large focal length is placed with its convex surface in contact with a plane glass plate because locus of equal path difference in this case is a circle.

. 21. The convex surface of radius 200 cm of a plano-convex lens rests on a concave spherical surface of radius 400 cm and Newton's rings are viewed with reflected light of wavelength 600 nm. Calculate the diameter of the 8th bright ring.

In. The thickness of the air-film at B as shown in below figure.



$$t = BC = BD - CD = \frac{\rho^2}{2R_1} - \frac{\rho^2}{2R_2} = \frac{\rho}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

where $\rho = OD$, $R_1 =$ Radius of curvature of OB and $R_2 =$ Radius of curvature of OC. Thus, the condition of the mth bright fringe is

$$\rho_m^2 = \frac{\left(m - \frac{1}{2}\right)\frac{\lambda}{n}}{\frac{1}{R_1} - \frac{1}{R_2}}$$

In the present problem, $m = 8$, $\lambda = 600$ nm, $n = 1$, $R_1 = 200$ cm, $R_2 = 400$ cm. We find $2\rho_8 = 0.85$ cm.

9.12.8 Michelson's interferometer:

An instrument that is designed primarily to measure lengths or changes in length with degree of accuracy by means of interference fringes is known as an interferometer. One of the widely used interferometers is the one devised and built by A.A. Michelson.

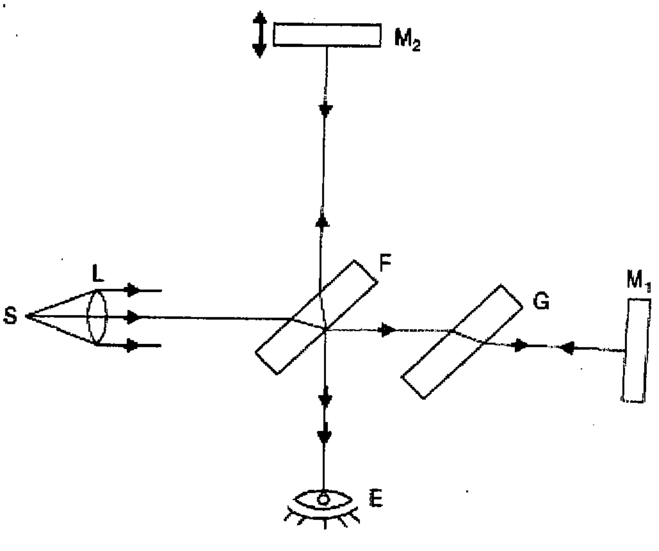


Figure **9.15**

The light from the source S falls on a beam splitter F (Fig. 9.15). F and G are two plane-parallel optical flats, which are usually cut from a single parallel plate so that they are identical. The surface of F nearest to G may be half silvered in order that the light falling upon it be 50% reflected and 50% transmitted, although this is not necessary for the operation of the interferometer. The mirror M_2 is mounted on a carriage, so that it can be moved along a precision-mechanical track. The motion of M_2 is controlled by a very fine micrometer screw. The mirror M_1 is held by springs against adjusting screws so that M_1 may be made exactly perpendicular to M_2, Light from the source S is rendered parallel by the lens L. After entering F the light is divided into two parts: one part proceeds to the mirror M_1 which returns the light to the eye after reflection at F; the other part, reflecting inside F, proceeds to M_2, which reflects it back through F to E. From a single source, by division of amplitude, two beams are produced and they are recombined for the formation of interference fringes.

The beam going towards the mirror M_2 and reflected back, has to pass twice through the glass plate F. Therefore to compensate for the path the plate G is used between the mirror M_1 and F. The light beam going towards the mirror M_1 and reflected back towards F also passes twice through the compensating plate G. Thus the paths of the two rays in glass are the same. Suppose the mirrors M_1 and M_2 are at distances d_1 and d_2 respectively from the glass plate F. In measuring d_1 and d_2 we do not consider the paths of the rays in glass. The path difference for the two waves when they recombine is $2d_2 - 2d_1$ and anything that changes this path difference will cause a change in the relative phase of the two waves as they enter the eye. If the mirror M_2 is moved by a distance $\frac{1}{2}\lambda$, the path difference changes by » and the observer will see the fringe pattern shift by one fringe. If the mirrors M_1 and M_2 are exactly perpendicular to each other, the system is essentially equivalent to the interference from an air film whose thickness is equal to $d_2 - d_1$. Suppose when the mirror M_2 is moved through a distance d, the number of fringes that cross the field of view is m. Then we have $d = m\frac{\lambda}{2}$ or $2d = m\lambda$.

Ex. 22. An airtight chamber 5 cm long with glass windows is placed in one arm of a Michelson's interferometer. Light of wavelength $\lambda = 480$ nm is used. The air is slowly evacuated from the chamber using a vacuum pump. While the air is being removed 61 fringes are observed to pass through the field of view. Find the index of refraction of air at atmospheric pressure.

Soln. In Michelson interferometer, light travels twice through the chamber. Thus, we have the formula,

$$2t = \frac{x\lambda}{(n-1)\beta} = \frac{61\beta\lambda}{(n-1)\beta} \text{ or } n-1 = \frac{61\lambda}{2t} = \frac{61\times 480\times 10^{-9}}{2\times 5\times 10^{-2}} = 0.00029$$

giving $n = 1.00029$.

Ex. 23. After obtaining the fringes in Michelson's interferometer with white light, the white light is replaced by sodium light. When one mirror of the interferometer is now moved through a distance of 0.15 mm the fringes are found to disappear. If the mean wavelength for the two components of the D lines of sodium light is 5893 Å, find the difference between their wavelengths.

Soln. We have $2d = m_1\lambda_1 = \left(m_1 + \frac{1}{2}\right)\lambda_2$, which gives $m_1 = \frac{\lambda_2}{2(\lambda_1 - \lambda_2)}$ and $\Delta\lambda = \lambda_1 - \lambda_2 = \frac{\lambda_1\lambda_2}{4d} = \frac{\lambda^2}{4d}$

$$= \frac{5893\times 5893\times 10^{-8}}{4\times 0.15\times 10^{-1}} \text{ Å} = 5.79 \text{ Å}.$$

Ex. 24. When a liquid is introduced between the lens and the plate in the Newton Ring apparatus, the diameter of the 15th ring changes from 2cm to 1.8cm. The refractive index of the liquid is **[DU 2015]**

(a) 1.05 (b) 1.11 (c) 1.23 (d) 1.33

ln. $D \propto \frac{1}{\sqrt{\mu}}$ $\mu \rightarrow$ refractive index

$$\Rightarrow \quad \frac{D_2}{D_1} = \sqrt{\frac{\mu_1}{\mu_2}} \Rightarrow \frac{1.8\,cm}{2cm} = \sqrt{\frac{1}{\mu}}$$

$$\Rightarrow \quad \mu = \frac{2^2}{1.8^2} = \left(\frac{10}{9}\right)^2 \approx 1.23$$

. 25. The spherial surface of a plano-convex lens of radius of curvature R = 1m is gently placed on a flat plate. The space between them is filled with a transparent liquid of refractive indices of the lens and the flat plate are 1.5 and 1.6

[IIT-JAM : 2006]

n. (a) Optical path difference = $2\mu t$, Where μ is refractive index of liquid. For destructive interference, and t is thickness of the film at A.

$$2\mu t = \left(n + \frac{1}{2}\right)\lambda \qquad \text{... (1)}$$

and $2t = \frac{\rho_n^2}{R}$ ... (2) Where ρ_n is radius of nth dark ring.

Substituting (2) into (1)

$$\frac{\mu\rho_n^2}{R} = \left(n + \frac{1}{2}\right)\lambda \qquad \Rightarrow \rho_n = \sqrt{\left(n + \frac{1}{2}\right)\frac{\lambda R}{\mu}}$$

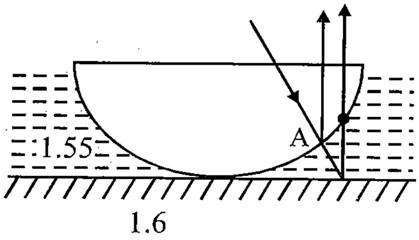
ρ_{16} = radius of 16th dark ring $= \sqrt{5} \times 10^{-3}$ m.

16th dark ring will correspond to n = 15.

$$\sqrt{5} \times 10^{-3} = \sqrt{\frac{31}{2} \times \frac{\lambda \times 1}{\mu}}$$

$$\lambda = \frac{5 \times 10^{-6} \times 1.55}{15.5} = 5 \times 10^{-7} \text{ met}$$

$$= 0.5\,\mu\text{m}$$



(b) When transparent liquid is removed, only ray 2 suffers a phase change of π, and path difference $\frac{\lambda}{2}$.

$$\text{Path difference} = \left(2t + \frac{\lambda}{2}\right)$$

For dark ring $2t + \frac{\lambda}{2} = \left(n + \frac{1}{2}\right)\lambda \qquad \Rightarrow 2t = n\lambda$ ... (3)

From (3), $\frac{\rho_n^2}{R} = n\lambda \qquad \Rightarrow \rho_n = \sqrt{nR\lambda}$

For 21th dark ring, n = 20 $\qquad \Rightarrow \rho_{20} = \sqrt{20 \times 5 \times 10^{-11} \times 1} = \sqrt{10}$ mm

SOLVED EXAMPLES

1. In a Lloyd's mirror experiment calculate the ratio of ratio of the intensities at the interference maxima and minima if the mirror reflects only 75% of the light incident upon it.

Soln. Reflectivity R = 75 / 100 = 3 /4. Amplitude reflection coefficient $r = \sqrt{R} = \sqrt{3}/2$. So a_1 be the amplitude of the direct beam then the amplitude of the reflected beam would be $a_2 = a_1 r = a_1\sqrt{3}/2$

$$\Rightarrow \quad \frac{I_{\max}}{I_{\min}} = \frac{(a_1 + a_2)^2}{\left(a_{1-a_2}\right)^2} = \frac{(1+r)^2}{(1-r)^2} = \left(\frac{1+\sqrt{3}/2}{1-\sqrt{3}/2}\right)^2 = 194$$

2. The acute angle of a biprism of refractive index 1.5 is 2°. A slit illuminated by a monochromatic light is placed 10 cm from the biprism. If the distance between the two dark fringes observed at a distance of 1 cm from the biprim is 0.18 mm, find the wavelength of light used.

Soln. We know that fringe width $\beta = \dfrac{\lambda D}{d}$

where the distance d between virtual sources is given by $d = 2a(n-1)\alpha$

Here a = 10 cm = 0.1 m, n = 1.5, $\alpha = 2^\circ = 2\lambda/180$ radian.

$$\Rightarrow \quad d = 2\times 0.1(1.5-1)\times 2\pi/180 = 0.349\times 10^{-2}\,\text{m}$$

Also, $\beta = 0.18\times 10^{-3}\,\text{m}$, D = 1m + 0.1m = 1.1m.

$$\Rightarrow \quad \lambda = \frac{0.18{\times}10^{-3}\times 0.349\times 10^{-2}}{1.1}\,\text{m} = 571.2\,\text{nm}$$

3. In Newton's ring experiment, the diameter of *m*th dark ring is 8 mm and the diameter of (m + 5)th dark ring is 12 mm. If the radius of curvature of the lower surface of the lens is 10 m, find the wavelength of light used.

Soln. We know that $\lambda = \dfrac{D_{m+p}^2 - D_m^2}{4pR}$

Here p = 5, R = 10 m, $D_m = 8{\times}10^{-3}\,\text{m}$

$D_{m+p} = 12\times 10^{-3}\text{m}$

$$\Rightarrow \quad \lambda = \frac{\left(12\times 10^{-3}\right)^2 - \left(8\times 10^{-3}\right)^2}{4\times 5\times 10}\,\text{m} = 400\,\text{nm}.$$

4. Newton's ring are formed in reflected light of wavelength 600 nm with a combination of plane plate of glass and a plano-convex lens of 1m radius of curvature. On introducing a liquid between the lens and the plate it is found that the diameter of the 7th dark ring decreases by 0.54 mm. Find the refractive index of the liquid.

Soln. The diameter D_m of *m*th dark ring is given by,

$D_m^2 = 4m\lambda R$ when the film is air.

$$\Rightarrow \quad D_7 = \sqrt{4\times 7\times 600{\times}\,10^{-9}\times 1}\,\text{m} = 4.1\,\text{mm}$$

When there is a liquid of refractive index n within the plate and lens, the diameter becomes (4.1 – 0.54) mm = 3.56 mm.

Now $\left(D_m^2\right)_{liquid} = \dfrac{4m\lambda R}{n}$

Therefore, refractive index

$$n = \frac{\left(D_m^2\right)_{air}}{\left(D_m^2\right)_{liquid}} = \frac{(4.1)^2}{(3.56)^2} = 1.326$$

5. In Young's double slit experiment the angular position of a point above the central maximum whose intensity is one-fourth of maximum intensity is

(a) $\sin^{-1}\left(\dfrac{\lambda}{d}\right)$ (b) $\sin^{-1}\left(\dfrac{\lambda}{2d}\right)$ (c) $\sin^{-1}\left(\dfrac{\lambda}{3d}\right)$ (d) $\sin^{-1}\left(\dfrac{\lambda}{4d}\right)$

Soln. $I = I_{\max}\cos^2\left(\dfrac{\delta}{2}\right)$ where $\delta = \dfrac{2\pi}{\lambda} d\sin\theta$

$\dfrac{1}{4} = \cos^2\dfrac{\delta}{2}$ or $\dfrac{\delta}{2} = \dfrac{\pi}{3}$ or $\delta = \dfrac{2\pi}{3}$.

Thus, $\dfrac{2\pi}{3} = \dfrac{2\pi}{\lambda} d\sin\theta$ **or** $\sin\theta = \dfrac{\lambda}{3d}$ **or** $\theta = \sin^{-1}\left(\dfrac{\lambda}{3d}\right)$.

Hence, correct option is (c).

In Newton's experiment with a light containing two wavelengths $\lambda_1 = 589$ nm and $\lambda_2 = 589.6$ nm. find the distance (from the point of contact of plano-convex lens with glass plate) at which the rings disappear. Take the radius of curvature of the curved surface as 1 m.

oln. The fringes will disappear if mth order bright ring of λ_1 overlaps with mth order dark ring of λ_2.
We know,

$$D_m^2(\text{bright}) = \frac{2(2m+1)\lambda, R}{n} \quad \text{...(i)}$$

$$D_m^2(\text{dark}) = \frac{4m\lambda R}{n} \quad \text{...(ii)}$$

$$D_m^2(\text{bright}) = D_m^2(\text{dark})$$

$$\Rightarrow \quad \frac{2(2m+1)\lambda_1 R}{n} = \frac{4m\lambda_2 R}{n}$$

$$\Rightarrow \quad (2m+1)\lambda_1 = 2m\lambda_2$$

$$\Rightarrow \quad \lambda_1 = 2m(\lambda_2 - \lambda_1)$$

$$\Rightarrow \quad m = \frac{\lambda_1}{2(\lambda_2 - \lambda_1)}$$

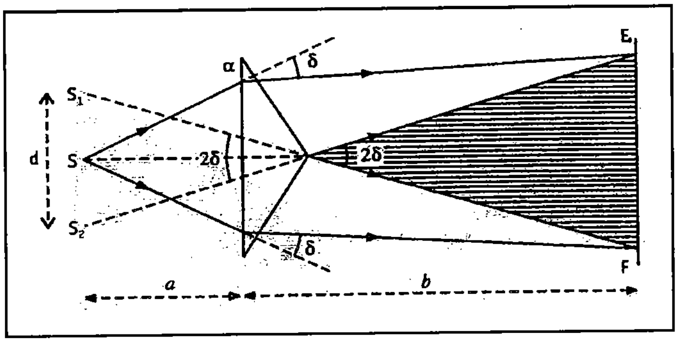
$$\Rightarrow \quad m = \frac{\lambda_1}{2(\lambda_2 - \lambda_1)} = \frac{589}{2 \times 0.6} = 490$$

Therefore, the distance at which the rings will disappear is

$$r_{490} = \sqrt{490 \times 589.6 \times 10^{-9} \times 1} \text{ m} \approx 1.7 \text{ m}$$

7. In a Fresnel's biprism experiment the base angle of the prism is 3° and the refractive index of glass is 1.5. Interference fringes are formed with a source of wavelength 600 nm located 10 cm from the biprism. If the screen is placed at a distance of 1 m from the source find the maximum number of fringes that can be observed.

Soln.



In Figure above, deviation $\delta = (n-1)\alpha$

Distance between virtual sources is $d = 2\delta . a = 2a(n-1)\alpha$

Fringe width $\beta = \dfrac{\lambda(a+b)}{2a(n-1)\alpha} = \dfrac{\lambda D}{d}$ $\quad [\because D = a+b]$

Overlapping region $EF = b.2\delta = 2b(n-1)\alpha$

$\therefore$ Maximum number of fringes observable is given by $\dfrac{EF}{\beta}$.

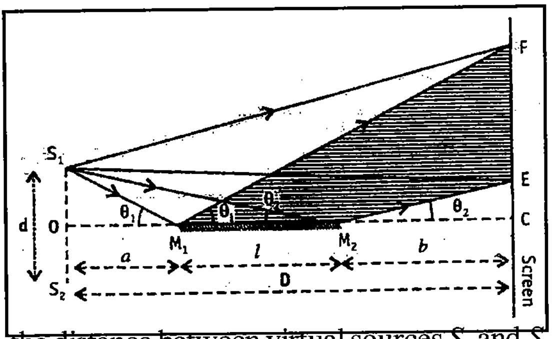
Here a = 10 cm, b = 90 cm, a + b = 1m, $\lambda = 600\times10^{-9}$ m, $n = 1.5$, $\alpha = 3° = \dfrac{3\times\pi}{180}$ radian.

$$\Rightarrow \quad \beta = \frac{600\times10^{-9}\times1\times180}{2\times0.1\times0.5\times3\lambda} \text{m} = 11.46\times10^{-5}\text{m}$$

Hence maximum number of fringes observable would be $\dfrac{EF}{\beta} = \dfrac{2b(n-1)\alpha}{\beta} = \dfrac{2\times0.9\times0.5\times3\pi}{11.46\times10^{-5}\times180} = 411$

8. A Lloyd's mirror of length 5 cm is illuminated with a light of wavelength 589 nm from a narrow slit 0.1 cm from its plane and 5 cm measured in that plane, from its near edge. Find the fringe width on a screen placed 1.2 m from the slit. Also, determine the maximum number of observable fringes.

Soln.



In Figure above the distance between virtual sources S_1 and S_2 is

$d = 2\times0.1$ cm $= 0.2$ cm $= 0.2\times10^{-2}$ m and D = 1.2 m

$\Rightarrow$ Fringe width $\beta = \dfrac{\lambda D}{d} = \dfrac{589\times10^{-9}\times1.2}{0.2\times10^{-2}}$ m $= 0.3534$ mm

In Figure above, EF is the region of overlap on the screen. Now

$$CE = b\tan\theta_2 = (D-a-l)\tan\theta_2 = (D-a-l).\frac{d/2}{(a+l)}$$

$$CF = (b+l)\tan\theta_1 = (D-a)\tan\theta_1 = (D-a).\frac{d/2}{a}$$

Here a = 0.05 m, l = length of mirror = 0.05 m, D = 1.2 m

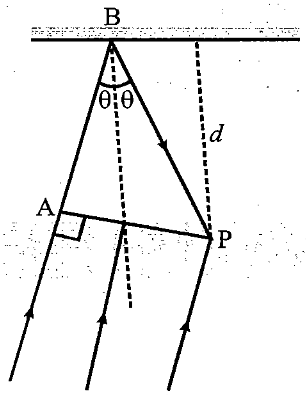
$$\Rightarrow \quad EF = CF - CE = (D-a)\frac{d}{2a} - (D-a-l).\frac{d}{2(a+l)}$$

$$= (1.2-0.05)\frac{0.002}{2\times0.05} - (1.2-0.05-0.05)\frac{0.002}{2\times0.1} = 0.012 \text{ m} = 1.2 \text{ cm}$$

Therefore, maximum number of fringes observable is $\dfrac{EF}{\beta} = \dfrac{12mm}{0.3534mm} = 34$

. A plane wavefront of light is incident on a plane mirror as shown in below figure. Show that the intensity is maximum at P when

$$\cos\theta = \frac{\lambda}{4d}, \frac{3\lambda}{4d}, \frac{5d}{4d}, \ldots.$$



oln. The path difference between the disturbance reaching at point P directly and after reflection is

$$AB + BP + \frac{\lambda}{2} = BP\cos2\theta + \frac{d}{\sin(90°-\theta)} + \frac{\lambda}{2} = \frac{d}{\cos\theta}(\cos2\theta+1) + \frac{\lambda}{2} = 2d\cos\theta + \frac{\lambda}{2}$$

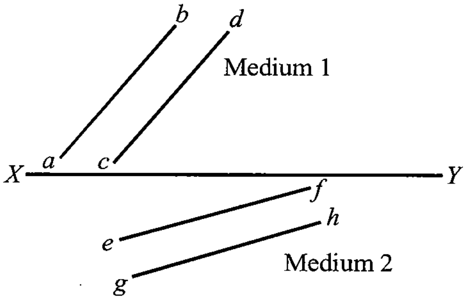
Here $\dfrac{\lambda}{2}$ is due to reflection from the denser medium.

For maximum intensity at P, we have

$$2d\cos\theta + \frac{\lambda}{2} = n\lambda,\ n = 1,2,3,\ldots$$

or $\cos\theta = \dfrac{(2n-1)\lambda}{4d},\ n = 1,2,3,\ldots$

10. The below figure shows a surface XY separating two transparent media, medium 1 and medium 2. The lines ab and cd represent wavefronts of a light wave travelling in medium 1 and incident on XY. The lines ef and gh represent wave fronts of the light wave in medium-2 after refraction.



(a) Light travels as a
(A) parallel beam in each medium
(B) convergent beam in each medium
(C) divergent beam in each medium
(D) divergent beam in one medium and convergent beam in the other medium.

(b) The phases of the light wave at c, d, e, f are ϕ_c, ϕ_d, ϕ_e and ϕ_f respectively. It is given that $\phi_c \neq \phi_f$. Then

(A) ϕ_c cannot be equal to ϕ_e
(B) ϕ_d cannot be equal to ϕ_e
(C) $(\phi_d - \phi_f)$ is equal to $(\phi_c - \phi_e)$
(D) $(\phi_d - \phi_c)$ is not equal to $(\phi_f - \phi_e)$

(c) Speed of light is
(A) the same in medium 1 and medium 2
(B) larger in medium 1 than in medium 2
(C) larger in medium 2 than in medium 1
(D) different at b and d.

Soln. (a) The ray of light is perpendicular to the wavefront.
Correct choice (A)

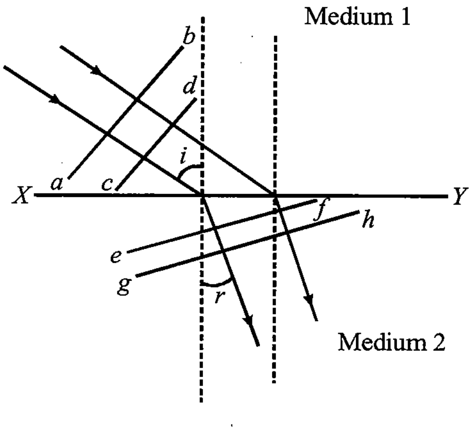
(b) The particles are in same phase on a wavefront.

Thus $\phi_c = \phi_d$ and $\phi_e = \phi_f$.

$$\phi_d - \phi_f = \phi_c - \phi_e$$

Correct choice (C)

(c) The ray of light goes towards normal after refraction (see below figure). Thus the medium 2 is denser than medium 1. The velocity of light decreases in the denser medium.



Correct choice (B).

1. In a double slit experiment instead of taking slits of equal widths, one slit is made twice as wide as the other, then in the interference pattern
(a) the intensities of both the maxima and the minima increases
(b) the intensities of the maxima decreases and the minima has zero intensity
(c) the intensity of maxima decreases and that of minima increases
(d) the intensity of maxima decreases and the minima has zero intensity

[IIT-JEE : 2000]

oln. (a)

In interference we know that

$$I_{\max} = \left(\sqrt{I_1} + \sqrt{I_2}\right)^2 \text{ and } I_{\min} = \left(\sqrt{I_1} \sim \sqrt{I_2}\right)^2$$

Under normal conditions (when the widths of both the slits are equal)

$$I_1 \approx I_2 = I \text{ (say)}$$

$$\therefore\ I_{\max} = 4I \text{ and } I_{\min} = 0$$

When the width of one of the slits is increased. Intensity due to that slit would increase, while that of the other will remain same. So, let :

$$I_1 = I \text{ and } I_2 = \eta I \qquad (\eta > 1)$$

Then, $I_{\max} = I\left(1 + \sqrt{\eta}\right)^2 > 4I$

and $I_{\min} = I\left(\sqrt{\eta} - 1\right)^2 > 0$

$\therefore$ Intensity of both maxima and minima is increased.

. Find the thickness and refractive index of an anti-reflection coating necessary for a glass lens ($n_g = 1.5$) in air at vacuum wavelength 590 nm.

ln. The thickness of the anti-reflection coating is chosen so that the wavelength in the dielectric material is one quarter the wavelength of the incoming wave. For a quarter wavelength antireflection coating of a transparent material with a refractive index n_c and light incident on the coating with a free-space wavelength λ_0, the thickness t which causes minimum reflection is calculated by $\frac{\lambda_c}{4} = \frac{\lambda_0}{4n_c}$

Reflection is further minimized if the refractive index of the anti-reflection coating is the geometric mean of that of the materials on either side; that is, glass or air. Refractive index of the coating material is given by

$$n_c = \sqrt{n_0 n_g} = \sqrt{1 \times 1.5} = 1.225$$

Thickness required is equal to

$$\Rightarrow t = \frac{\lambda_0}{4n_c} = \frac{590}{4 \times 1.225} \text{ nm} = 120.4 \text{ nm.}$$

Chapter 10

Diffraction of Light

10.1 FRAUNHOFER DIFFRACTION IN A SINGLE SLIT:

Let a plane wavefront AB of monochormatic light of wavelength λ propagating normal to a slit S_1S_2 be incident on it. According to Huygens-Fresnel principle each point of the wavefront in the plane of the slit may be regarded as the source of secondary spherical wavelets. The secondary wavelets travelling normal to the slit are brought to focus by a convex lens L on screen at C. The wavelets travelling at an angle θ with the normal are brought to focus at Q. Let us calculate the intensity of light at Q.

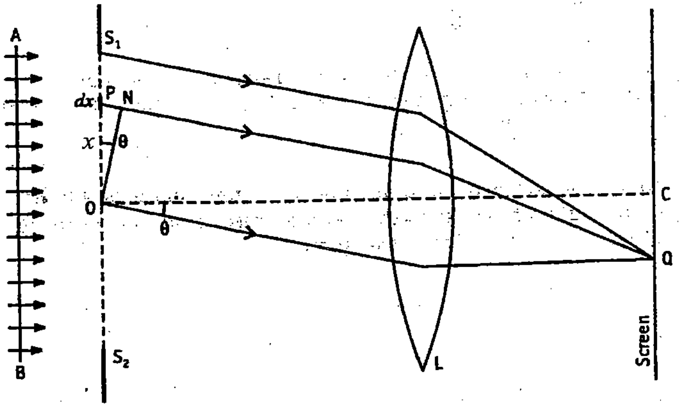


Figure **10.1**

10.1.1 Intensity of the diffraction spots:

Let the complex light disturbance at any instant due to secondary waves from the origin O (the mid point of the slit) be represented by $Ae^{i\omega t}$ where A is the amplitude and ω is the circular frequency of the wave. The phase difference between the waves at Q coming from O and from a point P at a distance x from O is given by

$$\frac{2\pi}{\lambda}\times PN = \frac{2\pi}{\lambda}x\sin\theta = lx \qquad \ldots(1)$$

where, $$l = \frac{2\pi}{\lambda}\sin\theta \qquad \ldots(2)$$

Hence the disturbance at Q due to secondary waves from P will be proportional to $e^{i(\omega t - lx)}$. The disturbance at Q to the diffracting element dx can be written as

$$dy = C.A.dx.e^{i(\omega t - lx)} \qquad \ldots(3)$$

where we assume the amplitude to be proportional to the width dx; C is a proportionality constant.

If a is the width of the slit then the resultant complex disturbance at Q due to the waves coming from all the diffracting elements will be given by

$$y = \int_{-\frac{a}{2}}^{+\frac{a}{2}} CAe^{i(\omega t - lx)}dx = CAe^{i\omega t}\left[\frac{e^{-ilx}}{-il}\right]_{-\frac{a}{2}}^{+\frac{a}{2}} = CAa.\frac{\sin\frac{la}{2}}{\frac{la}{2}}.e^{i\omega t} \qquad \ldots (4)$$

$$\left[\because \sin\theta = \frac{e^{i\theta} - e^{i\theta}}{2i}\right]$$

The resultant intensity I at Q is obtained by multiplying y by its complex conjugate y^*. Thus

$$I = y.y^* = I_0\frac{\sin^2\alpha}{\alpha^2} \qquad \ldots (5).$$

where $I_0 = (CAa)^2$ and $\alpha = \frac{la}{2} = \frac{\pi}{\lambda}.a\sin\theta$ ... (6)

10.1.2 Condition for maxima and minima:

The maxima or minima are given by $\frac{dI}{d\alpha} = 0$

$$\Rightarrow \quad \frac{2\sin\alpha\cos\alpha}{\alpha^2} - \frac{2\sin^2\alpha}{\alpha^3} = 0$$

$$\Rightarrow \quad \sin\alpha(\alpha\cos\alpha - \sin\alpha) = 0$$

Hence, either $\sin\alpha = 0$ ... (7)

or $\alpha = \tan\alpha$ ... (8)

The value of $\frac{d^2I}{d\alpha^2}$ is found to be positive for $\sin\alpha = 0$ and negative for $\alpha = \tan\alpha$. Hence the condition (7) will give minima and condition (8)will give maxima. Thus for minima,

$$\sin\alpha = 0$$

$$\Rightarrow \quad \alpha = m\pi,\ m = \pm 1, \pm 2, \pm 3......$$

$$\Rightarrow \quad a\sin\theta = m\lambda \qquad \ldots (9)$$

Here $m = 0$ is excluded because for this $\alpha = 0$ and since $\lim_{\alpha\to 0}\frac{\sin\alpha}{\alpha} = 1$, it actually gives a maximum.

The positions of maxima can be obtained by solving Equation (8) It is transcendental equation and can be solved graphically by plotting the curves $y = \alpha$ and $y = \tan\alpha$ and finding the point of intersections as shown in Figure (10.2). An inspection of the Figure (10.2) shows that one value is $\alpha = 0$ and other values of α which will give maxima are less than but gradually approaching towards $\pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}$ ... etc.

$\alpha = 0$ gives the position of principal maximum and the other values of α give secondary maxima. The intensity of principal maximum is

$$I = \lim_{\alpha\to 0} I_0\frac{\sin^2\alpha}{\alpha^2} = I_0 \qquad \ldots (10)$$

The intensity of first secondary maximum is

$$I_1 \approx I_0\frac{\sin^2\frac{3\pi}{2}}{\left(\frac{3\pi}{2}\right)^2} = \frac{4}{9\pi^2}I_0 \qquad \ldots (11)$$

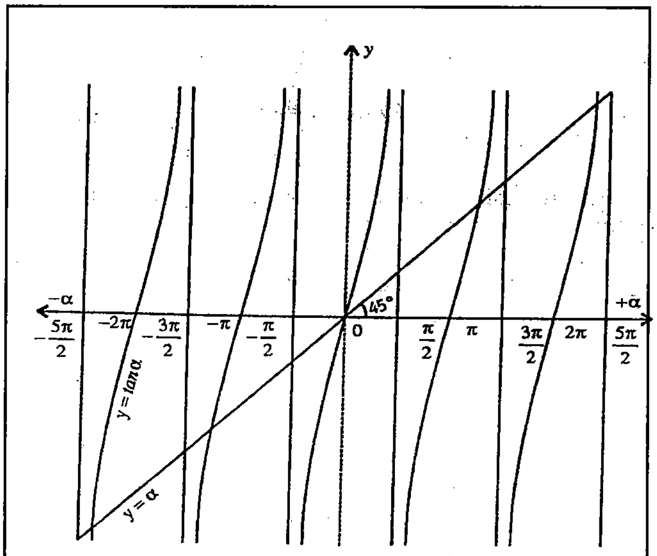


***Figure* 10.2**

The intensity of second secondary maximum is

$$I_2 \approx I_0 \frac{\sin^2 \frac{5\pi}{2}}{\left(\frac{5\pi}{2}\right)^2} = \frac{4}{25\pi^2} I_0 \qquad \text{... (12)}$$

Thus the secondary maxima are very feeble compared with the principal maximum and their intensities diminish very rapidly with the increase in order number.
The intensity distribution in the diffraction pattern due to a single slit is shown in Figure (10.3).

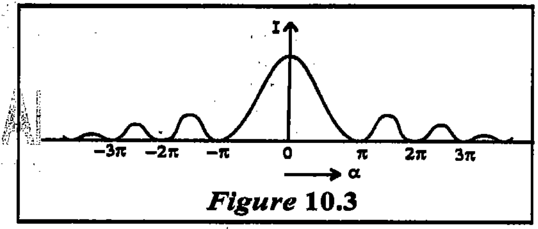


***Figure* 10.3**

The angle of the diffraction θ_1 for the first minima on either side of central maximum is given by Equation(9) as

$$a \sin\theta_1 = \lambda; \quad \Rightarrow \theta_1 \approx \sin\theta_1 = \frac{\lambda}{a}$$

Therefore, the angular width of principal maximum $2\theta_1 = 2\lambda / a$, is inversely proportional to the width (a) of the slit.

Ex. 1. In the Fraunhofer diffraction of light of wavelength λ at a slit of width a, the angular positions of different diffraction maxima (other than the central maximum) are given by

(a) $\sin\theta = \pm \frac{m\lambda}{a}$; $m = 0, 1, 2, 3, ...$

(b) $\sin\theta \cong \pm \left(m + \frac{1}{2}\right) \frac{\lambda}{a}$; $m = 0, 1, 2, 3, ...$

(c) $\sin\theta = \pm \frac{m\lambda}{a}$; $m = 1, 2, 3, ...$

(d) $\sin\theta \cong \pm \left(m + \frac{1}{2}\right) \frac{\lambda}{a}$; $m = 1, 2, 3, ...$

[IIT-JAM GP : 2005]

ln. (b)

2. Monochromatic light of wavelength 633nm falls on a slit. If the angular separation between the first minimum of intensity on either side of the central maximum is 19.5 degrees, the slit width is approximately,

(a) 3.7 mm (b) 37 μm (c) 3.7 μm (d) 37 nm

[IISc : 2006]

ln. (c)

Angular separation, $\theta = \frac{1}{2} \times 19.5 = 9.75°$

The position of first minima after central maxima is given by

$$\Rightarrow \; a = \frac{\lambda}{\sin\theta} = 3.7\ \mu\text{m}$$

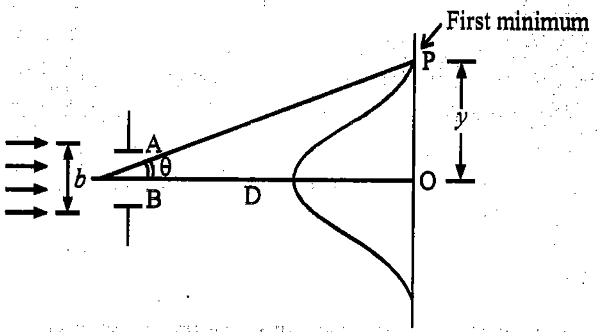
3. A parallel monochromatic beam of light is incident normally on a narrow slit. A diffraction pattern is formed on a screen placed perpendicular to the direction of the incident beam. At the first minimum of the diffraction pattern, the phase difference between the rays coming from the two edges of the slit is

(a) zero (b) π/2 (c) π (d) 2π

[IIT-JEE : 1998]

ln. (d)

At first minima, $b\sin\theta = \lambda$



$\because \sin\theta \simeq \theta$

$$\Rightarrow \; b\,\theta = \lambda \;\text{ or }\; b\left(\frac{y}{D}\right) = \lambda \;\text{ or }\; y = \frac{\lambda D}{b} \;\text{ or }\; \frac{yb}{D} = \lambda \qquad \ldots \text{(i)}$$

Now, at P (First minima) [path difference between the rays reaching from two edges (A and B) will be

$\Delta x = \frac{yb}{D}$ (Compare with $\Delta x = \frac{yd}{D}$ in YDSE)

or $\Delta x = \lambda$ [From equation (i)]

Corresponding phase difference (ϕ) will be

$$\phi = \left(\frac{2\pi}{\lambda}\right).\Delta x,\; \phi = \frac{2\pi}{\lambda}.\lambda = 2\pi$$

4. A beam of light of wavelength 600 nm from a distant source falls on a single slit 1.00 mm wide and the resulting diffraction pattern is observed on a screen 2m away. The distance between the first dark fringe on either side of the central maximum is **[DU 2015]**

(a) 0.6 mm (b) 1.2 mm (c) 1.8 mm (d) 2.4 mm

n. (b) X = 600nm, b = 1.00 mm, D = 2m

Since, we have for single slit

$\because\ b\sin\theta = \lambda$

$$\Rightarrow b\left(\frac{y}{D}\right) = \lambda \Rightarrow y = \frac{\lambda D}{b} = \frac{600\times10^{-9}\,m\times 2m}{1.0\times10^{-3}\,m} = 12\times10^{-4}\,m = 1.2\ mm$$

Ex. 5. Yellow light is used in a single slit diffraction experiment with slit width of 0.6 mm. If yellow light is replaced by X-rays, then the observed pattern will reveal

(a) that the central maximum is narrower
(b) more number of fringes
(c) less number of fringes
(d) no diffraction pattern

[IIT-JEE : 1999]

Soln. **(d)**

Diffraction is obtained when the slit width is of the order of wavelength of light (or any electromagnetic wave) used. Here, wavelength of X-rays (1-100 Å) << slit width (0.6 mm). Therefore, no diffraction pattern will be observed.

10.1.3 White light effect:

The condition for minima of rays incident normally on the slit is, $a\sin\theta = m\lambda$; $\Rightarrow \theta = \lambda/a$ [for first order dark band]. As $\lambda_r > \lambda_\upsilon$; $\theta_r > \theta_v$. If white light be employed, the central maximum becomes white, while other maxima will be coloured, and red maxima being farther apart than blue.

10.2 FRAUNHOFER DIFFRACTION IN A DOUBLE SLIT:

10.2.1 Introduction:

Let a parallel beam of monochromatic light of wavelength λ be incident normally on a surface containing two slits, each of width a and separated by an opaque feature of width b. The distance between any pair of corresponding points of the two slits is $d = a + b$. According to Huygens-fresnel principle each point of the incident wavefront in the plane of the two slits may be regarded as the source of secondary spherical wavelets. The secondary wavelets travelling normal to the slits are brought to focus by a convex lens L on the screen at C. The wavelets travelling at an angle θ with the normal are brought to focus at Q. Let us calculate the intensity of light at Q.

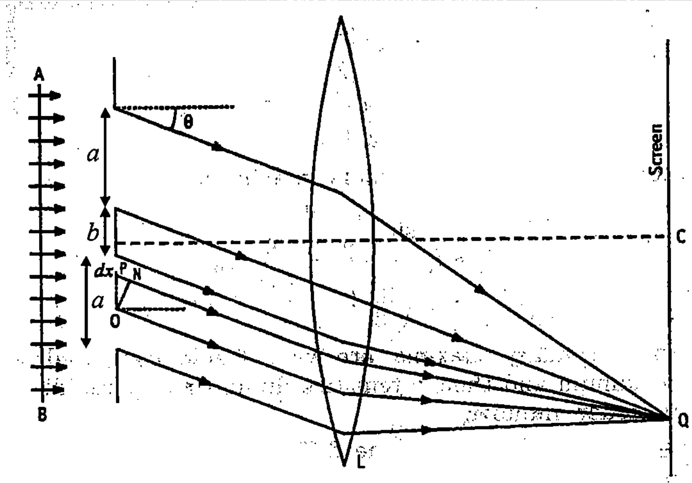


Figure 10.4

10.2.2 Intensity of the diffraction spots:

Let the complex light disturbance at any instant due to secondary waves from the origin O (mid point of the first slit) be represented by $Ae^{i\omega t}$ where A is the amplitude and ω is the circular frequency of the wave. The phase difference between the waves at Q coming from O and from a point P at a distance x from O is given by

$$\frac{2\pi}{\lambda} \times PN = \frac{2\pi}{\lambda} \times x\sin\theta = lx \qquad \ldots (13)$$

where, $l = \dfrac{2\pi\sin\theta}{\lambda}$

Hence the disturbance at Q due to secondary waves from P will be proportional to $e^{i(\omega t - lx)}$. The disturbance at Q due to the diffracting element dx can be written as

$$dy = CAdxe^{i(\omega t - lx)} \qquad \ldots (14)$$

where we assume the amplitude to be proportional to the width dx; C is a proportionality constant. The resultant complex disturbance at Q due to both the slits will be

$$y = \int_{-a/2}^{+a/2} CA.e^{i(\omega t - lx)}dx + \int_{d-a/2}^{d+a/2} CA.e^{i(\omega t - lx)}dx = CAe^{i\omega t}\left[\frac{e^{-ilx}}{-il}\right]_{-a/2}^{+a/2} + CAe^{i\omega t}\left[\frac{e^{-ilx}}{-il}\right]_{d-a/2}^{d+a/2}$$

$$= CAa.\frac{\sin\frac{al}{2}}{\frac{al}{2}}\left[1 + e^{-ild}\right].e^{i\omega t} \qquad \ldots (15)$$

The resultant intensity I at Q is obtained by multiplying y by its complex conjugate y^*. Thus

$$I = yy^* = (CAa)^2.\frac{\sin^2\frac{al}{2}}{\left(\frac{al}{2}\right)^2}\left(1 + e^{-ild}\right)\left(1 + e^{ild}\right)$$

$$= (CAa)^2.\frac{\sin^2\frac{al}{2}}{\left(\frac{al}{2}\right)^2}\left(2 + e^{ild} + e^{-ild}\right) = (CAa)^2.\frac{\sin^2\frac{al}{2}}{\left(\frac{al}{2}\right)^2}(2 + 2\cos ld)$$

$$= 4I_0\frac{\sin^2\alpha}{\alpha^2}.\cos^2\beta \qquad \ldots (16)$$

where $I_0 = (CAa)^2$, $\alpha = \dfrac{al}{2} = \dfrac{\pi}{\lambda}.a\sin\theta$ and $\beta = \dfrac{ld}{2} = \dfrac{\pi}{\lambda}(a + b)\sin\theta$

The resultant intensity thus depends on two factors : (i) $I_0\dfrac{\sin^2\alpha}{\alpha^2}$, which gives the diffraction pattern of a single slit and (ii) $\cos^2\beta$, which gives interference pattern of the diffracted light beams from the two slits.

10.2.3 Condition for minima:

The resultant intensity would be zero when either factor of Equation(16) becomes zero.

The factor $\dfrac{\sin^2\alpha}{\alpha^2} = 0$ when $\sin\alpha = 0$ but $\alpha \neq 0$. Therefore, $\alpha = m\pi$; $m = \pm 1, \pm 2, \ldots.$

$$\Rightarrow \qquad a\sin\theta = m\lambda \qquad \ldots (17)$$

These are position of diffraction minima.

The positions of minima due to the factor $\cos^2\beta$ are given by

$\beta = \pm(2s+1)\pi/2;\ s = 0, 1, 2, 3.....$

$$\Rightarrow \qquad (a+b)\sin\theta = \pm(2s+1)\lambda/2 \qquad ...(18)$$

These minima are known as the interference minima.

10.2. 4 Condition of maxima:

The positions of maxima due to the factor $\frac{\sin^2\alpha}{\alpha^2}$ are at $\alpha = 0$ and at values approaching $\pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}$ etc.

The positions of maxima due to the factor $\cos^2\beta$ given by $\beta = p\pi;\ p\pi = 0, \pm1, \pm2 \pm 3,...$

$$\Rightarrow \qquad (a+b)\sin\theta = p\lambda \qquad ...(19)$$

10.2.5 Missing order:

If the conditions for maxima of interference pattern equation (19) and minima of diffraction pattern equation(17) are simultaneously satisfied for a given value of θ then the corresponding interference maxima will be missing. If the slit width a is kept constant and separation b is varied then it is obvious from (19) the distance between consecutive interference maxima changes but the diffraction pattern due to single slit remains constant. Let for some value of θ conditions (17) and (19) are satisfied simultaneously. Then,

$$\frac{a+b}{a} = \frac{p}{m} \qquad ...(20)$$

Thus for missing order the ratio $\frac{a+b}{a}$ should be the ratio of two integer. For example, if $\frac{a+b}{a} = \frac{2}{1}$; or, $a = b$ then $p = 2m$. Hence 2, 4, 6 etc. orders of interferce maxima are absent which corresponds to 1, 2, 3 etc. order of diffraction dark bands. There will be interference maxima (corresponding to $p = 0, \pm1, \pm2$) within the central diffraction maximum.

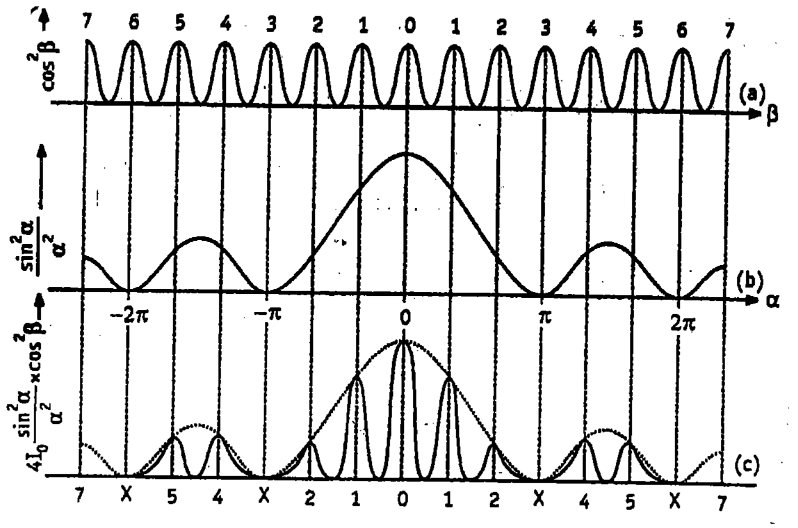


Figure **10.5**

Ex. 6. A parallel beam of light is incident normally on a narrow slit of width 0.22 mm. The Fraunhofer diffraction pattern is observed on a screen which is placed at the focal plane of a convex lens whose focal length is 70 cm. Calculate the distance between the first two secondary maxima on the screen. The wavelength of light = 600 nm and the lens is placed very close to the slit.

Soln. The first and second maxima occur at $\alpha = 1.43\,\pi$ and $2.46\,\pi$.

Thus, $a \sin\theta = 1.43\pi$ and 2.46λ

$$\text{or } \sin\theta = \frac{1.43\lambda}{a} = 3.9 \times 10^{-3} \text{ for 1st maximum}$$

$$\text{and } \sin\theta = \frac{2.46\lambda}{a} = 6.71 \times 10^{-3} \text{ for 2nd maximum}$$

Consquently the maxima will be separated on the screen by the distance given by

$(6.71 - 3.9) \times 10^{-3} \times 70 = 0.2$ cm.

10.3 PLANE DIFFRACTION GRATING:

10.3.1 Construction:

Plane diffraction grating consists of a number of parallel and equidistant lines ruled on an optically plane and parallel glass plate by a fine diamond point. The number of such ruled lines per mm is of the order of 100. Each ruled line behaves as an opaque line while the transparent portion between two consecutive ruled lines behaves as a slit. If a be the width of a clear space and b be the width of a ruled line, then the distance $(a + b)$ is called grating element or grating constant. The two points in the consecutive clear spaces whose distance of separation is $(a + b)$, are called corresponding points.

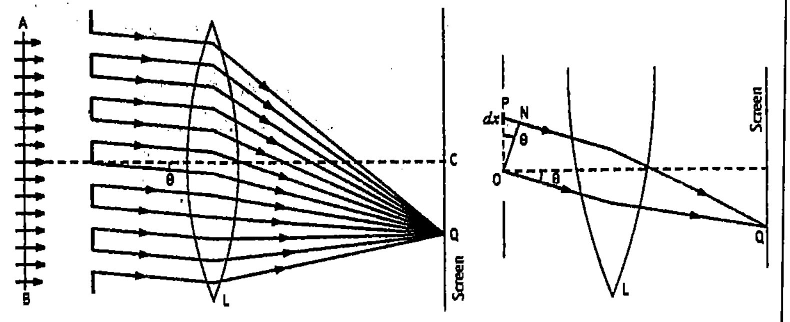


***Figure* 10. 6 (a)** ***Figure* 10. 6 (b)**

Drawing of exactly parallel and equidistant lines on a glass plate by a diamond point is an extremely difficult task and hence the grating becomes too costly. To make it comparatively cheaper, a cast of this ruled surface is made with it some transparent material and such casts are called replica gratings. Cellulose acetate, properly diluted, is poured on the surface of the master grating and then dried to a thin, though film which can be easily separated from the master grating under water. This film can then be mounted on a plane glass plate forming what is known as replica grating. Due to distortion and shrinkages of the seldom functions as well as its master.

10.3.2 Theory:

Let a parallel beam of monochromatic light of wavelength λ be incident normally on a plane diffraction grating consisting of N slits each of width a and with equal opaque space b between two successive slits. According to Huygens-Fresnel principle every point of the incident wavefront in the plane of the slits may be regarded as the origin of secondary spherical wavelets. The secondary wavelets travelling normal to the slits are brought to focus by a convex lens L on the screen at C. The wavelets travelling at an angle θ with the normal are brought to focus at Q. Let us calculate the intensity of light at Q. Let the complex light disturbance at Q due to secondary wavelets from O (the centre of the 1st slit) be represented by $Ae^{i\omega t}$ where A is the amplitude and ω is the circular frequency. The phase difference between the waves at Q [Figure 10.6 (b)] coming from O and from a point P at a distance x from O is given by

$$\frac{2\pi}{\lambda} \times PN = \frac{2\pi}{\lambda} \times x \sin\theta = lx \qquad \text{... (21)}$$

$$\text{where } l = \frac{2\pi}{\lambda}.\sin\theta$$

Hence the disturbance at Q due to secondary waves from P will be proportional to $e^{i(\omega t - lx)}$. The disturbance at Q due to the diffracting element dx can be written as

$$dy = CA\,dx.e^{i(\omega t - lx)} \qquad \text{... (22)}$$

where we assume the amplitude to be proportional to the width dx; C is a proportionality constant. The resultant complex disturbance at Q due to all the slits will be

$$y = \int_{-a/2}^{+a/2} CA.e^{i(\omega t - lx)}dx + \int_{d-a/2}^{d+a/2} CA.e^{i(\omega t - lx)}dx + ...\int_{(N-1)d-a/2}^{(N-1)d+a/2} CA.e^{i(\omega t - lx)}dx$$

where $d = a + b$ is the grating element.

$$\Rightarrow \quad y = CAe^{i\omega t}\left[\frac{e^{-ilx}}{-il}\right]_{-a/2}^{+a/2} + CAe^{i\omega t}\left[\frac{e^{-ilx}}{-il}\right]_{d-a/2}^{d+a/2} + ...+ CAe^{i\omega t}\left[\frac{e^{-ilx}}{-il}\right]_{(N-1)d-a/2}^{(N-1)d+a/2}$$

$$= CAa.\frac{\sin\frac{al}{2}}{\frac{al}{2}}\left[1 + e^{-ild} + ... + e^{-i(N-1)ld}\right].e^{i\omega t} = CAa\frac{\sin\frac{al}{2}}{\frac{al}{2}}.\frac{e^{iNld} - 1}{e^{-ild} - 1}.e^{i\omega t}$$

The resultant intensity I at Q is obtained by multiplying y with its complex conjugate y^*. Thus

$$I = yy^* = (CAa)^2.\frac{\sin^2\frac{al}{2}}{\left(\frac{al}{2}\right)^2}.\frac{e^{iNld} - 1}{e^{-ild} - 1}.\frac{e^{-iNld} - 1}{e^{+iNld} - 1} = (CAa)^2.\frac{\sin^2\frac{al}{2}}{\left(\frac{al}{2}\right)^2}.\frac{2 - e^{iNld} - e^{-iNld}}{2 - e^{ild} - e^{-ild}}$$

$$= (CAa)^2.\frac{\sin^2\frac{al}{2}}{\left(\frac{al}{2}\right)^2}.\frac{2 - 2\cos Nld}{2 - 2\cos ld} = I_0\frac{\sin^2\alpha}{\alpha^2}\frac{\sin^2 N\beta}{\sin^2\beta} \qquad \text{... (23)}$$

where $I_0 = (CAa)^2$, $\alpha = \frac{al}{2} = \frac{\pi}{\lambda}a\sin\theta$ and $\beta = \frac{\pi}{\lambda}d\sin\theta = \frac{\pi}{\lambda}(a + b)\sin\theta$

The resultant intensity thus depends on two factors : (i) $I_0\frac{\sin^2\alpha}{\alpha^2} = I_1$ which gives the diffraction pattern of a single slit and (ii) $\frac{\sin^2 N\beta}{\sin^2\beta} = I_2$ which gives the interference pattern of the diffracted light beams from N slits.

10. 3.3 Principal maxima:

If the slit width a is very small and observation is confined to the neighbourhood of the central pattern the variation of the factor $\frac{\sin^2\alpha}{\alpha^2}$ is small and under this condition the maxima will be solely controlled by the factor $I_2 = \frac{\sin^2 N\beta}{\sin^2 \beta}$. This factor is maximum when $\beta = m\pi$; $m = 0, \pm 1, \pm 2, ...$

$$\Rightarrow \qquad (a+b)\sin\theta = m\lambda \qquad ...(24)$$

These are known as principal maxima.

For $\beta = m\pi$, $I_2 = \frac{0}{0}$ and hence indeterminate. But in limit $\beta \to m\pi$ we get the maximum value of I_2.

$$\lim_{\beta \to n\pi} \frac{\sin N\beta}{\sin\beta} = \lim_{\beta \to n\pi} \frac{N\cos N\beta}{\cos\beta} = N \qquad \text{[using L Hospital's rule]}$$

This gives $I_2 = N_2$ and $I = I_{pm}$ (say) $= I_0 \frac{\sin^2\alpha}{\alpha^2} \times N^2 = I_1 \times N^2$ ...(25)

Thus the intensity of principal maxima increases as the number of slits (N) increases, but due to the presence of the factor $\frac{\sin^2\alpha}{\alpha^2}$, whose value decreases with the increase of the angle of diffraction (θ), the intensity of principal maxima decreases with the increase in the order number of bands.

10.3.4 Conditions for secondary minima and maxima:

The factor $I_2 = \frac{\sin^2 N\beta}{\sin^2\beta}$ depends on β and for maxima or minima we must have $\frac{dI_2}{d\beta} = 0$.

Now $$\frac{dI_2}{d\beta} = \frac{2N\sin N\beta . \cos N\beta}{\sin^2\beta} - \frac{2\sin^2 N\beta . \cos\beta}{\sin^3\beta}$$

$$= 2\frac{\sin^2 N\beta}{\sin^2\beta}\left(N\cot N\beta - \cot\beta\right)$$

Hence for maxima or minima, either $\frac{\sin N\beta}{\sin\beta} = 0$ or, $N\cot N\beta = \cot\beta$ ...(26)

(i) Secondary minima:

When $\sin N\beta = 0$ but $\sin\beta \neq 0$ then $\frac{\sin N\beta}{\sin\beta} = 0$ and hence intensity becomes zero (i.e., minimum).

Thus for minima, $N\beta = \pm s\pi$

$$\Rightarrow \qquad (a+b)\sin\theta = \pm\frac{s}{N}\lambda \qquad ...(27)$$

where 's' has integral values excepting *0, N, 2N, 3N*, etc. as for these values of s, $\sin\beta = 0$ and we obtain principal maxima. Thus it is evident from Equation (24) and (27) that between two consecutive principal maxima known are $(N-1)$ minima. Hence there will be $N-2$ other maxima known as secondary maxima between any two adjacent principal maxima.

(ii) Secondary maxima

The condition $N\cot N\beta = \cot\beta$ makes $\frac{dI_2}{d\beta} = 0$. Also, it can be shown that this condition makes $\frac{d^2 I_2}{d\beta^2}$ negative. Thus the values of β which satisfy the condition $N\cot N\beta = \cot\beta$ will give the positions of secondary maxima, except $\beta = m\pi$ which gives principal maxima.

Now to find the intensity of secondary maxima we note that $N\cot N\beta = \cot\beta$

$$\Rightarrow \qquad N^2\frac{\cos^2 N\beta}{\cos^2\beta} = \frac{\sin^2 N\beta}{\sin^2\beta}$$

$$\Rightarrow \qquad \frac{N^2\left(1-\sin^2 N\beta\right)}{1-\sin^2\beta} = \frac{N^2\sin^2 N\beta}{N^2\sin^2\beta} = \frac{N^2}{1+\left(N^2-1\right)\sin^2\beta} \qquad \left[\because \frac{a}{b}=\frac{c}{d}=\frac{a+c}{b+d}\right]$$

Hence the intensity of secondary maxima is given by,

$$I_{sm} = I_1\times\frac{N^2}{1+\left(N^2-1\right)\sin^2} = \frac{I_{pm}}{1+\left(N^2-1\right)\sin^2\beta}$$

$$\Rightarrow \qquad \frac{I_{sm}}{I_{pm}} = \frac{1}{1+\left(N^2-1\right)\sin^2\beta} \qquad \text{... (28)}$$

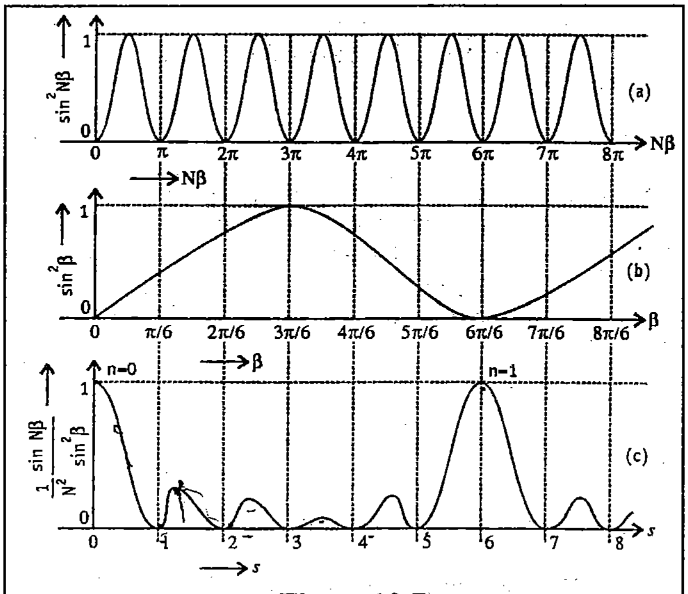


Figure **10.7**

10.3.5 Absent spectra and ghosts in a diffraction grating:

For the mth order principal maximum in the direction θ, we have the condition

$$(a+b)\sin\theta = m\lambda \qquad \text{... (29)}$$

Suppose the value of a is such that sth order diffraction minimum occurs in the same direction θ then

$$a\sin\theta = s\lambda \qquad \text{... (30)}$$

If these two conditions are satisfied simultaneously then mth order principal maximum will be absent from the resulting spectra. From equation (29) and (30) we have,

$$\frac{a+b}{a} = \frac{m}{s} \qquad \text{... (31)}$$

For example, if $a = b$ then $m = 2s$. Then $m = 2, 4, 6$, etc. order of principal maxima will be absent corresponding to the diffraction minima $s = 1, 2, 3$, etc.

Ghosts

In an ideal grating the rulings should be equally spaced. But in practice there remains some errors in the rulings. If the error is random the grating gives a continuous background illumination. If the error is progressive in nature the spectral lines becomes sharper in planes which are different from the focal plane of the optical system.

The most common error is periodic in nature. It arises from defects in the driving mechanism of the ruling machine. It gives rise to false lines accompanying the principal maxima of ideal grating. These additional false lines are known as ghosts.

10.3.6 Angular dispersive power of a grating:

The angular dispersive power of a grating is defined as the rate of change of the angle of diffraction (θ) with the change in wavelength i.e. $\frac{d\theta}{d\lambda}$.

If θ is the angle of diffraction for rays of wavelength λ which form the mth order bright band then we have

$$(a+b)\sin\theta = m\lambda$$

$$\Rightarrow \qquad \frac{d\theta}{d\lambda} = \frac{m}{(a+b)\cos\theta} \qquad \text{... (32)}$$

10.3.7 Comparision of grating and prism spectra:

(i) A prism produces only one spectrum but a grating produces a number of spectra. Consequently prism spectrum is brighter than the grating spectrum.

(ii) In grating spectra there is chance of overlapping of different spectral lines in higher order. As there is only one prismatic spectrum there is no such overlapping in prismatic spectrum.

(iii) In prism the deviation of violet colour is more than the deviation of red colour. But in grating spectrum the deviation of violet is smaller than that of red.

(iv) Resolving power of grating is much greater than that of the prism.

(v) Grating spectrum is nearly normal but the prism spectrum is nearer to normal.

(vi) Dispersion in the grating spectrum does not depend on the material of the grating, but dispersion in prism spectrum depends on the material of the prism.

(vii) Sometimes, due to periodic error in ruling additional false spectra known as ghosts are produced by the grating. But prism spectrum is free from such ghosts.

10.3.8 Determination of wavelength of light by using plane diffraction grating:

Since, $\quad (a+b)\sin\theta = m\lambda \quad \Rightarrow \quad \lambda = \frac{(a+b)\sin\theta}{m} = \frac{\sin\theta}{pm}$

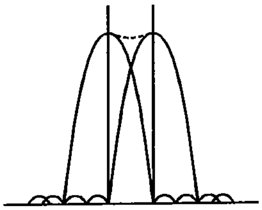
The quantity $p = \frac{1}{a+b}$ represents the number of rulings per unit length, which is usually supplied by the manufacturer of the grating. If not known, it can be easily determined by measuring the angle of diffraction θ for a light of known wavelength and then using the relation $p = \frac{\sin\theta}{m\lambda}$.

10.3.9 Resolution Power:

The resolution power of an optical instrument represent its ability to produce distinctly separate spectral lines of light having two or more close wavelength.

Rayleigh's criterion of resolution

Two spectral lines of equal intensities are just resolved by an optical instrument **when principal maximum of the diffraction pattern due to one falls on the first minimum of the diffraction pattern of the other.**



Just resolved

10.3.10 Resolution Power of a Grating

The resolution power of grating represents its ability to form separate spectral lines for wavelength very close together. It is measured by $\frac{\lambda}{d\lambda}$.

Where $d\lambda$ is the smallest wavelength difference that can be just resolved at wavelength λ.

It can be shown that for a grating resolving power is related with total number of ruling (N) on the grating and the order of spectrum by

$$R = \frac{\lambda}{d\lambda} = Nn$$

Resolving power for central principal maximum is zero as $n = 0$

$$R = Nn = \frac{N(a+b)\sin\theta n}{\lambda}$$

Here, $N(a+b)$ is total width of grating

So, resolving power of grating is directly proportional to total width of the ruled space on the grating.

Ex. 7. The second order maximum in the diffraction of X-rays of 0.20 nonometer wavelength from a simple cubic crystal is found to occur at an angle of thirty degrees with respect to the crystal plane. The distance between the lattice planes is

(a) 1 Angstroms (b) 2 Angstroms (c) 4 Angstroms (d) 8 Angstroms

[JEST : 2012]

Ans. (c)

Ex. 8. A diffraction grating of length 2.5×10^{-2} m is illuminated by a light with two wavelength 5997 A^0 and 6003 A^0. The maximum size of the grating element *d* (in μm) required to resolve the two wavelengths in the first order is

(a) 50 (b) 25 (c) 75 (d) 100

[IIT-JAM GP : 2012]

Soln. **(b)**

$$\text{Resolving power required} = \frac{\lambda}{d\lambda} = \frac{6000}{6} = 1000$$

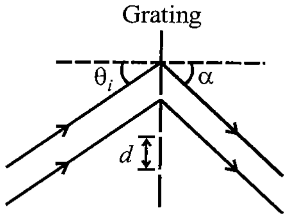
$R = Nn = 1000$

Where n = number of order and N and number of slits

So, $N = 1000$

$$\text{So, grating element } a + d = \frac{\text{length}}{\text{Number of slits}} = \frac{2.5\times10^{-2}}{1000} = 25\ \mu\text{m}$$

Ex. 9. A parallel beam of light of wavelength λ is incident on a transmission grating with groove spacing *d*, at an angle θ_i, as shown in the figure on the left. The plane of incidence is normal to the grooves. After diffraction, the transmitted beam is seen to be an angle α relative to the normal. Which of the following conditions must be satisfied for this to happen?



(a) $d(\sin\theta_i + \sin\alpha) = n\lambda$ (b) $2d\sin(\theta_i - \alpha) = n\lambda$

(c) $d(\sin\theta_i - \sin\alpha) = n\lambda$ (d) $2d\sin(\alpha + \theta_i) = n\lambda$

[TIFR : 2013]

ns. (a)

x. 10. A narrow beam of light of wavelength 589.3 nm from a sodium lamp is incident normally on a diffraction grating of transmission type. If the grating constant is 1000000 m^{-1}, the number of principal maxima observed in the transmitted light will be

(a) 7 (b) 5 (c) 3 (d) 1

[TIFR : 2010]

oln. **(d)**

Grating constant 10^6 per m.

$$= \frac{1}{10^6} = 10^{-6} m$$

$$\text{Highest order } n_{\max} = \left[\frac{a+b}{\lambda}\right] = \left[\frac{10^{-6}}{589.3\times10^{-9}}\right] = [1.697] = 1 \qquad [\because \sin\theta_{\max} = 1]$$

x. 11. What is the maximum width, a, of any clear space in a grating having 10^4 lines per inch, if complete second order visible spectrum is to be formed ?

oln. $d = \frac{2.54}{10^4} = 2.54\times10^{-4}$ cm.

The maximum angle θ in the second order visible spectrum $(\lambda = 700 \text{ nm})$ is given by

$$\sin\theta = \frac{2\times7.0\times10^{-5}}{2.54\times10^{-4}} = 0.5512$$

The half angular breadth of the central diffraction band must accommodate the complete second order spectrum, i.e.,

$$\sin\theta = \frac{\lambda}{a} = \frac{7.0\times10^{-5}}{a} = 0.5512 \quad \textbf{or} \quad a = \frac{7\times10^{-5}}{0.5512} \text{cm} = 1.27\times10^{-4} \text{ cm.}$$

Note: that this is just half the grating element d.

. 12. A beam of light is incident normally on a diffraction grating of width 2 cm. It is found that at 30°, n$^{\text{th}}$ order diffraction maximum for $\lambda_1 = 5000\mathring{A}$ is super-imposed on the $(n+1)^{th}$ order for $\lambda_2 = 4000\mathring{A}$. How many lines per cm does the grating have? Find out whether the first order spectrum from such a grating can be used to resolve the wavelengths $\lambda_3 = 5800\mathring{A}$ and $\lambda_4 = 5802\mathring{A}$?

[IIT-JAM : 2005]

ln. Condition for m$^{\text{th}}$ order maxima is $d\sin\theta = m\lambda$. Thus, $d\sin 30^0 = n\lambda_1$

And $d\sin 30 = (n+1)\lambda_2$

$\Rightarrow \quad n\lambda_1 = (n+1)\lambda_2\,;\; n = \dfrac{\lambda_2}{\lambda_1 - \lambda_2} = \dfrac{4000}{1000} = 4$

$\therefore \quad d\sin 30 = 4 \times 5000 \quad \Rightarrow d = 40000\ \mathring{A}$

Number of lines per cm, $N = \dfrac{1\,cm}{d} = \dfrac{1\times10^{-2}}{40000\times10^{-10}} = 2500$

Resolving power of grating $=$ Nm
For $m = 1$
Resolving power $= 2500 \times 1 = 2500$

Resolving power required to resolve to wavelengths λ_3 and λ_4 is given by $\dfrac{(\lambda_3+\lambda_4)/2}{\lambda_4-\lambda_3} = \dfrac{5801}{2} = 2900.5$

Since required resolving power is larger than resolving power of grating, these two wavelengths can not be resolved in first order spectrum.

Ex. 13. A parallel beam of light of diameter 1.8 cm contains two wavelengths 4999.75Å and 5000.25Å. The light is incident perpendicular on a large diffraction grating with 5000 lines per centimeter.
(a) Using Rayleigh criteriion, find the least order at which the two wavelengths are resoleved.
(b) What will be the angular separation (in radians) of the two wavelengths at order n = 2?

[IIT-JAM : 2009]

Soln. (a) Resolving power required to resolve two wavelenghts $= \dfrac{\lambda}{d\lambda} = \dfrac{5000}{0.5} = 10000$

So, resolving power of grating must be greater than 10000.

Resolving power of grating $R = Nn \geq 10000 \Rightarrow n \geq \dfrac{10000}{5000\times1.8} = \dfrac{10}{9} = 1.11;\ n_{\min} = 2$

[Total number of line = 5000 lines/cm×1.8 (diameter of parallel beam)]

(b) $(a+b)\sin\theta = n\lambda$ ... (1)

Differentiating, we get $(a+b)\cos\theta d\theta = nd\lambda$ and $(a+b) = \dfrac{10^{-2}}{5000} = 2\times10^{-6}\,m$ ... (2)

From (1), $\sin\theta = \dfrac{n\lambda}{(a+b)} = \dfrac{2\times5000\times10^{-10}}{2\times100} = \dfrac{1}{2};\ \theta = \dfrac{\pi}{6},\ \cos\theta = \dfrac{\sqrt{3}}{2}$

From (2), $2\times10^{-6}\times\dfrac{\sqrt{3}}{2}d\theta = 2\times0.5\times10^{-10}\,;\ \boxed{d\theta = \dfrac{1}{\sqrt{3}}\times10^{-4}\ \text{radian}}$

Ex. 14. A diffraction grating having N slits, each of width 'b' and period d, is illuminated normally by a monochromatic plane wave of wavelength λ.
(a) Obtain an expression for the highest diffraction order that can be observed. What is the phase difference between waves from first and N^th^ in the highest diffraction.
(b) If alternate slits are covered with a retarder that retards the wave by π, obtain an expression for the intensity distribution of the Fraunhofer diffraction pattern?

[IIT-JAM : 2011]

Soln. (a) Condition for maxima in case of grating $(\mathrm{b}+\mathrm{d})\sin\theta = \mathrm{m}\lambda$

The highest diffraction order can be observed when $\theta = \dfrac{\pi}{2},\ (\mathrm{b}+\mathrm{d}) = \max.\lambda$

The electric field due to 1st slit reaching P, $E_1 = A\frac{\sin\beta}{\beta}\sin(\omega t-\beta+0)$

Due to N$^{\text{th}}$ slit, $E_N = A\frac{\sin\beta}{\beta}\sin(\omega t-\beta+(N-1)\phi)$

The phase difference between 1$^{\text{st}}$ and N$^{\text{th}}$, $\Delta\phi=(N-1)\phi-0=(N-1)\frac{2\pi}{\lambda}(b+d)\sin\theta$

For highest diffraction order $\theta=\frac{\pi}{2}$. $\Rightarrow \Delta\phi=(N-1)2\pi\frac{(b+d)}{\lambda}$

(b) If alternate slits are covered with retarder which retards wave by π the resultant dielectric field at P is:

$$\vec{E}=\frac{A\sin\beta}{\beta}\left[\sin(\omega t-\beta)+\sin(\omega t-\beta+\phi+\pi)+.....\sin(\omega t-\beta+(N-2)\phi)+\sin(\omega t-\beta)+(N-1)\phi)\right]$$

$$\vec{E}=\frac{A\sin\beta}{\beta}\left[\sin(\omega t-\beta)-\sin(\omega t-\beta+\phi)+.....\sin(\omega t-\beta+(N-2)\phi)-\sin(\omega t-\beta)+(N-1)\phi\right]$$

$$E=\text{imaginary part of }\frac{A\sin\beta}{\beta}\left[e^{i(\omega t-\beta)}-e^{i(\omega t-\beta+\phi)}+......e^{i(\omega t-\beta+(N-2)\phi)}-e^{i(\omega t-\beta+(N-1)\phi)}\right]$$

$$=\text{Immaginary part of }\frac{A\sin\beta}{\beta}e^{i(\omega t-\beta)}\left[1-e^{i\phi}+e^{i2\phi}-......+e^{i(N-2)\phi}-e^{i(N-1)\phi}\right]$$

$$=\text{Immaginary part of }\frac{A\sin\beta}{\beta}e^{i(\omega t-\beta)}\left(1-e^{i\phi}\right)\left[1+e^{i2\phi}+....+e^{i(N-2)\phi}\right]$$

$$=\text{Immaginary part of }\frac{A\sin\beta}{\beta}e^{i(\omega t-\beta)}\left(1-e^{i\phi}\right)\left[\frac{1-e^{iN\phi}}{1-e^{i2\phi}}\right]$$

$$=\text{Immaginary part of }\frac{A\sin\beta}{\beta}e^{i(\omega t-\beta)}\left(1-e^{i\phi}\right)\frac{e^{\frac{iN\phi}{2}}}{e^{i\phi}}\left[\frac{e^{\frac{-iN\phi}{2}}-e^{\frac{iN\phi}{2}}}{e^{-i\phi}-e^{i\phi}}\right]$$

$$=\text{Immaginary part of }\frac{A\sin\beta}{\beta}e^{i\left[(\omega t-\beta)+\left(\frac{N}{2}-1\right)\phi\right]}\left(1-e^{i\phi}\right)\frac{\sin N\frac{\phi}{2}}{\sin\phi}$$

$$=\text{Immaginary part of }\frac{A\sin\beta}{\beta}\frac{\sin N\frac{\phi}{2}}{\sin\phi}\left[e^{i\left(\omega t-\beta+\left(\frac{N}{2}-1\right)\phi\right)}-e^{i\left(\omega t-\beta+\frac{N}{2}\phi\right)}\right]$$

$$=A\frac{\sin\beta}{\beta}\frac{\sin N\frac{\phi}{2}}{\sin\phi}\left[\sin\left(\omega t-\beta+\left(\frac{N-1}{2}\right)\phi\right)-\sin\left(\omega t-\beta+\frac{N}{2}\phi\right)\right]$$

$$=A\frac{\sin\beta}{\beta}\frac{\sin N\frac{\phi}{2}}{\sin\phi}2\cos\left(\omega t-\beta+\left(\frac{N-1}{2}\right)\phi\right)\sin\left(\frac{\phi}{4}\right)$$

Resutant intensity. $I=A^2\frac{\sin^2\beta}{\beta^2}\frac{\sin^2 N\gamma}{\sin^2 2\gamma}\sin^2\frac{\gamma}{2}$ $\therefore$ $\gamma=\frac{\phi}{2}$

SOLVED EXAMPLES

1. A convex lens of focal length 40 cm is employed to focus the Fraunhofer diffraction pattern of a single slit of 0.3 mm width. Calculate the linear distance of the first order dark band from the central band. Take the wavelength of light $\lambda = 589$ nm.

Soln. We know that for mth dark band, $a\sin\theta = m\lambda$

Here, $a = 0.3$ mm $= 0.3 \times 10^{-3}$ m; $m = 1$; $\lambda = 589 \times 10^{-9}$ m and $\sin\theta \approx \theta$ radian for θ is very small.

$$\Rightarrow \quad \theta = \frac{589 \times 10^{-9}}{0.3 \times 10^{-3}} \text{ radian} = 0.001963 \text{ radian} = 196.3 \times 10^{-5} \text{ radian}$$

Linear distance $= x = \theta \times f = 196.3 \times 10^{-5} \times 0.4\,\text{m} = 0.7852$ mm

2. Fraunhofer double slit diffraction pattern is observed in the focal plane of a lens of focal length 0.5 m. The wavelength of incident light is 500 nm. The distance between two maxima adjacent to the maximum of zero order is 5 mm and the fourth order maximum is missing. Find the width of each slit and the distance between their centres.

Soln. For missing of 4th order maximum we have $\frac{a+b}{a} = 4$; or, $b = 3a$

Now for 1st order maximum $(a+b)\sin\theta_1 = \lambda$

$$\text{or,} \quad \theta_1 \approx \frac{\lambda}{a+b} = \frac{\lambda}{4a}$$

Therefore, distance between two maxima adjacent to the maximum of zero order is

$$x = f \times 2\theta_1 = \frac{2f\lambda}{4a}$$

$$\Rightarrow \quad a = \frac{f\lambda}{2x} = \frac{0.5 \times 500 \times 10^{-9}}{2 \times 5 \times 10^{-3}} \text{ m} = 0.025 \text{ mm}, \; b = 3a = 0.075 \text{ mm}$$

3. How many orders would be visible, if the wavelength of incident light is 589 nm and the number of lines in the grating is 100 per mm?

Soln. For principal maxima, $(a+b)\sin\theta = m\lambda \quad \Rightarrow \sin\theta = pm\lambda$

Here $\theta = 90°$ (maximum); $\lambda = 589 \times 10^{-9}$ m; $p = \frac{1}{a+b} = 100 \times 10^{3}$ per m

$$\Rightarrow \quad m = \frac{a+b}{\lambda} = \frac{1}{10^5 \times 589 \times 10^{-9}} = 16.32$$

4. Light, which is a mixture of two wavelength 500 nm and 520 nm is incident normally on a plane transmission grating having 1000 lines / mm. A lens of focal length 1 m is used to observe the spectrum on a screen. Calculate the separation of the two lines in the first order spectrum.

Soln. For 1st line, $\sin\theta_1 = pm\lambda = 10^6 \times 1 \times 500 \times 10^{-9} = 0.50$

Thus, $\theta_1 = 30°$

For 2nd line, $\sin\theta_2 = 10^6 \times 1 \times 520 \times 10^{-9} = 0.52$

Thus $\theta_2 = 31°20'$. Hence the angular separation of the two lines is $\theta_2 - \theta_1 = 1°, 20' = (4/3)^{0}$. That is, angular separation $= \frac{\pi}{180} \times \frac{4}{3} = \frac{\pi}{135} = 0.02328$ radian. Hence the linear separation of the lines on a screen placed perpendicular to the deflected lines would be $x = 0.02328 \times 1\,\text{m} = 2.328$ cm.

5. The objective of a telescope has a diameter 2.54 m. Assuming the mean wavelength of incident light to be 550 nm calculate the least angular separation of two stars which can be resolved by it.

Soln. The least resolvable angle $= \theta = \frac{1.22\lambda}{D}$ radian.

Here $D = 2.54$ m, $\lambda = 550 \times 10^{-9}$ m.

$$\Rightarrow \quad \theta = \frac{1.22 \times 550 \times 10^{-9}}{2.54} = 2.642 \times 10^{-7} \text{ radian.}$$

$$\Rightarrow \quad \theta = \left(2.642 \times 10^{-7} \times \frac{180}{\pi} \times 3600\right)'' = (0.0545)''$$

A diffraction grating 2 cm wide is just able to resolve sodium D-lines in second order. Find the number of rulings per mm.

Soln. We have $\frac{\lambda}{d\lambda} = N.m$

Here m = 2, $\lambda = \frac{589.0 + 589.6}{2}$ nm $= 589.3$ nm, $d\lambda = 0.6$ nm

Therefore, $$N = \frac{\lambda}{m.d\lambda} = \frac{589.3}{2 \times 0.6} = 491$$

$\therefore$ Number of ruling per mm $= \frac{\text{Total number of lines}}{\text{width of grating in mm}} = \frac{491}{20} = 24.5$

Consider a grating of width 5 cm with slits of width a = 0.001 mm separated by a distance of 0.002 mm. How many orders would be visible at $\lambda = 550$ nm? Calculate the width of principal maxima. Would there be any missing orders?

oln. We know, $(a + b)\sin\theta = m\lambda$

Here a = 0.001 mm, b = 0.002 mm, $\lambda = 550 \times 10^{-6}$ mm

For $\theta = 90°$ (maximum), m (order no.) = 5.45

$\therefore$ Maximum number of observable order = 5

Angular width of principal maximum

$$= 2d\theta = \frac{2}{Nm\cot\theta} = \frac{2\lambda}{N(a+b)\cos\theta} = \frac{2 \times 550 \times 10^{-9}}{5 \times 10^{-2}} = 2.2 \times 10^{-5} \text{ radian}$$

where we put, N(a + b) = width of grating $= 5 \times 10^{-2}$ m and $\cos\theta \approx 1$.

Now, $$\frac{a+b}{a} = \frac{3}{1}$$

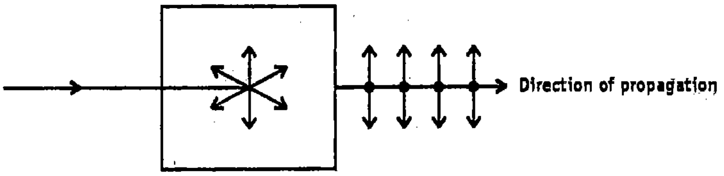
Hence, 3rd, 6th, 9th .... etc. order maxima will be missing.

Chapter 11

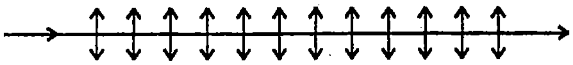
Polarisation of Light

11.1 INTRODUCTION:

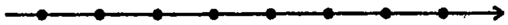
The phenomena of interference and diffraction of light show that the light is a wave motion which may be longitudinal or transverse in nature. It is the phenomenon of polarisation which for its explanation requires that the light must be a transverse wave. In fact in the electromagnetic wave consisting of vibrating electric and magnetic vectors at right angles to each other and also at right angles to the direction of propagation of light.



(a) Unpolarised light



(b) Plane polarised light with its vibration in the plane of the paper



(c) Plane polarised light with its vibration in the plane perpendicular to the plane of the paper

Figure 11.1

The electric vector is called light vector because it is responsible for the sensation of vision. In an ordinary beam of light with millions of waves the light vectors of component waves will remain in all possible directions on a plane drawn at right angles to the direction of propagation. This happens so due an ordinary beam of light with the electric atoms or molecules in the source. Such an ordinary beam of light with the electric vectors arranges symmetrically about the direction of propagations is called an unpolarised light. We may assume all these vibrations to be resolved into two rectangular components of equal amplitude but having a relative phase difference which varies rapidly and randomly. Now, if by some means one of these rectangular vibrations is cut off we get vibrations of all the component having $\vec{E}$ field in a particular direction. Such a light is said to be plane polarised light (also called p-state light).

The electric vector of a plane polarised light is confined to a particular plane, known as the plane of vibration. This plane contains the $\vec{E}$ -vector and the propagation vector $\vec{k}$. The plane perpendicular to the plane of vibration is known as plane of polarisation.

11.2 POLARISATION BY REFLECTION AND BREWSTER'S LAW:

If an ordinary beam of light is incident on a glass plate at a particular angle $i_p \approx 57°$ it is found that the reflected light becomes almost wholly a plane polarised light, the plane of vibration being perpendicular to the plane of incidence. For angles other than i_p, the reflected light is partially polarised. This particular angle of incidence at which the reflected light is almost wholly polarised is called the angle of polarisation for the given surface. It depends on the nature of the reflecting medium and the wavelength of light.

Brewster discovered that at the polarising angle the reflected and the refracted rays are just 90° apart. Therefore, the angle of refraction is $90° - i_p$. Hence the refractive index of glass with respect to air is,

$$n = \frac{\sin i_p}{\sin\left(90° - i_p\right)} = \tan i_p \qquad \ldots (1)$$

Thus the tangent of the angle of polarisation is equal to the refractive index of the reflecting medium. This is known as Brewster's law.

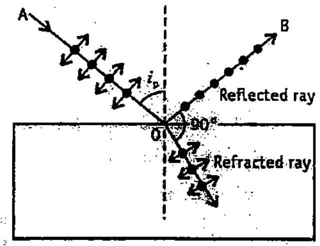


Figure 11.2

x. 1. Parallel light is incident from air on a surface of glass plate a Brewster angle. Which of the following statement is CORRECT ?
(a) Incident and reflected rays are right angle to each other
(b) Incident and refracted rays are parallel to each other
(c) Refracted and reflected rays are at right angle to each other
(d) There is no refracted ray in the glass medium

[IIT-JAM GP : 2007]

ln. (c)

x. 2. Unpolarized light is incident at Brewster's angle on the surface of a medium. Which of the following statements is FALSE ?
(a) The parallel component of the light is completely reflected
(b) The reflected light is completely polarized
(c) Some of the incident light is reflected and some is refracted
(d) The reflected light is perpendicular to the refracted light

[IIT-JAM GP : 2009]

ln. (a)

. 3. Sunlight is incident on the surface of the sea at an angle θ. A sailor viewing through a rotating polaroid can almost block the reflected light by adjusting the angle of the polaroid. The value of θ is about (Refractive index of water is 1.33): **[DU 2014]**

(a) 57° (b) 53° (c) 49° (d) 41°

ln. (b) Since the polanoid is able to almost block the reflecting sunlight completely, the reflected sunlight must be lie plane polarized.

From Brewster's law,

$$\mu = \tan\theta$$

$$\Rightarrow 1.33 = \tan\theta$$

$\Rightarrow$ $\theta = \tan^{-1}(1.33) \approx 53°$

where, μ = refractive index of water.

Ex. 4. Longitudinal waves will not show: **[DU 2014]**

(a) Interference (b) Diffraction (c) Polarisation (d) Total internal reflection

Soln. (c) Polarisation is a phenomenon possible only for transverse wave and not logitudinal waves.

Ex. 5. An unpolarized ray of light in a medium with refractive index n_1 is incident on a plane glass surface with refractive index $n_2(n_2 > n_1)$. The intensity of the reflected light is minimum when the angle of incidence i is given by **[DU 2015]**

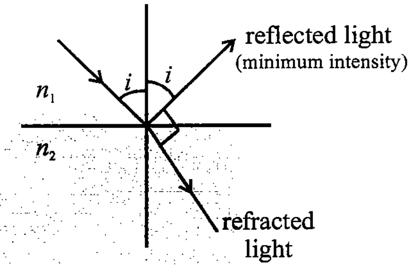
(a) $\tan(i) = n_1/n_2$ (b) $\tan(i) = n_2/n_1$

(c) $\tan(i) = 1/n_1 n_2$ (d) $\tan(i) = n_1 n_2$

Soln. **(b)** $n_1 \sin i = n_2 \sin(90° - i)$

$$\Rightarrow \frac{n_2}{n_1} = \frac{\sin i}{\cos i} = \tan i$$

$$\Rightarrow \boxed{\tan(i) = \frac{n_2}{n_1}}$$



11.3 POLARISATION BY REFRACTION:

If a beam of ordinary light is incident on a glass plate at the polarising angle i_p for glass, the transmitted light will be partially plane polarised though the reflected light will be almost wholly plane polarised. If a number of parallel glass plates are used then at each reflection, the rays whose vibrations are perpendicular to the plane of incidence are reflected. The transmitted light contains vibrations in the plane of incidence as well as a reduced amount of vibrations perpendicular to the plane of incidence. Thus with a large number of parallel plates the transmitted light loses more and more of its perpendicular vibrations at each reflection and ultimately becomes wholly polarised whose vibrations remain in the plane of incidence.

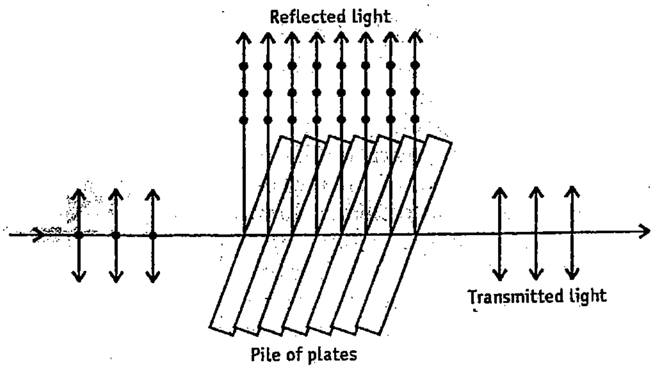


Figure 11.3

11.4 DOUBLE REFRACTION:

When an ordinary ray of unpolarised light is made incident on a crystal like calcite, quartz etc. it is refracted into two rays. One of these rays obeys the laws of refraction of

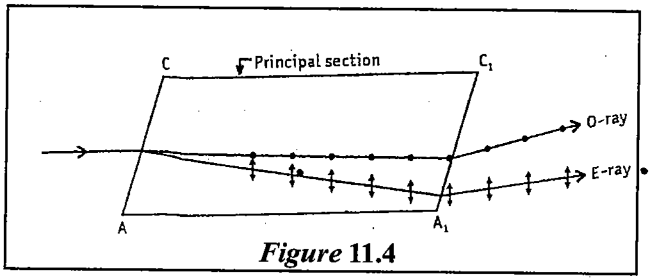


Figure 11.4

light and is called the ordinary ray (O-ray) while the other ray does not, in general, obey those laws and is called the extraordinary ray (E-ray). Both of these E and O-rays are plane polarised whose vibrations are along and at right angles to the principal section. This phenomenon in which a single incident ray is refracted into two rays is called double refraction or birefringence refracting crystals or birefringent.

11.5 OPTIC AXIS AND PRINCIPAL SECTION OF A CRYSTAL:

Optic axis of crystal is defined as the direction through it, along which if a ray travels then there will be no double refraction of the ray and both the ordinary and extraordinary rays travel with equal velocity along this direction.

A doubly refracting crystal possessing only one optic axis is called a uniaxial crystal.

Principal section of a crystal is its section defined by a plane which passes through the optic axis of the crystal and is perpendicular to its two opposite refracting faces.

11.6 NICOL PRISM:

Nicol prism is an optical device designed from calcite and is used for the production and analysis of plane polarised light. A beam of ordinary light entering the calcite crystal breaks up into E-rays and O-rays by double refraction. The O-ray is cut off by internal refraction while the E-ray is allowed to pass through.

To construct a Nicol prism a calcite rhombohedron of length three times its breadth is produced from natural crystal by cleavage. The end faces are then cut artifically so that the acute angles of the principal section is reduced from 71° to 68°

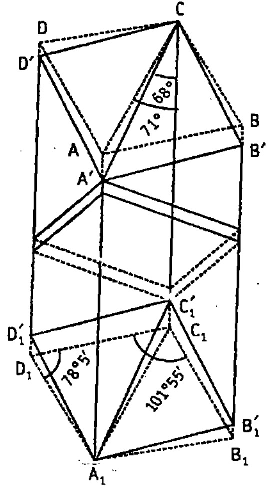


Figure 11.5 (a)

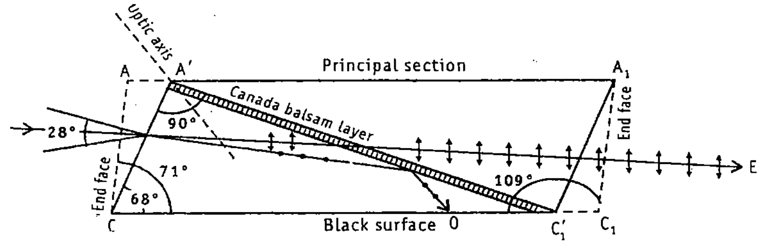
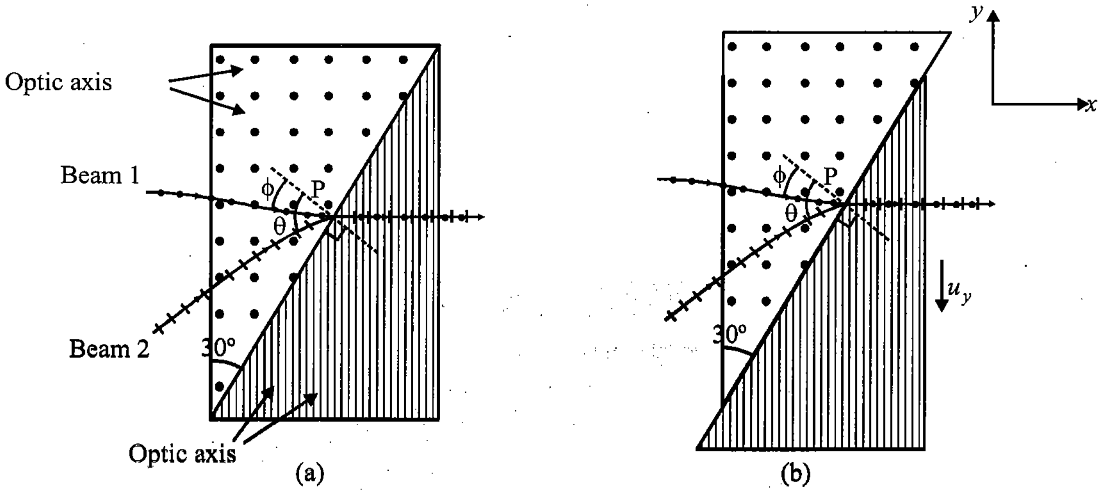


Figure 11.5 (b)

The resulting crystal is then cut into two portions by a plane perpendicular to both the principal section and the end faces of the rhomb. The cut surfaces are highly polished and cemented together by a layer of Canada-balsam. The sides of the prism are coated with lamp-black and are kept covered by a brass tube. The two end faces are kept open so that the light may be incident on one end face and may emerge out of the other end face.

A section of the resulting crystal parallel to the principal section is shown in figure 11.5 (b).

Ex. 6. Two orthogonally polarized beams (each of wavelength 0.5 μm and with polarization marked in the figure) are incident on a two-prism assembly and emerge along x-direction, as shown. The prisms are of identical and n_0 and n_e are the refractive indices of the o-ray and e-ray, respectively. Use $\sin\phi = \dfrac{\sin\theta}{3}$, and $n_0 = \dfrac{\sqrt{3}+1}{4}$



(a) (b)

(a) Find the value of θ and n_e. **[Marks-09]**

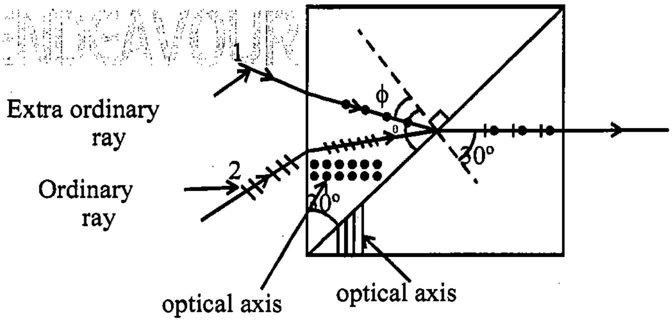
(b) If the right hand side prism starts sliding down with the vertical component of the velocity $u_y = 1\ \mu\text{m/s}$, what would be the minimum time after which the state of polarization of the emergent beam would repeat itself?

[IIT-JAM : 2012]

Soln. For beam 1 (at the interface of two crystals)

$$\frac{\sin i}{\sin r} = \frac{\mu_2}{\mu_1}$$

$$\frac{\sin\phi}{\sin 30} = \frac{n_o}{n_e} \qquad \text{\_\_\_\_\_\_ (i)}$$



$$\therefore n_e = \frac{n_o}{2\sin\phi} \qquad \text{\_\_\_\_\_\_ (ii)}$$

for beam 2 (At the interface of two crystals)

$$\frac{\sin i}{\sin r} = \frac{\mu_2}{\mu_1};\ \frac{\sin\theta}{\sin 30} = \frac{n_e}{n_o} \qquad \text{\_\_\_\_\_\_ (iii)}$$

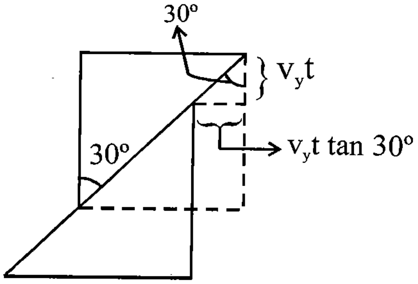
$$\text{(i)} \times \text{(ii) gives, } \sin\phi.\sin\theta = \frac{1}{4}$$

$$\text{Since } \sin\phi = \frac{\sin\theta}{3} \Rightarrow \sin\phi\cdot\sin\theta = \frac{\sin^2\theta}{3} = \frac{1}{4} \Rightarrow \sin\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = 60^\circ$$

From (ii) $$n_e = \frac{n_o}{2\times\frac{\sin\theta}{3}} = \sqrt{3}\,n_o = \sqrt{3}\left(\frac{\sqrt{3}+1}{4}\right) = \frac{3+\sqrt{3}}{4}$$

(b) State of polarization will repeate itself when path difference between e – ray and o – ray changes by λ

In time t the right crystal moves down by $v_y t$ distance.



Therefore, change in width of medium in time t = $v_y t \tan 30°$

$\therefore$ change in optical path $\Delta x = |n_0 - n_e|\, v_y t \tan 30°$

$$\because \quad |n_o - n_e| = \frac{3+\sqrt{3}-\sqrt{3}-1}{4} = \frac{1}{2}$$

$\therefore$ state of polarization will repeate in time t if

$$\Delta x = \frac{1}{2} v_y t \tan 30° = \lambda$$

$$\Rightarrow t = \frac{2\lambda}{v_y \tan 30°} = \frac{2\times 0.5\times 10^{-6}}{1\times 10^{-6}\times\frac{1}{\sqrt{3}}} = \sqrt{3}\text{ Sec.}$$

Due to sliding of right prism, length of medium traversed by the two beams changes, this leads to change in optical path difference.

11.7 MALUS'S LAW:

Malus experimentally obtain a relation which shows how the intensity of light transmitted by the analyser varies with the angle that its plane of transmission makes with that of the polariser. Suppose that the angle between the two planes of transmission be θ at any instant, as shown in Figure 11.6. The light vector $AP = E_0$ in the plane polarised light emerging from the polariser may be resolved into two components $AE = E_0\cos\theta$ and $AO = E_0\sin\theta$ respectively, along and perpendicular to the plane of transmission of the analyser. The perpendicular component $E_0\sin\theta$ is eliminated while the parallel component is freely transmitted through the analyser. Hence the intensity of light that emerges from the analyser is given by,

$$I = E_0^2\cos^2\theta = I_0\cos^2\theta \qquad \dots (2)$$

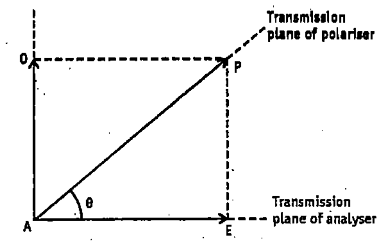

Transmission plane of polariser
O
P
θ
A
E
Transmission plane of analyser

Figure 11.6

where I_0 is the intensity of the polarised light incident on the analyser. This is, of course, one-half of the intensity of the unpolarised light incident on the polariser.

7. Unpolarized light of intensity I_0 passes successively through two identical linear polarisers A and B, placed such that their polarization axes are at an angle of 45° (see figure) with respect to one another. Assuming A and B to be perfect polarisers (i.e., no absorption losses), the intensity of the transmitted light will be $I_T =$

(a) $\frac{I_0}{4}$ (b) $\frac{I_0}{2\sqrt{2}}$ (c) $\frac{I_0}{2}$ (d) $\frac{I_0}{\sqrt{2}}$

[TIFR : 2011]

Soln. **(a)**

When an unpolarized light of intensity I_0 passes through a polariser intensity of transmitted light is equal to $I_0/2$.

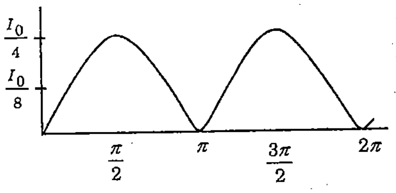
Intensity after $A = \frac{I_0}{2}$

Intensity of light after B

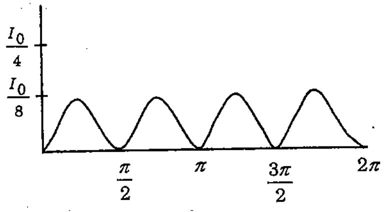
$$I = \frac{I_0}{2}\cos^2\theta = \frac{I_0}{4}$$

Ex. 8. Three polarizers P, Q and R are placed parallel to each other with their planes perpendicular to the z-axis. Q is placed between P and R. Initially the polarizing directions of P and Q are parallel, but that of R is perpendicular to them. In this arrangement when unpolarized light of intensity I_0 is incident on P, the intensity coming out of R is zero. The polarizer Q is now rotated about the z-axis. As a function of angle of rotation, the intensity of light coming out of R is best represented by

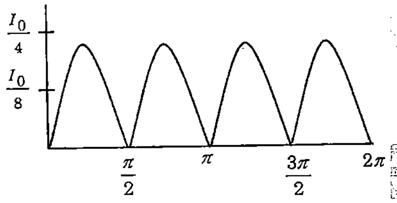
(a)



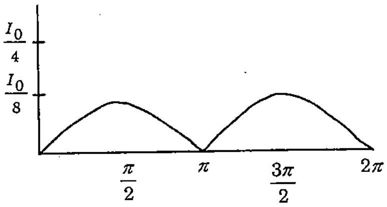
(b)



(c)



(d)



[IIT-JAM : 2007]

Soln. **(b)** $I_P = \frac{I_0}{2} \quad \Rightarrow \quad I_Q = I_P\cos^2\theta = \frac{I_0}{2}\cos^2\theta$, and

$$I_R = I_Q\cos^2(90-\theta) \Rightarrow I_R = \frac{I_0}{2}\cos^2\theta\sin^2\theta$$

11.8 SUPERPOSTION OF TWO PLANE POLARISED WAVES VIBRATING IN TWO MUTUALLY PERPENDICULAR PLANES:

Let a plane polarised monochromatic light be incident normally on a calcite plate cut with faces parallel to optic axis (Figure-11.7). Suppose the electric vector makes an angle θ with the optic axis. On entering the crystal the amplitude 'A' of the incident wave may be assumed to be resolved into two components one along the optic axis forming E-wave of amplitude $a = A\cos\theta$ and the other normal to the optic axis forming O-wave of amplitude $b = A\sin\theta$. If the incident wave is represented by $E = A\cos(\omega t - kz)$ then the component waves may be represented as

$$x = a\cos(\omega t - kz) \text{ and } y = b\cos(\omega t - kz) \qquad \ldots (3)$$

Now the E- and O-rays travel through the crystal with different velocities ($v_e > v_o$ in calcite) and therefore, they emerge from the crystal with certain relative phase difference, say, δ. So the emergent rays may be represented as

$$\Rightarrow \quad x = a\cos(\omega t - kz + \delta) \qquad \text{... (4)}$$

$$\Rightarrow \quad y = b\cos(\omega t - kz) \qquad \text{... (5)}$$

$$\Rightarrow \quad \frac{x}{a} = \cos(\omega t - kz)\cos\delta - \sin(\omega t - kz)\sin\delta$$

$$\Rightarrow \quad \frac{x}{a} = \frac{y}{b}.\cos\delta - \sqrt{1 - \frac{y^2}{b^2}}.\sin\delta$$

$$\Rightarrow \quad \left[\frac{x}{a} - \frac{y}{b}\cos\delta\right]^2 = \left(1 - \frac{y^2}{b^2}\right)\sin^2\delta$$

$$\Rightarrow \quad \frac{x^2}{a^2} - \frac{2xy\,.\cos\delta}{ab} + \frac{y^2}{b^2} = \sin^2\delta \qquad \text{... (6)}$$

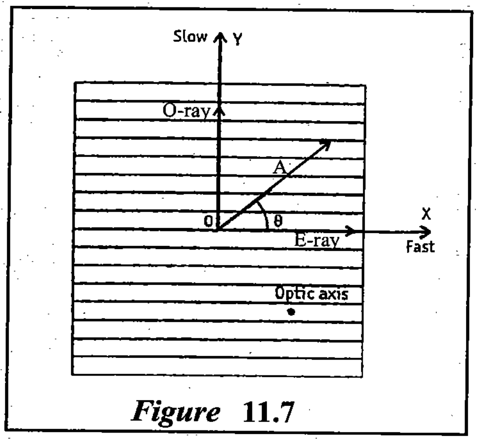


Figure **11.7**

This equation in general, represents an ellipse confined within a rectangle of sides $2a$ and $2b$.

11.8.1 Special cases:

(i) If $\delta = 2m\pi$, $m = 0, 1, 2, 3, ...$, Equation (6) simplifies to

$$y = \frac{b}{a}x = \tan\theta . x$$

In this case the emergent light is plane polarised, the plane of vibration making an angle θ with the x-axis.

(ii) If $\delta = (2m + 1)\pi$, $m = 0, 1, 2, 3, ...$ then Equation (6) simplifies to

$$y = -\frac{b}{a}x = -\tan\theta . x$$

The emergent light is again plane polarised with the direction of vibration making an angle $-\theta$ with the x-axis.

(iii) If $\delta = (2m + 1)\pi / 2$, $m = 0, 1, 2, 3, ...$ Equation(6) reduces to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

In this case the emergent light is elliptically polarised.

If $\delta = \pi/2, 5\pi/2, 9\pi/2$..... etc. then from Equations (4) and (5) we can write

$$x = -a\sin\omega t\,,\ y = b\cos\omega t \qquad \text{... (7)}$$

where we assume $z = 0$ as the plane of observation.

Equation (7) describe an ellipse in counter-clockwise direction (left-handed) with respect to an observer towards whom the wave is travelling. (Note that the vector $\vec{r} = \hat{i}x + \hat{j}y$ rotates counter-clockwise with increasing time).

If $\delta = 3\pi/2, 7\pi/2$,... etc. then from Equations (4) and (5) we get at $z = 0$,

$$x = a\sin\omega t\,;\ y = b\cos\omega t$$

It describes an ellipse in clockwise direction (right-handed). Here the vector $\vec{r} = \hat{i}x + \hat{j}y$ rotates clockwise with increasing time.

(iv) If $\delta = (2m+1)\pi/2$ and $\theta = 45°$ then a = b and the ellipse reduces to a circle, $x^2 + y^2 = a^2$

The emergent light is then said to be circularly polarised.

As above, for $\delta = \pi/2, 5\pi/2$.... etc. we get left circularly polarised light and $\delta = 3\pi/2, 7\pi/2$,.. etc. we get right circularly polarised light.

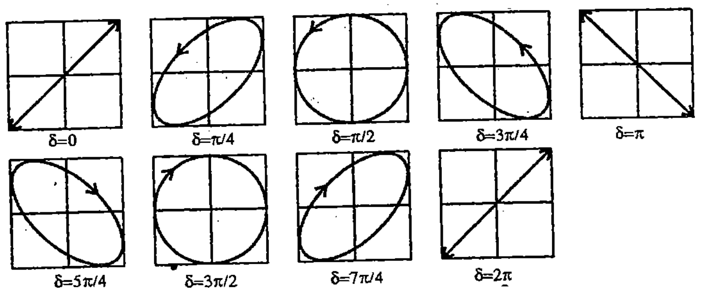


Figure **11.8**

11.9 RETARDATION PLATES:

A plate made of doubly refracting crystal with its refracting faces cut parallel to the optic axis and employed to introduce a given phase difference between *E*-ray and *O*-ray is called a retardation plate. If d be the thickness of the plate, n_o and n_e be the refractive indices for *O*-ray and *E*-ray then the path difference introduced by the plate is given by,

$$(n_0 \sim n_e).d \qquad \text{... (8)}$$

Corresponding phase difference is given by $\delta = \dfrac{2\pi}{\lambda}(n_0 \sim n_e)d$

QUARTER-WAVE PLATE

If the thickness 'd' of the plate is such that a **path difference of λ/4 or a phase difference of π/2** is introduced between O-ray and E-ray then the plate is called a **quarter-wave plate**. Thus the thickness of a quarter wave plate is given by,

$$(n_o \sim n_c)d = \lambda/4\,; \quad \Rightarrow d = \frac{\lambda}{4(n_o - n_e)} \qquad \text{... (9)}$$

HALF-WAVE PLATE

On the other hand, if the thickness d of the plate is such that a **path difference of λ/2 or a phase difference of π** is introduced between O- and E-waves then the plate is called a **half-wave plate**. The thickness of a half-wave plate is given by,

$$(n_o \sim n_e).d = \lambda/2 \quad \Rightarrow d = \frac{\lambda}{2(n_o \sim n_e)} \qquad \dots (10)$$

x. 9. Consider a beam of light of wavelength λ incident on a system of a polarizer and an analyzer. The analyzer is oriented at 45° to the polarizer. When an optical component is introuduced between them, the output intensity becomes zero. (Light is incident normally on all components). The optical component is:

(a) A full-wave plate (b) A half-wave plate

(c) A quarter-wave plate (d) An ordinary glass plate

[IIT-JAM : 2005]

oln. **(b) When half wave plate is introduced, the plane of polarization is rotates by** $(\pi - 2\theta)$. Since the intensity of light after analyser is zero. Plane of polarization rotates by an angle of 45°. Then

$$\pi - 2\theta = \frac{\pi}{4}, \quad \theta = \frac{3\pi}{8}$$

x. 10. $\vec{E}(x, y, z, t) = A(3\hat{i} + 4\hat{j})\exp[i(\omega t - kz)]$ represents an electromagnetic wave. Possible directions of the fast axis of a quarter wave plate which converts this wave into a circularly polarized wave are

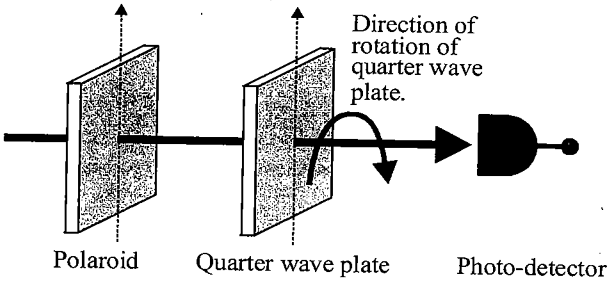
(a) $\frac{1}{\sqrt{2}}[7\hat{i} + \hat{j}]$ and $\frac{1}{\sqrt{2}}[-\hat{i} + 7\hat{j}]$ (b) $\frac{1}{\sqrt{2}}[3\hat{i} + 4\hat{j}]$ and $\frac{1}{\sqrt{2}}[4\hat{i} - 3\hat{j}]$

(c) $\frac{1}{\sqrt{2}}[3\hat{i} - 4\hat{j}]$ and $\frac{1}{\sqrt{2}}[4\hat{i} + 3\hat{j}]$ (d) $\frac{1}{\sqrt{2}}[7\hat{i} - \hat{j}]$ and $\frac{1}{\sqrt{2}}[\hat{i} + 7\hat{j}]$

[IIT-JAM : 2006]

ln. **(a)** Directions of vibration of $\vec{E}$ wave lie in direction of $(3\hat{i} + 4\hat{j})$. Unit vector in this direction is $\hat{n} = \frac{1}{5}[3\hat{i} + 4\hat{j}]$. In order to convert the plane light to circularly polarized, fast axis of qualter wave plate must make 45° angle with $\hat{n}$.

x. 11. A quarter-wave plate is placed in between a polarizer and a photo-detector. When the optic axis of the quarter-wave plate is kept initially parallel to the pass axis of the polarizer and perpendicular to the direction of light propagation. The intensity of light passing through the quarter-wave plate is measured to be I_0 (see figure). If the quarter wave plate is now rotated by 45° about an axis parallel to the light propagation, what would be the intensity of the emergent light measured by the photo-detector?



(a) $\frac{I_0}{\sqrt{2}}$ (b) $\frac{I_0}{2}$ (c) $\frac{I_0}{2\sqrt{2}}$ (d) I_0

[IIT-JAM : 2010]

n. **(d)**

. 12. A quater wave plate is inserted in between two crossed polarizers. The polarizer pass axes make an angle 45° with the fast and slow axes of the wave plate. If an unpolarized light beam is passed through the system, the transmission would be

(a) 0% (b) 25% (c) 50% (d) 100%

[IIT-JAM GP : 2011]

Soln. **(b)**

After passing through first polarizer, the unpolarized light will become plane polarized with intensity getting half. When plane polarized light passes through half wave plate, it becomes circularly polarized. After passing through the second polarizer, the intensity again becomes half.

So, final intensity is 25% of intensity of unpolarized beam.

Ex. 13. A half wave plate is an anisotropic optical element which introduces a retardation of $\lambda/2$ in the optical path length for the electric field component parallel to its optical axis, where λ is the wavelength of the incident on a half-wave plate with its polarization at 45° to the optical axis, the transmitted light is (compared to the initial polarization)

(a) linearly polarized and rotated by 45° (b) linearly polarized and rotated by 90°

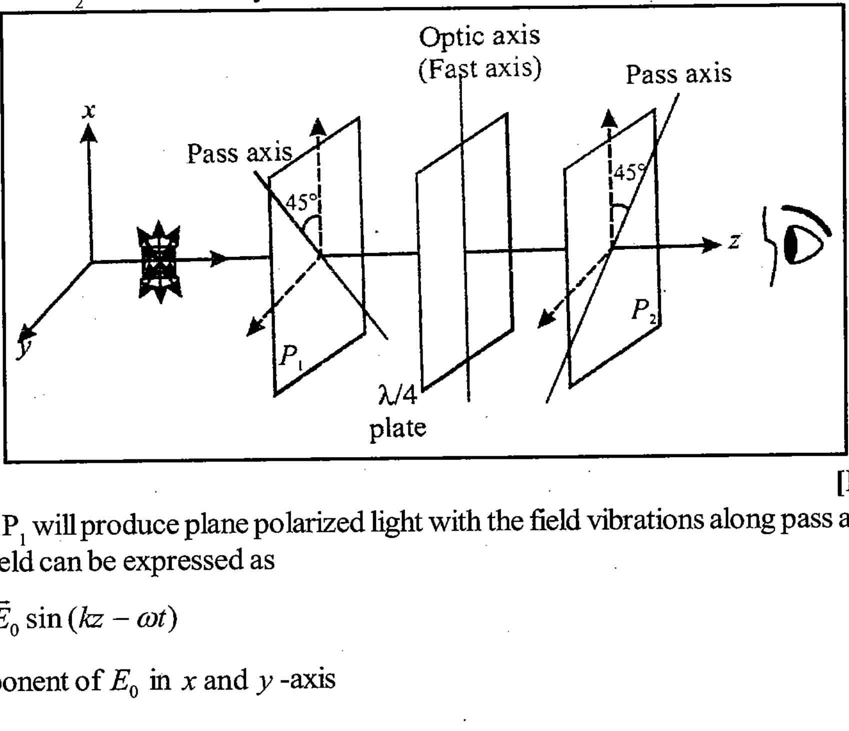
(c) left circularly polarized (d) right circularly polarized

[IISc : 2009]

Soln. **(b)**

When a linearly polarized light with plane of polarization making an angle θ with the optic axis of half wave plate falls on it, the emergent light remains linearly polarized with plane of polarization making an angle of $\pi - \theta$ with the optic axis. So, the amount of rotation is equal to $\pi - 2\theta$.

Ex. 14. A Calcite $\lambda/4$ plate is introduced between the two crossed polarizers (P_1 and P_2) as shown in the figure. An unpolarized beam is incident normal to P_1 from left hand side. Write down the expressions for the electric field associated with the beam after P_1, after $\lambda/4$ plate and after P_2. Also find the state-of-polarization after P_1, after $\lambda/4$ plate and after P_2 as observed by an observer shown in the figure.



[IIT-JAM GP : 2013]

Soln. The polarizer P_1 will produce plane polarized light with the field vibrations along pass axis

So, electric field can be expressed as

$$\vec{E} = \vec{E}_0 \sin(kz - \omega t)$$

Writing component of E_0 in x and y -axis

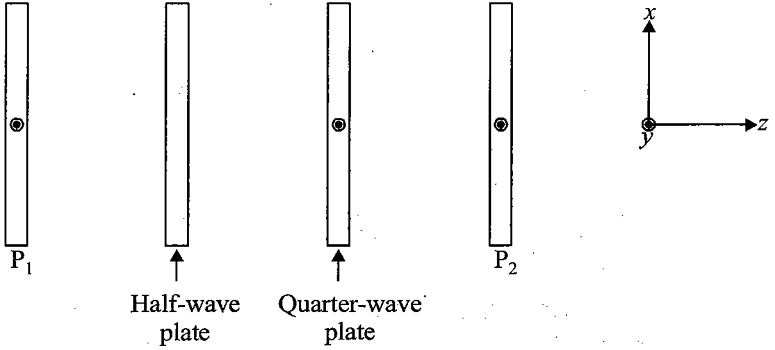
$$\vec{E} = E_0\left(\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}\right)\sin(kz - \omega t)$$

The plane polarized beam of light will fall on the $\lambda/4$ plate with its field vibration making angle of 45° w.r.t. optics axis. So, the $\lambda/4$ plate will form circularly polarized wave the tip of field vector moving in a circle given by $E_x^2 + E_y^2 = E_1^2$ where $E_1 = \frac{E_0}{\sqrt{2}}$, $E_x = \frac{E_0}{\sqrt{2}} \sin(kz - \omega t)$. The polarizer P_2 will produce plane polarized wave with its field vibration along pass axis and amplitude $\frac{E_0}{\sqrt{2}}$.

$$\vec{E} = \frac{E_0}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}\right)\sin(kz - \omega t)$$

$$= \frac{E_0}{2}\left(\hat{i} - \hat{j}\right)\sin(kz - \omega t).$$

x. 15. A half-wave plate and a quarter-wave plate are placed between a polarizer P_1 and an analyzer P_2. All of these are parallel to each other and perpendicular to the direction of propagation of unpolarized incident light (see figure). The optic-axis of the half wave plate makes an angle of 30° with respect to the pass-axis of P_1 and that of the quarter wave plate is parallel to the pass-axis of P_1.



(a) Determine the state of polarization for the light after passing through (i) the half wave plate and (ii) the quarter wave plate.

(b) What should be the orientation of the pass-axis P_2 with respect to that of P_1 such that the intensity of the light emerging from P_2 is maximum?

[IIT-JAM : 2008]

oln. (a) Let us assume that pass axis of polariser and quarter wave plate is along x-axis.

Let the equation of polarised light be $x = A\sin\omega t$, it will split into two components

$$x' = A\cos\left(\frac{\pi}{6}\right)\sin\omega t = \frac{A\sqrt{3}}{2}\sin\omega t \qquad \text{(Extraordinary ray)}$$

$$y' = A\sin\left(\frac{\pi}{6}\right)\sin\omega t = \frac{A}{2}\sin\omega t \qquad \text{(Ordinary ray)}$$

After passing through half-wave plate. Extra ordinary wave will suffer an additional phase change π with respect to ordinary wave

$$x' = \frac{A\sqrt{3}}{2}\sin(\omega t + \pi) = -\frac{A\sqrt{3}}{2}\sin\omega t \quad \Rightarrow y' = \frac{A}{2}\sin\omega t$$

Equation of lissajous figure, $y' = -\frac{1}{\sqrt{3}}x'$

After passing through half wave plate, the light will remain plane polarises but, the plane of vibration will now shift from $\frac{\pi}{6}$ to $\frac{5\pi}{6}$ with respect to optic axis.

Again, the plane polarises light will go under double refraction and split into ordinary and extraordinary light.

$$y = A\cos\frac{\pi}{6}\sin\omega t \qquad \text{(ordinary light)}$$

$$= A\frac{\sqrt{3}}{2}\sin\omega t$$

$$x = -A\sin\frac{\pi}{6}\sin\omega t \qquad \text{(extraordinary light)}$$

$$= -\frac{A}{2}\sin\omega t$$

The extra-ordinary light will suffer an additional phase

difference of $\frac{\pi}{2}$ with respect to ordinary light.

So, $\quad x = -\frac{A}{2}\sin\left(\omega t + \frac{\pi}{2}\right) = -\frac{A}{2}\sin\omega t \qquad \Rightarrow y = \frac{\sqrt{3}}{2}A\sin\omega t$

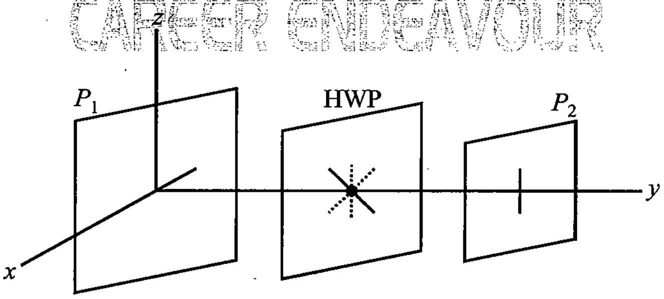
Lissagous figure $\dfrac{x^2}{\left(\frac{A}{2}\right)^2} + \dfrac{y^2}{\left(\frac{\sqrt{3}}{2}A\right)^2} = 1$ is an ellipse with major axis along y-axis.

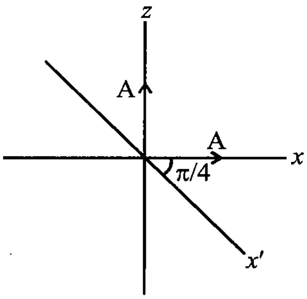
(b) The pass of axis of P_2 should be along the major axis of ellipse i.e. y-axis. So as to get maximum intensity of light.

Ex. 16. A thin rectangular polarizer P_1 with x-axis as pass axis is kept at the origin. Another identical polarizer P_2 with z-axis as the pass is kept parallel to P_1 at $y = d$. A Half Wave Plate (HWP) is introduced between P_1 and P_2 with its optic-axis making 45° with the pass axis of P_1. An unpolarized light propagating along y-axis is incident normally on P_1 from left hand side. What would be the state of polarization after P_1, half wave plate and P_2?

[IIT-JAM GP : 2007]

Soln. When an unpolarized light is incident normally on P_1 the light get plane polarized with vibration only x-axis. So, light will get plane polarized after P_1, when this plane polarized light fall on half wave plate. It breaks into the refracted rays–ordinary and extraordinary ray. After passing through half wave a phase difference of π will be introduced and plane of vibration will change from π/4 to π – π/4 w.r.t. optic axis. So, after half wave plate, the light will remain plane polarized but the plane of vibration which earlier was at angle π/4 w.r.t. optic axis will now be at angle 3π/4 w.r.t. optic axis i.e. along z-direction.





Now, when this plane polarized light falls on P_2 the pass axis of P_2 is parallel to direction of vibration. So, light will completely emerge out of P_2 as plane polarized and without any change in intensity.

11.10 PRODUCTION OF ELLIPTICALLY AND CIRCULARLY POLARISED LIGHT:

11.10.1 Elliptically polarised light:

An elliptically polarised light can be produced by superposing two mutually perpendicular coherent linear vibrations of unequal amplitudes but differing in phase by $\pi/2$. For example, the linear optical vibrations (at z = 0), $x = a\cos(\omega t + \pi/2)$, $y = b\cos\omega t$ ... (11)

yield an elliptic vibration given by, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ... (12)

Two such linear vibrations can be obtained by allowing a beam of plane polarised light to be incident normally on a quarter-wave plate with the direction of vibration making an angle θ other than 45° with the optic axis of the plate. The incident light of amplitude 'A' can be resolved into two components, one along the optic axis forming E-wave of amplitude $a = A\cos\theta$ and the other normal to the optic axis forming wave of amplitude $b = A\sin\theta$. Let these two components (at z = 0) be represented by,

$$x = a\cos\omega t \text{ and } y = b\cos\omega t. \qquad \text{... (13)}$$

On passing through the $\lambda/4$ plate a relative phase difference of $\pi/2$ will be introduced between these two waves.

11.10.2 Circularly polarised light:

A circularly polarised light can be produced by superposing two mutually perpendicular coherent linear vibrations of equal amplitudes but differing in phase by $\pi/2$. For example, the linear optical vibrations (at z = 0),

$$x = a\cos(\omega t + \pi/2) \text{ and } y = a\cos\omega t \qquad \text{... (14)}$$

yield a circular optical vibration $x^2 + y^2 = a^2$ ... (15)

Two such linear vibrations can be obtained by allowing a beam of plane polarised light to be incident normally on a $\lambda/4$ plate with the direction of vibration making an angle $\theta = 45°$ to the optic axis of the plate. The incident light of amplitude 'A' can be resolved into two components, one along the optic axis forming E-wave of amplitude $A\cos 45° = A/\sqrt{2}$ and the other normal to the optic axis forming E-wave of amplitude $A\sin 45° = A/\sqrt{2}$. Let these two components (at z = 0) be represented as,

$$x = \frac{A}{\sqrt{2}}\cos\omega t \text{ and } y = \frac{A}{\sqrt{2}}\cos\omega t \qquad \text{... (16)}$$

Now on passing through the $\lambda/4$ plate a relative phase difference of $\pi/2$ will be introduced between these two waves.

x. 17. Which of the following is a circularly polarized standing wave ?

(a) $\Psi(z,t) = (\hat{i}A_1 + \hat{j}A_2)\sin(kz)\cos(\omega t)$

(b) $\Psi(z,t) = (\hat{i}A_1 + \hat{j}A_2)\cos(\omega t - kz)$

(c) $\Psi(z,t) = [\hat{i}\cos(\omega t) + \hat{j}\cos(\omega t - \pi/2)]A\sin(kz)$

(d) $\Psi(z,t) = A\{\hat{i}\cos(\omega t - kz) + \hat{j}\cos[(\omega t - \pi/2) - kz]\}$

[IIT-JAM GP : 2011]

Soln. (c)

Ex. 18. Circularly polarized light consists of two perpendicular electromagnetic plane waves of equal amplitude, but differing by a phase θ. The magnitude of θ must be,

(a) $3\pi/4$ (b) π (c) $\pi/2$ (d) $\pi/4$

[IISc : 2006]

Soln. (c)

Since, for circular polarization, two waves must have equal amplitude and phase difference must be $\pi/2$.

11.11 DETECTION AND ANALYSIS OF POLARISED LIGHT BY USING NICOL PRISM (OR POLAROID) AND $\lambda/4$ PLATE:

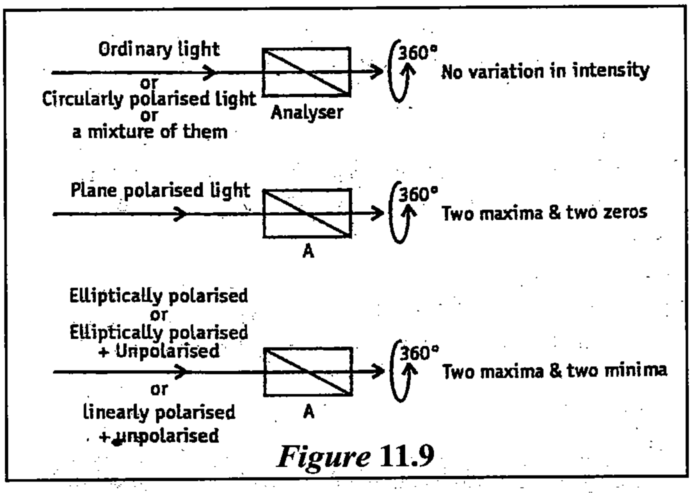

Ordinary light
or
Circularly polarised light
or
a mixture of them
Analyser
360°
No variation in intensity
Plane polarised light
A
360°
Two maxima & two zeros
Elliptically polarised
or
Elliptically polarised
+ Unpolarised
or
linearly polarised
+ unpolarised
A
360°
Two maxima & two minima

Figure 11.9

Distinction between circularly polarised light, unpolarised and a mixture of these two:

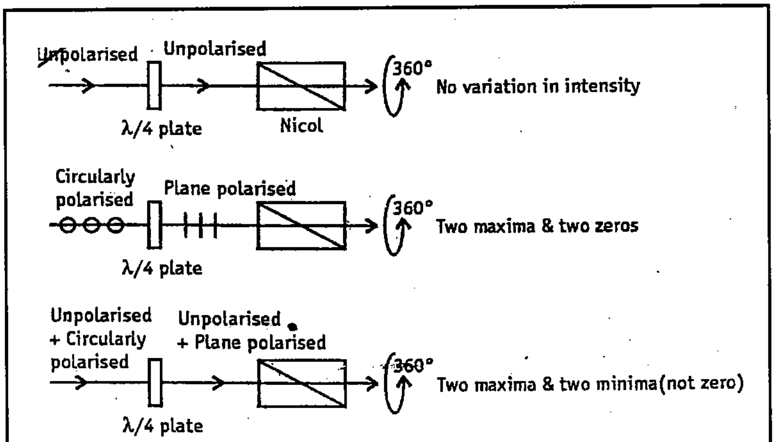

Unpolarised
Unpolarised
λ/4 plate
Nicol
360°
No variation in intensity
Circularly polarised
Plane polarised
λ/4 plate
360°
Two maxima & two zeros
Unpolarised + Circularly polarised
Unpolarised + Plane polarised
λ/4 plate
360°
Two maxima & two minima(not zero)

Figure 11.10

Distinction between elliptically polarised light, mixture of elliptically polarised and unpolarised light and mixture of plane and unpolarised light:

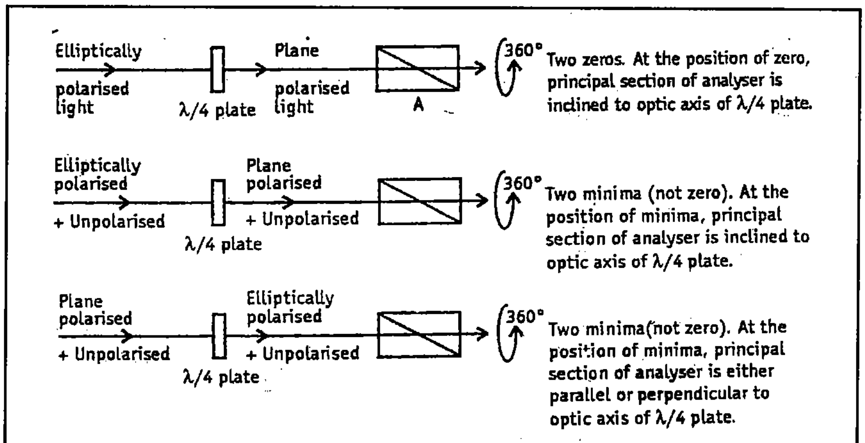
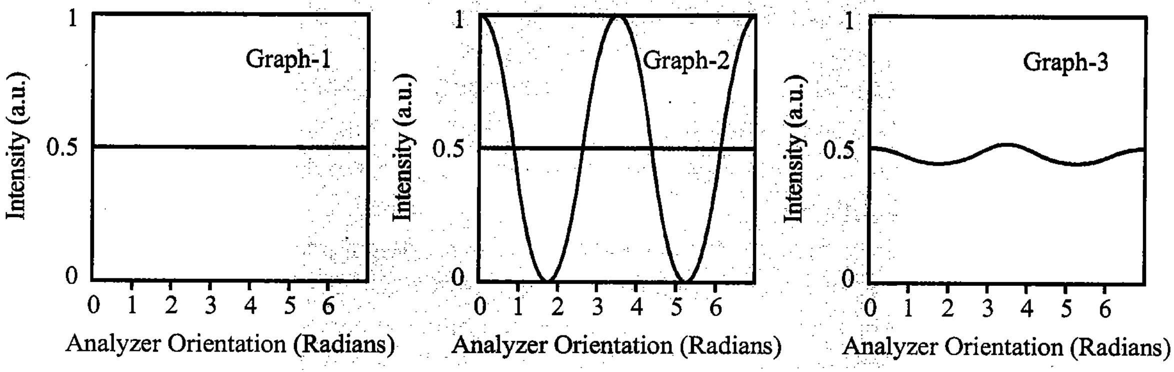

Elliptically polarised light
λ/4 plate
Plane polarised light
A
360°
Two zeros. At the position of zero, principal section of analyser is inclined to optic axis of λ/4 plate.
Elliptically polarised + Unpolarised
λ/4 plate
Plane polarised + Unpolarised
360°
Two minima (not zero). At the position of minima, principal section of analyser is inclined to optic axis of λ/4 plate.
Plane polarised + Unpolarised
λ/4 plate
Elliptically polarised + Unpolarised
360°
Two minima(not zero). At the position of minima, principal section of analyser is either parallel or perpendicular to optic axis of λ/4 plate.

Figure **11.10**

Ex. 19. Intensity of three different light beams after passing through an analyzer is found to vary as shown in the following graphs. Identify the option giving the correct states of polarization of the incident beams from the graphs.



(a) Graph 1: Linear Polarization, Graph 2: Circular Polarization, Graph 3: Elliptic Polarization
(b) Graph 1: Circular Polarization, Graph 2: Linear Polarization, Graph 3: Elliptic Polarization
(c) Graph 1: Unpolarization, Graph 2: Circular Polarization, Graph 3: Linear Polarization
(d) Graph 1: Unpolarization, Graph 2: Elliptic Polarization, Graph 3: Circular Polarization

[IIT-JAM PH : 2011]

Ans. (b), For explanation, see section 11.11.

Ex. 20. A linearly polarized electromagnetic wave is incident on a quarter wave plate. The emerge wave will in general be **[DU 2015]**

(a) linearly polarized
(b) elliptically polarized
(c) unpolarized
(d) mixture of linearly polarized and unpolarized wave

Soln. **(b)**

11.11.1 Fast and slow axes of $\lambda/4$ plate:

A quarter-wave plate ($\lambda/4$ plate) has two principal direction, one parallel and another perpendicular to the optic axis which is lying on its surface. The vibrations of the E-ray are along the optic axis while those of O-ray are perpendicular to the optic axis. In the case of a positive crystal like quartz, the *O*-ray travels faster than E-ray in the direction perpendicular to the optic axis and it is customary to call the direction perpendicular to the optic of positive crystal as the fast axis while the axis of positive crystal as the slow axis. In the case of a negative crystal like calcite, O-rays travel slower than E-ray in the direction perpendicular to the axis and for this the axis perpendicular to the optical axis of a negative crystal is called slow axis while the axis of the negative crystal is called fast axis.

11.12 OPTICAL ACTIVITY:

If plane polarised light is passed through some crystals like quartz along the optic axis the plane of vibration gradually undergoes rotation about the optical axis. The angle of rotation is found to depend on the thickness and nature of the crystal and also on the wavelength of the light employed. This phenomenon which rotates is called optical activity or rotatory polarisation and the substance which rotates the direction of vibration of the incident polarised light is called optically active substance.

There are two classes of active substances. One class of substance rotates the line of vibration of the incident light towards right and the substances belonging to this class are called **dextro-rotatory** substances. Another class of substance rotates the line of vibration towards left and the substances which belong to this class are called **Laevo-rotatory** substances.

11.13 BIOT'S LAWS : MEANING OF SPECIFIC ROTATION, MOLECULAR ROTATION AND ROTATORY DISPERSION:

(1) The angle (θ) of rotation produced by an active substance is proportional to the length (l) of the substance traversed by the ray and is not affected by turning the substance around, consequently the rotation will be annulled if the ray is reflected back through the substance.

(2) The combined rotation produced by two different substanc having different thickness is the algebraic sum of the rotation caused by each separately.

(3) The amount of rotation (θ) in the case of a solution, is proportional to the amount of active substance (m) present per c.c. of the solution.

(4) The amount of rotation varies with the wavelength of the incident light and with temperature of the substance. The rotation is approximately proportional to the inverse square of the wavelength.

In case of a solution, the first and third laws may be combined to have the relation $\theta = s/m$, where s is a constant and is called rotatory power or specific rotation of the solution. Thus specific rotation of a solution is defined as the amount of rotation produced by a 10 cm column of liquid containing 1 gm of the active substance per c.c. of the solution. Thus for a solution containing m mg. of active substance per c.c. the rotation (θ) for a path length l cm is given by,

$$\theta = s.m.(l/10) \qquad \ldots(17)$$

where s is the specific rotation. For a pure liquid m is simply the density.

In the case of optically active crystals specific rotation is defined as the rotation produced by a crystal plate of thickness 1 mm.

Molecular rotation is obtained by taking the product of the specific rotation and the molecular weight of the substance. Rotation of a plane polarised light produced by an optically active substance is found to be approximately proportional to the inverse square of the wavelength. So if a plane polarised white light is passed through an optically active substance it will decompose into constituent colours. This phenomenon is known as rotatory dispersion.

11.14 FRESNEL'S EXPLANATION OF ROTATORY POLARISATION

According to Fresnel a plane polarised light may be considered as the superposition of two equal and oppositely circularly polarised light. In case of optically active quartz the secondary wavelets due to O- and E-rays do not touch each other along the optic axis. As a result the two equal and opposite circular vibrations travel along the axis with unequal speed. On passing through the crystal some amount of phase difference (say δ) will be introduced between them. So emergent circular vibrations with plane of vibration is rotated by certain amount $(\delta/2)$ with respect to the incident vibration. This can be explained as, let vibration at z = 0 be represented by,

$$x = 2a\cos\omega t \qquad \dots(18)$$

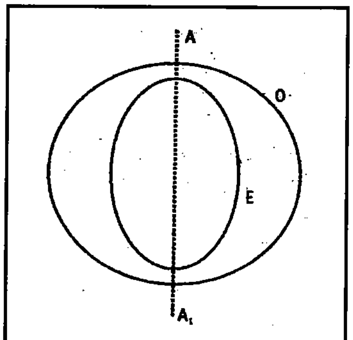


Figure **11.11**

This can be resolved into two circular motions:

$$\left.\begin{array}{l} x_1 = a\cos\,\omega t \\ y_1 = a\sin\,\omega t \end{array}\right\} \text{left circular} \quad \left.\begin{array}{l} x_2 = a\cos\,\omega t \\ y_2 = -a\sin\,\omega t \end{array}\right\} \text{right circular} \dots(19)$$

These vibration will travel with different velocities and as a result some phase difference will be introduced between them. Let the right circular motion travels faster than the left circular motion and there is a relative phase advancement of right circular motion by δ. Since only relative phases matter the emergent light can be represented by,

$$\left.\begin{array}{l} x_1 = a\cos\,\omega t \\ y_1 = a\sin\,\omega t \end{array}\right\} \text{and} \quad \left.\begin{array}{l} x_2 = a\cos\,(\omega t + \delta) \\ y_2 = -a\sin\,(\omega t + \delta) \end{array}\right\} \qquad \dots(20)$$

Thus the resultant vibrations along the x and y directions, on emergence are respectively given by,

$$x = x_1 + x_2 = a\left[\cos\omega t + \cos(\omega t + \delta)\right] = 2a\cos\frac{\delta}{2}.\cos\left(\omega t + \frac{\delta}{2}\right) \qquad \dots(21)$$

$$y = y_1 + y_2 = a\left[\sin\omega t - \sin(\omega t + \delta)\right] = -2a\sin\frac{\delta}{2}.\cos\left(\omega t + \frac{\delta}{2}\right) \qquad \dots(22)$$

Taking the ratio of x and y we get

$$y = \tan\left(-\frac{\delta}{2}\right)x$$

So the emergent light is plane polarised whose direction of vibration makes an angle $-\delta/2$ with the x-axis (which is the direction of vibration of incident plane polarised light)

If υ_L and υ_R be respectively the velocities of the left and right handed circular motion within the active substance $(\upsilon_R > \upsilon_L)$ then their time difference $= t = \left(l/\upsilon_L - l/\upsilon_R\right)$

[where l = thickness of active substance].

Therefore, Phase difference between the two circular vibrations is given by,

$$\delta = \frac{2\pi}{T}\left(\frac{l}{\upsilon_L} - \frac{l}{\upsilon_R}\right) \qquad \dots(23)$$

[$\because$ for periodic time difference T, the phase difference is 2π]

$$\Rightarrow \quad \delta = \frac{2\pi l}{cT}\left(\frac{c}{\upsilon_L} - \frac{c}{\upsilon_R}\right) = \frac{2\pi l}{\lambda}(n_L - n_R) \quad \text{... (24)}$$

where n_L and n_R are respectively the refractive indices of the active substance for left and right circularly polarised light along the direction of optic axis.

Again, $$\delta = 2\pi l\left(\frac{1}{\lambda_L} - \frac{1}{\lambda_R}\right) \quad [\because \upsilon_L T = \lambda_L;\ \upsilon_R T = \lambda_R]$$

$$\Rightarrow \quad \delta = \frac{2\pi l \cdot \Delta\lambda}{\lambda^2} \text{ (approx.)} \quad \text{... (25)}$$

Thus the rotation of the plane of vibration is given by

$$\phi = \delta/2 = \frac{\pi l}{\lambda}(n_L - n_R)$$

$$\Rightarrow \quad \phi = \delta/2 = \frac{\pi l \cdot \Delta\lambda}{\lambda^2} \quad \text{... (26)}$$

Thus the rotation varies as the thickness l and inversely as the square of the wavelength λ.

SOLVED EXAMPLES

1. An unpolarised plane light wave of intensity 10 mW/cm² passes through two Nicols with their principal sections at 30° to each other. Calculate the intensity of the transmitted wave.

Soln. The intensity (I_1) of the polarised light produced by the 1st Nicol is one half on the incident intensity (I_0). This is so because the incident unpolarised light can be resolved into two orthogonal incoherent linear vibrations each of intensity $I_0/2$ and only one component parallel to the transmission axis will pass through the 1st Nicol.

Thus, $$I_1 = \frac{1}{2}I_0 = 5 \text{ mW/cm}^2$$

Now according to the Malus's law the intensity of light transmitted through the 2nd Nicol is given by,

$$I_2 = I_1\cos^2\theta = 5 \times \cos^2 30° \text{ mW/cm}^2 = 3.75 \text{ mW/cm}^2$$

2. A plane polarised light of wavelength 600 nm changes to a circularly polarised light on passing through a quartz crystal cut parallel to optic axis. Calculate the minimum thickness to produce such effect. Given $n_e - n_o = 0.005$

Soln. The thickness d should be such that it will introduce a phase difference of $\pi/2$ between E- and O-rays. Thus,

$$\phi = \frac{\pi}{2} = \frac{2\pi}{\lambda} \times (n_e - n_o) \times d\ ;\ \Rightarrow d = \frac{\lambda}{4(n_e - n_o)} = \frac{600 \times 10^{-9}}{4 \times 0.005} \text{ m} = 0.003 \text{ mm}$$

3. A left circularly polarised light is passed through a half-wave plate. What would be the state of polarisation of the emergent light?

or

How can you convert a left-handed circularly polarised light into a right-handed circularly polarised light?

Soln. Let the incident left circularly polarised light be represented by

$$E_x = E_0\cos(\omega t - kz), E\text{-wave}$$

$$E_y = E_0\sin(\omega t - kz)$$

It represents a right circularly polarised light.

Find the state of polarisation when the x and y components of the electric field are given by

(i) $E_x = E_0 \sin(\omega t + kz)$ $\qquad E_y = E_0 \cos(\omega t + kz)$

(ii) $E_x = E_0 \cos(\omega t + kz)$ $\qquad E_y = \dfrac{E_0}{\sqrt{2}} \cos(\omega t + kz + \pi)$

oln. (i) $E_x^2 + E_y^2 = E_0^2$. It represent a circular vibration. At any particular position, say, z = 0, the vector $\vec{E} = \hat{i}E_x + \hat{j}E_y = \hat{i}E_0 \sin \omega t + \hat{j}E_0 \cos \omega t$ rotates in clockwise direction. Thus the light under question is a right circularly polarised light.

(ii) $E_y = \dfrac{E_0}{\sqrt{2}} \cos(\omega t + kz + \pi) = -\dfrac{1}{\sqrt{2}} E_x$

It represents a linearly polarised light making an angle $\tan^{-1}\left(-\dfrac{1}{\sqrt{2}}\right)$ with the x-axis.

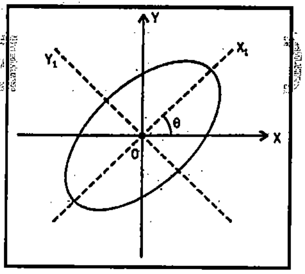
Elliptically polarised light is incident normally on a quarter-wave plate. Find the state of polarisation of the emergent light if the major axis of the quarter-wave plate makes an angle θ with the optic axis (i) θ = 0°, (ii) θ = 90°, (iii) θ = 30°.

oln. Suppose the incident light be a left-handed elliptic vibration represented by $x_1 = a\cos \omega t,\ y_1 = b \sin \omega t$ where the major and minor axis of the elliptic vibration are chosen as X_1 and Y_1 axes respectively. Let the major axis (OX_1) makes an angle θ with the optic axis (OX) of the crystal (see below figure). Hence the elliptic vibration can be resolved into components along OX and OY as,

$$x = x_1 \cos\theta + y_1 \cos(90° + \theta)$$

$$y = x_1 \sin\theta + y_1 \sin(90° + \theta)$$

On passing through the quarter-wave plate a relative phase difference of $\pi / 2$ will be introduced between x- and y-components of the electric field. Let the emergent light be represented by



$$\left.\begin{aligned} x &= a\cos(\omega t + \pi/2)\cos\theta + b\sin(\omega t + \pi/2)\cos(90° + \theta), \\ y &= a\cos\omega t \sin\theta + b\sin\omega t \sin(90° + \theta) \end{aligned}\right\} \quad \ldots(1)$$

(i) Now if $\theta = 0$, then $x = -a\sin\omega t,\ y = b\sin\omega t$

$$\Rightarrow \qquad y = -\frac{b}{a}x$$

It represents a linearly polarised light making an angle $\tan^{-1}\left(-\dfrac{b}{a}\right)$ with the optic (X) axis.

(ii) If $\theta = 90^\circ$ then $x = -b\cos\omega t,\ y = a\cos\omega t$

$$\Rightarrow \quad y = -\frac{a}{b}x$$

The emergent light is again plane polarised with the plane of vibration making an angle $\tan^{-1}\left(-\frac{a}{b}\right)$ with the optic axis.

(iii) If $\theta = 30^\circ$ then from Equation (1)

$$x = -a\sin\omega t.\frac{\sqrt{3}}{2} - b\cos\omega t.\frac{1}{2}$$

$$y = a\cos\omega t.\frac{1}{2} + b\sin\omega t.\frac{\sqrt{3}}{2}$$

Eliminating t from these equation we get,

$$\frac{(bx+ay)^2}{\left(a^2-b^2\right).\left(\frac{1}{2}\right)^2} + \frac{(ax+by)^2}{\left(b^2-a^2\right)\left(\frac{\sqrt{3}}{2}\right)^2} = 1$$

It indicates that the emergent vibration will be elliptic.

6. Write an expression for a linearly polarised light wave of angular frequency ω propagating in the +z direction with its plane of vibration making an angle 60° to the *xz* plane.

Soln. If E_0 be the amplitude of the wave, its x- and y-components are,

$$E_{0x} = E_0\cos 60^\circ = E_0/2 \text{ and } E_{0y} = E_0\sin 60^\circ = \frac{\sqrt{3}}{2}E_0$$

Hence the required expression may be of the form

$$\vec{E}_{(z,t)} = \hat{i}\frac{E_0}{2}\cos(kz-\omega t) + \hat{j}\frac{\sqrt{3}}{2}E_0\cos(kz-\omega t)$$

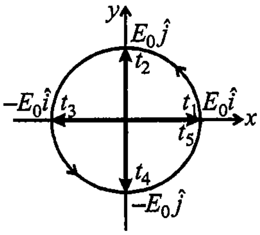
7. Describe the state of polarisation of the wave represented by

$$\vec{E}_{(z,t)} = \hat{i}E_0\cos(kz-\omega t) - \hat{j}E_0\sin(kz-\omega t)$$

Soln. The magnitude of $\vec{E}$ is E_0 which is constant, but the direction of $\vec{E}$ is a function of *z* and *t*. Let us examine its behaviour at a fixed space point *z = 0* and at different times.

$$\vec{E}(0,t) = \hat{i}E_0\cos\omega t + \hat{j}E_0\sin\omega t$$

At $t_1 = 0$	$\vec{E} = E_0\hat{i}$
At $t_2 = \frac{T}{4}$	$\vec{E} = +E_0\hat{j}$
At $t_3 = \frac{T}{2}$	$\vec{E} = -E_0\hat{i}$
At $t_4 = \frac{3T}{4}$	$\vec{E} = -E_0\hat{j}$
At $t_5 = T$	$\vec{E} = E_0\hat{i}$



This shows that the electric field vector (E_0) rotates anticlockwise, since the magnitude is constant the tip of $\vec{E}$ sweeps out a circle at a fixed space point.
Thus the wave is left circularly polarised (L-state).

The specific rotation of the quartz for $\lambda = 508.6$ nm is 29.73 deg/mm. Calculate the difference between the refractive indices for left and right circularly polarised light for quartz.

ln. Angle of rotation of the vibration of a plane polarised light by a quartz plate of thickness l for wavelength λ is given by, $\theta = \frac{\delta}{2} = \frac{\pi l}{\lambda}(n_L - n_R)$

$$\Rightarrow \quad n_L - n_R = \frac{\theta.\lambda}{\pi\lambda} = \frac{\left(29.73 \times \frac{\pi}{180}\right) \times 508.6 \times 10^{-9}}{\pi \times 1 \times 10^{-3}} = 8.4 \times 10^{-5}$$

A quartz plate cut with its surface perpendicular to optic axis is required to annul completely the rotation of the plane of polarisation of red light ($\lambda = 760$ nm) by a 26.7 cm length of lactose solution containing 100 g of active substance per litre of the solution. Find the thickness of the plate. Given specific rotation of lactose = 52.5°dm^{-1} . g^{-1}. cm^{-3} and for quartz $n_L = 1.53920$ and $n_R = 1.53914$.

n. Rotation produced by quartz of thickness l is

$$\theta = \frac{\delta}{2} = \frac{\pi l}{\lambda}(n_L - n_R) = \frac{\pi l}{760 \times 10^{-9}}(1.53920 - 1.53914) \times \frac{180}{\pi}\text{deg}$$

and rotation produced by the lactose solution $= s.m.\frac{l'}{10} = 52.5 \times \frac{100}{1000} \times \frac{26.7}{10}\text{deg}$

Equating these two rotations we get, $l = 09864 \times 10^{-3}\,\text{m} = 0.9864$ mm

Which of the following phenomenon does NOT produce completely polarized light from unpolarized light?

(a) Absorption (b) Refraction (c) Scattering (d) Reflection

[IIT-JAM GP : 2007]

1. (a)

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5. Optics - Jenkins and White
6. Optics - Ghatak

The cover page shows the phenomenon of dispersion of light. This proved that the white light is a collection of multiple wavelengths.

ELECTROMAGNETIC THEORY

for

M.Sc. Entrance

IIT-JAM | JNU | TIFR | JEST | BHU | ISM | JNCASR | DU

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Chapter 1

Mathematics Involved in Electromagnetic Theory

Gradiant of a scalar function:

If $\phi(x, y, z)$ be a scalar function then $\hat{i}\frac{\partial \phi}{\partial x}+\hat{j}\frac{\partial \phi}{\partial y}+\hat{k}\frac{\partial \phi}{\partial z}$ is called the gradient of the scalar function ϕ.

And is denoted by grad ϕ.

Thus, $$grad\ \phi=\hat{i}\frac{\partial \phi}{\partial x}+\hat{j}\frac{\partial \phi}{\partial y}+\hat{k}\frac{\partial \phi}{\partial z}$$

$$grad\ \phi=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\phi(x, y, z)$$

$$grad\ \phi=\nabla \phi$$

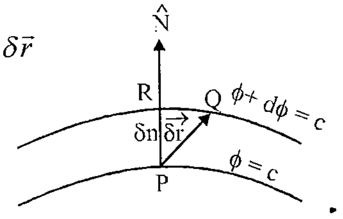
Geometrical Meaning of Gradient, Normal:

If a surface $\phi(x, y, z)=c$ passes through a point P. The value of the function at each point on the surface is the same as at P. Then such a surface is called a level surface through P. For example, if $\phi(x, y, z)$ represents potential at the point P, then equipotential surface $\phi(x, y, z)=c$ is a level surface.

Two level surfaces can not intersect.

Let the level surface pass through the point P at which the value of the function is ϕ. Consider another level surface passing through q, where the value of the function is $\phi+d\phi$.

Let $\vec{r}$ and $\vec{r}+\delta\vec{r}$ be the position vector of P and Q then $\overrightarrow{PQ}=\delta\vec{r}$



$$\vec{\nabla}\phi \cdot d\vec{r}=\left(\hat{i}\frac{\partial \phi}{\partial x}+\hat{j}\frac{\partial \phi}{\partial y}+\hat{k}\frac{\partial \phi}{\partial z}\right)\cdot\left(\hat{i}dx+\hat{j}dy+\hat{k}dz\right)$$

$$=\frac{\partial \phi}{\partial x}dx+\frac{\partial \phi}{\partial y}dy+\frac{\partial \phi}{\partial z}dz=d\phi \qquad \ldots (1)$$

If Q lies on the level surface of P, then $d\phi=0$

Equation (1) becomes, $\vec{\nabla}\phi \cdot d\vec{r}=0$, then $\vec{\nabla}\phi$ is perpendicular to $d\vec{r}$ (tangent)

Hence, $\vec{\nabla}\phi$ is normal to the surface $\phi(x, y, z)=c$

The Divergence:

From the definition of $\vec{\nabla}$ we construct the divergence:

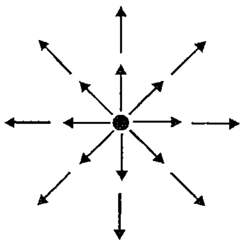
$$\vec{\nabla}\cdot\vec{v}=\left(\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}+\hat{z}\frac{\partial}{\partial z}\right)\cdot\left(v_x\hat{x}+v_y\hat{y}+v_z\hat{z}\right)$$

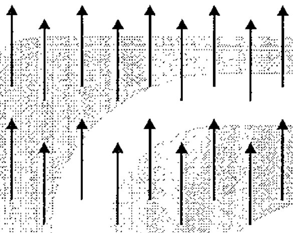
$$=\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}$$

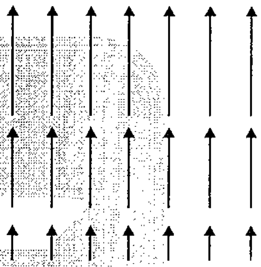
Observe that the divergence of a vector function $\vec{v}$ is itself a scalar $\vec{\nabla}.\vec{v}$

Gemetrical interpretation :

The name divergence is well chosen, for $\vec{\nabla}.\vec{v}$ is a measure of how much the vector $\vec{v}$ spreads out (diverges) from the point in question. For example, the vector function in figure (a) has a large (positive) divergence (if the arrows pointed in, it would be a large negative divergence), the function in figure (b) has zero divergence, and the function in figure (c) again has a positive divergence.


(a)


(b)


(c)

The Curl:

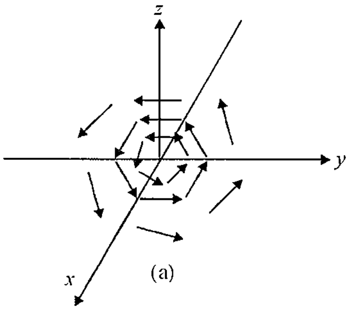
From the defintion $\vec{\nabla}$ we construct the curl:

$$\vec{\nabla}\times\vec{v}=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ v_x & v_y & v_z\end{vmatrix}=\hat{x}\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right)+\hat{y}\left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right)+\hat{z}\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)$$

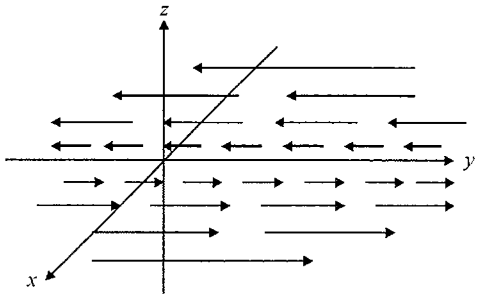
Notice that the curl of a vector function $\vec{v}$ is, like any cross product, a vector. Carl of a scalar does not exist.

Geometrical interpretation :

The name curl is also well chosen, for $\vec{\nabla}\times\vec{v}$ is a measure of how much the vector $\vec{v}$ 'curls around' the point in question. Thus the two functions in figure (a) have a substantial curl, pointing in the z-direction, as the natural right-hand rule would suggest. Imagine (again) you are standing at the edge of a pond. Float a small paddlewheel (a cork with toothpicks pointing out radially would do); if it starts to rotate, then you placed it at a point of non-zero curl. A whirlpool would be a region of large curl.



(a)



Important Identities:

(i) $\vec{\nabla}(fg) = f\,\vec{\nabla}g + g\vec{\nabla}f$

(ii) $\vec{\nabla}(\vec{A}.\vec{B}) = \vec{A}\times(\vec{\nabla}\times\vec{B}) + \vec{B}\times(\vec{\nabla}\times\vec{A}) + (\vec{A}.\vec{\nabla})\vec{B} + (\vec{B}.\vec{\nabla})\vec{A}$,

(iii) $\vec{\nabla}.(f\vec{A}) = f(\vec{\nabla}.\vec{A}) + \vec{A}.(\vec{\nabla}f)$

(iv) $\vec{\nabla}.(\vec{A}\times\vec{B}) = \vec{B}.(\vec{\nabla}\times\vec{A}) - \vec{A}.(\vec{\nabla}\times\vec{B})$,

(v) $\vec{\nabla}\times(f\vec{A}) = f(\vec{\nabla}\times\vec{A}) - \vec{A}\times(\vec{\nabla}f)$

(vi) $\vec{\nabla}\times(\vec{A}\times\vec{B}) = (\vec{B}.\vec{\nabla})\vec{A} - (\vec{A}.\vec{\nabla})\vec{B} + \vec{A}(\vec{\nabla}.\vec{B}) - \vec{B}(\vec{\nabla}.\vec{A})$

The gradient, the divergence, and the curl are the only first derivatives we can make with ∇; by applying ∇ twice we can construct five species of second derivatives. The gradient ∇T is a vector, so we can take the divergence and curl of it.

(1) Divergence of gradient: $\vec{\nabla}.(\vec{\nabla}T) = \nabla^2 T$ (Laplacian of T)

(2) Curl of gradient: $\vec{\nabla}\times(\vec{\nabla}T) = 0$

(3) Gradient of divergence: $\vec{\nabla}(\vec{\nabla}.\vec{v})$ (This is not same as laplacian)

(4) Divergence of curl: $\vec{\nabla}.(\vec{\nabla}\times\vec{v}) = 0$

(5) Curl of curl: $\vec{\nabla}\times(\vec{\nabla}\times\vec{v}) = \vec{\nabla}(\vec{\nabla}.\vec{v}) - \nabla^2\vec{v}$

Gauss' Divergence Theorem:

This theorem is applicable only for closed surfaces and this theorem converts surface integral into volume integral and vice versa.

If V is the volume bounded by a closed surface S and $\vec{A}$ is a vector function of position with continuous derivatives, then.

$$\oint\!\!\oint_S \vec{A}\cdot\vec{ds} = \oint\!\!\oint_S \vec{A}\cdot\hat{n}ds = \iiint_V (\vec{\nabla}\cdot\vec{A})dV$$

where $\hat{n}$ is the outward normal to 'S' indicating the positive direction of S.

Stokes Theorem:

This theorem is applicable only for open surfaces and this theorem converts surface integral into line integral and vice versa.

If S is an open, two-sided surface bounded by a closed, non-intersecting curve C, and $\vec{A}$ is a vector function of position with continuous derivatives; then

$$\oint_C \vec{A}\cdot\vec{dr} = \iint_S (\vec{\nabla}\times\vec{A}).\hat{n}ds = \iint_S (\vec{\nabla}\times\vec{A})\cdot\vec{ds}$$

where C is traversed in the positive (counterclockwise) direction.

Coordinate System:

Coordinate system	$\hat{u}_1$	$\hat{u}_2$	$\hat{u}_3$	h_1	h_2	h_3	$d\ell$	ds	dv
Cartesian	$\hat{x}$	$\hat{y}$	$\hat{z}$	1	1	1	dx or dy or dz	$dxdy$ or $dydz$ or $dzdx$	$dxdydz$
Spherical	$\hat{r}$	$\hat{\theta}$	$\hat{\phi}$	1	r	$r\sin\theta$	$rd\theta$	$r^2 d\theta\sin\theta d\phi$	$r^2 dr\sin\theta d\theta d\phi$
Cylindrical	$\hat{r}$	$\hat{\phi}$	$\hat{z}$	1	r	1	$rd\phi$	$rd\phi dz$	$rdrd\phi dz$

(i) $\vec{\nabla}=\frac{1}{h_1}\frac{\partial}{\partial u_1}\hat{e}_1+\frac{1}{h_2}\frac{\partial}{\partial u_2}\hat{e}_2+\frac{1}{h_3}\frac{\partial}{\partial u_3}\hat{e}_3$

(ii) $\vec{\nabla}\cdot\vec{A}=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3A_1)+\frac{\partial}{\partial u_2}(h_1h_3A_2)+\frac{\partial}{\partial u_3}(h_1h_2A_3)\right]$

(iii) $\vec{\nabla}\times\vec{A}=\frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}$

(iv) $\nabla^2\phi=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\phi}{\partial u_1}\right)+\frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\phi}{\partial u_2}\right)+\frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\phi}{\partial u_3}\right)\right]$

Formulas for gradient divergence and curl for spherical coordinates (r,ϕ,θ)

Gradient: $\vec{\nabla}T=\frac{\partial T}{\partial r}\hat{r}+\frac{1}{r}\frac{\partial T}{\partial\theta}\hat{\theta}+\frac{1}{r\sin\theta}\frac{\partial T}{\partial\phi}\hat{\phi}$

Divergence: $\vec{\nabla}.\vec{v}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2v_r\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, v_\theta\right)+\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$

Laplacian: $\nabla^2T=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2T}{\partial\phi^2}$

Formulas for gradient divergence and curl for cylindrical coordinates (r,ϕ,z)

Gradient: $\vec{\nabla}T=\frac{\partial T}{\partial r}\hat{r}+\frac{1}{r}\frac{\partial T}{\partial\phi}\hat{\phi}+\frac{\partial T}{\partial z}\hat{z}$

Divergence: $\vec{\nabla}.\vec{v}=\frac{1}{r}\frac{\partial}{\partial r}(rv_r)+\frac{1}{r}\frac{\partial}{\partial\phi}(v_\phi)+\frac{\partial v_z}{\partial z}$

Laplacian: $\nabla^2T=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2T}{\partial\phi^2}+\frac{\partial^2T}{\partial z^2}$

Example-1: Calculate divergence of following vectors.

(i) $\vec{A} = k\left(3x^2y\hat{i} + 2yx\hat{j} + 4xyz^2\hat{k}\right)$ at point $(1,1,1)$.

(ii) $\vec{A} = k\,r\sin\theta\,\hat{\phi}$ where r, θ, ϕ are spherical cordinates

(iii) $\vec{A} = kz^2\cos\phi\,\hat{r}$ where r, ϕ, z are cylindrical coordinates.

Soln. (i) $\vec{\nabla}\cdot\vec{A} = \dfrac{\partial A_x}{\partial x} + \dfrac{\partial A_y}{\partial y} + \dfrac{\partial A_z}{\partial z}$

$$= k\left[6xy + 2x + 8xyz\right]$$

at (1, 1, 1) $\vec{\nabla}\cdot\vec{A} = k\left[6+2+8\right] = 16k$

(ii) In spherical polar coordinates

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(A_\phi\right)$$

Here $\vec{A} = kr\sin\theta\hat{\phi}$ therefore

$$A_r = 0, \quad A_\theta = 0, \quad A_\phi = kr\sin\theta$$

$$\therefore \quad \vec{\nabla}\cdot\vec{A} = 0 + 0 + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(kr\sin\theta\right) = 0 \quad \Rightarrow \vec{\nabla}\cdot\vec{A} = 0$$

(iii) In cylindrical coordinate.

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r}\frac{\partial}{\partial r}\left(rA_r\right) + \frac{1}{r}\frac{\partial}{\partial\phi}\left(A_\phi\right) + \frac{\partial A_z}{\partial z}$$

Here, $\vec{A} = kz^2\cos\phi\,\hat{r}$

$$\therefore \quad A_r = kz^2\cos\phi, \quad A_\phi = 0, \quad A_z = 0$$

$$\therefore \quad \vec{\nabla}\cdot\vec{A} = \frac{1}{r}\frac{\partial}{\partial r}\left(rkz^2\cos\phi\right) + 0 + 0$$

$$= \frac{kz^2\cos\phi}{r}$$

Example-2: Calculate Laplacian $\left(\nabla^2\right)$ of following

(i) $A = k\left(x^2yz^3\right)$

(ii) $A = kr^3, \quad r = \sqrt{x^2+y^2+z^2}$

(iii) $A = k\hat{r}, \quad r = \sqrt{x^2+y^2}$

Soln. (i) $\nabla^2 A = \left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}\right)A$

$$= k\left[2yz^3 + 0 + 6x^2yz\right]$$

$$= 2kyz\left(z^2 + 3x^2\right)$$

(ii) $A = kr^3, r$ is in 3 dimension, so let us use spherical polar coordinates.

$$\nabla^2 A=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial A}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial A}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 A}{\partial\phi^2}$$

A is function of r only so last two terms are zero.

$$\therefore\ \nabla^2 A=\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\left(kr^3\right)\right]=\frac{1}{r^2}\frac{\partial}{\partial r}\left[3kr^4\right]=12\,kr$$

(iii) $A=kr$, r is in x-y plane so it is easier to use cylindrical coordinates.

$$\nabla^2 A=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 A}{\partial\phi^2}+\frac{\partial^2 A}{\partial z^2}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}(kr)\right)+0+0=\frac{k}{r}$$

Example-3: Claculate curl of following vectors.

(i) $\vec{A}=c\left[x\hat{j}-y\hat{i}+z\hat{k}\right]$

(ii) $\vec{A}=kr\hat{\phi}$ in spherical polar coordinate

(iii) $\vec{A}=\frac{k}{r}\hat{\phi}$ in cylindrical coordinate.

n. (i) $$\vec{\nabla}\times\vec{A}=c\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ -y & x & z\end{vmatrix}=c\left[\hat{i}(0-0)-\hat{j}(0-0)+\hat{k}(1+1)\right]=2c\hat{k}$$

(ii) $$\vec{\nabla}\times\vec{A}=\frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi}\\ A_r & rA_\theta & r\sin\theta A_\phi\end{vmatrix}=\frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi}\\ 0 & 0 & kr^2\sin\theta\end{vmatrix}$$

$$=\frac{1}{r^2\sin\theta}\left[\hat{r}\left(kr^2\cos\theta-0\right)-r\hat{\theta}\left(2kr\sin\theta-0\right)+r\sin\theta\hat{\phi}(0-0)\right]$$

$$=\frac{1}{r^2\sin\theta}\left[kr^2\cos\theta\hat{r}-2kr^2\sin\theta\hat{\theta}\right]=k\left[\cot\theta\hat{r}-2\hat{\theta}\right]$$

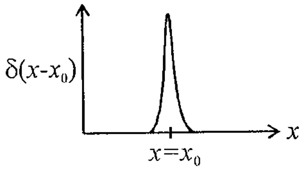
(iii) $$\vec{\nabla}\times\vec{A}=\frac{1}{r}\begin{vmatrix}\hat{r} & r\hat{\phi} & \hat{z}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z}\\ A_r & rA_\phi & A_z\end{vmatrix}$$

$$=\frac{1}{r}\begin{vmatrix}\hat{r} & r\hat{\phi} & \hat{z}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z}\\ 0 & r\cdot\frac{k}{r} & 0\end{vmatrix}=\frac{1}{r}\begin{vmatrix}r & r\hat{\phi} & \hat{z}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z}\\ 0 & k & 0\end{vmatrix}=0$$

Dirac-delta function:

(A) One dimensional delta function:

$$\delta(x-x_0)=\begin{cases}0 & for\ x\neq x_0\\ \infty & for\ x=x_0\end{cases}$$



Properties of delta function:

(1) The delta function is even $\delta(-x)=\delta(x)$

(2) $\delta(ax)=\frac{1}{|a|}\delta(x)\quad (a\neq 0)$

(3) $\int_{-\infty}^{\infty}\delta(x)\,dx=1$

(4) $\int_a^b f(x)\delta(x-x_0)\,dx=\begin{cases}f(x_0) & if\ a<x_0<b\\ 0 & \text{elsewhere}\end{cases}$

(5) $\int\delta(a-x)\delta(x-b)\,dx=\delta(a-b)$

(6) $\delta\left[(x-a)(x-b)\right]=\frac{1}{|a-b|}\left[\delta(x-a)+\delta(x-b)\right]\ (a\neq b)$

(7) $\delta(x^2-a^2)=\frac{1}{2|a|}\left[\delta(x-a)+\delta(x+a)\right]\ (a\neq 0)$

(8) $\int_a^b f(x)\delta^n(x-x_0)\,dx=(-1)^n f^n(x_0)\quad a<x_0<b$

(B) Three dimensional delta function:

(1) The three dimensional form delta function in cartesian coordinates is

$$\delta(\vec{r}-\vec{r}_0)=\delta(x-x_0)\delta(y-y_0)\delta(z-z_0)$$

(2) In spherical coordinate

$$\delta(\vec{r}-\vec{r}_0)=\frac{1}{r^2\sin\theta}\delta(r-r_0)\delta(\theta-\theta_0)\delta(\phi-\phi_0)$$

(3) $\int_V\delta(\vec{r}-\vec{r}_0)\,dV=\begin{cases}1 & if\ r_0\ \text{is in V}\\ 0 & if\ r_0\ \text{is not in V}\end{cases}$

(4) $\int_V f(\vec{r})\,\delta(\vec{r}-\vec{r}_0)\,dV=\begin{cases}f(r_0) & if\ r_0\ \text{is in V}\\ 0 & if\ r_0\ \text{is not in V}\end{cases}$

$$\vec{\nabla}\cdot\frac{\hat{r}}{r^2}=4\pi\delta(\vec{r})$$

$$\nabla^2\frac{1}{r}=-4\pi\delta(\vec{r})$$

Useful Integration:

$$\int \sin x dx = -\cos x \qquad \int \sinh x dx = \cosh x$$

$$\int \cos x dx = \sin x \qquad \int \cosh x dx = \sinh x$$

$$\int \sec^2 x dx = \tan x \qquad \int \sec h^2 x dx = \tanh x$$

$$\int \text{cosec}^2 x dx = -\cot x \qquad \int \text{cosec} h^2 x dx = -\coth x$$

$$\int \sec x \tan x dx = \sec x \qquad \int \sec hx \tanh x dx = -\sec hx$$

$$\int \text{cosec}\, x \cot x dx = -\text{cosec}\, x \qquad \int \text{cosec} hx \coth x dx = -\text{cosec} hx$$

$$\int e^x dx = e^x \qquad \int a^x dx = \left(a^x\right) / \log_e a$$

$$\int x^n dx = \frac{x^{n+1}}{n+1}, n \neq -1 \qquad \int \{f(x)\}^n f'(x) dx = \frac{\{f(x)\}^{n+1}}{n+1}, n \neq -1$$

$$\int \frac{1}{x} dx = \log x \qquad \int \frac{f'(x)}{f(x)} dx = \log x$$

$$\int \tan x dx = \log \sec x = -\log \cos x \qquad \int \cot x dx = \log \sin x$$

$$\int \text{cosec}\, x\, dx = \log \tan(x/2) = \log(\text{cosec}\, x - \cot x)$$

$$\int \sec x dx = \log \tan(\pi / 4 + x / 2) = \log(\sec x + \tan x)$$

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}; \ \int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \frac{x - a}{x + a}, \ x > a; \ \int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \frac{a + x}{a - x}, x < a$$

$$\int \frac{dx}{x\left(x^2 - a^2\right)^{1/2}} = \frac{1}{a} \sec^{-1} \frac{x}{a} \qquad \int \frac{dx}{\left(a^2 - x^2\right)^{1/2}} = \sin^{-1} \frac{x}{a}$$

$$\int \frac{dx}{\left(a^2 + x^2\right)^{1/2}} = \sinh^{-1} \frac{x}{a} \qquad \int \frac{dx}{\left(x^2 - a^2\right)^{1/2}} = \cosh^{-1} \frac{x}{a}$$

$$\int \left(a^2 - x^2\right)^{1/2} dx = (x / 2) \times \left(a^2 - x^2\right)^{1/2} + \left(a^2 / 2\right) \times \sin^{-1}(x / a)$$

$$\int \left(a^2 + x^2\right)^{1/2} dx = (x / 2) \times \left(a^2 + x^2\right)^{1/2} + \left(a^2 / 2\right) \times \log\left\{x + \left(a^2 + x^2\right)^{1/2}\right\}$$

$$\int \left(x^2 - a^2\right)^{1/2} dx = (x / 2) \times \left(x^2 - a^2\right)^{1/2} - \left(a^2 / 2\right) \times \log\left\{x + \left(x^2 - a^2\right)^{1/2}\right\}$$

$$\int \cos(ax + b) dx = \frac{\sin(ax + b)}{a} \qquad \int e^{ax+b} dx = \frac{e^{ax+b}}{a}$$

$$\int e^{ax} \sin bx dx = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2} \qquad \int_0^\infty e^{-ax} \sin bx dx = \frac{b}{a^2 + b^2}$$

$$\int e^{ax} \cos bx dx = \frac{e^{ax}(a \cos bx + b \sin bx)}{a^2 + b^2} \qquad \int_0^\infty e^{-ax} \cos bx dx = \frac{a}{a^2 + b^2}$$

$$\int f_1(x) f_2(x) dx = f_1(x)\left\{\int f_2(x) dx\right\} - \int\left[\left\{\frac{d}{dx} f_1(x)\right\} \times \left\{\int f_2(x) dx\right\}\right] dx$$

PRACTICE SET

1. Find the gradients of the following functions:
   (a) $f(x,y,z)=x^2+y^3+z^4$ (b) $f(x,y,z)=x^2y^3z^4$ (c) $f(x,y,z)=e^x\sin(y)\ln(z)$

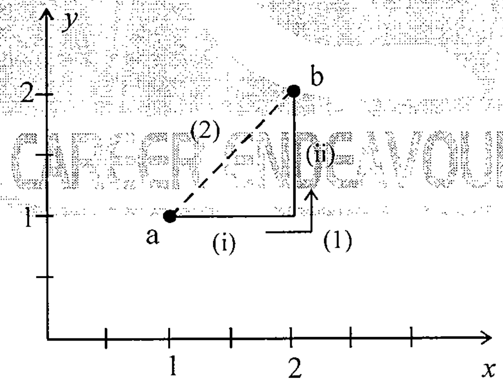
2. Let 'r' be the separation vector from a fixed point (x',y',z') to the point (x,y,z) and let 'r' be its length. Show that
   (a) $\vec{\nabla}(r^2)=2\vec{r}$ (b) $\vec{\nabla}(1/r)=-\hat{r}/r^2$ (c) what is the general formula $\vec{\nabla}(r^n)$?

3. Calculate the divergence of the following vector functions
   (a) $\vec{v}_a=x^2\hat{x}+3xz^2\hat{y}-2xz\hat{z}$ (b) $\vec{v}_b=xy\hat{x}+2yz\hat{y}+3zx\hat{z}$
   (c) $\vec{v}_c=y^2\hat{x}+(2xy+z^2)\hat{y}+2yz\hat{z}$

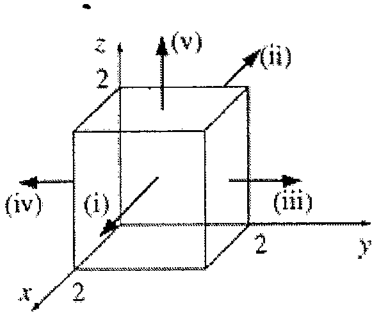
4. The vector function, $v=\dfrac{\hat{r}}{r^2}$ compute its divergence.

5. Calculate the Laplacian of the following functions:
   (a) $T_a=x^2+2xy+3z+4$ (b) $T_b=\sin x\sin y\sin z$
   (c) $T_c=e^{-5x}\sin 4y\cos 3z$ (d) $v=x^2\hat{x}+3xz^2\hat{y}-2xz\hat{z}$

6. Calculate the line integral of the function $\vec{v}=y^2\hat{x}+2x(y+1)\hat{y}$ from the point a = (1, 1, 0) to the point b = (2, 2, 0), along the paths (1) and (2) in the following figure. What is $\oint v.dl$ for the loop that goes from 'a' to 'b' along (1) and returns to 'a' along (2)?



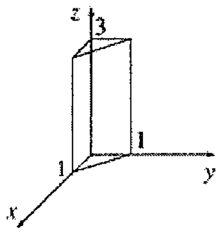
7. Calculate the surface integral of $v=2xz\hat{x}+(x+2)\hat{y}+y(z^2-3)\hat{z}$ over five sides (excluding the bottom) of the cubical box (side 2) in figure. Let "upward and outward" be the positive direction, as indicated by the arrows.



Calculate the volume integral of $T = xyz^2$ over the prism in figure.

n. You can do the three integrals in any order. Let's do 'x' first: it runs from 0 to (1–y); then y(it goes from 0 to 1); and finally z (0 to 3):

$$\int T d\tau = \int_0^3 z^2 \left\{\int_0^1 y\left[\int_0^{1-y} x dx\right]dy\right\} dz = \frac{1}{2}\int_0^3 z^2 dz\int_0^1 \left(1-y^2\right)^2 y dy = \frac{1}{2}\times 9\times\frac{1}{12} = \frac{3}{8}$$



Calculate the line integral of the function $v = x^2\hat{x} + 2yz\,\hat{y} + y^2\hat{z}$ from the origin to the point (1, 1, 1) by three different routes:

(a) $(0,0,0)\to(1,0,0)\to(1,1,0)\to(1,1,1)$ (b) $(0,0,0)\to(0,0,1)\to(0,1,1)\to(1,1,1)$

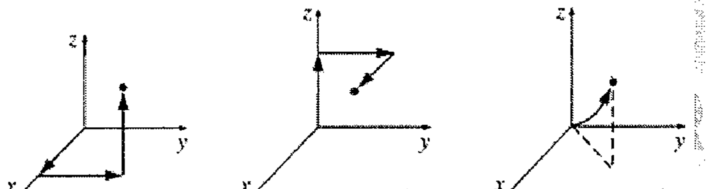
(c) The direct straight line.

Let $T = xy^2$, and take point 'a' to be the origin (0, 0, 0) and 'b' the point (2, 1, 0). Check the fundamental theorem for gradients.

Check the fundamental theorem for the gradients, using $T = x^2 + 4xy + 2yz^3$, the points $a = (0,0,0)$, $b = (1,1,1)$ and the three paths in figures (a), (b) and (c) are

(a) $(0,0,0)\to(1,0,0)\to(1,1,0)\to(1,1,1)$ (b) $(0,0,0)\to(0,0,1)\to(0,1,1)\to(1,1,1)$

(c) The parabolic path $z = x^2;\ y = x$

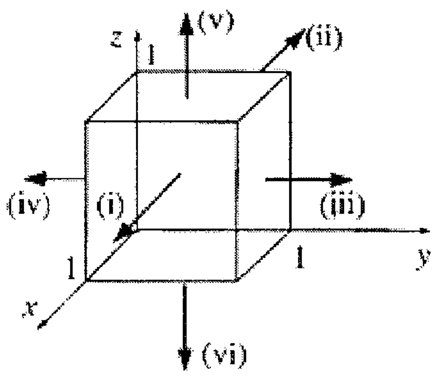


(a) (b) (c)

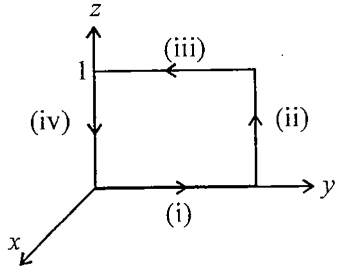
Check the divergence theorem using the function

$$\vec{v} = y^2\hat{x} + \left(2xy + z^2\right)\hat{y} + (2yz)\hat{z}$$

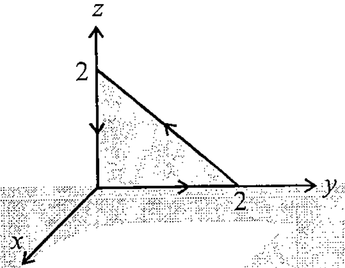
and the unit cube situated at the origin has shown in the following figure.



13. Suppose $\vec{v} = \left(2xz + 3y^2\right)\hat{y} + \left(4yz^2\right)\hat{z}$. Check Stokes' theorem for the square surface shown in figure.



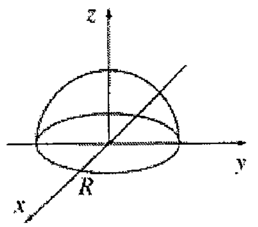
14. Test Stokes' theorem for the function $\vec{v} = (xy)\hat{x} + (2yz)\hat{y} + (3zx)\hat{z}$, using the triangular shaded area of figure.



15. Evaluate the integral $\int_0^\infty x\, e^{-x} dx$

16. Compute the divergence of the function $\vec{v} = (r\cos\theta)\hat{r} + (r\sin\theta)\hat{\theta} + (r\sin\theta\cos\phi)\hat{\phi}$

Check the divergence theorem for this function, using as your volume the inverted hemispherical bowl of radius R, resting on the xy-plane and centered at the origin (Figure)

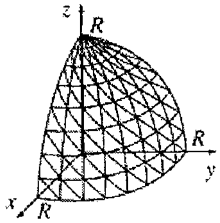


17. Find the formulas for r, θ, ϕ in terms of $\hat{x}, \hat{y}, \hat{z}$.

18. Express the unit vector, $\hat{r}, \hat{\theta}, \hat{\phi}$ in terms of $\hat{x}, \hat{y}, \hat{z}$.

19. Express the cylinderical unit vectors $\hat{s}, \hat{\phi}, \hat{z}$ in terms of $\hat{x}, \hat{y}, \hat{z}$. Invert your formulas to get $\hat{x}, \hat{y}, \hat{z}$ in terms of $\hat{s}, \hat{\phi}, \hat{z}$.

20. Check the divergence theorem for the function $\vec{v} = r^2\cos\theta\,\hat{r} + r^2\cos\phi\,\hat{\theta} - r^2\cos\theta\sin\phi\,\hat{\phi}$

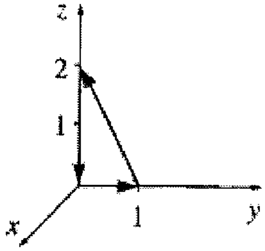


using as your volume one octant of the sphere of radius R. Make sure you include the entire surface.

Ans. $\pi R^4/4$.

Compute the line integral of

$$\vec{v} = 6\hat{x} + yz^2\hat{y} + (3y+z)\hat{z}$$



along the triangular path shown in figure. Check your answer using Stoke's theorem.

Ans. 8/3.

Chapter 2

Coulomb Force, Field & Gauss Laws

Charge: It is a physical quantity and it is a fundamental property of matter.

- Taking charge on electron to be negative and charge on proton to be positive is a convention introduced by Benjamin franklin.
- Charge of an isolated system is always conserved
- In 1909 R.A. Millikan showed using his oil drop experiment that charge is always quantized.

  i.e. Q = ne, where $n = 0, \pm 1, \pm 2 ----$

  e = charge of an electron $= 1.6\times 10^{-19}\,C$. Charge of an isolated system is always found to be quantized however charge of quarks are not quantized. But this doesn't violate quantization of charge. Because quarks do not exist separately.

Charge Density:

Case-I : Linear charge distribution, $\lambda = \dfrac{dq}{d\ell} \Rightarrow dq = \lambda d\ell$ or $q = \int \lambda d\ell$

Case-II : Surface charge distribution, $\sigma = \dfrac{dq}{ds} \Rightarrow dq = \sigma ds$ or $q = \iint \sigma ds$

Case-III: Volume charge distribution, $\rho = \dfrac{dq}{dv} \Rightarrow dq = \rho dv$ or $q = \iiint \rho dv$

Density of a point charge located at $\vec{r}_0$ is $q\delta\left(\vec{r} - \vec{r}_0\right)$

Example: Suppose the electron cloud around the nucleus of charge $+e$ in an atom can be represented by the following spherical charge distributors.

$$\rho(r) = \frac{-e}{\pi r_0} \cdot e^{-2r/r_0}$$

where r_0 is called the Bohr radius. Calculate the total charge of the cloud.

Soln. $Q = \int \rho dV = \iiint \rho r^2 dr \sin\theta d\theta d\phi = \int_0^\infty \rho(r) r^2 dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi$

$$Q = \int_0^\infty \rho(r) 4\pi r^2 dr = \frac{-e}{\pi r_0} \cdot 4\pi \int_0^\infty r^2 e^{-2r/r_0} dr$$

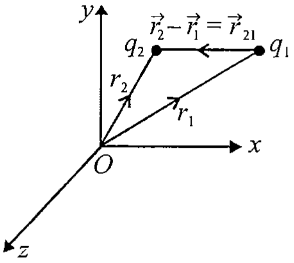
Putting $\dfrac{2r}{r_0} = x$, we get $Q = \dfrac{-e}{2}\displaystyle\int_0^\infty x^2 \cdot e^{-x} dx = \dfrac{-e}{2}\Gamma(3) = \dfrac{-e}{2}\times 2 = -e$

Coulomb's law:

The force between two stationary point charge is proportional to the product of charges, inversely proportional to the square of the distance between them and it acts along the line joining the charges.

Let $\vec{r}_1$ and $\vec{r}_2$ are position of two changes q_1 and q_2.

Therefore, according to Coulomb law the force on q_2 due to q_1 is



$$\vec{F}_{21} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{21}^2}\hat{r}_{21} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{21}^3}\vec{r}_{21} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{|\vec{r}_2 - \vec{r}_1|^3}(\vec{r}_2 - \vec{r}_1) \quad \text{... (1)}$$

The force on q_1 due to q_2 is

$$\vec{F}_{12} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}^2}\hat{r}_{12} = -\frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{21}^2}\hat{r}_{21} \quad \text{... (2)}$$

where, ε_0 is dielectric permittivity of free space

$$\varepsilon_0 = 8.85 \times 10^{-12}\ C^2/N\text{-}m^2$$

From equation (1) and (2), we get

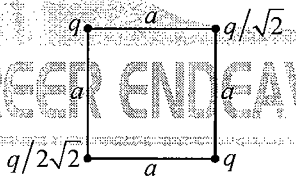
$$\boxed{\vec{F}_{21} = -\vec{F}_{12}}$$

Principle of Superposition:

Let we have N point charges $q_0, q_1, q_2, q_3 ......q_N$. The total force on q_0 due to all other charges.

$$\vec{F}_0 = \vec{F}_{01} + \vec{F}_{02} + \vec{F}_{03} + ... + \vec{F}_{0N} = \sum_{j=1}^{N}\vec{F}_{0j}$$

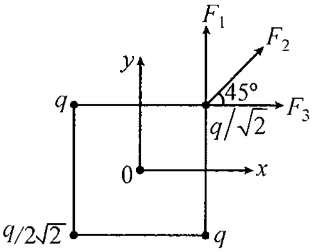
Example: Four charges q, $q/\sqrt{2}$, $q/2\sqrt{2}$ and q are placed at corners of a square of side a. Calculate net force on charge $q/\sqrt{2}$.



n. Let us choose coordinate system as shown in figure. Three forces acting on $q/\sqrt{2}$ due to other three charges are also shown in the figure.

Therefore, $\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3$

$$= \frac{q^2/\sqrt{2}}{4\pi\varepsilon_0 a^2}\hat{i} + \frac{q^2/4}{4\pi\varepsilon_0\left(a\sqrt{2}\right)^2}\left(\cos 45^\circ\,\hat{i} + \sin 45^\circ\,\hat{j}\right) + \frac{q^2/\sqrt{2}}{4\pi\varepsilon_0 a^2}\hat{j}$$



$$= \frac{q^2}{4\sqrt{2}\pi\varepsilon_0 a^2}\left[\hat{i} + \frac{1}{8}(\hat{i} + \hat{j}) + \hat{j}\right]$$

$$= \frac{9q^2}{32\sqrt{2}\pi\varepsilon_0 a^2}\left[\hat{i} + \hat{j}\right] \Rightarrow \left|\vec{F}\right| = \frac{9q^2}{32\pi\varepsilon_0 a^2}$$

Example: Three charges $q, -q$, and $2q$ are placed at (a, a, a) $(a, -a, a)$ and $(a, -a, -a)$ respectively. Calculate net force on $2q$.

Soln. $2q$ will experience force due to two other charges.

Let $q_1 = q,\ q_2 = -q,\ q_3 = 2q$

$$\vec{r}_1 = a\left(\hat{i}+\hat{j}+\hat{k}\right),\ \vec{r}_2 = a\left(\hat{i}-\hat{j}+\hat{k}\right),\ \vec{r}_3 = a\left(\hat{i}-\hat{j}-\hat{k}\right)$$

Force on $2q$

$$\vec{F} = \vec{F}_{31} + \vec{F}_{32}$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{q_3 q_1}{r_{31}^3}\vec{r}_{31} + \frac{q_3 q_2}{r_{32}^3}\vec{r}_{32}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{2q^2}{\left|\vec{r}_3-\vec{r}_1\right|^3}(\vec{r}_3-\vec{r}_1) - \frac{2q^2}{\left|\vec{r}_3-\vec{r}_2\right|^2}(\vec{r}_3-\vec{r}_2)\right] = \frac{q^2}{2\pi\varepsilon_0}\left[\frac{a(-2\hat{j}-2\hat{k})}{16\sqrt{2}a^3} - \frac{a(-2\hat{k})}{8a^3}\right]$$

$$= \frac{q^2}{16\pi\varepsilon_0 a^2}\left[-\frac{1}{\sqrt{2}}\left(\hat{j}+\hat{k}-2\sqrt{2}\hat{k}\right)\right]$$

$$= \frac{-q^2}{16\sqrt{2}\pi\varepsilon_0 a^2}\left[\hat{j}+(1-2\sqrt{2})\hat{k}\right] = \frac{q^2}{16\sqrt{2}\pi\varepsilon_0 a^2}\left[\left(2\sqrt{2}-1\right)\hat{k}-\hat{j}\right].$$

Electric field:

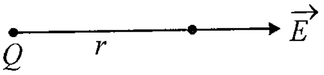
If we keep a charge in the space, the region within which other charges feel force is called electric field. And the force that acts on the unit charge at a particular point is called electric field intensity $\left(\vec{E}\right)$ at that point.

If $\vec{F}$ be force on charge q due to some other charge then field at location of q due to other charge is

$$\vec{E} = \frac{\vec{F}}{q}$$

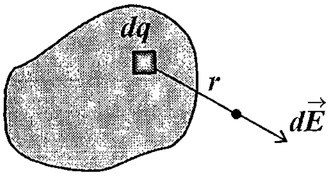
Field due to a point charge is given

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}$$



If $\vec{r}_q$ be position vector of q and $\vec{r}_p$ is position vector of point p where field is to be calculated, then

$$\vec{E}\left(\vec{r}=\vec{r}_p\right) = \frac{q\left(\vec{r}_p-\vec{r}_q\right)}{4\pi\epsilon_0\left|\vec{r}_p-\vec{r}_q\right|^3}$$



For continuous charge distribution,

$$\vec{E} = \int d\vec{E} = \frac{1}{4\pi\varepsilon_0}\int\frac{dq}{r^3}\vec{r}$$

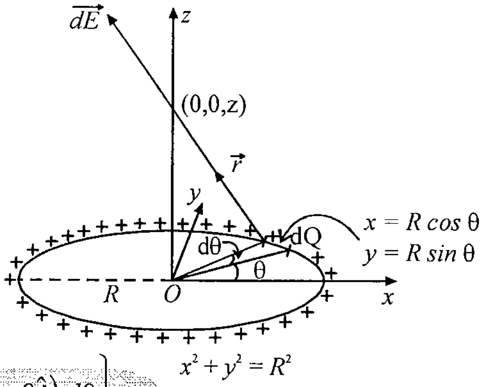
Electric field due to uniformly charged ring at axial point:

Let R be the radius of ring and Q be total charge on the ring. We are assuming that Q is uniformly distributed. Electric field at (0, 0, z) due to elementary portion shown in figure is

$$d\vec{E}=\frac{dQ}{4\pi\varepsilon_0 r^2}\hat{r}=\frac{\lambda\,dl\,\vec{r}}{4\pi\varepsilon_0 r^3}$$

Here $\lambda=\dfrac{Q}{2\pi R}$, $dl=Rd\theta$

$$r=\sqrt{R^2+z^2},\quad \vec{r}=z\hat{k}-\vec{R}=z\hat{k}-R\left(\cos\theta\hat{i}+\sin\theta\hat{j}\right)$$



$$\therefore\ d\vec{E}=\frac{Q\left[z\hat{k}-R\left(\cos\theta\hat{i}+\sin\theta\hat{j}\right)\right]d\theta}{8\pi^2\varepsilon_0\left(R^2+z^2\right)^{3/2}}$$

$$\therefore\ \vec{E}=\int d\vec{E}=\frac{Q}{8\pi^2\varepsilon_0\left(R^2+z^2\right)^{3/2}}\left\{z\hat{k}\int_0^{2\pi}d\theta-R\int_0^{2\pi}\left(\cos\theta\hat{i}+\sin\theta\hat{j}\right)d\theta\right\}$$

$$\therefore\ \vec{E}=\frac{Q}{8\pi^2\varepsilon_0\left(R^2+z^2\right)^{3/2}}\left\{z\hat{k}\cdot2\pi-R(0)\right\}$$

or $\boxed{\vec{E}=\dfrac{Qz}{4\pi\varepsilon_0\left(R^2+z^2\right)^{3/2}}\hat{k}}$ i.e. in axial direction

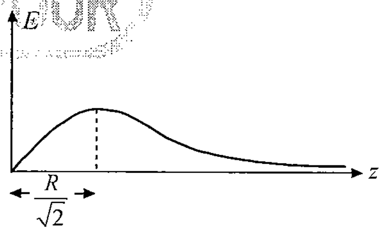
for $z<<R$ (near center), $z^2+R^2\approx R^2$

$$\boxed{E=\frac{Qz}{4\pi\varepsilon_0 R^3}}$$

for $z>>R$ (for from center), $z^2+R^2\sim z^2$

$\boxed{E=\dfrac{Q}{4\pi\varepsilon_0 z^2}}$ (like a point charge)

electric field varies with axial distance as



To find the position where field is extremum use $\dfrac{dE}{dz}=0$

or $\dfrac{d}{dz}\left[\dfrac{z}{\left(R^2+z^2\right)^{3/2}}\right]=0\Rightarrow\dfrac{1.\left(R^2+z^2\right)^{3/2}-z.\frac{3}{2}\left(R^2+z^2\right)^{1/2}2z}{\left(R^2+z^2\right)^{3/2}}=0$

$$\left(R^2+z^2\right)^{1/2}\left[R^2+z^2-3z^2\right]=0$$

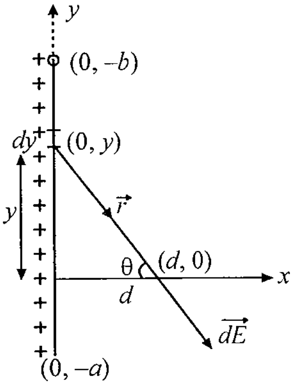
or $\left(R^2+z^2\right)^{1/2}\left(R^2-2z^2\right)=0$

$$\therefore\ z=\pm\frac{R}{\sqrt{2}}$$

Electric field due to straight wire :

Consider a uniformly charged wire of length l and charge density λ.
Let us find electric field due to this wire at a distance 'd' from it
Field due to elementary portion

$$d\vec{E} = \frac{dQ}{4\pi\varepsilon_0 r^2}\hat{r} = \frac{\lambda dy\vec{r}}{4\pi\varepsilon_0 r^3}$$



Here $\vec{r} = d\hat{i} - y\hat{j}$

$$r = \sqrt{d^2 + y^2}$$

$$\therefore \quad d\vec{E} = \frac{\lambda dy\left(d\hat{i} - y\hat{j}\right)}{4\pi\varepsilon_0\left(\sqrt{d^2 + y}\right)^3}$$

$$\therefore \quad \vec{E} = \int d\vec{E} = \frac{\lambda}{4\pi\varepsilon_0}\left\{d\hat{i}\int\frac{dy}{\left(d^2+y^2\right)^{3/2}} - \hat{j}\int y\frac{dy}{\left(d^2+y^2\right)^{3/2}}\right\}$$

Let the ends of wire are at $(0, -a)$ and $(0, b)$

$$\therefore \quad \vec{E} = \frac{\lambda}{4\pi\varepsilon_0}\left\{d\hat{i}\int_{-a}^{b}\frac{dy}{\left(d^2+y^2\right)^{3/2}} - \hat{j}\int_{-a}^{b}\frac{y\,dy}{\left(d^2+y^2\right)^{3/2}}\right\}$$

To integrate put $y = d\tan\theta,\ dy = d\sec^2\theta\, d\theta$

$$\vec{E} = \frac{\lambda}{4\pi\varepsilon_0}\left\{\frac{\hat{i}}{d}[\sin\theta] + \frac{\hat{j}}{d}[\cos\theta]\right\} = \frac{\lambda}{4\pi\varepsilon_0}\left\{\frac{\hat{i}}{d}\left[\frac{y}{\sqrt{y^2+d^2}}\right]_{-a}^{b} + \frac{\hat{j}}{d}\left[\frac{d}{\sqrt{y^2+d^2}}\right]_{-a}^{b}\right\}$$

$$\vec{E} = \frac{\lambda}{4\pi\varepsilon_0 d}\left[\hat{i}\left(\frac{b}{\sqrt{b^2+d^2}} + \frac{a}{\sqrt{a^2+d^2}}\right) + \hat{j}\left(\frac{d}{\sqrt{b^2+d^2}} - \frac{d}{\sqrt{a^2+d^2}}\right)\right]$$

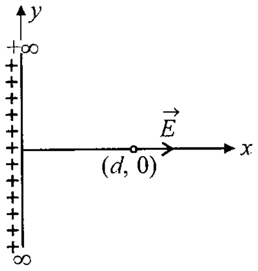
Special Cases.

- Field near mid point of infinite wire

  $a \to \infty, b \to +\infty$

  $$\vec{E} = \frac{\lambda}{4\pi\varepsilon_0 d}\left\{\hat{i}(1+1) + \hat{j}(0-0)\right\}$$

  $$\vec{E} = \frac{\lambda}{2\pi\varepsilon_0 d}\hat{i}$$

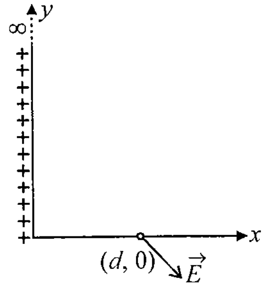


- Field near end of infinite wire

  $a = 0,\ b \to \infty,$

  $$\vec{E} = \frac{\lambda}{4\pi\varepsilon_0 d}\left[\hat{i}(1+0) + \hat{j}(0-1)\right]$$

  $$\vec{E} = \frac{\lambda}{4\pi\varepsilon_0 d}\left(\hat{i} - \hat{j}\right)$$



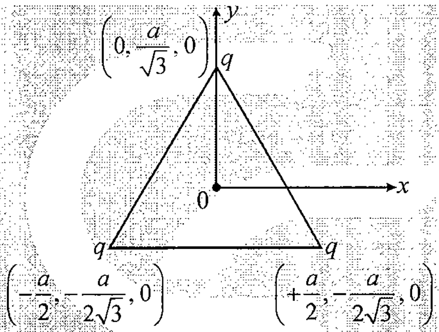
Example : Two charges q and $-q$ are placed at $-(a,0,0)$ and $(a,0,0)$. Calculate electric field at point (x, y, z).

n. Position vector of charges q and $-q$ respectively are $\vec{r}_1=-a\hat{i},\ \vec{r}_2=a\hat{i}$, position vector of space point where we have to find field is $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$.

Therefore, $$\vec{E}=\frac{q}{4\pi\varepsilon_0}\frac{\vec{r}-\vec{r}_1}{|\vec{r}-\vec{r}_1|^3}+\frac{-q}{4\pi\varepsilon_0}\frac{\vec{r}-\vec{r}_2}{|\vec{r}-\vec{r}_2|^3}$$

$$\vec{E}=\frac{q}{4\pi\varepsilon_0}\left[\frac{(x+a)\hat{i}+y\hat{j}+z\hat{k}}{\left((x+a)^2+y^2+z^2\right)^{3/2}}-\frac{(x-a)\hat{i}+y\hat{j}+z\hat{k}}{\left[(x-a)^2+y^2+z^2\right]^{3/2}}\right]$$

Example: Three point charges each equal to q are placed at vertices of an equilateral triangle of side 'a'. Calculate electric field at a distance $a\sqrt{2/3}$ above the centroide of the triangle.



y
$\left(0, \frac{a}{\sqrt{3}}, 0\right)$
q
x
0
q
q
$\left(-\frac{a}{2}, -\frac{a}{2\sqrt{3}}, 0\right)$
$\left(+\frac{a}{2}, -\frac{a}{2\sqrt{3}}, 0\right)$

n. Let us choose a coordinate system as shown in the figure.
Therefore, coordinates of charges become

$$\vec{r}_1=\left(0,\frac{a}{\sqrt{3}},0\right),\ \vec{r}_2=\left(\frac{a}{2},\frac{-a}{2\sqrt{3}},0\right),\ \vec{r}_3=\left(\frac{-a}{2},\frac{-a}{2\sqrt{3}},0\right)$$

Coordinate of the point where, we have to find electric field is $\vec{r}=\left(0,0,\sqrt{\frac{2}{3}}a\right)$

Therefore, $$\vec{E}=\frac{q}{4\pi\varepsilon_0}\cdot\frac{r-\vec{r}_1}{|\vec{r}-\vec{r}_1|^3}+\frac{q}{4\pi\varepsilon_0}\frac{\vec{r}-\vec{r}_2}{|\vec{r}-\vec{r}_2|^3}+\frac{q}{4\pi\varepsilon_0}\frac{\vec{r}-\vec{r}_3}{|\vec{r}-\vec{r}_3|^3}$$

$$=\frac{q}{4\pi\varepsilon_0}\left[\frac{\sqrt{\frac{2}{3}}a\hat{k}-\frac{a}{\sqrt{3}}\hat{j}}{\left(\frac{2}{3}a^2+\frac{1}{3}a^2\right)^{3/2}}+\frac{\sqrt{\frac{2}{3}}a\hat{k}-\frac{a}{2}\hat{i}+\frac{a}{2\sqrt{3}}\hat{j}}{\left(\frac{2}{3}a^2+\frac{a^2}{4}+\frac{a^2}{12}\right)^{3/2}}+\frac{\sqrt{\frac{2}{3}}a\hat{k}+\frac{a}{2}\hat{i}+\frac{a}{2\sqrt{3}}\hat{j}}{\left(\frac{2}{3}a^2+\frac{a^2}{4}+\frac{a^2}{12}\right)^{3/2}}\right]$$

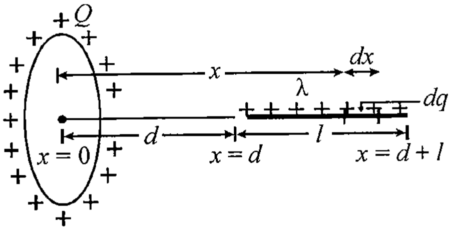
$$\vec{E}=\frac{q}{4\pi\varepsilon_0 a^3}\left[\sqrt{\frac{2}{3}}a\hat{k}-\frac{a}{\sqrt{3}}\hat{j}+\sqrt{\frac{2}{3}}a\hat{k}-\frac{a}{2}\hat{i}+\frac{a}{2\sqrt{3}}\hat{j}+\sqrt{\frac{2}{3}}a\hat{k}+\frac{a}{2}\hat{i}+\frac{a}{2\sqrt{3}}\hat{j}\right]$$

$$=\frac{q}{4\pi\varepsilon_0 a^3}\left[3\sqrt{\frac{2}{3}}a\hat{k}\right]\Rightarrow\vec{E}=\frac{\sqrt{6}q}{4\pi\varepsilon_0 a^2}\hat{k}.$$

Exaxmple: A thin wire of length *l* and uniform charge density λ is placed along the axis of a ring of radius *R* and uniform total charge *q*. If distance between the nearer end of wire and center of ring is '*d*'. Calculate force on the wire.

Soln. The two objects are not point charges, therefore we cannot get force directly from coulomb's law. If $\vec{E}$ be field of ring near elementary charge *dq* of wire then force on wire is

$$\vec{F} = \int \vec{E}\, dq$$



$$= \int_d^{d+l} \frac{Q\hat{i}}{4\pi\varepsilon_0 (R^2+x^2)^{3/2}} \cdot \lambda\, dx$$

$$= \frac{Q\lambda\hat{i}}{4\pi\varepsilon_0} \int_d^{d+l} \frac{dx}{(R^2+x^2)^{3/2}}$$

Substitute, $x = R\tan\theta$, to integrate

$$dx = R\sec^2\theta\, d\theta$$

$$\int \frac{dx}{(R^2+x^2)^{3/2}} = \int \frac{R\sec^2\theta\, d\theta}{R^3\sec^3\theta} = \frac{1}{R^2}\int \cos\theta\, d\theta$$

$$= \frac{1}{R^2}[\sin\theta] = \frac{1}{R^2}\left[\frac{x}{\sqrt{R^2+x^2}}\right]$$

$$\therefore \quad \vec{F} = \frac{Q\lambda\hat{i}}{4\pi\varepsilon_0 R^2}\left[\frac{x}{\sqrt{R^2+x^2}}\right]_d^{d+l}$$

$$\vec{F} = \frac{Q\lambda\hat{i}}{4\pi\varepsilon_0 R^2}\left[\frac{d+l}{\sqrt{R^2+(d+l)}} - \frac{d}{\sqrt{R^2+d^2}}\right]$$

Example: A thin conducting ring of radius *r* has an electric charge +*Q*. What would be the increase in the tension of the wire, if a point charge +*q* is placed at the centre of the ring?

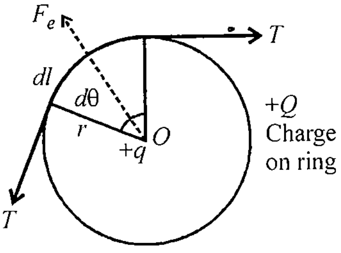
Soln. The situation is shown in figure. Charge on a small element *dl* of the ring $dQ = \frac{Q}{2\pi r}dl$, outward electric force on this element $F_e = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q\,dl}{2\pi r}\right)\left(\frac{q}{r^2}\right)$.

Let the tension be increased by *T*, to balance this force F_e. The increase in tension is given by

$$= 2T\sin\left(\tfrac{d\theta}{2}\right)$$

$$\cong 2T\left(\tfrac{d\theta}{2}\right)$$

$$\cong T\left(\tfrac{dl}{r}\right) \qquad (\because dl = r\,d\theta)$$



Hence, $T\frac{dl}{r} = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q\,dl}{2\pi r}\right)\left(\frac{q}{r^2}\right)$

Therefore, $T = \frac{Qq}{8\pi^2\varepsilon_0 r^2}$.

Electric line of force:

An electric line of force is an imaginary curve drawn in such a way that tangent to this curve at any point gives the direction of electric field at the point.

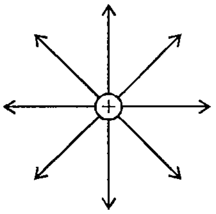


Figure (1)

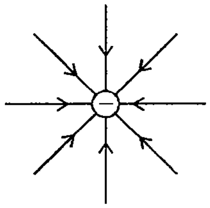


Figure (2)

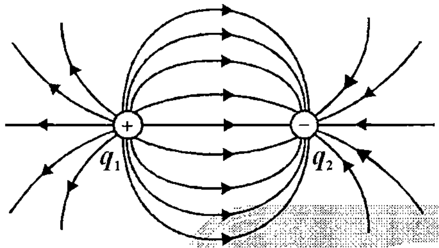


Figure (3)

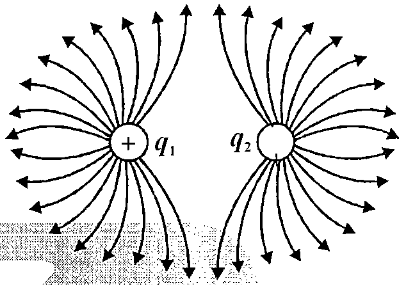


Figure (4)

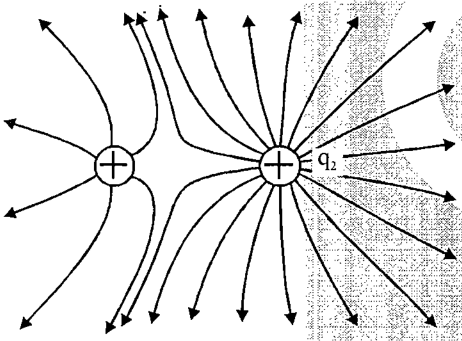


Figure (5)

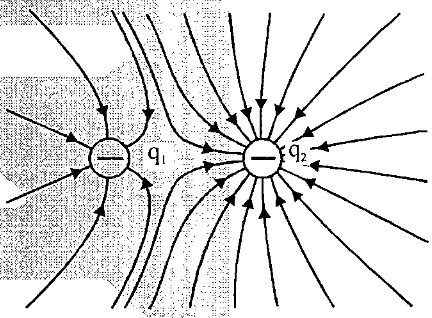


Figure (6)

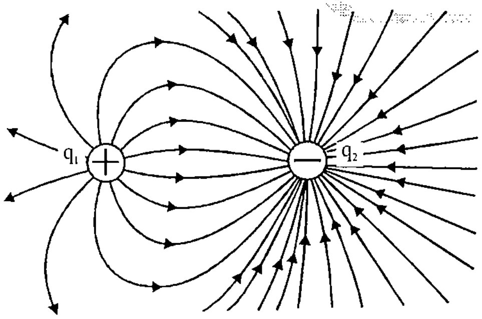


Figure (7)

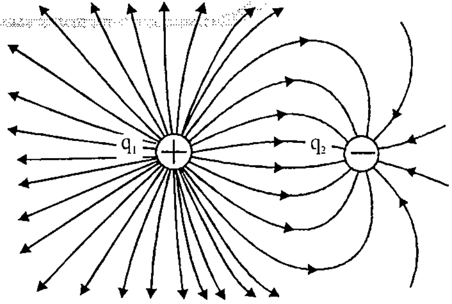


Figure (8)

(1) For single positive charge

(2) For single negative charge

(3) $q_1 = q_2$ and q_1 positive and q_2 negative.

(4) $q_1 = q_2$ and q_1 = positive, q_2 = positive

(5) $q_1 < q_2$ and both is positive

(6) $q_1 < q_2$ and both is negative

(7) $q_1 < q_2$ and q_1 = positive, q_2 = negative

(8) $q_1 > q_2$ and q_1 is positive and q_2 is negative.

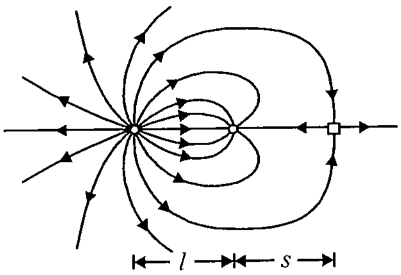


Figure. Field lines for two rods of charge 2λ and –λ; the rods are normal to the page. The box denotes the point where the field is zero.

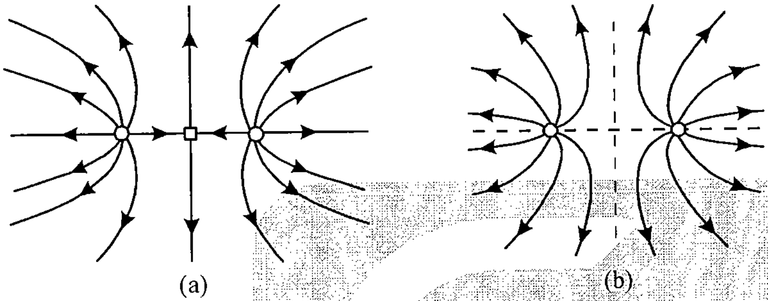

(a) (b)

Figure. Representations of the field lines due to two equal positively charged rods normal to the page. (a) Representation with field lines going from one rod toward the other. These field lines appear to end at the origin (surrounded by a box), with two lines leaving vertically. Thus the net number of field lines that do not go from one rod

Properties of field lines:

(1) Field lines start from a positive charge and end in a negative charge.

(2) Since tangent of field line represent the direction of electric field and at a particular point the electric field do not have direction more than one. So, two field lines can not cross each other.

(3) Fields lines are continuous curves if there is no conductor.

Differential equation for force lines:

Suppose $\vec{r} = \vec{r}(s)$ represents force lines $\frac{d\vec{r}}{ds}$ is tangent to this curve at every point. Since tangent represent the direction of electric field.

$$\therefore \quad \frac{d\vec{r}}{ds} = a\vec{E}(r) \quad (a \text{ is a constant})$$

$$\therefore \quad \frac{dx}{ds} = aE_x, \ \frac{dy}{ds} = aE_y, \ \frac{dz}{ds} = aE_z$$

$$\therefore \quad \boxed{\frac{dx}{E_x} = \frac{dy}{E_y} = \frac{dz}{E_z}}$$

Gauss' law:

The total electric flux through any closed surface is equal to $\frac{1}{\varepsilon_0}$ times the total charge enclosed by the surface.

For a closed surface 's' enclosing J number of point charges $q_1......q_j$ the mathematical formula is

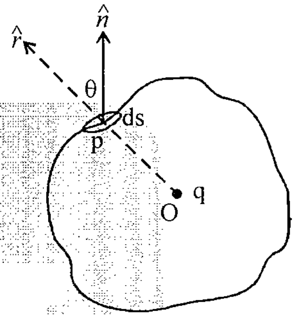
$$\boxed{\oint \vec{E}\cdot d\vec{s} = \frac{1}{\varepsilon_0}\sum_{i=1}^{j} q_i}$$

Proof of Gauss' law:

Suppose a point charge q located at O and we consider a closed surface enclosing q. Let us consider an elementary area $d\vec{s} = \hat{n}ds$ at distance r from O.

Therefore, the electric field at P is

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$



Therefore, the flux through the $d\vec{s}$ is

$$\vec{E}\cdot d\vec{s} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}\cdot\hat{n}ds$$

Therefore, total flux $= \oint \vec{E}\cdot d\vec{s}$

$$= \frac{q}{4\pi\varepsilon_0}\oint \frac{\hat{r}\cdot\hat{n}}{r^2}ds$$

$$= \frac{q}{4\pi\varepsilon_0}\oint \frac{ds\cos\theta}{r^2}, \oint \frac{ds\cos\theta}{r^2} = \text{solid angle} = 4\pi$$

$$= \frac{q}{4\pi\varepsilon_0}4\pi = \frac{q}{\varepsilon_0}$$

Differential form of Gauss' laws:

We know that Gauss's law is,

$$\oint \vec{E}\cdot d\vec{s} = \frac{q}{\varepsilon_0}$$

For continuous change distribution

$$\oint \vec{E}\cdot d\vec{s} = \int \frac{\rho dv}{\varepsilon_0}$$

$$\Rightarrow \qquad \int\left(\vec{\nabla}.\vec{E}\right)dv = \int \frac{\rho dv}{\varepsilon_0}$$

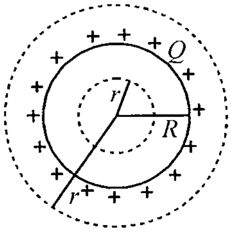
$$\therefore \qquad \boxed{\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}}$$

This equation is the differential form of Gauss's law.

Application of Gauss' Law for field calculation:

Electric field can be calculate using Gauss Law only for few symmetric cases, like spherical cylindrical, and plane symmetry.

Electric field due to uniformly charged spherical shell.



Consider a uniformly changed shell having radius R and total charge Q.

To calculate electric field

Let us draw spherical Gaussian surfaces as shown by dotted lines in figure.

Due to symmetry to electric field lines are in radial direction

Gauss Law $\oint \vec{E}\cdot d\vec{s} = \frac{Q_{enclosed}}{\varepsilon_0}$

$$\oint E\hat{r}\cdot ds\hat{r} = \frac{Q_{enclosed}}{\varepsilon_0}$$

$$\oint Eds = \frac{Q_{enclosed}}{\varepsilon_0}$$

Since all points on Gaussian sphere are at equal distance from center. Therefore.

$$E\oint ds = \frac{Q_{enclosed}}{\varepsilon_0}$$

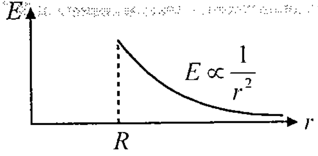
$$E4\pi r^2 = \frac{Q_{enclosed}}{\varepsilon_0}$$

$$E = \frac{Q_{enclosed}}{4\pi\varepsilon_0 r^2}$$

$$Q_{enclosed} = 0 \text{ for } R < R \ \therefore\ E = 0$$

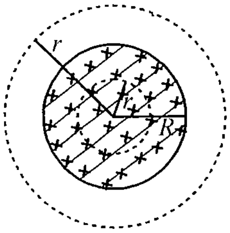
$$Q_{enclosed} = Q \text{ for } r > R \ \therefore\ E = \frac{Q}{4\pi\varepsilon_0 r^2}$$

Thus a sphere behaves like a point charge for outside point.



Electric field due sphere having uniform volume charge density.

Consider uniformly charged sphere of radius R and total charge Q.



Volume charge density, $\rho = \frac{Q}{\frac{4}{3}\pi R^3} =$ Constant

To calculate electric field draw Gaussian surfaces as shown by dotted lines in figure

Proceeding similar to shell case we get

$$E = \frac{Q_{enclosed}}{4\pi\varepsilon_0 r^2}$$

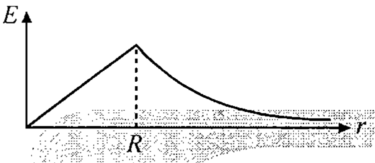
for $r < R$ $$Q_{enc}=\int\rho d\tau=\rho\int_0^r d\tau=\rho\frac{4}{3}\pi r^3$$

$$\therefore \quad \boxed{E=\frac{\rho r}{3\varepsilon_0}=\frac{Qr}{4\pi\varepsilon_0 R^3}}$$

for $r > R$ $$Q_{enc}=\int\rho d\tau=\rho\int_0^R d\tau=\rho\cdot\frac{4}{3}\pi R^3$$

$$\therefore \quad \boxed{E=\frac{\rho R^3}{3\varepsilon_0 r^2}=\frac{Q}{4\pi\varepsilon_0 r^2}}$$ (same as point charge)

Variation of electric field with radial distance is

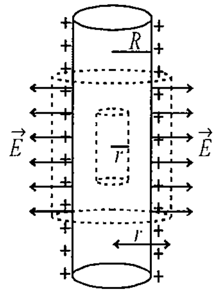


Electric field due to uniformly changed long hollow cylinder

Consider a long hollow cylinder of radius R and surface charge density σ.
To calculate electric field let us draw cylindrical Gaussian surfaces as shown in the figure

In this type cases electric field can be calculated only near the mid region because in the mid region field lines are in radial direction

Gauss' Law



$$\oint\vec{E}\cdot d\vec{s}=\frac{Q_{enclosed}}{\varepsilon_0}$$

Here flux through flat surfaces of cylinder is zero.

$$\therefore \quad \int_{curved}E\hat{r}\cdot ds\hat{r}=\frac{Q_{enclosed}}{\varepsilon_0}$$

$$\int_{curved}Eds=\frac{Q_{enclosed}}{\varepsilon_0}$$

$$E\int ds=\frac{Q_{enclosed}}{\varepsilon_0}$$

$$E\cdot 2\pi rh=\frac{Q_{enclosed}}{\varepsilon_0}\quad\Rightarrow\quad\boxed{E=\frac{Q_{enc}}{2\pi\varepsilon_0 rh}}$$

for $r < R$ $Q_{enclosed}=0,\ \ E=0$

for $r > R$ $Q_{enclosed}=\sigma\, 2\pi Rh$

$$\therefore \quad E=\frac{\sigma R}{\varepsilon_0 r}$$

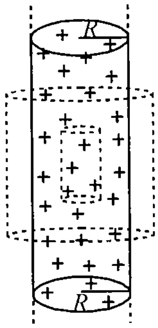
Electric field due to long cylinder having uniform volume charge density :

Proceeding similar to previous case we get $E = \dfrac{Q_{enclosed}}{2\pi\varepsilon_0 rh}$

for $r < R$ $Q_{enclosed} = \int_0^r \rho d\tau = \rho\int_0^r d\tau$

$\therefore$ $Q_{enclosed} = \rho\pi r^2 h$

$\therefore$ $\boxed{E = \dfrac{\rho r}{2\varepsilon_0}}$



for $r > R$

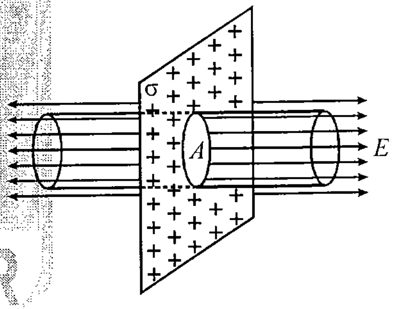
$Q_{enclosed} = \int \rho d\tau = \rho\int d\tau = \rho\pi R^2 h$

$\therefore$ $\boxed{E = \dfrac{\rho R^2}{2\varepsilon_0 r}}$

Electric field due to uniformly charged sheet : (Non-conducting)

Let σ be surface charge density of a large sheet, since the sheet is non conducting we assume that charge is only on one layer.

To calculate electric field let us draw a cylindrical Gaussian surface as shown in the figure. Due to symmetry field near the center of sheet is perpendicular to the sheet.



Gauss, law $\oint \vec{E}\cdot d\vec{s} = \dfrac{Q_{\text{enclosed}}}{\varepsilon_0}$

Flux is only through the flate surface of cylindrical Gaussian surface,

Therefore, $\int_{\text{flat}} \vec{E}\cdot d\vec{s} = \dfrac{Q_{\text{enclosed}}}{\varepsilon_0}$

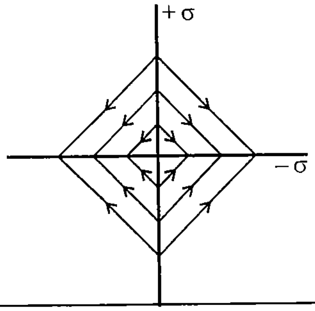
$2EA = \dfrac{\sigma A}{\varepsilon_0}$. Therefore, $E = \dfrac{\sigma}{2\varepsilon_0}$

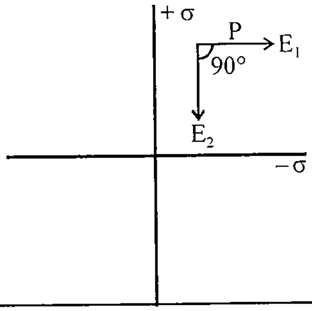
Conducting sheet : In conducting sheet charge resides on both sides therefore each layer gives a field $\sigma/2\varepsilon_0$.

Therefore, net field due to conducting sheet is $E = \dfrac{\sigma}{\varepsilon_0}$.

Example: Two uniformly charged infinite planes of charged density $+\sigma$ and $-\sigma$ interest each other at right angles. Find the electric field at any point and hence sketch the field lines.

Soln.





Let any point P. The electric field due to planes are $E_1 = \frac{+\sigma}{2\varepsilon_0}$ and $E_2 = \frac{-\sigma}{2\varepsilon_0}$ and the angle between E_1 and E_2 is 90°

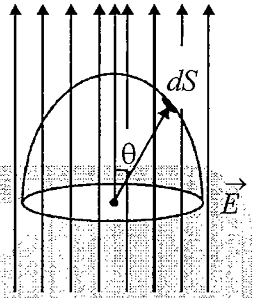
$$\therefore \text{ Total field } E = \sqrt{E_1^2 + E_2^2}$$

$$= \sqrt{\left(\frac{\sigma}{2\varepsilon_0}\right)^2 + \left(-\frac{\sigma}{2\varepsilon_0}\right)^2} = \frac{\sqrt{2}\,\sigma}{2\varepsilon_0} = \frac{\sigma}{\sqrt{2}\,\varepsilon_0}$$

Example : A hemispherical surface of radius R is in a constant electric field $\vec{E}$ directed perpendicular to the base as shown in the figure. Determined the flux through the curved surface of the hemisphere.

n. The net flux through the closed surface is zero.

Flux through the curve surface:



Let angle between E_0 and $\hat{n}$ is θ.

$$\phi_E = \int \vec{E}\cdot d\vec{S} = \int E\cos\theta dS$$

$$= \int_0^{\pi/2}\int_0^{2\pi} ER^2\cos\theta\sin\theta d\theta d\phi \qquad dS = R^2\sin\theta d\theta d\phi$$

$$= \frac{1}{2}ER^2\left(-\frac{\cos 2\theta}{2}\right)_0^{\pi/2}\cdot 2\pi = E\pi R^2$$

OR

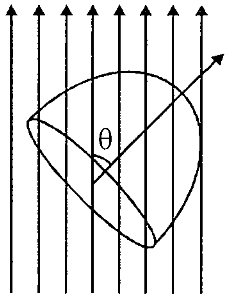
$$\phi_{total} = \phi_{in} + \phi_{out} = 0 \quad \text{(for closed surface)}$$

$$\therefore \qquad -\phi_{in} = \phi_{out} \qquad \Rightarrow \boxed{\phi_{out} = -\phi_{in}}$$

$$\therefore \qquad \phi_{in} = \int \vec{E}\cdot d\vec{S} = E\cdot\pi R^2 \text{ (flux in through the circular surface)}$$

$$\therefore \qquad \phi_{out} = E\cdot\pi R^2 \text{ (flux. out through the curve surface)}$$

Special Case:



If θ be the angle between $\vec{E}$ and normal to the circular face.

The flux out through the curve surface.

$$\boxed{\phi = \cos\theta E\pi R^2}$$

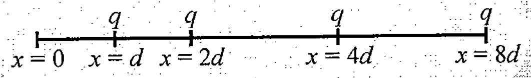
Example : A certain charge distribution has a region within which the charge has everywhere the same density ρ. A spherical cavity is made in this region by removing the charge originally present at the locaton of the cavity wihout disturbing the rest of the charge. The cavity is centered at the point where the electric field originally was E_c. Find how the presence of the cavity affects this field.

Soln. The effect of the cavity can be determined by regarding the zero charge density in the cavity as being made up of two equally large charge densities of opposite polarity, ρ and $-\rho$. The field at the center of the cavity is then the sum of three fields: (a) the field due to the charge of density ρ located within the cavity, (b) the field due to the charge of density $-\rho$ located within the cavity, and (c) the field due to the charge located outside the cavity. Now, the sum of the first and the third fields is just E_c, since if the cavity were filled with charge of density ρ, the original charge distribution is restored. The second field is zero, since by symmetry, the field at the center of a spherical charge distribution of uniform density is zero. The sum of all three fields is thus E_c, so that a spherical cavity made in a region of uniform charge density does not affect the field at the point at which the cavity is centered. This, incidentally, allows one to make the so-called *cavity definition* of the electric field inside a charge-filled region. According to this definition the field in a charge-filled region is the field measured at the center of a spherical cavity whose dimensions are small compared to the distance over which the charge density changes appreciably.

ADDITIONAL SOLVED PROBLEMS

1. An infinite number of charges each equal to q are placed along the x-axis at $x = d$, $x = 2d$, $x = 4d$, $x = 8d$,.... and so on. Find electric field at the point $x = 0$ due to this set of charges.
What will be electric field if in the above set up the consecutive charge have opposite sign ?

Soln. The set of charges is shown in figure :



(i) As the point charges are along the same straight line, hence the intensities at $x = 0$ are also along the x-axis.

$$E = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{(d)^2} + \frac{q}{(2d)^2} + \frac{q}{(4d)^2} + \frac{q}{(8d)^2} + \cdots + \infty\right]$$

$$= \frac{q}{4\pi\varepsilon_0 d^2}\left[\frac{1}{1} + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots + \infty\right]$$

$$= \frac{q}{4\pi\varepsilon_0 d^2}\left[\frac{1-(1/2)^\infty}{1-(1/4)}\right] = \frac{q}{4\pi\varepsilon_0 d^2}\left[\frac{1}{(3/4)}\right]$$

$$= \frac{q}{3\pi\varepsilon_0 d^2}$$

(ii) When the consecutive charges are negative, then the electric intensity at $x = 0$ is given by

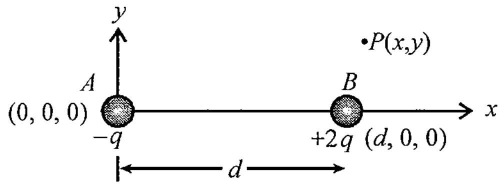
$$E = \frac{1}{4\pi\varepsilon_0 d^2}\left[\frac{q}{(1)^2} - \frac{q}{(2)^2} + \frac{q}{(4)^2} - \frac{q}{(8)^2} + \frac{q}{(16)^2} - \frac{q}{(32)^2} + \cdots \infty\right]$$

$$= \frac{q}{4\pi\varepsilon_0 d^2}\left[\left(1+\frac{1}{16}+\frac{1}{256}+\cdots\right)-\left(\frac{1}{4}+\frac{1}{64}+\frac{1}{1024}+\cdots\right)\right]$$

$$= \frac{q}{4\pi\varepsilon_0 d^2}\left[\left(\frac{1}{1-(1/16)}\right)-\frac{1}{4}\left(\frac{1}{1-(1/16)}\right)\right] = \frac{q}{4\pi\varepsilon_0}\left[\frac{16}{15}-\frac{1}{4}\left(\frac{16}{15}\right)\right] = \frac{q}{5\pi\varepsilon_0 d^2}$$

Two points charges $+2q$ and $-q$ are placed distance d apart. find the locus of all points at which the resultant electric field is parallel to the line joining the charges ?

n. Let the two point charges $-q$ and $+2q$ are placed at point A and B respectively which are d distance apart as shown in figure.



Net electronic field at point $P(x,y,0)$ due to this charge distribution is

$$\vec{E} - = -\frac{q}{4\pi\varepsilon_0}\left[\frac{x\hat{i}+y\hat{j}}{\left(x^2+y^2\right)^{3/2}}+2\frac{(x-d)\hat{i}+y\hat{j}}{\left[(x-d)^2+y^2\right]^{3/2}}\right]$$

If $\vec{E}$ is parallel to line AB

$$\vec{E}\cdot\hat{j}=0$$

$$\frac{y}{\left(x^2+y^2\right)^{3/2}}-\frac{2y}{\left[(x-d)^2+y^2\right]^{3/2}}=0$$

$$(x-d)^2+y^2=2\left(x^2+y^2\right)$$

$$x^2+y^2+2xd+d^2=0$$

$$(x+d)^2+y^2=0$$

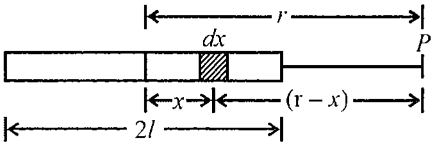
$$y=\pm(x+d)$$

At every point on this straight line $\vec{E}$ is parallel to line AB.

A thin straight rod of length 2ℓ carrying a uniform distributed charge q is located in vacuum. Find the magnitude of the electric field strength as a function of the distance r from the rod's centre along the straight line coinciding with the rod's direction (at the points lying outside the rod).

n. Here, $dE=\frac{1}{4\pi\varepsilon_0}\left(\frac{q\,dx}{2\ell}\right)\times\frac{1}{(r-x)^2}$

If the element lies on outer side, then

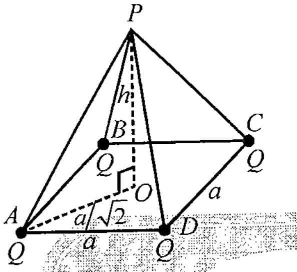


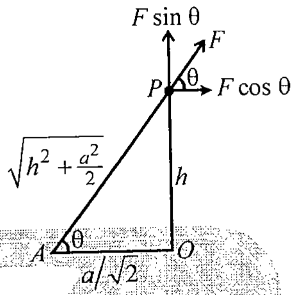
$$\therefore \quad E=\int dE=\int_{-\ell}^{+\ell}\frac{1}{4\pi\varepsilon_0}\left(\frac{q\,dx}{2\ell}\right)\times\frac{1}{(r-x)^2}$$

Integrating this expression, we get $E=\dfrac{q}{4\pi\varepsilon_0}\times\dfrac{1}{(r^2-\ell^2)}$.

4. Four equal positive charges each of value Q are fixed at the four corners of a square of side a. A unit positive charge mass m is placed at P, at a height h above the centre of the square. What should be the value of Q in order that this unit charge is in equilibrium.

Soln. The situation is shown in figure :





Force experienced by unit positive charge placed at P due to a charge Q at A is given by

$$F=\frac{1}{4\pi\varepsilon_0}\frac{Q\times 1}{\left(h^2+\dfrac{a^2}{2}\right)}$$

Similarly, equal forces act on unit positive charge at P due to charges at B, C and D. When these forces are resolved in horizontal and vertical directions, the horizontal components $(F\cos\theta)$ cancel each other and the net vertical force is $4F\sin\theta$.

Therefore, now upward force $=4\dfrac{1}{4\pi\varepsilon_0}\dfrac{Q}{\left(h^2+\dfrac{a^2}{2}\right)}\cdot\sin\theta$

For the equilibrium of unit positive charge at P, Upward force = Weight of unit charge.

$$4\frac{1}{4\pi\varepsilon_0}\frac{Q}{\left(h^2+\dfrac{a^2}{2}\right)}\cdot\sin\theta=mg$$

From figure, $\sin\theta=\left\{\dfrac{h}{\sqrt{(h^2+a^2/2)}}\right\}$

Therefore, $\dfrac{1}{\pi\varepsilon_0}\dfrac{Qh}{\left(h^2+\dfrac{a^2}{2}\right)^{3/2}}=mg$ **or** $Q=\dfrac{mg\,\pi\varepsilon_0}{h}\left(h^2+\dfrac{a^2}{2}\right)^{3/2}$.

A pendulum bob of mass 80 mg and carrying a charge of 4×10^{-8} coulomb is at rest in a horizontal uniform electric field of 20,000 Vm^{-1}. Find the tension in the thread of the pendulum and the angle it makes with the vertical (Take $g = 10$ m/s^2)

ln. The different forces on the bob are shown in figure :

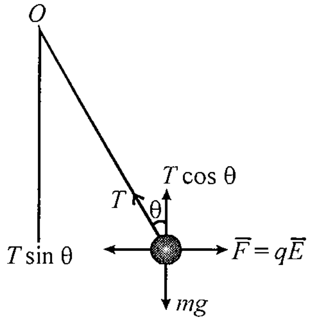
Here, $m = 80\,mg = 80\times10^{-6}$ kg, $q = 2\times10^{-8}$ C and $\mathbf{E} = 2\times10^4$ Vm^{-1}

Resolving T, we have vertical component $= T\cos\theta$ and horizontal component $= T\sin\theta$

For the equilibrium of the bob

$$T\cos\theta = mg \qquad \dots(1)$$

and $$T\sin\theta = qE \qquad \dots(2)$$



Dividing equation (2) by equation (1), we get

$$\tan\theta = \frac{qE}{mg} = \frac{4\times10^{-8}\times2\times10^4}{80\times10^{-6}\times10} = 1 \Rightarrow \theta = 45^\circ$$

Square and add equations (1) and (2), to get

$$\sqrt{(mg)^2+(qE)^2}$$

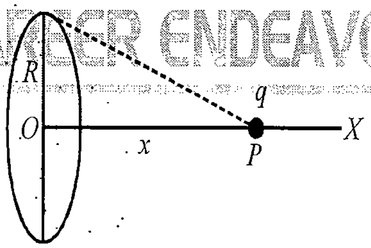
$$8\sqrt{2}\times10^{-4}\ N$$

A thin fixed ring of radius 1 metre has a positive charge $Q = 1\times10^{-5}$ coulomb uniformly distributed over it. A particle of mass $m = 0.9$ gm and having a negative charge of $q = 1\times10^{-6}$ coulomb is placed on the axis at a distance of 1 cm from the centre of the ring. Show that the motion of the negatively charged particle is approximately simple harmonic. Calculate the time period of oscillations.

ln. Let the charge is placed at distance x from center of ring as shown in figure.

We know that field near center of ring is

$$E = \frac{Qx}{4\pi\varepsilon_0 R^3} \text{ (see the field due to ring topic)}$$



where, $x << 1$

Therefore, force on the point charge is,

$$F = qE = -\frac{qQx}{4\pi\varepsilon_0 R^3} = \frac{-1\times10^{-6}\times10^{-5}\times10^9}{1^3}x$$

$$F = -9\times10^{-2}x$$

$$\because \qquad F \propto -x$$

Therefore, motion is S.H.M. (comparing with $F = -kx$)

We get $k = 9\times10^{-2}$

Therefore, time period $= 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{0.9\times10^{-3}}{9\times10^{-2}}} = \frac{2\pi}{10} = \frac{\pi}{5}$

$$= 2\pi\sqrt{\left(\frac{0.9\times10^{-3}\times[(0.01)^2+1]^{3/2}}{(9\times10^9)(10^{-5})(10^{-6})}\right)}$$

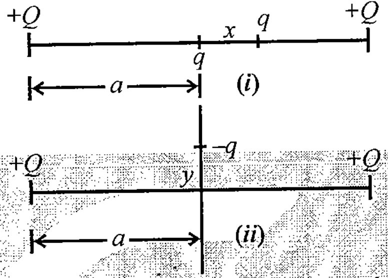
$$= \frac{\pi}{5}\text{ seconds.}$$

7. Two identical positive point charges Q each are fixed at a distance $2a$ apart. A point charge q lies midway between the fixed charges. Show that

(i) for small displacement (relative to a) along line joining the fixed charges, the charge q executes SHM if it is positive and

(ii) for small lateral displacement, it executes SHM if it is negative. Compare the frequencies of oscillation in the two cases.

Soln. The two situations are shown in figure :



(i) Let x be the displacement of the charge $+q$ from the mean position. Now net force acting on the charge q to bring in its original position.

$$F = -\frac{Qq}{4\pi\varepsilon_0(a-x)^2} + \frac{Qq}{4\pi\varepsilon_0(a+x)^2}$$

$$= -\frac{Qq}{4\pi\varepsilon_0}\frac{4ax}{(a^2-x^2)} \approx -\frac{Qq}{4\pi\varepsilon_0}\cdot\frac{4ax}{a^4} = \frac{Qq}{4\pi\varepsilon_0}\cdot\frac{4x}{a^3}$$

Restoring force $= F = -\dfrac{4Qq}{4\pi\varepsilon_0 a^3}x = -K_1 x$, (where, m is the mass of the charge)

As acceleration is directly proportional to displacement, hence the motion is SHM. Its time period T_1 is given by

$$\omega_1 = \sqrt{\frac{k_1}{m}} = \sqrt{\frac{4Qq}{4\pi\varepsilon_0 m a^3}}$$

(ii) Restoring force on $-q$ towards Q is given by

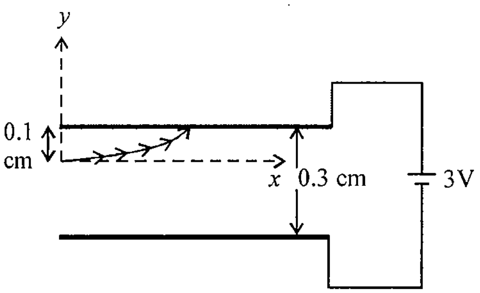
$$F = \frac{-2Qq}{4\pi\varepsilon_0(a^2+y^2)}\cdot\frac{y}{(a^2+y^2)} = \frac{-2Qqy}{4\pi\varepsilon_0(a^2+y^2)^{3/2}} \approx \frac{-2Qqy}{4\pi\varepsilon_0 a^3} = -k_2 y$$

$$\omega_2 = \sqrt{\frac{k_2}{m}} = \sqrt{\frac{2Qq}{4\pi\varepsilon_0 m a^3}} \Rightarrow \frac{\omega_1}{\omega_2} = \sqrt{2}$$

8. A particle of mass 9×10^{-31} kg and a negative charge of 1.6×10^{-19} coulomb projected horizontally with a velocity of 10^5 m/s into a region between two infinite horizontal parallel plates of metal. The distance between the plates is 0.3 cm and the particles enters 0.1 cm below the top plate. The top and bottom plates are connected respectively to the positive and negative terminals of a 30 volt battery. Find the component of the velocity of the particle in direction of field just before it hits one on the plates.

Soln. Electric field between the plates is

$$E = \frac{\text{potential difference}}{\text{separation}} = \frac{30}{0.3\times10^{-2}} = 10^4\ V/m$$



Acceleration of particle, $\vec{a} = \frac{\vec{F}}{m} = \frac{q\vec{E}}{m}$

Therefore, $\vec{a} = \frac{(1.6\times10^{-15})\hat{j}}{(9\times10^{-31})} = 1.77\times10^{15}\ \text{m/sec}^2$

For motion in y-direction

$$y = u_y t + \frac{1}{2}a_y t^2$$

$$0.1\times10^{-2} = 0 + \frac{1}{2}\left(1.77\times10^{-15}\right)t^2$$

Solving we get, $t = 1.063\times10^{-10}$ second

Component of velocity in the direction of field is given by

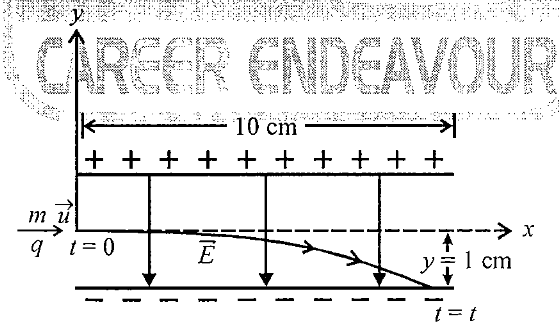
$$v_y = u_y + a_y t$$

$$= 0 + (1.77\times10^{15})\,(1.063\times10^{-10})$$

$$= 1.881\times10^4\ \text{m/s.}$$

A particle having a charge of 1.6×10^{-19} C enters midway between the plates of a parallel plate condenser. The initial velocity of particle is parallel to the plates. A potential difference of 300 volts is applied to the capacitor plates. If the length of the capacitor plate is 10 cm and they are separated by 2 cm, calculate the greatest initial velocity for which the particle will not be able to come out of the plates. The mass of the particle is 12×10^{-24} kg.

ln. The situation is shown in figure :



Here, $E = \frac{\text{Potential difference}}{\text{separation}} = \frac{300}{2\times10^{-2}} = 1.5\times10^4\ \frac{\text{V}}{\text{m}}.$

For maximum speed if the particle does not come out, then its path will be as shown in figure

At $t = t$, $x = 100$ cm, $y = -1$ cm

For x-direction,

$$x = u_x t + \frac{1}{2}a_x t^2 \Rightarrow 0.1 = u_{\max}t + \frac{1}{2}(0)t^2$$

$$t = \frac{0.1}{u_{\max}}$$

For y-direction,

$$y = u_y t + \frac{1}{2} a_y t^2 = 0 + \frac{1}{2}\left(\frac{F_y}{m}\right) t^2$$

Put t, we get

or $$u_{\max}^2 = \frac{1}{2} \cdot \frac{q E_y}{m y} (0.1)^2$$

$$= \frac{1}{2} \frac{(1.6 \times 10^{-19})(15000)}{(12 \times 10^{-24})(10^{-2})} \left(\frac{1}{10}\right)^2 = 10^8$$

$$\therefore \quad u_{\max} = 10^4 \text{ m/s.}$$

10. The distance between two plates of a cathode ray oscillograph is 1 cm and potential difference between them is 1200 volt. If an electron of energy 2000 eV enters at right angles to the field, what will be its deflection is the plates are 1.5 cm long ?

Soln. Electric field between the plates, $E = V/d$.

Therefore, $E = \dfrac{1200 \text{ volt}}{1 \times 10^{-2} \text{ metre}} = 1.2 \times 10^5$ volt/metre.

Kinetic energy of entering electron (k)

$$= 2000 \text{ eV} = 2000 \times (1.6 \times 10^{-19}) \text{ joule.} = 3.2 \times 10^{-16} \text{ joule} = \frac{1}{2} m v^2 \Rightarrow v^2 = \frac{2k}{m}$$

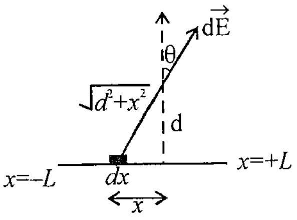
For x-direction, $x = u_x t + \dfrac{1}{2} a x t^2 \Rightarrow \ell = vt + 0 \Rightarrow t = \dfrac{\ell}{v}$

For y-direction, $y = \dfrac{e E l^2}{2 m v^2} = \dfrac{e E l^2}{4K} = u_y t + \dfrac{1}{2} a_y t^2 = 0 + \dfrac{1}{2}\left(\dfrac{qE}{m}\right) \cdot \left(\dfrac{1}{V}\right)^2$

Therefore, $y = \dfrac{(1.6 \times 10^{-19}) \times (1.2 \times 10^5)(1.5 \times 10^{-2})^2}{4 \times (3.2 \times 10^{-16})} = \mathbf{0.34 \times 10^{-2}}$ **metre = 0.34 cm.**

11. Find the electric field at a distance 'd' above the midpoint of straight line segment of length 2L, carryings uniform lines charge λ.

Soln. Let us take an elementary length dx at a distance x from the centre. The field at the point 'P' due to elementary length is



$$dE = \frac{1}{4\pi\varepsilon_0} \frac{dx\lambda}{(d^2 + x^2)}$$

$$\tan\theta = \frac{x}{d}$$

Net Field will be along the symmetric line

$$dx = d \sec^2\theta \, d\theta$$

$$\vec{E} = \int dE \cos\theta = \int \frac{1}{4\pi\varepsilon_0} \frac{dx\lambda}{(d^2 + x^2)} \cos\theta$$

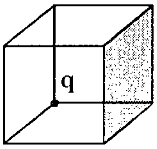
$$\cos\theta = \frac{d}{\sqrt{(d^2 + x^2)}}$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\int \frac{z\cos\theta\sec^2\theta d\theta}{\left(x^2+d^2\right)}$$

$$\Rightarrow \qquad \frac{\lambda}{4\pi\varepsilon_0 d}\int\cos\theta d\theta = \frac{\lambda}{4\pi\varepsilon d}\left[\sin\theta\right]_{-\theta}^{\theta} = \frac{2\lambda}{4\pi\varepsilon_0 d}\sin\theta = \frac{2\lambda}{4\pi\varepsilon_0}\frac{L}{d\sqrt{d^2+L^2}}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{2\lambda L}{d\sqrt{d^2+L^2}}\hat{z}$$

A point charge 'q' sits at a corner of a cube of side 'a', as shown in the figure on the right. The flux of the electric field vector through the shaded side is **[TIFR 2013]**



(a) $\frac{q}{8\varepsilon_0}$ (b) $\frac{q}{16\varepsilon_0}$ (c) $\frac{q}{24\varepsilon_0}$ (d) $\frac{q}{6\varepsilon_0}$

In. Let consider a Gaussian surface in shape of cube whose sides are $2a$. Now charge is at the centre

Therefore, flux through the area $24a^2$ is $\frac{q}{\varepsilon_0}$

Therefore, flux through the area a^2 is $\frac{q}{24\varepsilon_0}$

Correct option is (c)

A charge distribution has the charge density given by $\rho = Q\{\delta(x-x_0)-\delta(x+x_0)\}$. For this charge distribution the electric field at $(2x_0,0,0)$

(a) $\frac{2Q\hat{x}}{9\pi\epsilon_0 x_0^2}$ (b) $\frac{Q\hat{x}}{4\pi\epsilon_0 x_0^3}$ (c) $\frac{Q\hat{x}}{4\pi\epsilon_0 x_0^2}$ (d) $\frac{Q\hat{x}}{16\pi\epsilon_0 x_0^2}$

In. We have one dimension charge density

$$\rho = Q\left\{\delta\left(x-x_0\right)-\delta\left(x+x_0\right)\right\}$$

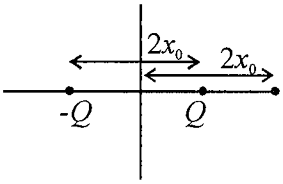
Therefore, the total charge,

$$Q = \int\rho dx = Q\int\delta\left(x-x_0\right)dx - Q\int\delta\left(x+x_0\right)dx$$

$$= Q\left(\text{at } x = x_0\right)+\left(-Q\right)\left(\text{at } x=-x_0\right)$$

Therefore, the field at $\left(2x_0,0,0\right)$ point is

$$E = \frac{1}{4\pi\varepsilon_0}\left\{\frac{Q}{x_0^2}-\frac{Q}{\left(3x_0\right)^2}\right\}$$



$$= \frac{Q}{4\pi\varepsilon_0}\left[\frac{9-1}{9x_0^2}\right] = \frac{2Q}{9\pi\varepsilon_0 x_0^2}\hat{x}$$

Correct option is (a)

14. Consider a particle of electric charge 'e' and mass 'm' moving under the influence of constant horizontal electric field E and constant vertical gravitational field described by acceleration due to gravity 'g'. If the particle starts from rest, what will be its trajectory ? **[JEST 2012]**

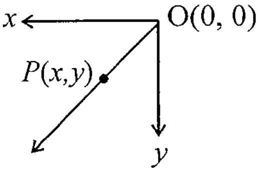
(a) Parabolic (b) elliptic (c) straight line (d) circular

Soln. Let at $t = 0$, electron at origine (0, 0) at $t = t$ the particle is at (x, y).

Acceleration along x-axis is $\frac{eE}{m}$ and acceleration along y-axis is g.

Therefore, the position of at a time t is $x = \frac{1}{2}\frac{eE}{m}t^2$ and $y = \frac{1}{2}gt^2$

$$\therefore \quad \frac{y}{x} = \left(\frac{gm}{eE}\right)$$



$$y = \left(\frac{gm}{eE}\right)x$$

It is the equation of straight line.

15. An ideal dipole is placed inside the sphere. Find the electric flux across the sphere.

Soln. The total charge inside sphere is

$$q_T = -q + q = 0$$

Therefore, the electric flux through the closed surface

$$\oint \vec{E}\cdot d\vec{s} = \frac{q_{enclosed}}{\varepsilon_0} = 0$$

16. The electric field in a region is given by $\vec{E} = \frac{3}{5}E_0\hat{i} + \frac{4}{5}E_0\hat{j}$ with $E_0 = 2.0\times10^3\ NC^{-1}$. Find the flux of this field through a rectangular surface of area 0.2 m^2 parallel to the y-z plane.

Soln. We know electric flux is given by

$$\phi = \int \vec{E}\cdot d\vec{s} = \vec{E}\cdot\vec{s}$$

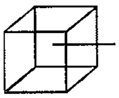
$$= \left(\frac{3}{5}E_0\hat{i} + \frac{4}{5}E_0\hat{j}\right)\cdot(0.2)\hat{i} \qquad \left(\vec{s} = 0.2\hat{i}\right)$$

$$= \frac{3}{5}E_0(0.2) = \frac{3}{25}E_0 = \frac{3}{25}\times2\times10^3 = \frac{6}{25}\times10^3 = 240\ Nm^2c^{-1}$$

17. A charge Q is uniformly distributed over a rod of length l. Consider a hypothetical cube of edge l with the centre of the cube at one end of the rod. Find the minimum possible flux of the electric field through the entire surface of the cube.

Soln. We know that electric flux

$$\phi = \vec{E}\cdot d\vec{s} = \frac{Q_{enclosed}}{\varepsilon_0}$$



$$\Rightarrow \quad \phi = \frac{Q/2}{\varepsilon_0}$$ [Since half of the rod inside and other half out side]

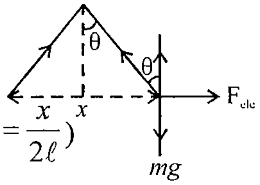
$$\Rightarrow \quad \phi = \frac{Q}{2\varepsilon_0}$$

Two small equally charged spheres, each of mass 'm' are suspended from the same point by silk threads of length l. The distance between the sphere $x << l$. Find the rate $\frac{dq}{dt}$ with which the charge leaks each sphere if their approach velocity varies as $v = \frac{a}{\sqrt{x}}$, where 'a' is a constant.

In. Consider equilibrium of sphere

$$T\cos\theta = mg$$

$$T\sin\theta = F_{coulomb}$$



$$\Rightarrow \quad \tan\theta = \frac{F_{coulomb}}{mg} \qquad \text{(From Geometry } \tan\theta = \frac{x}{2\ell}\text{)}$$

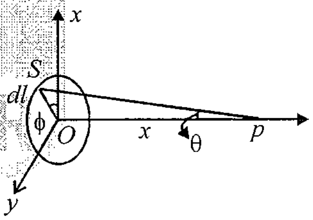
$$\therefore \quad \frac{x}{2\ell} = \frac{F_{coulomb}}{mg} \Rightarrow F_{coulomb} = \frac{mgx}{2\ell} \qquad \Rightarrow \frac{1}{4\pi\varepsilon_0}\frac{q^2}{x^2} = \frac{mgx}{2\ell}$$

$$\Rightarrow \quad q = \sqrt{\frac{2\pi\varepsilon_0 mg}{\ell}}\, x^{3/2}$$

$$\Rightarrow \quad \frac{dq}{dt} = \sqrt{\frac{2\pi\varepsilon_0 mg}{\ell}}\,\frac{3}{2}\sqrt{x}v = \frac{3}{2}\sqrt{\frac{2\pi\varepsilon_0 mgx}{\ell}}\,\frac{a}{\sqrt{x}} = \frac{3a}{2}\sqrt{\frac{2\pi\varepsilon_0 mg}{\ell}}$$

. A thin non-conducting ring of radius R has a linear charge density $\lambda = \lambda_0\cos\phi$, where λ_0 is a cosntant, ϕ is azimuthal angle. Find the magnitude of the electric field strength.
(a) at the centre of the ring.
(b) on the axis of the ring as a function of the distance x from its centre.

In. (a) Let consider an elementary length dl on the ring.
Therefore, the field at the centre due to elementary length dl is given by



$$E = \frac{1}{4\pi\varepsilon_0}\frac{\lambda d\ell}{R^2} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda R d\phi}{R^2} = \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi}\frac{\lambda_0\cos^2\phi}{R}d\phi$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{\lambda_0\pi}{R} = \frac{\lambda_0}{4\varepsilon_0 R}$$

Take an elementary length dl at an azimuthal angle ϕ from the x-axis. The field at P due to the elementary length dl is

$$dE = \frac{\lambda_0\cos\phi\, d\phi\, R}{4\pi\varepsilon_0\left(x^2+R^2\right)} \text{ along } \overrightarrow{SP}$$

It has two components

$dE\cos\theta$ along $\overrightarrow{OP}$ and $dE\sin\theta$ along OS.

The component along $\overrightarrow{OP}$ will give zero contribution.

Now the component along OS can be broken into the parts along $\overrightarrow{ox}$ and $\overrightarrow{oy}$ and given by

$dE\sin\theta\cos\phi$ along $\overrightarrow{ox}$ and

$dE\sin\theta\sin\phi$ along $\overrightarrow{oy}$

$$\therefore \quad E_x = \int dE\sin\theta\cos\phi = \frac{\lambda_0 R^2}{4\pi\varepsilon_0\left(x^2+R^2\right)^{3/2}}\int_0^{2\pi}\cos^2\phi\, d\phi$$

$$= \frac{\lambda_0 R^2}{4\pi\varepsilon_0\left(x^2+R^2\right)^{3/2}} = \frac{\lambda_0 R^2}{4\varepsilon_0\left(x^2+R^2\right)^{3/2}}$$

$$E_y = \int dE \sin\theta \sin\phi = \frac{\lambda_0 R^2}{4\pi\varepsilon_0\left(x^2+R^2\right)^{3/2}} \int_0^{2\pi} \cos\phi \sin\phi \, d\phi = 0$$

$$\therefore \quad \vec{E} = \frac{\lambda_0 R^2}{4\varepsilon_0\left(x^2+R^2\right)^{3/2}} \hat{x}$$

20. Find the electric field strength vector inside of a ball of radius R with volume charge density $\rho = ar$, where a is a constant vector.

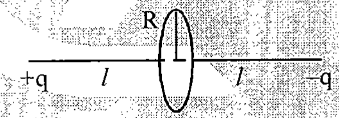
Soln. According to Gauss's law for spherically symmetric case

$$E = \frac{Q_{enclosed}}{4\pi\varepsilon_0 r^2}$$

$$\Rightarrow \quad Q_{enclosed} = \int \rho d\tau = \int 4\pi a r r^2 dr = 4ar^4$$

$$\Rightarrow \quad E = \frac{ar^4}{4\varepsilon_0 r^2} \Rightarrow E = \frac{ar^2}{4\varepsilon_0}$$

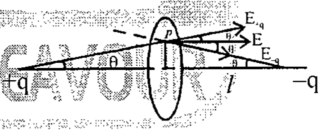
21. Two point charge q and $-q$ are separated by the distance $2l$. Find the flux of the electric field strength vector across a circle of radius R.



Soln. Let consider an elementary area on the circle at (r,ϕ) $rdrd\phi$

Therefore, the field at the point P due charge $+q$

$$\frac{1}{4\pi\varepsilon_0} \frac{q}{\left(r^2+\ell^2\right)}$$



Now, the field normal to the surface is

$$\frac{1}{4\pi\varepsilon_0} \frac{2q}{\left(r^2+\ell^2\right)} \cdot \cos\theta = \frac{1}{4\pi\varepsilon_0} \frac{2q\ell}{\left(r^2+\ell^2\right)^{3/2}}$$

Therefore, flux $\phi = \vec{E} \cdot d\vec{s} = \iint \frac{1}{4\pi\varepsilon_0} \frac{2q\ell}{\left(r^2+\ell^2\right)^{3/2}} r \, dr \, d\phi$

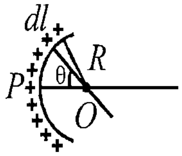
$$= \int_0^R \frac{q\ell r dr}{\varepsilon_0\left(r^2+\ell^2\right)^{3/2}} = \frac{q\ell}{\varepsilon_0} \int_0^R \frac{rdr}{\left(r^2+\ell^2\right)^{3/2}}$$

$$= \frac{q\ell}{\varepsilon_0} \int_{\ell}^{\sqrt{R^2+\ell^2}} \frac{zdz}{z^3} = -\frac{q\ell}{\varepsilon_0}\left[\frac{1}{z}\right]_{\ell}^{\sqrt{R^2+\ell^2}} = \frac{q}{\varepsilon_0}\left[1 - \frac{\ell}{\sqrt{R^2+\ell^2}}\right]$$

A thin halfring of radius R = 20 cm is uniformly charged with a total charge q = 0.70 nC. Find the field at the centre of curve.

ln. Line charge density

$$\lambda = \frac{q}{\pi R}$$



Therefore, the field at centre due to elementary length

$$dE = \frac{1}{4\pi\varepsilon_0}\frac{\lambda d\ell}{R^2}$$

It has two components, $dE\cos\theta$ and $dE\sin\theta$ but due to symmetry $dE\sin\theta$ will be cancelled out.

$$\therefore \quad E = \frac{1}{4\pi\varepsilon_0}\int_{-\pi/2}^{\pi/2}\frac{\lambda R d\theta}{R^2}\cos\theta = \frac{q}{4\pi^2\varepsilon_0 R^2}\left[\sin\theta\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{q}{4\pi^2\varepsilon_0 R^2}2 = \frac{q}{2\pi^2\varepsilon_0 R^2}$$

$$= \frac{0.70\times10^{-9}}{2\times(3.14)^2\times8.85\times(0.2)^2} = 0.10\,\frac{kV}{m}$$

Chapter 3

Electric Potential

Electric Potential:

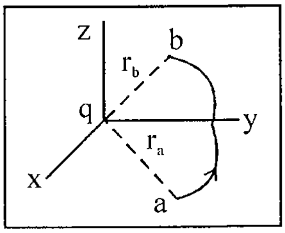
Curl of electric field is zero therefore, electric field is conservative field. Scalar potential is defined only for conservative field. In the following we first show that curl of electric field is zero.

We know field of a point charge at the origin

$$\bar{E}=\frac{1}{4\pi\in_0}\frac{q}{r^2}\hat{r}$$

$$\int_a^b \bar{E}.d\bar{l}=\frac{1}{4\pi\in_0}\int_a^b\frac{q}{r^2}\hat{r}\cdot\left(\hat{r}dr+\hat{\theta}rd\theta+\hat{\phi}r\sin\theta d\phi\right)$$

$$=\frac{1}{4\pi\in_0}\int_a^b\frac{q}{r^2}dr=\frac{1}{4\pi\in_0}\left(\frac{q}{r_a}-\frac{q}{r_b}\right)$$



If we take the integral along a closed path $(r_a=r_b)$ then

$$\oint \bar{E}.d\bar{l}=0$$

Applying Stokes theorem $\oint \bar{E}.d\bar{l}=\int_S(\bar{\nabla}\times\bar{E}).d\bar{S}=0$

Hence, $\bar{\nabla}\times\bar{E}=0$

This result can be generalised for any charge distribution using the principle of superposition. The general result is $\boxed{\bar{\nabla}\times\bar{E}=0}$

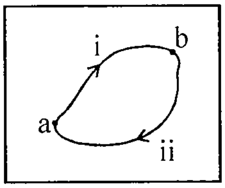
Path independence of line integral of electric field:

We have proved for any closed path the integral $\oint \bar{E}.d\bar{l}=0$. Take the closed path in figure above. Suppose

you go from a to b by path 1 and come back from b to a by path 2. The integral $\oint \bar{E}.d\bar{l}=0$

$$\int_{a\,(path\,i)}^{b}\bar{E}.d\bar{l}+\int_{b\,(path\,ii)}^{a}\bar{E}.d\bar{l}=0$$

Or, $$\int_{\substack{a\\path\,i}}^{b}\bar{E}.d\bar{l}+\int_{\substack{b\\path\,ii}}^{a}\bar{E}.d\bar{l}=0$$



Or, $\int\limits_{\substack{a \\ path\ i}}^{b} \bar{E}.d\bar{l} = \int\limits_{\substack{a \\ path\ ii}}^{b} \bar{E}.d\bar{l}$

Therefore, the line integral $\int_a^b \bar{E}.d\bar{l}$ is same for any path joining (a) and (b).

Definition of Electric Potential:

Since, Line integration of electric field between two points is indepedent of path followed

Therefore, we define a function

$$\boxed{V(\bar{r}) = -\int_O^r \bar{E}.d\bar{l}}$$

Here, O is a reference point which in generally taken to be at infinity.

The potential difference between 2 points

$$V(\bar{b}) - V(\bar{a}) = -\int_\infty^b \bar{E}.d\bar{l} + \int_\infty^a \bar{E}.d\bar{l}$$

$$= -\int_\infty^b \bar{E}.d\bar{l} - \int_a^\infty \bar{E}.d\bar{l} = -\int_a^b \bar{E}.d\bar{l}$$

From, fundamental theorem of gradient

$$V(\bar{b}) - V(\bar{a}) = \int_a^b (\bar{\nabla} V).d\bar{l} = -\int_a^b \bar{E}.d\bar{l}$$

Hence, $\boxed{\bar{E} = -\bar{\nabla} V}$

We could have deduced it in other ways also. {Directly from vector calculas→ As $\bar{\nabla} \times \bar{E} = 0$ and $\bar{\nabla} \times (\bar{\nabla}\phi) = 0$, ϕ be any scalar function- so $\bar{E} = \nabla\phi$; for same scalar function ϕ; easily understood].

Remember:

(i) Potential and potential energy are different

(ii) A surface over which the potential is constant is called an equipotential.

(iii) Potential is a scalar quantity

(iv) The reference point in general taken to be at infinity; but sometimes different reference points can also be chosen

(v) Potential due to charges obeys the superposition principle.

$$V = V_{q_1} + V_{q_2} + \ldots\ldots\ldots\ldots$$

(vi) Unit of electric field is newton/coulomb or volt/meter

Unit of potential is newton-meters/coulomb or jules /coulomb or volt

Electric Potential due to point charge:

Electric field of a point charge q at the origin is

$$\bar{E} = \frac{1}{4\pi \in_0} \frac{q}{r^2} \hat{r}$$

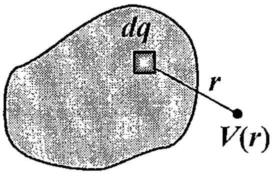
We know that, $dV = -\vec{E} \cdot d\vec{\ell} = -\frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r} \cdot \left(dr\, \hat{r} + r d\theta \hat{\theta} + r \sin\theta d\phi \hat{\phi} \right)$

$$\Rightarrow \qquad V = -\frac{q}{4\pi\varepsilon_0}\int_{\infty}^{r}\frac{1}{r^2}dr = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r}-\frac{1}{\infty}\right] = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

In general the potential of a point charge q at the distance r away from its position is

$$\boxed{V(r) = \frac{1}{4\pi\epsilon_0}\frac{q}{r}}$$

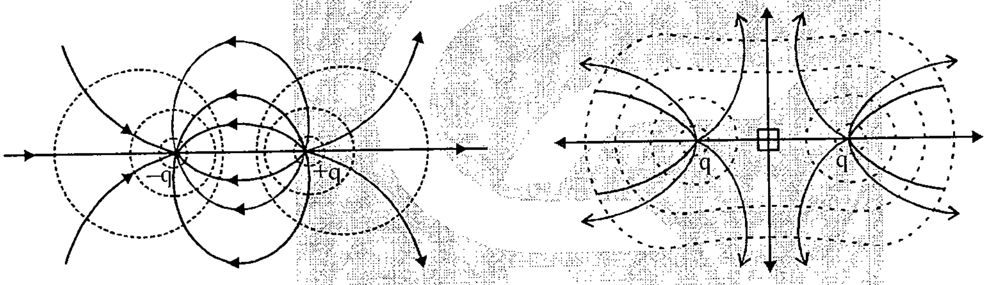
For a continuous charge distribution



$$\boxed{V(r) = \frac{1}{4\pi\epsilon_0}\int\frac{1}{r}dq}$$

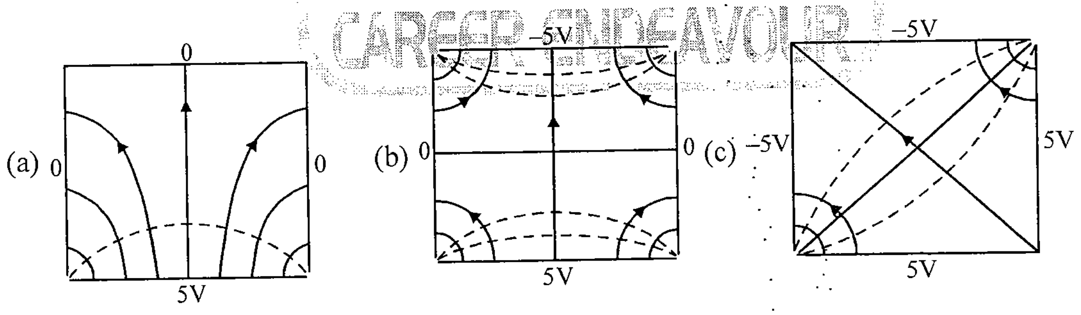
So what is the physical meaning of potential? It is the work done to move a unit positive charge from infinity to the point where the potential is to be measured.

Equipotential lines/surfaces : The lines or surfaces on which potential is constant everywhere are called equipotential. Electric field is perpendicular to equipotential line/surface. In the figure shown below equipotential lines (dotted) and field lines of two point charges are given.



(a) (b)

Electric field lines near some equipotential lines are shown below

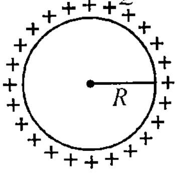


Electric potential due to uniformly charged thin shell :

We know that field due to shell is

$E_{\text{in}} = 0 \qquad r < R$

$\vec{E}_{\text{out}} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r} \qquad r > R$



If V be electric potential then $\vec{E} = -\vec{\nabla}V$ or $E = -\frac{\partial V}{\partial r}$

Therefore, for $r < R$ $\quad \dfrac{\partial V_{\text{in}}}{\partial r} = 0 \Rightarrow V_{\text{in}} = C_1$ $\quad$ ... (i)

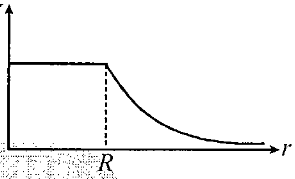
and $\quad$ for $r > R$ $\quad -\dfrac{\partial V_{\text{out}}}{\partial r} = \dfrac{Q}{4\pi\varepsilon_0 r^2} \Rightarrow V_{\text{out}} = \dfrac{Q}{4\pi\varepsilon_0 r} + C_2$ $\quad$ ... (ii)

If we take potential at infinity to be zero then from (ii), we get $C_2 = 0$.

Therefore, $\boxed{V_{\text{out}} = \dfrac{Q}{4\pi\varepsilon_0 r}}$ for $r \geq R$.

Since, potential is always continuous, therefore at $r = R$ potential inside and outside must be same.

i.e., $\quad V_{\text{in}} = V_{\text{out}}\big|_{r=R}$



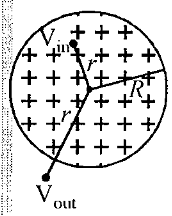
$\therefore \quad C_1 = \dfrac{Q}{4\pi\varepsilon_0 R}$

$\therefore \quad$ for $r \leq R$ $\quad \boxed{V_{\text{in}} = \dfrac{Q}{4\pi\varepsilon_0 R}}$

Electric potential due to sphere having uniform volume charge density:

Let Q be total charge and R be the radius of sphere.

Therefore, charge density, $\rho = \dfrac{Q}{\frac{4}{3}\pi R^3}$



We know that field in this case is given as : $\vec{E}_{\text{in}} = \dfrac{\rho r}{3\varepsilon_0}\hat{r}$; $\vec{E}_{\text{out}} = \dfrac{\rho R}{3\varepsilon_0 r^2}\hat{r}$

Therefore, using relation $\vec{E} = -\vec{\nabla}V$ or $E = -\dfrac{\partial V}{\partial r}$, we get

$$-\frac{\partial V_{\text{in}}}{\partial r} = \frac{\rho r}{3\varepsilon_0} \Rightarrow V_{\text{in}} = -\frac{\rho r^2}{6\varepsilon_0} + C_1 \quad \text{... (i)}$$

and $$-\frac{\partial V_{\text{out}}}{\partial r} = \frac{\rho R^3}{3\varepsilon_0 r^2} \Rightarrow V_{\text{out}} = \frac{\rho R^3}{3\varepsilon_0 r} + C_2 \quad \text{... (ii)}$$

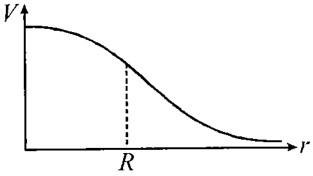
If we take potential at infinity to be zero then $C_2 = 0$.

Therefore, $\boxed{V_{\text{out}} = \dfrac{\rho R^3}{3\varepsilon_0 r} = \dfrac{Q}{4\pi\varepsilon_0 r}}$.

Since, potential is continuous function

Therefore, $V_{\text{in}}\big|_{r=R} = V_{\text{out}}\big|_{r=R}$

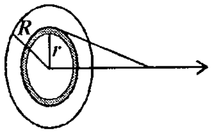
$$-\frac{\rho R^2}{6\varepsilon_0} + C_1 = \frac{\rho R^2}{3\varepsilon_0} \Rightarrow C_1 = \frac{\rho R^2}{2\varepsilon_0}$$



Therefore, from (i), $V_{\text{in}} = -\dfrac{\rho r^2}{6\varepsilon_0} + \dfrac{\rho R^2}{2\varepsilon_0} = \dfrac{\rho}{6\varepsilon_0}(3R^2 - r^2)$

Therefore, $\boxed{V_{\text{in}} = \dfrac{\rho}{6\varepsilon_0}(3R^2 - r^2) = \dfrac{Q}{8\pi\varepsilon_0 R^3}(3R^2 - r^2)}$.

The field and potential due to a circlar surface with uniform surface charge density σ.



Let us take an circular segment of radius r and $r + dr$.
The potential due to this element is

$$d\phi = \frac{1}{4\pi\varepsilon_0}\frac{2\pi r dr\sigma}{\sqrt{r^2+z^2}}$$

$$\therefore \quad \phi = \frac{1}{4\pi\varepsilon_0}2\pi\sigma\int_0^R \frac{2rdr}{\sqrt{z^2+r^2}} = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{z^2+r^2}\right]_0^R$$

$$= \frac{\sigma}{2\varepsilon_0}\left(\sqrt{z^2+R^2} - |z|\right)$$

$$\therefore \quad \vec{E} = -\frac{d\phi}{dz} = \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{z^2+R^2}}\right) \text{ for } z > 0$$

$$= -\frac{\sigma}{2\varepsilon_0}\left(1 + \frac{z}{\sqrt{z^2+R^2}}\right) \text{ for } z < 0$$

Example: Charges q and $-q$ are alternately placed at $x = x_0,\ x = 3x_0,\ x = 5x_0,\ x = 7x_0 \ldots\ldots\infty$. Find potential at $x = 0$.

Soln. Electric potential due to a point charge q at distance r from it is $q/4\pi\varepsilon_0 r$.

$$\underset{x=0}{} \quad \underset{x=x_0}{\overset{q}{\bullet}} \quad \underset{x=3x_0}{\overset{-q}{\bullet}} \quad \underset{x=5x_0}{\overset{q}{\bullet}} \quad \underset{x=7x_0\,\ldots\ldots\infty}{\overset{-q}{\bullet}}$$

Therefore, potential at $x = 0$ is given as

$$V = \frac{q}{4\pi\varepsilon_0 x_0} - \frac{q}{4\pi\varepsilon_0(3x_0)} + \frac{q}{4\pi\varepsilon_0(5x_0)} - \frac{q}{4\pi\varepsilon_0(7x_0)} \cdots\infty$$

$$= \frac{q}{4\pi\varepsilon_0 x_0}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}\cdots\infty\right]$$

$$= \frac{q}{4\pi\varepsilon_0 x_0} \times \frac{\pi}{4}\left\{\because \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}\cdots\infty \text{ this is known as Madhava's series}\right\}$$

$$\therefore V(x=0) = \frac{q}{16\varepsilon_0 x_0}.$$

Example: Two charges q and $-q$ are placed at $(a, 0, 0)$ and $(-a, 0, 0)$ respectively. Calculate electric field at $(0, a, 0)$ using electric potential.

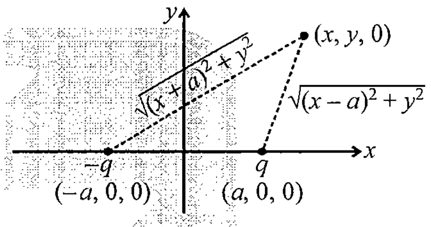
In. Electric potential at $(0, a, 0)$ is zero due to this charge distribution. Therefore, field cannot be calculated using potential of particular point.

Therefore, we first find potential at a general point $(x, y, 0)$.

$$V(x, y, 0) = \frac{q}{4\pi\varepsilon_0\sqrt{(x-a)^2+y^2}} - \frac{q}{4\pi\varepsilon_0\sqrt{(x+a)^2+y^2}}$$

$$V(x, y, 0) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{(x-a)^2+y^2}} - \frac{1}{\sqrt{(x+a)^2+y^2}}\right]$$

$$E_x = -\left.\frac{\partial V}{\partial x}\right|_{x=0,\, y=a} = -\frac{q}{4\pi\varepsilon_0}\left[-\frac{(x-a)}{((x-a)^2+y^2)^{3/2}} + \frac{(x+a)}{[(x+a)^2+y^2]^{3/2}}\right]_{x=0,\, y=a}$$



$$= -\frac{q}{4\pi\varepsilon_0}\left[\frac{a}{(2a^2)^{3/2}} + \frac{a}{(2a)^{3/2}}\right]$$

$$= -\frac{q}{4\pi\varepsilon_0 a^2}\cdot\frac{1}{\sqrt{2}} = -\frac{q}{4\sqrt{2}\pi\varepsilon_0 a^2}$$

$$E_y = -\frac{\partial V}{\partial y} = -\frac{q}{4\pi\varepsilon_0}\left[-\frac{y}{[(x-a)^2+y^2]^{3/2}} + \frac{y}{[(x+a)^2+y^2]^{3/2}}\right]_{x=0,\, y=a}$$

$$= -\frac{q}{4\pi\varepsilon_0}\left[-\frac{a}{(2a^2)^{3/2}} + \frac{a}{(2a^2)^{3/2}}\right]$$

$$E_y = 0$$

$$\therefore \quad \vec{E} = E_x\,\hat{i} + E_y\,\hat{j} \quad \Rightarrow \vec{E} = -\frac{q}{4\sqrt{2}\pi\varepsilon_0 a^2}\hat{i}.$$

Example: Electric field in a region is $\vec{E} = A\left[(x^2y+x^3)\,\hat{i} + \frac{1}{3}x^3\,\hat{j}\right]$. Calculate potential difference between points $(1, 1, 1)$ and $(-1, 1, -1)$.

In. Potential difference between two points is given as :

$$V_1 - V_2 = \int_{P_1}^{P_2} \vec{E}\cdot d\vec{l} = \int_{P_1}^{P_2} (E_x\,dx + E_y\,dy)$$

$$= A\int\left[(x^2y+x^3)\,dx + \frac{1}{3}x^3\,dy\right] = A\int_{1,1,1}^{-1,1,-1} d\left(\frac{1}{3}x^3y\right) + \int_{1,1,1}^{-1,1,-1} x^3\,dx$$

$$= A\left\{\left[\frac{1}{3}x^3y\right]_{1,1,1}^{-1,1,-1} + \left[\frac{x^4}{4}\right]_1^{-1}\right\} = A\left\{\frac{1}{3}(-1-1)+0\right\} = -\frac{2A}{3}.$$

Poisson's and Laplace's equation from Gauss' law:

We know that, $E = -\nabla\phi$, put this in differential form of Gauss' law $\left(\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}\right)$ we get

$$-\vec{\nabla}\cdot\vec{\nabla}\phi = \frac{\rho}{\varepsilon_0} \Rightarrow \boxed{\nabla^2\phi = -\frac{\rho}{\varepsilon_0}}$$

This equation is called Poisson's equation.

If we put $\rho = 0$ then we get

$$\boxed{\nabla^2\phi = 0}$$

It is known as Laplace equation.

Laplace equation in different dimension and coordinate systems

The vector equation $\nabla^2 V = 0$ radius to

1-D cartesian $\Rightarrow \frac{\partial^2 V}{\partial x^2} = 0$ ...(3)

2-D cartesian $\Rightarrow \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$ ...(4)

3-D cartesian $\Rightarrow \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$ ...(5)

Laplace equation in I-D cartesian:

$$\frac{\partial^2 V}{\partial x^2} = 0$$

General solution $V(x) = mx + b$ (m and b are constant)

The solution is a straight line. The boundary condition of the specific problem fixes the constants (m and b).
Suppose it is given that V=4 at x = 1 and V= 0 at x = 5

Using this two information you can find $m = -1, b = 5$.

Laplace equation in 2-D cartesian:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

We will solve the Laplace equation by separation of variables. You must have known the technique of separation of variable. Still we will revise briefly.

Laplace equation is, $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$ ...(8)

Let's take $V(x, y)$ as a product of two functions each of them depends on x and y only

$$V(x, y) = X(x)\,Y(y) \qquad ...(9)$$

Putting $V(x, y)$ on Eq. (8) we get

$$Y\frac{\partial^2 X}{\partial x^2}+X\frac{\partial^2 Y}{\partial y^2}=0$$

(as X & Y are function of x & y only we can replace partial derivative operator into normal derivative)

$$\Rightarrow \qquad Y\frac{d^2X}{dx^2}+X\frac{d^2Y}{dy^2}=0$$

$$\Rightarrow \qquad \frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}=0$$

Now since each side depends on single variable only and X and Y are independent, we can take each side as constant.

$$\frac{1}{X}\frac{d^2X}{dx^2}=-\frac{1}{Y}\frac{d^2Y}{dy^2}=-k^2 \qquad (k=\text{constant})$$

$$\Rightarrow \qquad \frac{d^2X}{dx^2}=k^2X \text{ and } \frac{d^2y}{dy^2}=-k^2Y \qquad ...(10)$$

We will obtain the solution

$$\left.\begin{array}{l} X=Ae^{kx}+Be^{-kx} \\ Y=C\sin ky+D\cos ky \end{array}\right\} \qquad ...(10)$$

Hence, $V(x,y)=\left(Ae^{kx}+Be^{-kx}\right)\left(C\sin ky+D\cos ky\right)$

A, B, C, D are constants and depends on specific problems. You can of course take

$X=A\cos kx+B\sin kx$ and $Y=Ce^{-ky}+De^{-ky}$

That also depends on the given problem. You need to carefully choose which part will be exponential and which part will be sinusoidal.

Example: Show that a unidirectional electrostatic field E cannot vary in a direction normal to the direction of the field.

In. Let us assume that the field is directed along the *x*-axis of the rectangular system of coordinates, so that

$$\vec{E}=E_x\hat{i}$$

Since, the curl of an electrostatic field is always zero, we have

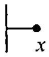
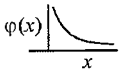
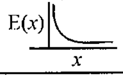
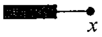
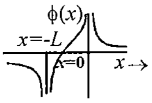
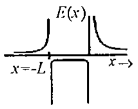
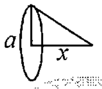
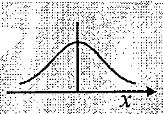
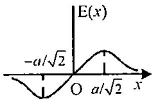
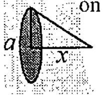
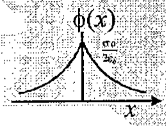
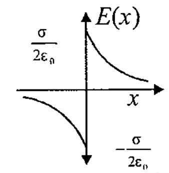
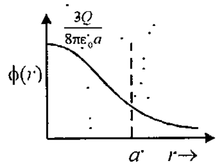
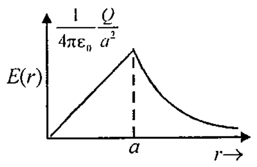
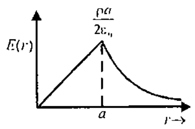
$$\vec{\nabla}\times\vec{E}=\vec{\nabla}\times\left(E_x\hat{i}\right)=\hat{j}\frac{\partial E_x}{\partial z}-\hat{k}\frac{\partial E_x}{\partial y}=0,$$

and since a vector may be equal to zero only if all its components are equal to zero, we obtain

$$\frac{\partial E_x}{\partial z}=0 \text{ and } \frac{\partial E_x}{\partial y}=0.$$

These two equations show that E_x and therefore E, which is equal to $E_x i$, cannot vary in the direction of either *z* or *y*, thus proving that a unidirectional field E cannot vary in a direction normal to the direction of the field.

STANDARD TABLE OF FIELD & POTENTIAL

S.No.	Shape	Position of point	Potential	Field
(1)	Uniformly line charge distribution	 at a distance x from midpoint of line charge x >> L	$\frac{1}{4\pi\varepsilon_0}\frac{Q}{x}$  	$\frac{1}{4\pi\varepsilon_0}\frac{Q}{x^2}$ 
	 	at a distance x from the one end of	$\frac{1}{4\pi\varepsilon_0}\frac{Q}{L}\ln\left(1+\frac{L}{x}\right)$  	$\frac{1}{4\pi\varepsilon_0}\frac{Q}{x(L+x)}$ 
(2)	Uniformly charge ring	 on the axis at distance x from centre	$\frac{1}{4\pi\varepsilon_0}\frac{Q}{\sqrt{a^2+x^2}}$  	$\frac{1}{4\pi\varepsilon_0}\frac{Qx}{(a^2+x^2)^{3/2}}x$ 
(3)	Uniformly charged disc	 on the axial at distance x from centre	$\frac{\sigma}{2\varepsilon_0}\left[\sqrt{a^2+x^2}-\lvert x\rvert\right]$  	$\frac{\sigma}{2\varepsilon_0}\left[1-\frac{x}{\sqrt{a^2+x^2}}\right]x$ for $x>0$ $-\frac{\sigma}{2\varepsilon_0}\left[1+\frac{x}{\sqrt{a^2+x^2}}\right]x$ for $x<0$ 
(4)	Uniformly charged sphere	Outside the sphere inside the sphere	$\frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$ $\frac{Q}{4\pi\varepsilon_0 a}\left[\frac{3}{2}-\frac{r^2}{2a^2}\right]$  	$\frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}x$ for $r>a$ $\frac{Q}{4\pi\varepsilon_0}\frac{Qr}{a^3}x$ for $r<a$ 
(5)	Uniformly charged long cylinder	Inside the cylinder Outside the cylinder	$\frac{a^2\rho}{2\varepsilon_0}\ln\left(\frac{r_0}{a}\right)+\frac{\rho}{4\varepsilon_0}(a^2-r^2)$ $\frac{a^2\rho}{2\varepsilon_0}\ln\left(\frac{r_0}{r}\right)$ r_0 is the reference point where potential is zero	$\frac{r\rho}{2\varepsilon_0}r$ for $r<a$ $\frac{a^2\rho}{2\varepsilon_0}\frac{r}{r}$ for $r>a$ 

Example: A certain charge distribution has a region within which the charge has everywhere the same density ρ. A spherical cavity of radius a is made in this region, the center of the cavity being at the point where the potential originally was φ_e. Find the potential at this point after the cavity is made.

In. The zero charge density in the cavity can be regarded as two equally large charge densities of opposite polarity, ρ and $-\rho$.
potential at the center of the cavity is then

$$\varphi = \frac{1}{4\pi\varepsilon_0}\int_{\substack{\text{External}\\ \text{region}}} \frac{\rho_e}{r}dv' + \frac{1}{4\pi\varepsilon_0}\int_{\text{Cavity}} \frac{\rho}{r}dv' - \frac{1}{4\pi\varepsilon_0}\int_{\text{Cavity}} \frac{\rho}{r}dv',$$

where the first integral is taken over the region of space external to the cavity (ρ_e is the density of charge in this region). The first two terms of this expression give just the potential φ_e associated with the initial charge distribution. We can therefore write for the potential at the center of the cavity

$$\varphi = \varphi_e - \frac{\rho}{4\pi\varepsilon_0}\int_0^a \frac{4\pi r^2}{r}dr,\quad \varphi = \varphi_e - \frac{\rho a^2}{2\varepsilon_0}.$$

It is interesting to note that although the cavity affects the potential at the center of the cavity, it does not affect the electric field there.

Example: Show that the electrostatic potential of a point charge (Coulomb's potential) satisfies Laplace's equtaion,

In. The point charge potential is, $\varphi = \dfrac{q}{4\pi\varepsilon_0 r} + \varphi_0$

Using the expression for the Laplacian in spherical coordinates and observing that the point charge potential is a function of r only, we have

$$\nabla^2\varphi = \nabla^2\frac{q}{4\pi\varepsilon_0 r} = \frac{q}{4\pi\varepsilon_0}\nabla^2\cdot\frac{1}{r} = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\right]$$

$$= -\frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{1}{r^2}\right) = 0$$

(observe that, by the definition of a point charge, r is always larger than zero, so that the effect of $r = 0$ on the above expression need not be investigated).

Example: Two conducting spheres of radius a and b, each carrying a charge q, are separated by a distance $R >> a, b$. What, approximately, will be the potential and the final charge on each sphere after they are connected by a fine, conducting wire?

In. We shall find an approximate solution assuming the following idealized conditions : (a) the charge distribution and the field of each sphere are radially symmetric; (b) each sphere can be regarded as a point charge from the location of the other sphere; (c) no charge is residing on the wire. With these assumptions an approximate solution of the problem may be obtained as follows.
In the final state the potentials on the surfaces of both spheres must be the same,

$$\varphi_a = \varphi_b,$$

because only then will there be no voltage along the wire and therefore no current in the wire (initially the potentials of the spheres are different; after the spheres are connected, a certain amount of charge will move from one sphere to the other until the whole system becomes an equipotential system). the potential at the surface of each sphere may be regarded as the sum of two partial potentials: the potential due to the charge residing on the same sphere and the potential due to the charge residing on the other sphere.

$$\varphi_a = \frac{q_a}{4\pi\varepsilon_0 a} + \frac{q_b}{4\pi\varepsilon_0 R}$$

$$\varphi_b = \frac{q_b}{4\pi\varepsilon_0 b} + \frac{q_a}{4\pi\varepsilon_0 R}.$$

Furthermore, since the total charge is conserved, we have according to the assumption (c),

$$q_a + q_b = 2q .$$

Combining these four equations and solving for q_a, q_b, φ_a, and φ_b, we obtain

$$q_a = \frac{2q(R-b)a}{(R-b)a+(R-a)b},$$

$$q_b = \frac{2q(R-a)b}{(R-b)a+(R-a)b},$$

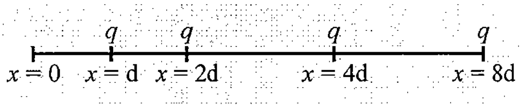
$$\varphi_a = \varphi_b = \frac{2q(R^2-ab)}{4\pi\varepsilon_0 R\left[(R-b)a+(R-a)b\right]}.$$

PROBLEMS ON ELECTRIC FIELD AND POTENTIAL

An infinite number of charges each equal to q are placed along the x-axis at $x = 1$, $x = 4$, $x = 8$,.... and so on. Find the potential at the point $x = 0$ due to this set of charges.

What will be potential if in the above set up the consecutive charge have opposite sign ?

In. The set of charges is shown in figure :



(i) The potential at $x = 0$ due to all charges is given by

$$V = \frac{1}{4\pi\varepsilon_0 d}\left(\frac{q}{1}+\frac{q}{2}+\frac{q}{4}+\frac{q}{8}+\cdots+\infty\right)$$

$$= \frac{q}{4\pi\varepsilon_0 d}\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\infty\right)$$

$$= \frac{q}{4\pi\varepsilon_0 d}\left[\frac{1-\left(\frac{1}{2}\right)^{\infty}}{1-\left(\frac{1}{2}\right)}\right] = \frac{q}{4\pi\varepsilon_0 d}\left[\frac{1-0}{\left(\frac{1}{2}\right)}\right] = \frac{q}{2\pi\varepsilon_0 d}$$

(ii) When the consecutive charges are negative, then

$$V = \frac{1}{4\pi\varepsilon_0 d}\left[\frac{q}{1}-\frac{q}{2}+\frac{q}{4}-\frac{q}{8}+\frac{q}{16}-\frac{q}{32}+\cdots\infty\right]$$

$$= \frac{1}{4\pi\varepsilon_0 d}\left[\left(\frac{q}{1}+\frac{q}{4}+\frac{q}{16}+\cdots\infty\right)-\left(\frac{q}{2}+\frac{q}{8}+\frac{q}{32}+\cdots\infty\right)\right]$$

$$= \frac{q}{4\pi\varepsilon_0 d}\left[\left(1+\frac{1}{4}+\frac{1}{16}+\cdots\infty\right)-\left(\frac{1}{2}+\frac{1}{8}+\frac{1}{32}+\cdots\infty\right)\right]$$

$$= \frac{q}{4\pi\varepsilon_0 d}\left[\left(\frac{1}{1-\left(\frac{1}{4}\right)}\right)-\frac{1}{2}\left(\frac{1}{1-\left(\frac{1}{4}\right)}\right)\right]$$

$$= \frac{q}{4\pi\varepsilon_0 d}\left[\frac{4}{3}-\frac{2}{3}\right] = \frac{2q}{4\pi\varepsilon_0 d\times 3} = \frac{q}{6\pi\varepsilon_0 d}$$

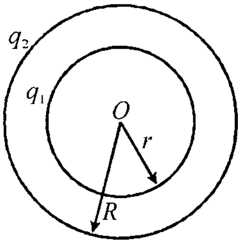
A charge Q is distributed over two concentric hollow spheres of radii r and R ($> r$) such the surface densities are equal. Find the potential at the common centre.

In. Let q_1 and q_2 be the charges on inner and outer sphere. Then.

$$q_1 + q_2 = Q$$

As the surface densities are equal, hence $\frac{q_1}{4\pi r^2} = \frac{q_2}{4\pi R^2}$ ($\because$ Surface density = charge/area)

Therefore, $q_1 R^2 = q_2 r^2$



Since, $q_2 = (Q - q_1)$, hence $q_1 R^2 = (Q - q_1) r^2$; $q_1 (R^2 + r^2) = Q r^2$

Therefore, $q_1 = \dfrac{Q r^2}{R^2 + r^2}$ and $q_2 = Q - q_1 = \dfrac{Q R^2}{R^2 + r^2}$

Now potential at O is given by $V = \dfrac{1}{4\pi\varepsilon_0} \dfrac{q_1}{r} + \dfrac{1}{4\pi\varepsilon_0} \dfrac{q_2}{R}$

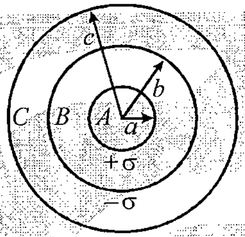
$$= \frac{1}{4\pi\varepsilon_0} \frac{Q r^2}{(R^2 + r^2) r} + \frac{1}{4\pi\varepsilon_0} \frac{Q R^2}{(R^2 + r^2) R} = \frac{Q}{4\pi\varepsilon_0} \frac{(r + R)}{(R^2 + r^2)}.$$

3. Three concentric metallic shells A, B and C of radii a, b, and c $(a < b < c)$ have surface charge densities $\sigma, -\sigma$ and σ respectively.

(i) Find the potentials of three shells A, B and C

(ii) If the shells A and C are at the same potential, obtain the relation between the radii a, b and c.

Soln. The three shells are shown in figure :



(1) Potential of A

= (Potential of A due to $+\sigma$ on A) + (Potential of A due to $-\sigma$ on B) + (Potential of A due to $+\sigma$ on C)

$$= \frac{1}{4\pi\varepsilon_0} \left[\frac{4\pi a^2 \sigma}{a} - \frac{4\pi b^2 \sigma}{b} + \frac{4\pi c^2 \sigma}{c} \right]$$

$$= \frac{\sigma}{\varepsilon_0} [a - b + c]$$

(2) Potential of B

= (Potential due to $+\sigma$ on A) + (Potential due to $-\sigma$ on B) + (Potential due to $+\sigma$ on C)

$$= \frac{1}{4\pi\varepsilon_0} \left[\frac{4\pi a^2 \sigma}{b} - \frac{4\pi b^2 \sigma}{b} + \frac{4\pi c^2 \sigma}{c} \right]$$

$$= \frac{\sigma}{\varepsilon_0} \left[\frac{a^2}{b} - b + c \right]$$

(3) Potential of C

= (Potential due to $+\sigma$ on A) + (Potential due to $-\sigma$ on B) + (Potential due to $+\sigma$ on C)

$$= \frac{1}{4\pi\varepsilon_0} \left[\frac{4\pi a^2 \sigma}{c} - \frac{4\pi b^2 \sigma}{c} + \frac{4\pi c^2 \sigma}{c} \right]$$

$$= \frac{\sigma}{\varepsilon_0} \left[\frac{a^2}{c} - \frac{b^2}{c} + c \right]$$

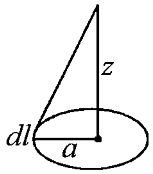
(ii) Given that $V_A = V_C$

$$\therefore \quad \frac{\sigma}{\varepsilon_0}[a-b+c]=\frac{\sigma}{\varepsilon_0}\left[\frac{a^2}{c}-\frac{b^2}{c}+c\right] \text{ or } a-b-c=\frac{a^2}{c}-\frac{b^2}{c}+c$$

Solving we get $c=(a+b)$.

Find the potential and field along the axis of circular ring with uniform charge density λ.

ln. Let consider as elementary length on the surface.
Therefore, the potential at the point P due to elementary length dl is



$$d\phi=\frac{1}{4\pi\varepsilon_0}\frac{\lambda d\ell}{\sqrt{z^2+a^2}}$$

$$\therefore \quad \phi=\frac{\lambda}{4\pi\varepsilon_0}\frac{1}{\sqrt{z^2+a^2}}\int d\ell$$

$$=\frac{\lambda}{4\pi\varepsilon_0}\frac{2\pi a}{\sqrt{z^2+a^2}}=\frac{1}{4\pi\varepsilon_0}\frac{2\pi\lambda a}{\sqrt{z^2+a^2}}$$

$$\therefore \quad \vec{E}=-\frac{d\phi}{dz}=\frac{1}{4\pi\varepsilon_0}\frac{2\pi a\lambda z}{\left(z^2+a^2\right)^{3/2}}\hat{z}$$

If the electrostatic potential were given by $\phi=\phi_0\left(x^2+y^2+z^2\right)$, where ϕ_0 is constant, then the charge density giving rise to the above potential would be :

(a) 0 (b) $-6\phi_0\varepsilon_0$ (c) $-2\phi_0\varepsilon_0$ (d) $-\frac{6\phi_0}{\varepsilon_0}$

ln. We have, $\phi=\phi_0\left(x^2+y^2+z^2\right)$
We know that Poisson's equation is

$$\nabla^2\phi=-\frac{\rho}{\varepsilon_0}$$

$$\Rightarrow \quad \rho=-\varepsilon_0\nabla^2\phi=-\varepsilon_0\phi_0\left(2+2+2\right)=-6\phi_0\varepsilon_0$$

Correct option is (b)

If $\vec{E}_1=xy\hat{i}+2yz\hat{j}+3xz\hat{k}$ and $\vec{E}_2=y^2\hat{i}+(2xy+z^2)\hat{j}+2yz\hat{k}$ then **[JEST 2013]**

(a) Both are impossible electrostatic fields.
(b) Both are possible electrostatic fields.
(c) Only $\vec{E}_1$ is a possible electrostatic field.
(d) Only $\vec{E}_2$ is a possible electrostatic field.

ln. We have, $\vec{E}_1=xy\hat{i}+2yz\hat{j}+3xz\hat{k}$

$$\vec{E}_2=y^2\hat{i}+\left(2xy+z^2\right)\hat{j}+2yz\hat{k}$$

$$\vec{\nabla}\times\vec{E}_1=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ xy & 2yz & 3xz\end{vmatrix}$$

$$=\hat{i}(0-2y)+\hat{j}(0-3z)+\hat{k}(0-x)$$

$$=-2y\hat{i}-3z\hat{j}-x\hat{k}\neq 0$$

Therefore, $\vec{E}_1$ is not electrostatic field.

$$\vec{\nabla}\times\vec{E}_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ y^2 & (2xy+z^2) & 2yz\end{vmatrix}$$

$$=\hat{i}(2z-2z)+\hat{j}(0-0)+\hat{k}(2y-2y)=0$$

Therefore, $\vec{E}_2$ is electrostatic field.

Correct optin is (d)

7. For a scalar function φ satisfying the Laplace equation, $\nabla\varphi$ has

(a) zero curl and non-zero divergence (b) non-zero curl and zero divergence

(c) zero curl and zero divergence (d) non-zero curl and non-zero divergence

Soln. If a scalar function ϕ satisfy the laplace equation then it obey the conditions

(i) $\vec{\nabla}\times\vec{\nabla}\phi=0$ (ii) $\vec{\nabla}\cdot\vec{\nabla}\phi=0$

Correct option is (c)

8. An electric field in a region is given by $\vec{E}(x,y,z)=ax\hat{i}+cz\hat{j}+6by\hat{k}$. For which of a, b, c does this represent an electrostatic field ? **[JEST 2012]**

(a) 13, 1, 12 (b) 17, 6, 1 (c) 13, 1, 6 (d) 45, 6, 1

Soln. We have $\vec{E}=ax\hat{i}+cz\hat{j}+6by\hat{k}$

For electrostatic field

$\vec{\nabla}\times\vec{E}$ should be zero.

$\therefore \quad \vec{\nabla}\times\vec{E}=0$

$$\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ ax & cz & 6by\end{vmatrix}=0 \Rightarrow 6b=c$$

Only option (c) satisfy the condition.

Correct option is (c)

9. The electrostatic potential due to a charge distribution is given by $V(r)=A\frac{e^{-\lambda r}}{r}$, where A and λ are constants. The total charge enclosed within a sphere of radius $1/\lambda$, with its origin at $r=0$ is given by,

(a) $\frac{8\pi\varepsilon_0 A}{e}$ (b) $\frac{4\pi\varepsilon_0 A}{e}$ (c) $\frac{\pi\varepsilon_0 A}{e}$ (d) 0 **[JEST 2015]**

Soln. We have, $V(r)=\frac{Ae^{-\lambda r}}{r}$

$$\therefore \quad E=-\frac{dV}{dr}=-\hat{r}\frac{d}{dr}\left[\frac{e^{-\lambda r}}{r}\right]A=\left[\frac{\lambda e^{-\lambda r}}{r}+\frac{e^{-\lambda r}}{r^2}\right]A\hat{r}$$

According to Poisson's law,

$$\vec{\nabla}\cdot\vec{E}=\frac{\rho}{\varepsilon_0}$$

$$\Rightarrow \quad \rho=\varepsilon_0\,\vec{\nabla}\cdot\vec{E}=\varepsilon_0\nabla\cdot\left[\frac{\lambda e^{-\lambda r}}{r^2}\vec{r}+\frac{e^{-\lambda r}}{r^3}\vec{r}\right]A=\varepsilon_0\lambda A\,\nabla\cdot\left(\frac{e^{-\lambda r}}{r^2}\vec{r}\right)+\varepsilon_0\,A\nabla\cdot\left(\frac{e^{-\lambda r}}{r^3}\vec{r}\right)$$

$$=\varepsilon_0\lambda A\;e^{-\lambda r}\left(\vec{\nabla}\cdot\frac{\vec{r}}{r^2}\right)+\frac{\vec{r}}{r^2}\varepsilon_0\lambda A\cdot\left(\nabla\,e^{-\lambda r}\right)+\varepsilon_0 Ae^{-\lambda r}\left(\vec{\nabla}\cdot\frac{\vec{r}}{r^3}\right)+\left(A\varepsilon_0\frac{\vec{r}}{r^3}\cdot\vec{\nabla}\,e^{-\lambda r}\right)$$

$$=\varepsilon_0\lambda A\,e^{-\lambda r}\frac{1}{r^2}-A\varepsilon_0\lambda^2\frac{1}{r}e^{-\lambda r}+A\varepsilon_0\;e^{-\lambda r}4\pi\delta(r)-A\varepsilon_0\lambda\frac{1}{r^2}e^{-\lambda r}$$

$$=4\pi A\varepsilon_0\left[\delta(r)-\frac{\lambda^2}{4\pi}\frac{1}{r}\right]e^{-\lambda r}$$

$$\therefore \quad q=4\pi A\varepsilon_0\int\delta(r)\,e^{-\lambda r}dV-\frac{4\pi A\varepsilon_0\lambda^2}{4\pi}\int\frac{1}{r}e^{-\lambda r}4\pi r^2dr$$

$$=4\pi\varepsilon_0 A-4\pi\varepsilon_0 A\int_0^1 e^{-z}zdz$$

$$=4\pi\varepsilon_0 A-4\pi\varepsilon_0 A\left[-e^{-z}z-e^{-z}\right]_0^1$$

$$=4\pi\varepsilon_0 A-4\pi\varepsilon_0 A\left[-e^{-1}-e^{-1}+1\right]=\frac{8\pi\varepsilon_0 A}{e}$$

Correct option is (a)

. The potential of a certain electrostatic field has the form $\phi=a\left(x^2+y^2\right)+bz^2$ where a and b are constants. Find the magnitude and direction of field strength vector, what shape have the equipotential surfaces in the following cases:

(a) $a>0, b>0$ (b) $a>0, b<0$

In. $\vec{E}=-\vec{\nabla}\phi$

$$=-2a\left(x\hat{i}+y\hat{j}\right)-2bz\hat{k}=-2ax\hat{i}-2ay\hat{j}-2bz\hat{k}$$

$$|E|=2\sqrt{a^2\left(x^2+y^2\right)+b^2z^2}$$

Therefore, the direction,

$$\hat{n}=-\frac{ax\hat{i}+y\hat{j}+bz\hat{k}}{\sqrt{a^2\left(x^2+y^2\right)+b^2z^2}}$$

(a) $\phi=a\left(x^2+y^2\right)+bz^2$

$$\Rightarrow \quad \frac{x^2+y^2}{(\phi/a)}+\frac{z^2}{(\phi/b)}=1$$

If $a > \phi$ and $b > 0$ then the equipotential surface will be ellipsoid.

(b) if $a > 0, b < 0$ then

$$\frac{x^2+y^2}{(\phi/a)} - \frac{z^2}{(\phi/b)} = 1$$

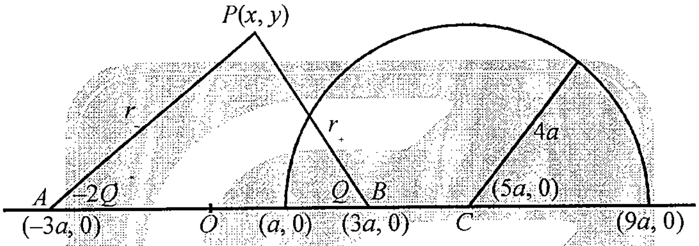
The equipotential surface will be hyperboloid.

11. Two fixed charges $-2Q$ and Q are located at the points with coordinates $(-3a, 0)$ and $(+3a, 0)$ respectively in the xy-plane.

(a) Show that all the points in the xy-plane where the electric potential due to the two charges is zero, lie on a circle. Find its radius and the location of its centre.

(b) Give the expression for the potential $V(x)$ at a general point on the x-axis and sketch the function $V(x)$ on the whole x-axis.

Soln. (a) The situation is shown in figure :



Let electric potential zero at point P with coordinates (x, y). The electric potential at P is given by

$$V = \frac{1}{4\pi\varepsilon_0}\left[-\frac{2Q}{r_-} + \frac{Q}{r_+}\right] = 0$$

where r_1 and r_2 are the distances of P from $-2Q$ and Q respectively. Then,

$$r_- = \sqrt{\left[(3a+x)^2 + y^2\right]} \text{ and } r_+ = \sqrt{\left[(3a-x)^2 + y^2\right]}$$

Therefore, $V = \dfrac{1}{4\pi\varepsilon_0}\left[-\dfrac{2Q}{\sqrt{\left[(3a+x)^2+y^2\right]}} + \dfrac{Q}{\sqrt{\left[(3a-x)^2+y^2\right]}}\right] = 0$

OR $$\frac{2}{\sqrt{\left[(3a+x)^2+y^2\right]}} = \frac{1}{\sqrt{\left[(3a-x)^2+y^2\right]}}$$

Solving this equation, we get $x^2 + y^2 - 10ax = 9a^2 = 0$

OR $(x-5a)^2 + y^2 = (4a)^2$

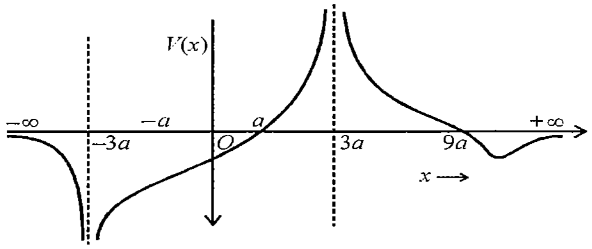
This is an equation of a circle in XY plane having a radius $4a$ and centred at $(5a, 0)$.

(b) The potential at any general point on X-axis is given by,

$$V(x) = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{(3a-x)} - \frac{2Q}{(3a+x)}\right] \text{ for } 0 < x < 3a$$

$$V(x) = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{(x-3a)} - \frac{2Q}{(3a+x)}\right] \text{ for } x > 3a$$

Sketch of potential $V(x)$ versus distance x is shown in figure :



We now sketch the potential function $V(x)$ on the whole X-axis. It is obvious from part first that the circle of zero potential cuts the x-axis at $(a, 0)$ and $(9a, 0)$ respectively. Hence,

(i) $V(x) = 0$ for $x = a$ and $x = 9a$

(ii) From the expression of $V(x)$, it is clear that $V(x) \to \infty$ as $x \to 3a$ and $V(x) \to -\infty$ when $x \to -3a$.

(iii) $V(x) \to 0$ as $x \to \pm\infty$

(iv) In general $V(x)$ varies as $\frac{1}{x}$

The sketch is shown in figure :

PRACTICE SET

1. Electrostatic field is always

(a) solenoidal (b) harmonic in character
(c) irrotational (d) none of the above

2. The unit of electric flux is

(a) gauss (b) $N.m^2.C^{-1}$ (c) weber (d) $N.C^{-1}$

3. Work done in displacing a charge 2e through a distance of 0.1m on an equipotential surface is

(a) 0.2J (b) zero (c) 20J (d) none of these

4. If a proton, a deutron and an α -particle are all accelerated through the smae potential difference, their kinetic energies would be in the ratio of

(a) 1 : 2 : 4 (b) 2 : 2 : 1 (c) 1 : 1 : 1 (d) 1 : 1 : 2

5. If the electric field and potential at a point are $\vec{E}$ and ϕ, respectively then

(a) for $\phi = 0, \vec{E}$ must be zero (b) for $\vec{E} \neq 0, \phi \neq 0$
(c) for $\vec{E} = 0$, ϕ must be zero (d) none of the above

6. An electric fipole is placed at the centre of a sphere. Then

(a) electric field is zero at every point on the sphere
(b) electric field is zero everywhere within the sphere
(c) electric field is zero outside the sphere
(d) total electric flux through the surface is zero

7. An electron travelling in a region of elctrostatic potential V_1 passes into a region of higher potential V_2. The change in kinetic energy of the electron is proportional to

(a) $\sqrt{V_2 - V_1}$ (b) $(V_2 - V_1)^2$ (c) $V_2 - V_1$ (d) none of these

8. The integral $\oint \vec{E} \cdot d\vec{\ell} = 0$ if the electric field $\vec{E}$ is caused by

(a) a static charge (b) both (a) and (b)
(c) a time varying magnetic field (d) none of these

9. A shpere of radius R carrries a volume charge density $\rho = \rho_0 r$, where ρ_0 is a constant and r is the radial distance from the centre of the charged sphere. The intensity of the electric field is proportional to:

(a) r^2 for $r \leq R$ and $1/r^2$ for $r \geq R$ (b) r for $r \leq R$ and $1/r^2$ for $r \geq R$
(c) $1/r^2$ for both $r \leq R$ and $r \geq R$ (d) $1/r$ for $r \leq R$ and $1/r^2$ for $r \geq R$

10. A uniformly charged thin spherical shell of radius R carries a total charge q. At a point inside (r < R) the spherical shell, if E and V are intensity of the electric field and electric potential respectively then :

(a) Both E and V are zero.
(b) E is zero but V is a nonzero constant.
(c) E is proportional to $1/r^2$ and V is proportional to $1/r$.
(d) Both E and V are nonzero constants.

ANSWER KEY					
Questions	1	2	3	4	5
Option	(c)	(b)	(b)	(d)	(d)
Questions	6	7	8	9	10
Option	(d)	(c)	(a)	(a)	(b)

Chapter 4

lectrostatic Energy

Electrostatic Energy:

The work necessary to assemble a system of charges against coulomb forces is stored in the field as potential energy. This is known as electrostatic energy.

Electrostatic energy system of point charges:

Let at any time are place a point charge q_1 at $\vec{r}_1$. To place the charge q_1 at the position $\vec{r}_1$, we require no work because there is no interacting coulomb field.

$$\therefore \qquad U_1 = 0$$

But to bring the charge q_2 to the position $\vec{r}_2$ we require work done against coulomb repulsion due to q_1.

This work $$U_2 = -\frac{1}{4\pi\varepsilon_0}\int_{\infty}^{r_{12}} \frac{q_1 q_2}{r^2}\,dr = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{|r_1 - r_2|} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}}$$

Therefore for two point charges,

$$\boxed{u = \frac{q_1 q_2}{4\pi\varepsilon_0 r_{12}}}$$

To bring another charge, we need work done.

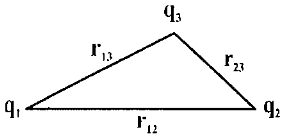
$$U_3 = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1\mu_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right]$$

Therefore, net work done

$$U = U_1 + U_2 + U_3 = 0 + \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right]$$

For N point charged,

$$\boxed{U = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{1}{4\pi\varepsilon_0}\frac{q_i q_j}{r_{ij}}}$$



We know electrostatic potential

$$\phi_i = \sum_{\substack{j=1 \\ i \neq j}}^{N}\frac{1}{4\pi\varepsilon_0}\frac{q_j}{r_{ij}}$$

$$\therefore \quad \boxed{U = \frac{1}{2}\sum_{i=1}^{N} q_i \phi_i}$$

Potential energy of interaction of a charge 'q' with other charges

$u = qV$, where V is potential at location of q due to other charges.

For continuous charge distribution :

For continous charge distribution u is given by

$$\boxed{U = \frac{1}{2}\int \rho(\vec{r})\phi(\vec{r})\,dr} \qquad [\rho(\vec{r}) \text{ is volume charge distribution}]$$

Electrostatic Energy in terms of field :

Suppose we have a finite region of space V in a dielectric medium of permitivity ε and the volume charge density ρ.

Therefore, the electrostatic energy of the system is given by

$$U = \frac{1}{2}\varepsilon_0 \int \rho(r)\phi(r)\,dV = \frac{1}{2}\int \left(\vec{\nabla}\cdot\vec{E}\right)\phi\,dV \qquad [\text{Gauss's law } \varepsilon_0 \vec{\nabla}\cdot\vec{E} = \rho]$$

$$= \frac{1}{2}\varepsilon_0 \int \left[\left(\vec{\nabla}\cdot\phi\vec{E}\right) - \vec{\nabla}\phi\cdot\vec{E}\right] dV$$

$$= \frac{1}{2}\varepsilon_0 \int \left(\vec{\nabla}\cdot\phi\vec{E}\right) dV - \frac{1}{2}\int \left(\vec{\nabla}\phi\cdot\vec{E}\right) dV$$

$$= \frac{1}{2}\varepsilon_0 \int \left(\phi\vec{E}\right) dS + \frac{1}{2}\left(\vec{E}\cdot\vec{E}\right) dV$$

At large distance from the charge distribution the first term will be vanish.

$$\therefore \quad \boxed{U = \frac{1}{2}\varepsilon_0 \int E^2 dV}$$

Therefore, the energy density, $\boxed{u = \frac{1}{2}\varepsilon_0 E^2}$

Forces and Torques from electrostatic energy:

We know work done = force . displacement

$$\Rightarrow \qquad dW = \vec{F}\cdot d\vec{r}$$

Also for isolated system, the work done

$$dW = -dU = -\vec{\nabla}U\cdot d\vec{r}$$

$$\therefore \quad \boxed{\vec{F} = -\vec{\nabla}U}, \; u = -\int \vec{F}\cdot d\vec{r}$$

If the part of the system under consideration is constrained to rotate about an axis and it undergoes a small rotation do under the action of a torque $\vec{\tau}$ then,

$$dW = \vec{\tau}\cdot d\vec{\theta} = -dU = -\nabla_\theta U\cdot d\theta$$

$$\therefore \quad \boxed{\vec{\tau} = -\nabla_\theta U} \quad \text{and } u = \int \vec{F}\cdot d\vec{r}$$

The electrostatic energy associated with a uniform spherical charge distribution of total charge q and radius R.

The electric field of this distribution is

$$E = E_1 = \frac{qr}{4\pi\varepsilon_0 R^3} \text{ for } r \le R$$

$$E = E_2 = \frac{q}{4\pi\varepsilon_0 R^2} \text{ for } r \ge R$$

Since, the field is radially symmetric, the volume element may be expressed as $dv = 4\pi r^2 dr$, so that the energy is

$$U = \frac{\varepsilon_0}{2}\int E^2 dv = \frac{\varepsilon_0}{2}\int_0^R E_1^2 4\pi r^2 dr + \frac{\varepsilon_0}{2}\int_R^\infty E_2^2 4\pi r^2 dr$$

$$= \frac{1}{2}\int_0^R \frac{q^2 r^2}{16\pi^2\varepsilon_0 R^6} 4\pi r^2 dr + \frac{1}{2}\int_R^\infty \frac{q^2}{16\pi^2\varepsilon_0 r^4} 4\pi r^2 dr$$

$$= \frac{q^2}{40\pi\varepsilon_0 R} + \frac{q^2}{8\pi\varepsilon_0 R}, \quad \boxed{U = \frac{3q^2}{20\pi\varepsilon_0 R}}$$

The electrostatic energy of a uniformly charged spherical shell of total charge q and radius a.

The electric field of the shell is

$$E = 0, \qquad \text{for } r \le R$$

$$E = \frac{q}{4\pi e_0 r^2} \qquad \text{for } r \ge R$$

The energy is therefore,

$$U = \frac{\varepsilon_0}{2}\int E^2 dv = \frac{1}{2}\int_R^\infty \frac{q^2}{16\pi^2\varepsilon_0 r^4} 4\pi r^2 dr$$

$$U = \frac{q^2}{8\pi\varepsilon_0 R}$$

Example: Consider a spherical charge distribution with a volume density

$$\rho(r) = \begin{cases} \rho_0 \cdot \dfrac{a}{r} & \text{for } 0 < r < a \\ 0 & \text{for } r > a \end{cases}$$

Show that the total electrostatic energy of the system is $\dfrac{Q^2}{6\pi \in_0 a}$, where Q is the total charge in the system.

ln. Total charge in the system is

$$Q = \int_0^a \rho(r)\cdot 4\pi r^2 dr = 4\pi\rho_0 a\int_0^a r\,dr = 2\pi\rho_0 a^3$$

The electric field at any outside point can be obtained from Gauss's law as

$$E=\frac{Q_{enclosed}}{4\pi\varepsilon_0 r^2}\ \text{or}\ Q_{enclosed}=\int\rho d\tau=4\pi\rho_0 a\int_0^a r\,dr=2\pi\rho_0 a^3$$

For an internal point,

$$\int\vec{E}_i\cdot d\vec{S}=\frac{Q_r}{\epsilon_0}=\frac{1}{\epsilon_0}\int_0^r\rho(r)\cdot 4\pi r^2dr=\frac{2\pi\rho_0 ar^2}{\epsilon_0}$$

$$\therefore\quad E_i=\frac{Q_{enclosed}}{4\pi\varepsilon_0 r^2}\ \text{or}\ E_i=\frac{Q}{4\pi\epsilon_0 a^2}$$

For external point, $E_{out}=\dfrac{Q_{enclosed}}{4\pi\varepsilon_0 r^2}=\dfrac{Q}{4\pi\varepsilon_0 r^2}$

Now, the total energy in the system is

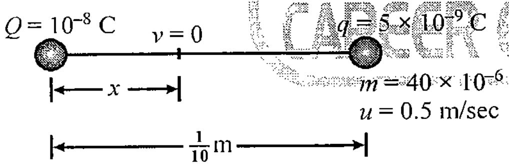
$$U=\int_0^a\frac{1}{2}\epsilon_0 E_i^2\cdot 4\pi r^2dr+\int_a^\infty\frac{1}{2}\epsilon_0 E_{out}^2\cdot 4\pi r^2 dr$$

$$\therefore\quad =2\pi\epsilon_0\int_0^a\left(\frac{Q}{4\pi\epsilon_0 a^2}\right)^2r^2dr+2\pi\epsilon_0\int_a^\infty\left(\frac{Q}{4\pi\epsilon_0}\right)^2\frac{dr}{r^2}=\frac{Q^2}{6\pi\epsilon_0 a}$$

ADDITIONAL SOLVED PROBLEMS

1. A particle of mass 40 *mg* and carrying a charge 5×10^{-9} C is moving directly towards a fixed positive point charge of magnitude 10^{-8} C. When it is at a distance of 10 cm from the fixed positive point charge it has a velocity of 50 cm sec^{-1}. At what distance from the fixed point charge will the particle come momentarily to rest? Is the acceleration constant during motion ?

Soln. Let x be the distance of q from Q when it comes to rest as shown in figure:



Use conservation of energy (K.E. + P.E.)$_{\text{initial}}$ = (K.E. + P.E.)$_{\text{final}}$

$$\frac{1}{2}\times40\times10^{-6}\times(0.5)^2+\frac{10^{-8}\times5\times9\times10^9}{\left(\frac{1}{10}\right)}=0+\frac{10^{-8}\times5\times10^{-9}\times9\times10^9}{x}$$

Solving we get, $r=4.737\times10^{-2}$ m.

The force obeys inverse square law i.e., it increases as distance decreases. Hence, acceleration also increases.

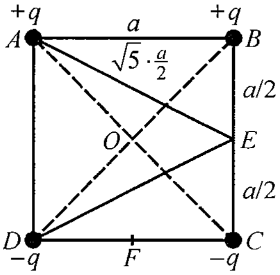
2. Four charges $+q,+q,-q$ and $-q$ are placed respectively at the corners A, BC and D of a square of side a, arranged in the given order. If E and F are mid points of side BC and CD respectively, what will be workdone in carrying a charge 'e' from O to E and from O to F.

Soln. The charge configuration is shown in figure :

Workdone in carrying a charge = change in potential energy = q(V$_{\text{final}}$ – V$_{\text{initial}}$)

$$AE^2 = a^2 + \left(\frac{a}{2}\right)^2 = \frac{5a^2}{4} \textbf{ or } AE = \frac{\sqrt{5}\cdot a}{2}$$

Potential at E i.e., $V_E = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{5}\cdot\frac{a}{2}} - \frac{q}{\sqrt{5}\cdot\frac{a}{2}} + \frac{q}{a/2} - \frac{q}{a/2}\right] = 0$



Potential at F i.e., V_F is given by

$$V_F = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{5}\cdot\frac{a}{2}} + \frac{q}{\sqrt{5}\cdot\frac{a}{2}} - \frac{q}{a/2} - \frac{q}{a/2}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{4q}{\sqrt{5}\cdot a} - \frac{4q}{a}\right]$$

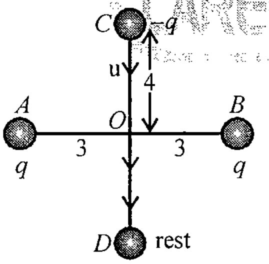
$$= \frac{1}{4\pi\varepsilon_0}\cdot\frac{4q}{a}\left[\frac{1}{\sqrt{5}} - 1\right]$$

Now, $W_{OE} = e\,(V_E - V_O) = e\,(0 - 0) = 0$ and $W_{OF} = e(V_F - V_O)$

$$= e\cdot\left[\frac{1}{4\pi\varepsilon_0}\cdot\frac{4q}{a}\left\{\frac{1}{\sqrt{5}} - 1\right\} - 0\right] = \frac{1}{4\pi\varepsilon_0}\cdot\frac{4eq}{a}\cdot\left[\frac{1}{\sqrt{5}} - 1\right].$$

Two fixed, equal positive charges, each of magnitude 5 × 10^{-5} coul, are located at points A and B separated at a distance of 6 m. An equal and opposite charge moves towards them along the line COD, the perpendicular bisector of the line AB. The moving charge, when reaches the point C at a distance of 4 m from O, has a kinetic energy of 4 joules. Calculate the distance of the farthest point D at which the negative charge will reach before returning towards C.

In. Applying conservation of energy, $(K.E. + P.E.)_{\text{at C}} = (K.E. + P.E.)_{\text{at D}}$



$$\frac{1}{2}mu^2 - \frac{q^2}{4\pi\varepsilon_0}\left(\frac{1}{AC} + \frac{1}{BC}\right) = 0 - \frac{q^2}{4\pi\varepsilon_0}\left(\frac{1}{AD} + \frac{1}{BD}\right)$$

Put $q = 3\times10^{-5}c$

$\therefore \quad \frac{1}{2}mu^2 = 44J$

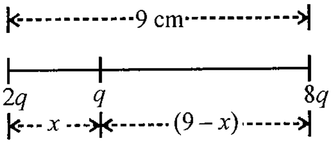
$AC = BC = \sqrt{3^2 + 4^2} = 5$ and solve to get

AD = BD = 9(m)

$\therefore \quad OD = \sqrt{(9)^2 - 3^2} = 6\sqrt{2}\ m$

4. Three charges, q, $2q$ and $8q$ are to be placed on 9 cm long straight line. Find the positions where the charges should be placed such that the potential energy of this system is minimum.

Soln. For minimum interaction energy, the charges of greater magnitudes should be placed at the extreme ends. The situation is shown in figure. Let the distance of charge $2q$ from charge q be x cm.



The potential energy of the system is given by

$$U=\frac{1}{4\pi\varepsilon_0}\left[\frac{2q\times q}{x}+\frac{q\times 8q}{(9-x)}+\frac{2q\times 8q}{9}\right]$$

For minimum potential energy, $\partial U/\partial x=0$

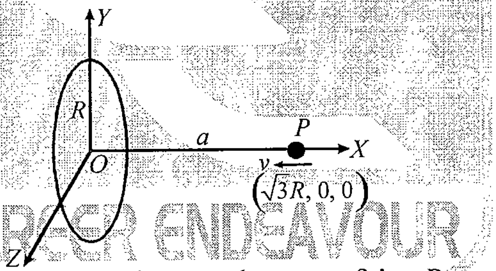
$$\therefore\quad \frac{\partial U}{\partial x}=\frac{1}{4\pi\varepsilon_0}\Rightarrow\frac{\partial}{\partial x}\left[\frac{2q^2}{x\times10^{-2}}+\frac{8q^2}{(9-x)10^{-2}}+\frac{16q^2}{9\times10^{-2}}\right]=0$$

Solving we get $x=3$ cm.

Thus the charge q should be placed at a distance of 3 cm from the charge $2q$.

5. A circular ring of radius R with uniform positive charge density λ per unit length is located in the YZ-plane with its centre at the origin O. A particle of mass m and positive charge q is projected from the point $P\left(R\sqrt{3},0,0\right)$ on the positive X-axis directly towards O, with initial velocity v. Find the smallest (non-zero) value of the speed v such that the particle does not return to P.

Soln. The situation is shown in figure :



Particle will not return to P if it manages to at least reach center of ring. Potential at axial point of ring (charge $=Q$, radius $=R$) is $V_{ring}=\dfrac{Q}{4\pi\varepsilon_0\sqrt{R^2+x^2}}$

Potential energy of charge and ring $=q\,V_{ring}$

Appling conservation of energy

$$(K.E.+P.E.)_{x=\sqrt{3}R}=(K.E.+P.E.)_{x=0}$$

$$\frac{1}{2}mv^2+\frac{qQ}{4\pi\varepsilon_0\sqrt{R^2+\left(\sqrt{3}\,R\right)^2}}=0+\frac{qQ}{4\pi\varepsilon_0\sqrt{R^2+0^2}}\Rightarrow v=\sqrt{\frac{qQ}{4\pi\varepsilon_0 Rm}}\text{, since } Q=\lambda 2\pi R$$

$$\therefore\quad \boxed{v=\frac{\lambda q}{2\varepsilon_0 m}}$$

A conducting sphere S_1, of radius r is attached to an insulating handle. Another conducting sphere S_2 of radius R is mounted as an insulating stand. S_2 is initially uncharged. S_1 is given a charge Q, brought into contact with S_2 and removed. S_1 is recharged such that the charge on it is again Q and is again brought into contact with S_2 and removed. This procedure is repeated n times.

(a) Find the electrostatic energy of S_2 after n such contacts with S_1.

(b) What is the limiting value of this energy as $n \to \infty$.

In. (a) Let Q_2 be the charge on sphere S_2 after S_1 is removed first Q_1 be charge on S_1.

$$\therefore \qquad Q_1 + Q_2 = Q$$

Since the potential at the surface of each sphere is the same, we have

$$\frac{Q_2}{4\pi\varepsilon_0 R}=\frac{Q_1}{4\pi\varepsilon_0 r} \Rightarrow \frac{Q_2}{R}=\frac{Q-Q_2}{r} \text{ i.e. } Q_2=\frac{1}{1+\frac{r}{R}} \text{ times, the total charge } Q_2=\frac{Q}{1+\left(\frac{r}{R}\right)}$$

Now S_1 is recharged such that the charge on it is again Q. Charge on S_2 is $\frac{Q}{\left(1+\frac{r}{R}\right)}$. They are brought in contact.

Total charge on S_1 and S_2 $=\left[Q+\left\{\frac{Q}{\left(1+\frac{r}{R}\right)}\right\}\right]$

Therefore, now total charge on Q_2 will be

$$=\frac{1}{1+\frac{r}{R}} \text{ time total charge}$$

$$=\frac{1}{\left(1+\frac{r}{R}\right)}\cdot\left[Q+\frac{Q}{1+\frac{r}{R}}\right]=\frac{Q}{1+\frac{r}{R}}+\frac{Q}{\left(1+\frac{r}{R}\right)^2}$$

Similarly, after n repeatation, total charge on S_2 becomes,

$$=\frac{Q}{\left\{1+\left(\frac{r}{R}\right)\right\}}+\frac{Q}{\left\{1+\left(\frac{r}{R}\right)\right\}^2}+\cdots+\frac{Q}{\left\{1+\left(\frac{r}{R}\right)\right\}^n}$$

$$=\frac{Q}{1+\left(\frac{r}{R}\right)}\left[\frac{1-1/\left(1+\frac{r}{R}\right)^n}{1-1/\left(1+\frac{r}{R}\right)}\right]$$

Electrostatic energy of S_2

$$=\frac{Q_{\text{final}}^2}{8\pi\varepsilon_0 R}=\frac{Q^2}{8\pi\varepsilon_0 R\left(1+\frac{r}{R}\right)^2}\times\left[\frac{1-1/\left(1+\frac{r}{R}\right)^n}{1-1/\left(1+\frac{r}{R}\right)}\right]^2$$

(b) Limiting energy at $n \to \infty$

$$E = \frac{Q^2}{8\pi\varepsilon_0 R\left(1+\frac{r}{R}\right)^2}\left[\frac{1-0}{1-1/\left(1+\frac{r}{R}\right)}\right]^2$$

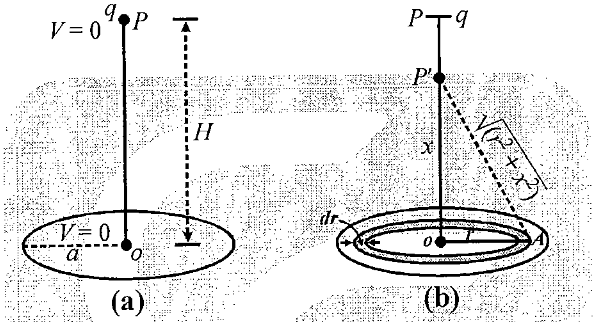
$$= \frac{Q^2}{8\pi\varepsilon_0 R}\left(\frac{R}{r}\right)^2 = \frac{Q^2 R}{8\pi\varepsilon_0 r^2}.$$

7. A non-conducting disc of radius a and uniform surface charge density σ is placed on the ground, with its axis vertical. A particle of mass m and positive charge q is dropped, along the axis of the disc, from a height H with zero initial velocity. The particle has $q/m = 4\pi\varepsilon_0\, g/\sigma$.

(a) Find the value of H if the particle just reaches the disc.

(b) Sketch the potential energy of the particle as a function of its height and find its equilibrium position.

Soln. See below figure.



We know that, $U = qV_{disc}$

$$= \frac{\sigma q}{2\varepsilon_0}\left[(a^2+x^2)^{1/2} - x\right] \qquad \dots(1)$$

(a) Applying the principle of conservation of energy at points P and O, we have

$(\text{K.E.} + \text{P.E.})_{\text{at P}} = (\text{K.E.} + \text{P.E.})_{\text{at O}}$

$$0 + mgH + \frac{\sigma q}{2\varepsilon_0}\left[\sqrt{(a^2+H^2)} - H\right] = 0 + \frac{\sigma q}{2\varepsilon_0}\left[\sqrt{(a^2+0^2)} - 0\right]$$

Given that $q = (4\varepsilon_0\, g\, m/\sigma)$

$$\therefore \quad mgH + \frac{\sigma(4\varepsilon_0\, g\, m/\sigma)}{2\varepsilon_0}\left[\sqrt{(a^2+H^2)} - H\right] = \frac{\sigma(4\pi\varepsilon_0\, g\, m/\sigma)}{2\varepsilon_0}a$$

or $\quad mgH + 2mg\left[\sqrt{(a^2+H^2)} - H\right] = 2mga$

$$H + 2\sqrt{(a^2+H^2)} - 2H = 2a$$

$$2\sqrt{(a^2+H^2)} = (2a+H)$$

or $\quad 4(a^2+H^2) = (2a+H)^2$

Solving we get $H = (4a/3)$

(b) Total potential energy at a height x above centre

$$\text{P.E.} = \frac{\sigma q}{2\varepsilon_0}\left[\sqrt{(a^2+x^2)} - x\right] + mgx$$

Substituting the value of q, we get

$$\text{P.E.} = 2mg\left[\sqrt{(a^2+x^2)} - x\right] + mgx$$

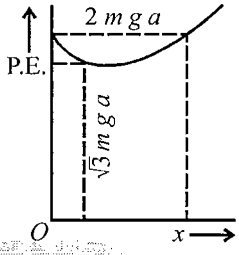
$$U = mg\left[2\sqrt{(a^2+x^2)} - x\right]$$

For equilibrium position, $\dfrac{dU}{dx} = 0$

$$\therefore \quad \frac{dU}{dx} = 0 = mg\left[\frac{2.2x}{2\sqrt{(a^2+x^2)}} - 1\right]$$

$$\text{or} \quad \frac{2x}{\sqrt{(a^2+x^2)}} = 1 \quad \text{or} \quad 4x^2 = (a^2+x^2)$$

$$\therefore \quad x = \frac{a}{\sqrt{3}}$$



The variation of potential energy with x is shown in figure.

Chapter 5

Electric Dipole & Dielectrics

Electric dipolemoment:

Dipole moment for a collection of charges, q_i having position vectors $\vec{r}_i$ is

$$\vec{p} = \sum_i q_i \vec{r}_i$$

For continuous system the dipole moment is written as

$$\vec{p} = \int_v \vec{r} \rho(\vec{r}) dv$$

Show that, if the total charge is zero then dipole moment $(\vec{p})$ is independent of the choice of origin at the coordinate system:

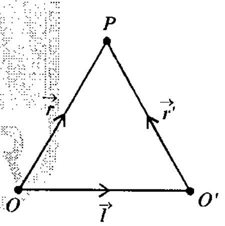
Suppose we consider a new coordinate system whose origin O' is at $\vec{\ell}$ with respect to the origin O of the old coordinate system. Let a point which is denoted by $\vec{r}'$ with respect to the old coordiinate system and with respect to the new coordinate system.

$$\vec{r}' = (\vec{r} - \vec{\ell})$$

Therefore, the dipole moment with respect to the old system is

$$\vec{p} = \int \vec{r} \rho dv = \int (\vec{r}' + \vec{\ell}) \rho dv$$

$$= \int \vec{r}' \rho dv + \vec{\ell} \int \rho dv = \int \vec{r}' \rho dv + \vec{\ell} Q$$



where, $Q = \int \rho dv$ = total charge

If $Q = 0$, then $\vec{p} = \int \vec{r}'' \rho dv$ dipole moment in the new system.

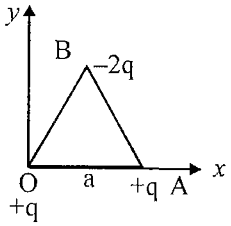
Example: Suppose three point charges +q, +q, –2q place at the vertices of an equilateral triangle.

Soln. Total charge, $Q = +q + q - 2q = 0$

Therefore, dipole moment is independent of choice of origin.

The dipole moment with respect to origin.

$$= q \times 0 + q(a\hat{i}) - 2q\left(\frac{a}{2}\hat{i} + \frac{\sqrt{3}}{2}a\hat{j}\right)$$



$$= 0 + qa\hat{i} - qa\hat{i} - q\sqrt{3}a\hat{j} = -\hat{j}\sqrt{3}aq = -\hat{j}\sqrt{3}aq$$

Example: A sphere with a charge distribution $\sigma(\theta) = \sigma_0 \cos\theta$. Find the dipole moment of the sphere.

Soln. Let elementary area,

$$ds = r^2 \sin\theta d\theta d\phi \qquad \text{and} \qquad \sigma(\theta_0) = \sigma_0 \cos\theta$$

$\therefore \qquad dq = \sigma ds = \sigma_0 \cos\theta = r^2 \sin\theta d\theta d\phi = 0$

The dipole moment is independent of choice of origin becomes q to total charge is zero.
Let any point on the surface element given by

$$\vec{r} = r\sin\theta\cos\phi\hat{i} + r\sin\theta\sin\phi\hat{j} + r\cos\theta\hat{k}$$

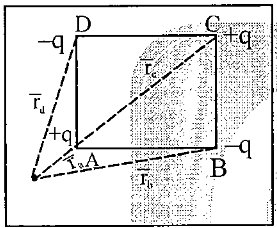
Therefore, the dipole moment,

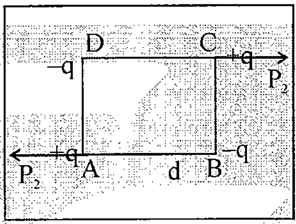
$$\vec{P} = \int \vec{r}dq = r^3\sigma_\theta \int_0^\pi \int_0^{2\pi} \left[\left(\sin^2\theta\cos\theta\cos\phi\right)\hat{i} + \left(\sin^2\theta\cos\theta\sin\theta\right)\hat{j} + \left(\cos\theta\sin\theta\right)\hat{k}\right] d\theta d\phi$$

$$= 0 + 0 + r^3\sigma_0 \int_0^\pi \cos^2\theta \sin\theta d\theta \int_0^\pi d\phi\hat{k} = \frac{4}{3}\pi r^3 \sigma_0 \hat{k}$$

Example : Consider Figure given below (Quadrapole). What is the dipole moment of system?

In. Take anywhere the origin. The position vectors of A, B, E and D are $\vec{r}_a, \vec{r}_b, \vec{r}_c$ and $\overline{\mathrm{r}}_d$ respectively. Then the dipole moment.





$$\vec{p} = \sum_i q_i \vec{r}_i = q\vec{r}_a + (-q)\vec{r}_b + q\left(\vec{r}_e\right) + (-q)\vec{r}_d$$

$$\vec{p} = q\left(\vec{r} - \vec{r}_b\right) + q\left(\vec{r}_e - \vec{r}_d\right) = q\left(\overrightarrow{BA}\right) + q\left(\overrightarrow{DC}\right) = q\left(-\vec{d} + \vec{d}\right) = 0$$

You can think it indifferent way too.
Let AB form a dipole and CD form another. The dipole moment of AB

$P_1 = qd$, Pointing leftward the dipole moment of CD

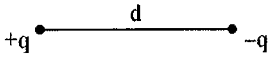
$P_2 = qd$, Pointing rightward. This adds up to zero.

Electric dipole:

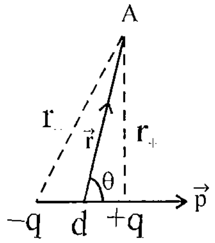
Two equal and opposite charges separated by a small distance is called electric dipole.

The multiple of one of it charge and distance between two point charges is called dipole moment. Direction of dipole moment is negative to positive charge.

$$\vec{P} = q\vec{d} \qquad \ldots (4)$$



Electric potential of a dipole :



Let r_+ and r_- are the distance from q_+ and q_- charge to the point A(where the potential needs to be evaluated). The mid-point of the line joining $+q$ and $-q$ taken to be origin.

Now, $V(\bar{r}) = \dfrac{1}{4\pi \in_0}\left(\dfrac{q}{r_+} - \dfrac{q}{r_-}\right)$ (combination force q_+ and q_- just added together)

From the law of cosine

$$r_+^2 = r^2 + \left(\frac{d}{2}\right)^2 - rd\cos\theta = r^2\left(1 - \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}\right)$$

$$r_-^2 = r^2 + \left(d/2\right)^2 + rd\cos\theta = r^2\left(1 + \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}\right)$$

We assume d to be very small compared to r. So we neglect $\dfrac{d^2}{4r^2}$ term

So, $$\frac{1}{r\pm} \cong \frac{1}{r}\left(1 \mp \frac{d}{r}\cos\theta\right)^{-1/2}$$

$$\cong \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right)$$ (expanding bionomically and taking the 1st term only)

Thus, $$\frac{1}{r_+} - \frac{1}{r_-} \cong \frac{d}{2r}\cos\theta$$

Hence, $$V(\bar{r}) \cong \frac{1}{4\pi \in_0}\frac{qd\cos\theta}{r^2}$$

$$\boxed{V_{dip}(r,\theta) = \frac{p\cos\theta}{4\pi \in_0 r^2} = \frac{1}{4\pi \in_0}\frac{\bar{p}.\hat{r}}{r^2}}$$

Electric field of a dipole:

$$E_r = -\frac{\partial V}{\partial r} = \frac{2p\cos\theta}{4\pi \in_0 r^3},\quad E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{p\sin\theta}{4\pi \in_0 r^3}$$

$$E_\phi = -\frac{1}{r\sin\theta}\cdot\frac{\partial V}{\partial \phi} = 0$$

$$\boxed{\bar{E}_{dip}(r,\theta) = \frac{p}{4\pi \in_0 r^3}\cdot\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)} \qquad ...(10)$$

This can be rewritten as

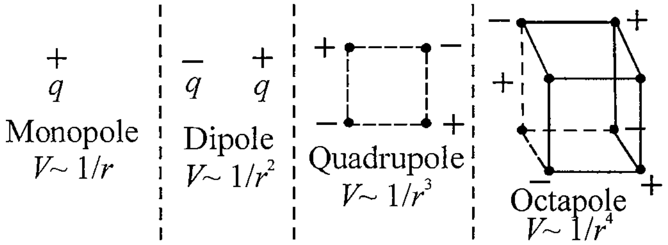
$$\boxed{\vec{E}_{dip} = \frac{1}{4\pi \in_0 r^3}\left[3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right]}$$

Asymptotic variation of potential and field:

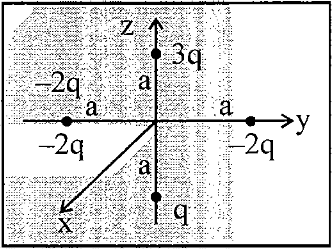
If total charge is non-zero, then $V \propto \frac{1}{r}, E \propto \frac{1}{r^2}$.

If total charge is zero and dipole moment is non-zero, then $V \propto \frac{1}{r^2}, E \propto \frac{1}{r^3}$

If total charge and dipole moment are zero and quadrupole term is non-zero then, $V \propto \frac{1}{r^3}, E \propto \frac{1}{r^4}$



Example : Four charge as shown in figure. Find a simple approximate formula for potential valid at points far from origin.



ln. Total charge, $Q_t = 3q + q - 2q - 2q = 0$

Therefore, the potential due to monopole is zero

The dipole moment w.r.t. origine is

$$\vec{p} = 3aq\hat{k} - aq\hat{k} + 2aq\hat{j} - 2aq\hat{j} = 2aq\hat{k}$$

Therefore, potential at a point $p(r\ \theta)$ is

$$\phi(r,\theta) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^3} = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{2aq\cos\theta}{r^2}$$

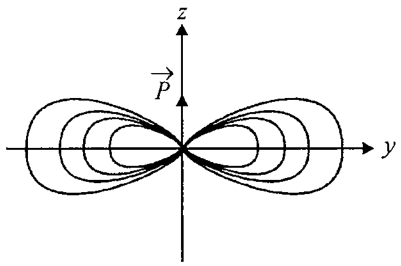
Equipotential surface and field lines of dipole:

If θ is the angle between dipole and r then the equation of field lines is

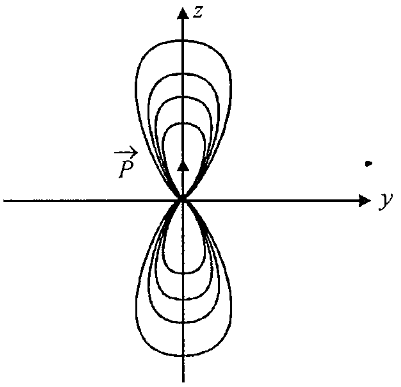
$\boxed{r = c\sin^2\theta}$ (where, c is a constant) ... (11)

The equation of equipotential line

$r(\theta) = A\sqrt{\cos\theta}$ (where, A is a constant) ... (12)



Field line of ideal dipole

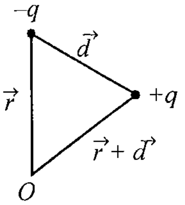


Equipotential line of ideal dipole

Potential energy of a dipole placed in an external electric field:

Let we put a dipole whose –q and +q respectively at $\vec{r}$ and $\left(\vec{r}+\vec{d}\right)$

The potential energy,



$$U=-qV\left(\vec{r}\right)+qV\left(\vec{r}+\vec{d}\right)$$

$$U=-qV\left(r\right)+qV\left(r\right)+qd\cdot\vec{\nabla}V\left(r\right) \quad \text{(using Taylor expansion)}$$

$$=qd\cdot\vec{\nabla}V\left(r\right)$$

$$\boxed{U=\vec{p}\cdot\vec{\nabla}V\left(r\right)} \qquad \ldots(13)$$

Or, $$\boxed{U=-\vec{p}\cdot\vec{E}\left(r\right)} \qquad \left(\because\ E\left(r\right)=-\nabla V\left(r\right)\right) \qquad \ldots(14)$$

Force and torque on a dipole placed in an electric field :

We know that,

$$\vec{F}=-\vec{\nabla}U=\nabla\left(\vec{p}\cdot\vec{E}\right)$$

$$=\left(\vec{p}\cdot\vec{\nabla}\right)\vec{E}+\left(\vec{E}\cdot\vec{\nabla}\right)\vec{p}+\vec{p}\times\left(\vec{\nabla}\times\vec{E}\right)+\vec{E}\times\left(\vec{\nabla}\times\vec{p}\right)$$

$$=\left(\vec{p}\cdot\vec{\nabla}\right)\vec{E}+0+0+0$$

$$\boxed{F=\left(\vec{p}\cdot\vec{\nabla}\right)\vec{E}} \qquad \ldots(15)$$

For uniform $\vec{E}$,

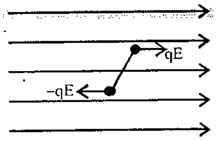
$$\vec{F}=\left(\vec{p}\cdot\vec{\nabla}\right)\vec{E}=0$$

that means there have no translational motion.

Torque on the dipole in uniform E field :

We know the potential energy of the dipole due external electric field is $u=\vec{p}\cdot\vec{E}=pE\cos\theta$

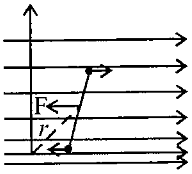
Therefore, the torque about its own centre is



$$\tau=-\frac{du}{d\theta}=pE\sin\theta$$

Or, $$\vec{\tau}=\vec{p}\times\vec{E}$$

Torque on the dipole in non-uniform electric field :



The torque on a small dipole about its own centre is still

$$\vec{\tau}=\vec{p}\times\vec{E}$$

But since there is a net force $\vec{F}$ on the dipole.

Therefore, an extra torque act on the dipole is $\vec{r}\times\vec{F}$ about any other point.

Therefore, total torque about any other point is $\boxed{\vec{\tau}=\vec{p}\times\vec{E}+\vec{r}\times\vec{F}}$ ...(16)

Mutual potential energy between two coplanar diples:

Let consider two diple of dipole moment $\vec{p}_1$ and $\vec{p}_2$ lying on a single plane, $\vec{r}_{21}$ is the position vector of dipole 2 with respect to 1.

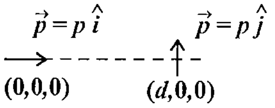
The electric field at the centre of dipole 2 is

$$\vec{E}_1 = \frac{1}{4\pi\varepsilon_0}\left[\frac{3(\vec{p}_1\cdot\vec{r}_{21})\vec{r}_{21}}{r_{21}^5} - \frac{\vec{p}_1}{r_{21}^3}\right]$$

The mutual potential energy,

$$U_{21} = -\vec{p}_2\cdot\vec{E}_1 = -\vec{p}_1\cdot\vec{E}_2 = \frac{1}{4\pi\varepsilon_0}\left[\frac{\vec{p}_1\cdot\vec{p}_2}{r_{21}^3} - \frac{3(\vec{p}_1\cdot\vec{r}_{21})(\vec{p}_2\cdot\vec{r}_{21})}{r_{21}^5}\right] \qquad \ldots(17)$$

Example: Two electric dipoles are placed as shown in figure. Calculate force of interaction on dipole 2 due to dipole 1.



ln. $\vec{F}_{21} = \vec{p}_2\cdot\vec{\nabla}\vec{E}_1$, consider field at $(x, y, 0)$

$$= p\hat{j}\cdot\vec{\nabla}\frac{1}{4\pi\varepsilon_0 r^3}(3(\vec{p}_1\cdot\hat{r})\hat{r} - \vec{p}_1)$$

$$= \frac{p}{4\pi\varepsilon_0}\frac{\partial}{\partial y}\left(\frac{3(\vec{p}_1\cdot\vec{r})\vec{r}}{r^5} - \frac{\vec{p}_1}{r^3}\right)$$

$$= \frac{p}{4\pi\varepsilon_0}\frac{\partial}{\partial y}\left[\frac{3px(x\hat{i}+y\hat{j})}{(x^2+y^2)^{5/2}} - \frac{p\hat{j}}{(x^2+y^2)^{3/2}}\right]$$

$$= \frac{p^2}{4\pi\varepsilon_0}\left[\frac{3(0\hat{i}+x\hat{j})\cdot(x^2+y^2)^{5/2} - 15xy(x\hat{i}+y\hat{j})(x^2+y^2)^{3/2}}{(x^2+y^2)^5} + \frac{3y\hat{i}}{(x^2+y^2)^{5/2}}\right]$$

Coordinate of p^2 is $(d, 0, 0)$

Therefore, $\vec{F}_{21} = \frac{p^2}{4\pi\varepsilon_0}\left[\frac{3d^6\hat{i} - 0}{d^{10}} + 0\right]$

$$\vec{F}_{21} = \frac{3p^2}{4\pi\varepsilon_0 d^4}\hat{i}.$$

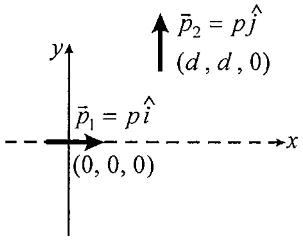
Example : Two electric dipoles are placed as shown in figure. Calculate torque on dipole 2 due to dipole 1.

ln. Torque on dipole 2 due to dipole 1 is given as :

$\vec{\tau}_{21} = \vec{p}_{21}\times\vec{E}_1$ (about center of 2)

$$= p\hat{j}\times\frac{1}{4\pi\varepsilon_0 r^3}[3(\vec{p}_1\cdot\hat{r})\hat{r} - \vec{p}_1]$$

Here, $r = \sqrt{2}d,\ \hat{r} = \frac{\hat{i}+\hat{j}}{\sqrt{2}}$



Therefore, $\vec{\tau}_{21} = \dfrac{p\hat{j}\times\left[3\left(p\hat{i}\cdot\frac{(\hat{i}+\hat{j})}{\sqrt{2}}\right)\frac{(\hat{i}+\hat{j})}{\sqrt{2}} - p\hat{i}\right]}{4\pi\varepsilon_0\left(\sqrt{2}d\right)^3}$

$$= \frac{p^2\hat{j}\times(\hat{i}+3\hat{j})}{16\sqrt{2}\pi\varepsilon_0 d^3} \Rightarrow \vec{\tau}_{21} = \frac{p^2(-\hat{k})}{16\sqrt{2}\pi\varepsilon_0 d^3}.$$

Example: Two electric dipoles $\vec{p}_1 = p\hat{i}$ and $\vec{p}_2 = p\hat{j}$ are located at (a, a, a) and $(-a, a, -a)$. Calculate interaction potential energy of the two dipoles.

Soln. $$U = \frac{1}{4\pi\varepsilon_0 r_{12}^3}[\vec{p}_1 \cdot \vec{p}_2 - 3(\vec{p}_1 \cdot \hat{r}_{12})(\vec{p}_2 \cdot \hat{r}_{12})]$$

$$\vec{r}_{12} = \vec{r}_1 - \vec{r}_2 = a(\hat{i} + \hat{j} + \hat{k}) - a(-\hat{i} + \hat{j} - \hat{k})$$

$$= 2a(\hat{i} + \hat{k})$$

$$\therefore \quad r_{12} = 2\sqrt{2}a$$

$$\hat{r}_{12} = \frac{1}{\sqrt{2}}(\hat{i} + \hat{j})$$

$$\therefore \quad U = \frac{1}{4\pi\varepsilon_0(16\sqrt{2}a^3)}\left[p\hat{i} \cdot p\hat{j} - 3p\hat{i} \cdot \frac{(\hat{i} + \hat{j})}{\sqrt{2}}\, p\hat{j} \cdot \frac{(\hat{i} + \hat{j})}{\sqrt{2}}\right]$$

$$U = \frac{-3p^2}{128\sqrt{2}\pi\varepsilon_0 a^3}.$$

Polar and non-polar molecules:

Polar: Centre of positive and negative charge do not coincide this type molecules are called polar molecules like H_2O, H_2S.

Non-Polar : If the centre of positive and negative charge coincide then it is called non-polar molecule, N_2, O_2, CCl_4.

Dielectric polarization :

When any dielectric is placed in an electrostate field the charge distribution within the molecules may become modified. The electrons are pulled in one direction and the nuclei in the opposite way. This causes a slight displacement of the centre of positive and negative charges. This phenomenon is known as electric polarization.

In case of polar dielectric in the absence of extrnal electric field the dipoles are randomly oriented due to thermal agitation and the net dipole moment per unit volume is zero. If an electric field is applied it tries to align the dipoles along the field direction while thermal agitation tries to oppose the alignment. In the equilibrium some net alignment hence some net dipole moment per unit volume is obtained. This phenomenon is known as orientational polarization.

Polarization $\vec{\mathbf{P}}$:

The electric dipole moment per unit volume is called polarization.

$$\vec{P} = \lim_{\Delta V \to 0} \frac{1}{\Delta V}\sum_i \vec{p}_i$$

Bound charge :

Due to polarization of any dielectric medium some kind of charges appear in the volume and on the surface of the dielectric medium. These charges are called bound charge. When external field vanish the bound charge all vanish.

The volume charge density, $\boxed{\rho = -\vec{\nabla} \cdot \vec{P}}$

The surface charge density, $\boxed{\sigma = \vec{P} \cdot \hat{n}}$

Gauss's law in dielectric medium: In dielectric medium the GAuss's law is

$$\boxed{\oint \vec{D}\cdot d\vec{S} = Q_f}$$

Where, $\vec{D} = \varepsilon_0\vec{E} + \vec{P}$ and Q_f = free charge

$$= \varepsilon\vec{E}$$

Poisson's and Laplace's equations in dielectrics:

We know $\oint \vec{D}\cdot d\vec{S} = Q_f$

$$\Rightarrow \quad \int\left(\vec{\nabla}\cdot\vec{D}\right)dv = \int \rho_f dv \Rightarrow \vec{\nabla}\cdot\vec{D} = \rho_f \Rightarrow \vec{\nabla}\cdot\vec{E} = \frac{\rho_f}{\varepsilon}$$

$$\Rightarrow \quad -\vec{\nabla}^2\phi = \frac{\rho_f}{\varepsilon} \qquad (\text{using } \vec{E} = -\vec{\nabla}\phi)$$

$$\Rightarrow \quad \boxed{\nabla^2\phi = -\frac{\rho_f}{\varepsilon}}$$ is known as Poisson's equation.

If medium is charge free then $\rho_f = 0$

$$\therefore \quad \boxed{\nabla^2\phi = 0}$$ is known as Laplace equation.

Susceptibility, permitivity and dielectric constant:

If the external electric field is not very strong, then

$$\vec{P} = \varepsilon_0\chi\vec{E}$$

χ is called electric susceptibility of the dielectric.

$$\therefore \quad \vec{D} = \varepsilon_0\vec{E} + \vec{P} = \varepsilon_0\vec{E} + \varepsilon_0\chi\vec{E} = \varepsilon_0(1+\chi)\vec{E} = \varepsilon\vec{E}$$

where, $\varepsilon = \varepsilon_0(1+\chi)$

$$\Rightarrow \quad \boxed{\frac{\varepsilon}{\varepsilon_0} = (1+\chi) = k}$$ is called dielectric constant.

For isotropic medium χ is scalar, but for anisotropic medium it is a tensor.

$$\begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix} = \varepsilon_0 \begin{pmatrix} \chi_{xx} & \chi_{xy} & \chi_{xz} \\ \chi_{xy} & \chi_{yy} & \chi_{yz} \\ \chi_{xz} & \chi_{yz} & \chi_{zz} \end{pmatrix}\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

Non-polar liquid dielectric and the Clausius-Mossotti Relation:

The electric field that is responsible for polarizing a molecule of the dielectric is called the molecular field or local field. For simple non-polar liquid dielectrics we may expect that a molecule finds itself, on the average, within a spherical hole. To calculate the molecular field at a point, let us imagine to remove a small sphere of polarized material from the region surrounding the point under consideration. Let $\vec{E}$ be the uniform macroscopic field within the dielectric before the sphere was removed. By the superposition principle $\vec{E}$ may be considered as the sum of the field $\vec{E}_1$ within the cavity due to all charges outside the cavity and the field $\vec{E}_2$ due to all charges within the removed polarized sphere, i.e.,

$$\vec{E} = \vec{E}_1 + \vec{E}_2$$

To find $\vec{E}_2$ we consider a uniformly polarized sphere as shown in figure. Consider an elementary area $d\vec{S} = \hat{n}dS$ at any point $A(r,\theta,\phi)$ on the surface of the sphere and take the z-axis along the direction of polarization $\vec{p}$. The surface density of polarization charge at A is $\vec{P}\cdot\hat{n} = P\cos\theta$. The field at O due to charges on dS is

$$d\vec{E}_2 = -\frac{1}{4\pi\varepsilon_0}\cdot\frac{P\cos\theta dS}{r^2}\hat{r}$$

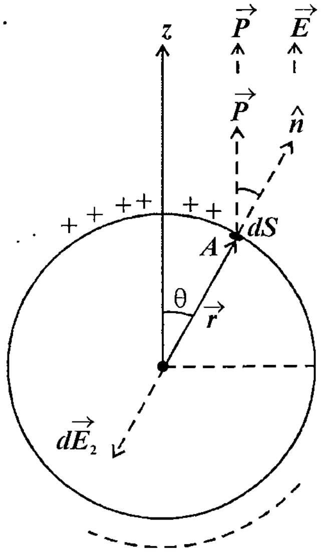
It is along the direction AO.

Because of symmetry about z-axis only the component of $d\vec{E}_2$ along the negative z-axis will contribute to the integral of equation (19) over the complete surface, the other component perpendicular to z-axis will give zero contribution when integrated over the whole surface. Since, $dS = r^2\sin\theta d\theta d\phi$.

$$\vec{E}_2 = -\frac{1}{4\pi\varepsilon_0}\int\frac{P\cos\theta dS}{r^2}\cos\theta\hat{z}$$

$$= -\frac{\vec{P}}{4\pi\varepsilon_0}\int_0^{2\pi}d\phi\int_0^{\pi}\cos^2\theta\sin\theta d\theta \quad \left[\because\ \vec{P} = P\hat{z}\right]$$

$$= -\frac{\vec{P}}{4\pi\varepsilon_0}\cdot 2\pi\cdot\frac{2}{3} = -\frac{\vec{P}}{3\varepsilon_0}$$



Thus, the field inside is uniform. It is a depolarizing field.

Now from equation (18) we get the field $\vec{E}_1$ within the cavity as

$$\vec{E}_1 = \vec{E} - \vec{E}_2 = \vec{E} + \frac{\vec{P}}{3\varepsilon_0}$$

For non-polar dielectrics with simple structures such as for most dielectrics liquids, the molecular field that polarizes a molecular may be taken as $\vec{E}_1$. Under the action of this field the molecule will be polarized. The dipole moment $\vec{p}$ induced in the molecule will be proportional to $\vec{E}_1$, i.e.,

$$\vec{p} = \sigma\vec{E}_1$$

where the constant of proportionality α is called the molecular polarizability. It is the dipole moment per unit polarizing field. Its value depends on the detailed structure of the molecule in question. If there are n molecules per unit volume of the dielectric then the polarization vector.

$$\vec{P} = n\vec{p} = n\alpha\vec{E}_1 = n\alpha\left(\vec{E} + \frac{\vec{P}}{3\varepsilon_0}\right)$$

Or, $$\vec{P} = \frac{n\alpha}{1 - n\alpha/3\varepsilon_0}\vec{E}$$

This equation may be written in terms of the dielectric constant K of the medium.

We have the relation

$$\vec{D} = \varepsilon_0\vec{E} + \vec{P} \text{ or } \vec{P} = \varepsilon\vec{E} - \varepsilon_0\vec{E} = \varepsilon_0(K-1)\vec{E} \qquad \left[\because\ K = \frac{\varepsilon}{\varepsilon_0}\right]$$

Now putting this value of $\vec{p}$, we get

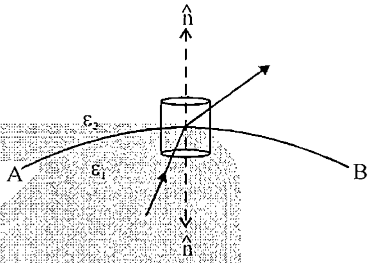
$$\varepsilon_0(K-1)=\frac{n\alpha}{1-\frac{n\alpha}{3\varepsilon_0}}$$

Or, $$\boxed{\alpha=\frac{3\varepsilon_0}{n}\cdot\frac{K-1}{K+2}}$$

This is known as Clausius-Mossotti equation.

Boundary condition:

Suppose AB be the boundary of two medium of dielectric constants ε_1 and ε_2. Now considers an Gaussian cylinder of cross-section ΔS.



Therefore, according to Gauss's law

$$\oint \vec{D}\cdot d\vec{s}=\sigma_{free}\cdot\Delta s$$

$$\Rightarrow\quad \vec{D}_1\cdot\Delta\vec{s}-\vec{D}_2\cdot\Delta\vec{s}=\sigma_{free}\cdot\Delta s$$

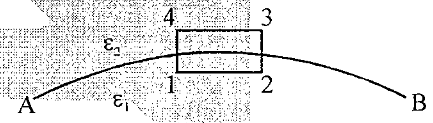
$$\Rightarrow\quad \vec{D}_2\cdot\hat{n}-\vec{D}_1\cdot\hat{n}=\sigma_{free}$$

$$\Rightarrow\quad \boxed{\varepsilon_1E_{2n}-\varepsilon_1E_{1n}=\sigma_{free}}$$

If the boundary is charge free then

$$\boxed{\varepsilon_2E_{2n}=\varepsilon_1E_{1n}}$$

Now, consider rectangular path 1234



$$\oint\left(\vec{\nabla}\times\vec{E}\right)\cdot ds=\oint E.d\ell=0$$

$$\Rightarrow\quad E_1\cdot\Delta\ell-E_2\cdot\Delta\ell=0$$

$$\Rightarrow\quad E_1\cdot\Delta\ell=E_2\cdot\Delta\ell$$

$$\Rightarrow\quad \boxed{E_{2t}=E_{1t}}$$

Electrostatic energy in dielectric:

$$\boxed{U=\frac{1}{2}\int\left(\vec{E}\cdot\vec{D}\right)dV}$$

Therefore, the energy density in dielectric, $\boxed{u=\frac{1}{2}\left(\vec{E}\cdot\vec{D}\right)}$

ADDITIONAL SOLVED PROBLEMS

1. A conducting sphere of radius a carries a charge q. It is coated with a linear dielectric material of permittivity ε out to a radious b. Find the potential at the centre of the conductor.

Soln. Applying Gauss's law to a spherical surface of radious r, where $a < r < b$

$$\oint \vec{D} \cdot d\vec{S} = q$$

$$\Rightarrow \quad D \cdot 4\pi r^2 = q$$

$$\Rightarrow \quad E = \frac{1}{4\pi\varepsilon}\frac{q}{r^2} \text{ for } a < r < b$$

Similarly, $E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} \qquad r > b$

Insider the metal surface the $\vec{E} = 0$

Therefore, the potential at the centre of sphere is

$$\phi = -\int_\infty^a E \cdot dr = -\int_\infty^b \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}dr - \int_b^a \frac{1}{4\pi\varepsilon_0}\frac{q}{r}dr$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{b} + \frac{q}{4\pi\varepsilon}\left[\frac{1}{a} - \frac{1}{b}\right]$$

This is the potential any point inside the metal sphere.

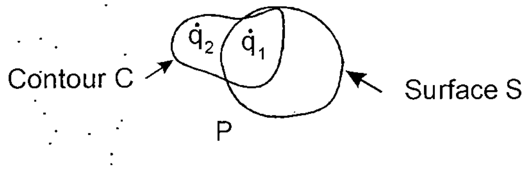
2. When a neutral dielectric is polarised, the polarization volume and surface charge appear. Show that total charge, however remain zero.

Soln. Total polarization charge

$$= \int \rho_p dV + \int \sigma_p dS = \int -\left(\vec{\nabla} \cdot \vec{p}\right) \cdot dV + \int \vec{p} \cdot \hat{n} dS$$

$$= -\int \vec{p} \cdot \hat{n} dS + \int \vec{p} \cdot \hat{n} dS = 0 \quad \text{[Using divergence theorem]}$$

3. Consider a set of two stationary point charges q_1 and q_2 as shown in the figure. Which of the following statements is correct?



(a) The electric field at P is independent of q_2

(b) The electric flux crossing the closed surface S is independent of q_2

(c) The line integral of the electric field $\vec{E}$ over the closed contour C depends on q_1 and q_2.

(d) $\vec{\nabla}.\vec{E} = 0$ everywhere

Soln. (i) The electric field at p depends on both change

(ii) The electric flux

$$\oint E \cdot dS = \frac{q_1}{\varepsilon_0}$$

Therefore, the electric flux crossing the closed surface S is independent of q_2.

Correct option is (b)

Statement for Linked Answer Q. 4 and Q.5 :

A dielectric sphere of radius R carries polarization $\vec{P} = kr^2\,\hat{r}$, wherer r is the distance from the centre and k is a constant. In the spherical polar coordinate system, $\hat{r}$, $\hat{\theta}$ and $\hat{\varphi}$ are the unit vectors.

The bound volume charge density inside the sphere at a distance r from the centre is :

(a) $-4kR$ (b) $-4kr$ (c) $-4kr^2$ (d) $-4kr^3$

ln. $\rho = -\vec{\nabla}\cdot\vec{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 kr^2\right) = -4kr$

Correct option is (b)

The electric field inside the sphere at a distance d from the centre is :

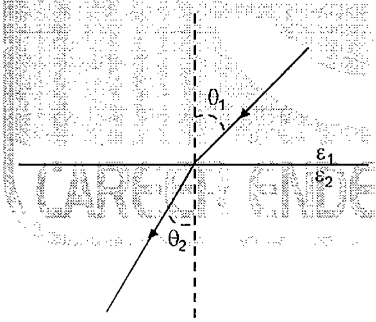
(a) $\frac{-kd^2}{\varepsilon_0}\hat{r}$ (b) $\frac{-kR^2}{\varepsilon_0}\hat{r}$ (c) $\frac{-kd^2}{\varepsilon_0}\hat{\theta}$ (d) $\frac{-kR^2}{\varepsilon_0}\hat{\theta}$

ln. $\oint E\cdot dS = \int\frac{\rho\cdot dv}{\varepsilon_0}$

$$\Rightarrow \quad E\cdot 4\pi d^2 = -\int\frac{4kr}{\varepsilon_0}r^2\sin\theta 2\pi d\theta dr \quad \Rightarrow \vec{E} = -\frac{kd^2}{\varepsilon_0}\hat{r}$$

Correct option is (a)

At the interface between two linear dielectrics (with dielectric constants (ε_1 and ε_2), the electric field lines bend, as shown in the figure. Assume that there are no free charges at the interface. The ratio ε_1 and ε_2 is :



(a) $\frac{\tan\theta_1}{\tan\theta_2}$ (b) $\frac{\cos\theta_1}{\cos\theta_2}$ (c) $\frac{\sin\theta_1}{\sin\theta_2}$ (d) $\frac{\cot\theta_1}{\cot\theta_2}$

ln. We know the boundary condition

$$D_2\cdot\hat{n} = D_1\cdot\hat{n}$$

Or, $D_1\cos\theta_2 = D_1\cos\theta_1$ ... (i)

And $E_{1\tau} = E_{2\tau}$

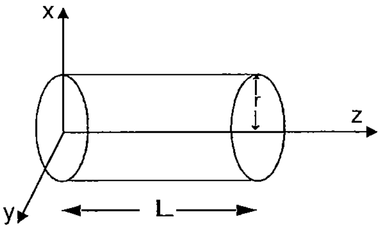
$\Rightarrow$ $E_2\sin\theta_2 = E_1\sin\theta_2$ ... (ii)

Therefore, (ii) divided by (i), we get

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\varepsilon_1}{\varepsilon_2}$$

Correct option is (a)

7. A cylindrical rod of length L and radius r, made of an inhomogeneous dielectric, is placed with its axis along the z direction with one end at the origin as shown below :



If the rod carries a polarization, $\vec{P}=\left(5z^2+7\right)\hat{k}$, the volume bound charge inside the dielectric is :

(a) Zero (b) $10\pi r^2L$ (c) $-5\pi r^2L$ (d) $-5\pi r^2L^2$

Soln. We have , $\vec{P}=\left(5z^2+7\right)\hat{k}$

Volume charge density,

$$\rho=-\vec{\nabla}\cdot\vec{p}=-\frac{\partial}{\partial z}\left(5z^2+7\right)=-10z$$

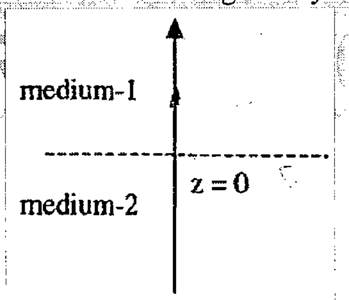
Therefore, the total volume charge inside the cylinder

$$Q_{total}=-10z\,2\pi r dr dz=-10\left[\frac{z^2}{2}\right]_0^L\frac{2\pi r^2}{2}=-5\pi r^2L^2$$

Correct option is (d)

8. Two infinitely extended homogeneous isotropic dielectric media (medium -1 and medium – 2 with dielectric constants $\frac{\varepsilon_1}{\varepsilon_0}=2$ and $\frac{\varepsilon_2}{\varepsilon_0}=5$, respectively) meet at the z = 0 plane as shown in the figure. A uniform electric field exists everywhere. For $z\geq 0$, the electric field is given by $\vec{E}_1=2\hat{i}-3\hat{j}+5\hat{k}$. The interface separating the two media is charge free.

The electric displacement vector in the medium-2 is given by



(a) $\overrightarrow{D_2}=\varepsilon_0\left[10\hat{i}+15\hat{j}+10\hat{k}\right]$ (b) $\overrightarrow{D_2}=\varepsilon_0\left[10\hat{i}-15\hat{j}+10\hat{k}\right]$

(c) $\overrightarrow{D_2}=\varepsilon_0\left[4\hat{i}-6\hat{j}+10\hat{k}\right]$ (d) $\overrightarrow{D_2}=\varepsilon_0\left[4\hat{i}+6\hat{j}+10\hat{k}\right]$

Soln. Since surface is charge free

$\therefore$ $\vec{D}_2\cdot\hat{n}=\vec{D}_1\cdot\hat{n}$

$\therefore$ $5\varepsilon_0E_{2n}=2\varepsilon_0 5$

$E_{2n}=2\hat{k}$

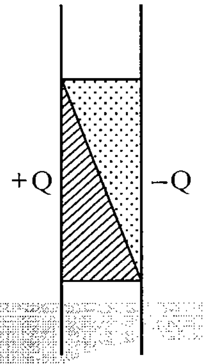
And $E_{2\tau}=E_{1\tau}$; $E_{2\tau}=\left(2\hat{i}-3\hat{j}\right)$

$\therefore \qquad \vec{E}_2 = \left(2\hat{i} - 3\hat{j} + 2\hat{k}\right)$

$\therefore \qquad \vec{D}_2 = \varepsilon_0 5\left(\vec{E}_2\right) = \varepsilon_0\left(10\hat{i} - 15\hat{j} + 10\hat{k}\right)$

Correct option is (b)

The space between two plates of a capacitor carrying charges +Q and –Q is filled with two different dielectric materials, as shown in the figure. Across the interface of the two dielectric materials, which one of the following statements is correct ?



(a) $\vec{E}$ and $\vec{D}$ are continuous

(b) $\vec{E}$ is continuous and $\vec{D}$ is discontinuous

(c) $\vec{D}$ is continuous and $\vec{E}$ is discontinuous

(d) $\vec{E}$ and $\vec{D}$ are discontinuous

ln. We know the boundary conditions,

$D_{2n} = D_{1n}$

$E_{2\tau} = E_{1\tau}$

$E_1 = E_{1n} + E_{1\tau} = \dfrac{\varepsilon_2}{\varepsilon_1} E_{2n} + E_{2\tau} \neq E_2 \quad (\because\ \varepsilon_1 \neq \varepsilon_2)$

$D_1 = D_{1n} + D_{1\tau} = D_{2n} + \varepsilon_0 E_{1\tau} + \varepsilon_0 \chi_1 E_{1\tau} \qquad \left(D_{1\tau} = \varepsilon_0 E_{1\tau} + \varepsilon_0 \chi_1 E_{1\tau}\ \text{for isotropic medium}\right)$

$\quad = D_{2n} + \varepsilon_0\left(1 + \chi_1\right) E_{2\tau} = D_{2n} + \varepsilon_1 E_{2\tau} \neq D_{2n} + D_{2\tau} \neq D_2$

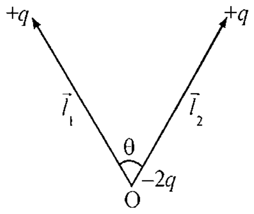
Therefore, D and E both are discontinuous.

Correct option is (d)

). Consider a system of three charges: $-2q$ charge at the origin, $+q$ at $\vec{l}_1$ and another charge $+q$ at $\vec{l}_2$. Find the dipole moment $\vec{p}$ of the system. What is the magnitude of $\vec{p}$ if $\left|\vec{l}_1\right| = \left|\vec{l}_2\right| = l$ and angle between $\vec{l}_1$ and $\vec{l}_2$ is θ.

ln. As the total charge is zero the dipole moment will be independent of the choice of origin. Now the dipole moment with respect to the origin is

$$\vec{p} = \int \vec{r}' dq = \vec{l}_1 \cdot (+q) + \vec{l}_2 \cdot (+q) + 0 \cdot (-2q) = q\left(\vec{l}_1 + \vec{l}_2\right).$$



Its magnitude

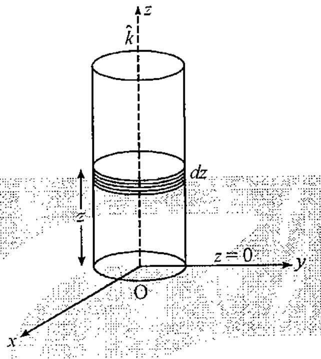
$$p = |\vec{p}| = q\sqrt{l_1^2 + l_2^2 + 2l_1 l_2 \cos\theta} = ql\sqrt{2(1+\cos\theta)}.$$

11. Consider a thin rod with its one end at $z=0$ and the other end at $z=l$. Its charge density is given by $\rho(z)=\alpha(z-l/2)$ for $0\le z\le l$. Area of cross-section of the rod is A. Find the total charge on the rod and dipole moment of the system.

Soln. Total charge on the rod is, $\int_0^l \rho(z)\cdot A dz = \alpha A\int_0^l\left(z-\frac{l}{2}\right)dz = 0.$

So the dipole moment is independent on the choice of the origin. Now the dipole moment with respect the origin O (Fig. 3.P-2) is

$$\vec p = \int \vec r' dq = \hat k\int_0^l z\cdot(\rho A dz) = \hat k\alpha A\int_0^l z\left(z-\frac{l}{2}\right)dz = \hat k\alpha A\cdot\frac{l^3}{12}.$$



12. Consider a circular disc of radius a in the xy-plane centered on the origin and containing a surface charge density $\sigma(r,\theta)=\sigma_0 r\cos\theta$, where θ is the angle made by the position vector with a fixed diameter chosen as x-axis. Find the dipole moment of the charge distribution.

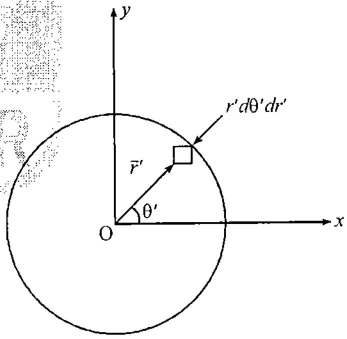
Soln. Considering a surface element $r'd\theta'\cdot dr'$ about the point $\vec r'$ we can write for the dipole moment about the origin as

$$\vec p = \int \vec r' dq = \int (x'\hat i + y'\hat j)\,\sigma\cdot r'd\theta' dr'$$

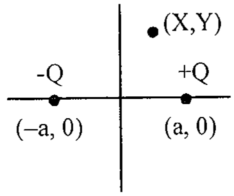
$$= \int (r'\cos\theta'\hat i + r'\sin\theta'\hat j)\,\sigma_0 r'\cos\theta'\cdot r'd\theta' dr'$$

$$= \hat i\sigma_0\int_0^a r'^3 dr'\int_0^{2\pi}\cos^2\theta' d\theta' + \hat j\sigma_0\int_0^a r'^3 dr'\int_0^{2\pi}\sin\theta'\cos\theta' d\theta'$$

$$= \hat i\sigma_0\frac{a^4}{4}\cdot\pi + 0 = \hat i\sigma_0\pi a^4/4.$$



13. Consider two charges +Q and –Q placed at the points (a, 0) and (–a, 0) in a plane, as shown in the figure on the right. If the origin is moved to the point (X, Y), the magnitude of the dipole moment of the given charge distribution with respect to this origin will be **[TIFR 2013]**



(a) $Q\sqrt{(a-X)^2+y^2}-Q\sqrt{(a+X)^2+y^2}$ (b) $Q(a-X)-Q(-a+X)$

(c) 2Qa (d) $2Q\sqrt{X^2+Y^2}$

ln. Total charge, $Q_T = -Q + Q = 0$

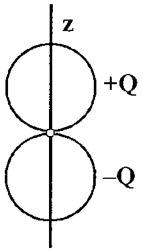
Since the total charge of the sytsem is zero.

Therefore, the dipole moment does not depends on origin.

So, the dipole moment with respect to the origin of old coordinate = $Q.2a$ = the dipole moment with respect to the origin of new coordinate.

Correct option is (c)

. An electric dipole is constructed by fixing two circular charged rings, each of radius 'a', with an insulating contact (see figure). One of these rings has total charge +Q and the other has total charge –Q. If the charge is distributed uniformly along each ring, the dipole moment about the point of contact will be



(a) $\frac{Qa}{\pi}\hat{z}$ (b) $4\pi Qa\hat{z}$

(c) $2Qa\,\hat{z}$ (d) Zero **[TIFR 2014]**

ln. Since charge is uniformly distributed over the surface of ring so we can treat them as a point charge at the center of teh ring.

The dipole moment with respect to the contact point is

$$P = Qa\hat{z} + (-a)(-Q)\hat{z} = 2aQ\hat{z}$$

. Near he point A lying on the boundary between glass and vacuum the electric field strength in vacuum is equalto $E_0 = 10\,V/m$ the angle between the vector E_0 and normal n of the boundary line being equal to $\alpha_0 = 30°$. Find the field strength E in glass near the point A. The angle α between the vector $\vec{E}$ and n, as well as the surface charge density.

ln. Boundary condtions,

(1) $E_{1\tau} = E_{2\tau}$

$\Rightarrow$ $E_{1\tau} = E_0 \sin\alpha_0$

(2) As there is no free charge at the boundary.

$\therefore$ $D_{1n} = D_{2n}$

$\Rightarrow$ $\varepsilon_1 E_{1n} = \varepsilon_0\, E_{2n}$

$\Rightarrow$ $\varepsilon_1 E_{1n} = \varepsilon_0\, E_0 \cos\alpha_0$

$$E_{1n} = \frac{\varepsilon_0 E_0 \cos\alpha_0}{\varepsilon}$$

So, $|E| = \sqrt{E_{1n}^2 + E_{1\tau}^2}$ and $\tan\alpha = \frac{E_{1\tau}}{E_{1n}} = \varepsilon\tan\alpha_0$

$\therefore$ $\alpha = \tan^{-1}(\varepsilon\tan\alpha_0)$

Now from the boundary condition for polarization vector P, we get

$P_{2n} - P_{1n} = -\sigma'$ (σ' is the surface bound charge density)

$\Rightarrow$ $0 - P_{1n} = -\sigma'$

$\Rightarrow$ $P_{1n} = \sigma'$

$$\Rightarrow \qquad \varepsilon_0(k-1)E_{1n}=\sigma' \quad \Rightarrow \varepsilon_0(k-1)\varepsilon_0\frac{E_0\cos\alpha_0}{\varepsilon}=\sigma'$$

$$\Rightarrow \qquad \sigma'=\varepsilon_0\frac{(k-1)}{k}E_0\cos\alpha_0$$

16. Near the plane surface of a uniform isotropic dielectric with permitivity ε the electric field strength in vacuum is equal to E_0, the vector E_0 forming an angle θ with the normal of the dielectrics surface. Assuming the field to be uniform both side. Find the flux of the vector E through a sphere of radius R with centre located at the surface of the dielectrics.

Soln. Boundary conditions.

(i) $E_{1\tau}=E_{2\tau}$

(ii) $D_{1n}=D_{2n}$ (if the surface is charge free)

(iii) $P_{2n}-P_{1n}=-\sigma'$ (σ' is bound surface charge density)

$$\therefore \qquad E_{1\tau}=E_0\sin\theta \quad \Rightarrow E_{1n}=\frac{\varepsilon_0 E_0\cos\theta}{\varepsilon}$$

$$0-P_{1n}=\sigma' \quad \Rightarrow \sigma'=\varepsilon_0(k-1)\frac{E_0\cos\theta}{k}$$

$$\therefore \qquad \oint \vec{E}\cdot d\vec{s}=\frac{Q}{\varepsilon_0}=\pi R^2\frac{\sigma'}{\varepsilon_0}=\pi R^2E_0\left(\frac{k-1}{k}\right)\cos\theta$$

$$\therefore \qquad \phi=\pi R^2E_0\frac{(k-1)}{k}\cos\theta$$

17. Under certain conditions the polarization of an infinite uncharged dielectric plate takes the form $\vec{P}=\vec{P}_0\left(1-\frac{x^2}{d^2}\right)$, where P_0 is a vector perpendicular to the plate, x is the distance from the middle of the plate, d is its half-thickness. Find the strength E of the electric field inside the plate and potential difference between its surface.

Soln. We know that, $\vec{\nabla}\cdot\vec{D}=\rho_f$

Since the medium is charge free

$$\vec{\nabla}\cdot\vec{D}=0 \qquad \Rightarrow \frac{dD_x}{dx}=0 \qquad \Rightarrow D_x=\text{constant}$$

But $x\to\infty$, $D_x=0$

$\therefore \quad D_x=0$ everyhere

$$\therefore \qquad \vec{D}=\varepsilon_0\vec{E}+\vec{P}$$

$$\Rightarrow \qquad \vec{E}=-\frac{\vec{P}}{\varepsilon_0}=-\frac{\vec{P}_0}{\varepsilon_0}\left(1-\frac{x^2}{d^2}\right)$$

$$\therefore \qquad \Delta V=\int dV=\int -E\cdot dx=\frac{P_0}{\varepsilon_0}\left[x-\frac{x^3}{3d^2}\right]_0^d=\frac{P_0}{\varepsilon_0}\left[d-\frac{d}{3}\right]=\frac{2P_0}{3\varepsilon_0}d$$

. Two identically charged spheres are suspended by strings of equal length. The strings make an angle of 30° with each other. When suspended in a liquid of density 0.8 gm/cm³ the angle remains the same. What is the dielectric constant of the liquid ? The density of the material of the sphere is 1.6 gm/cm³.

ln. The two situations are shown in figure (a) and (b) respectively.

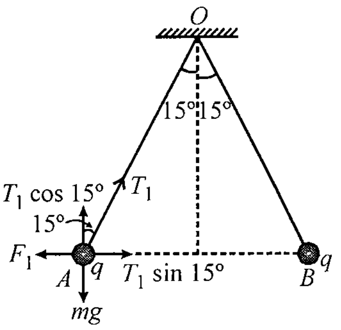


Figure : (a)

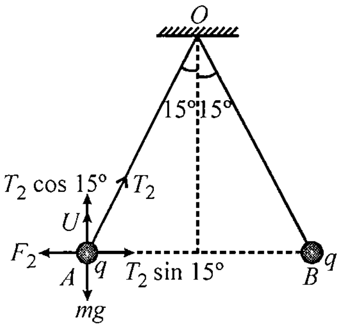


Figure : (b)

The force of repulsion, $F_1 = \dfrac{1}{4\pi\varepsilon_0}\cdot\dfrac{qq}{r^2}$

From figure (a), $T_1\sin 15° = F_1$ and $T_1\cos 15° = mg$

Therefore, $F_1 = mg\tan 15°$

Let V be the volume of each sphere, then $m = Vd$, where d is the density of the material of sphere

Therefore, $\dfrac{1}{4\pi\varepsilon_0}\dfrac{qq}{r^2} = V\,d\,g\tan 15°$ ... (1)

When the system is suspended in the liquid, then force of repulsion $F_2 = \dfrac{1}{4\pi\varepsilon_0 K}\cdot\dfrac{q^2}{r^2}$

From figure (b), $T_2\sin 15° = F_2$ and $mg = T_2\cos 15° + U$, where U = upthrust = $V\,d_1 g$ (d_1 = density of liquid).

So, $T_2\sin 15° = F_2$ and $T_2\cos 15° = mg - V\,d_1 g$

Therefore, $\tan 15° = \dfrac{F_2}{mg - V\,d_1\,g}$ Or $F_2 = (Vgd - V\,d_1 g)\tan 15°$

Or $\dfrac{1}{4\pi\varepsilon_0 K}\dfrac{q^2}{r^2} = (Vdg - V\,d_1 g)\tan 15°$ ... (2)

Dividing Eq. (1) by Eq. (2), we have

$$K = \frac{Vdg}{Vdg - Vd_1 g} = \frac{d}{d - d_1} \qquad \text{... (3)}$$

Substituting the values of d and d_1, we get $K = \dfrac{1.6}{1.6 - 0.8} = 2$.

PRACTICE SET

1. The electric field just outside a charged conductor having a surface density σ is:

(a) $\frac{\sigma}{\in_0}$ (b) $\frac{2\sigma}{\in_0}$ (c) $\in_0 \sigma$ (d) zero

2. Dipolar potential varies with distance r as

(a) $\frac{1}{r}$ (b) $\frac{1}{r^3}$ (c) $\frac{1}{r^2}$ (d) $\frac{1}{r^4}$

3. In a dielectric medium if $\vec{P}$ is the polarization vector and $\vec{E}$ is the electric field then in the relation $\vec{P} = \alpha \vec{E}$, α is in general a.

(a) scalar (b) tensor of rank 2 (c) vector (d) none of these

4. Gauss's law in a dielectric medium takes the form $\oint_S \vec{D} \cdot d\vec{S} = q$ where q is:

(a) total free charge enclosed (b) polarization charges
(c) total of both free and polarization charges (d) zero

5. Atomic polarizability α has the dimensions of

(a) $C.m^{-1}.V^{-1}$ (b) $C.m^{-2}$ (c) $C.m^2.V^{-1}$ (d) none of these

6. The interaction energy of an electric dipole of moment $\vec{p}$ in an external field $\vec{E}$ is

(a) $|\vec{p} \times \vec{E}|$ (b) $\vec{p} \cdot \vec{E}$ (c) $-|\vec{p} \times \vec{E}|$ (d) $-\vec{p} \cdot \vec{E}$

7. The net force on an electric dipole placed in a uniform electric field

(a) depends on the dipole moment (b) depends on the strength of electric field
(c) depends on the orientation of the dipole (d) is always zero

8. The bound volume charge density in a sphere of radius a carrying polarization $\vec{p} = k\vec{r}$ is

(a) kr (b) zero (c) k (d) $-3k$

9. A point charge q is held at a distance a/2 from the centre of a thin conducting shell of radius a. The potential of the shell is

(a) $\frac{1}{2\pi \in_0} \cdot \frac{q}{a}$ (b) $\frac{1}{8\pi \in_0} \cdot \frac{q}{a}$ (c) $\frac{1}{4\pi \in_0} \cdot \frac{q}{a}$ (d) zero

10. A medium in which the relations $\vec{\nabla} \cdot \vec{D} = \in \vec{\nabla} \cdot \vec{E}$ and $\vec{\nabla} \cdot \vec{J} = \sigma \vec{\nabla} \cdot \vec{E}$ are true, is said to be

(a) linear and inhomogeneous (b) linear and homogeneous
(c) linear and isotropic (d) homogeneous and isotropic

11. The electric field due to an electric quadrupole varies with distance r as

(a) $\frac{1}{r^2}$ (b) $\frac{1}{r^4}$ (c) $\frac{1}{r^3}$ (d) none of these

12. If the total charge of a system is zero then its dipole moment is

(a) zero (b) independent on the choice of origin
(c) dependent on the choice of origin (d) none of these

13. Four charges q, q, –2q and q are placed at the corners of a square of side a having coordinates (0, 0), (a, 0) (a, a) and (0, a), respectively. The electric dipole moment of the configuration about the origin is

(a) $qa(\hat{i} + \hat{j})$ (b) $qa(\hat{i} - \hat{j})$ (c) $-qa(\hat{i} + \hat{j})$ (d) $-qa(\hat{i} - \hat{j})$

. The electric field due to an electric dipole varies with distance r as

(a) $\frac{1}{r^2}$ (b) $\frac{1}{r^4}$ (c) $\frac{1}{r^3}$ (d) none of these

. For a spherically symmetric charge distribution quadrupole moment is

(a) always positive (b) always zero

(c) always negative (d) undefined

ANSWER KEY					
Questions	1	2	3	4	5
Option	(a)	(b)	(b)	(a)	(c)
Questions	6	7	8	9	10
Option	(d)	(d)	(d)	(a)	(b)
Questions	11	12	13	14	15
Option	(b)	(b)	(c)	(c)	(b)

Chapter 6

Conductors, Image Methods & Capacitors

Conductor:

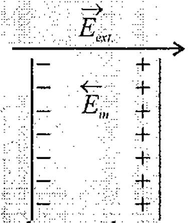
A conductor is a substance in which electrons can move freelly like Au, Ag, Cu

Electrostatic Properties of Conductors:

Electric field inside the conductor is zero:

Since a conductor have not any resistance, charge can move freelly inside the conductor. As a result when we applied external electric field the induce charge distributed such a way so that the total external field canceled out by the internal field.

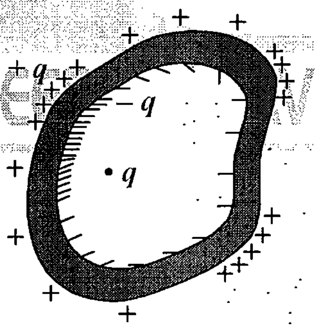
Charge density inside the conductors is zero.



We know that, $\vec{\nabla}\cdot\vec{E}=\dfrac{\rho}{\varepsilon_0}$

Since, $E_{in}=0$ Therefore, $\rho=0$

Due to this reason when a charge q is placed in the cavity of conductor then a charge $-q$ is induced on inner surface and a charge $+q$ is induced on outer surface of computer.



Free changes exist only on the surface of a conductor.

Conductor forms an equipotential region.

Let consider two point on the conductor surface A and B.

We assume that the potential at A and B are $V(A)$, $V(B)$ respectively.

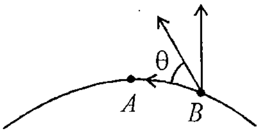
Therefore, the work done for remove a charge form A to B is given by

$$W=V(A)-V(B)=\int_B^A E\cdot d\ell=0 \qquad [\because\ E_{in}=0]$$

$$\Rightarrow \quad \boxed{V(A)=V(B)}$$

The electric field outside of a conductor is normal to the surface.

Let $\vec{E}$ field is not normal to the surface, then it has a tangential component $E\cos\theta$. Now, any charge bring from A to B.



Therefore, work done $W = E\cos\cdot\overline{AB}$

Since, surface is equipotential, work doen is zero.

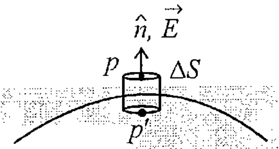
$$\vec{E}\cdot\overline{AB}\cos\theta = 0$$

$$\theta = \frac{\pi}{2} \quad \left(\vec{E} \neq 0 \neq \overline{AB}\right)$$

Therefore, the electric field is normal to the surface.

Electric field strength just outside a conductor:

Let σ be the surface change density on the surface and consider a cylindrical Gaussian surface whose cross-section is ΔS.



$$\therefore \qquad \vec{E}\cdot\hat{n}\Delta S = \frac{\sigma\Delta S}{\varepsilon_0}$$

$$\Rightarrow \qquad \boxed{\vec{E} = \frac{\sigma}{\varepsilon_0}\hat{n}}$$

Mechanical Pressure :

Electric field at any external point P just out side the conductor is given by

$$\vec{E} = \frac{\sigma}{\varepsilon_0}\hat{n}$$

This field may be considered as the sum of two fields.

$$\vec{E} = \vec{E}_1 + \vec{E}_2$$

Where E_1 is due to charges on a very small area ΔS around P.

E_2 is due to rest of the charges.

Let P' is internal point just opposite of P. The E_2 remain same but E_1 change such that total field become zero.

Therefore, $\quad 0 = -E_1 + E_2$

Or, $\quad \vec{E}_1 = \vec{E}_2 = \dfrac{\sigma}{2\varepsilon_0}\hat{n}$

The charge $\sigma\Delta S$ on ΔS cannot exert a force on itself but this charge $\sigma\Delta S$ feel a force due to $\vec{E}_2$.

Therefore, the force is $\vec{F} = \sigma\Delta S E_2 = \dfrac{\sigma^2\Delta S\hat{n}}{2\varepsilon_0}$

Therefore, pressure, $P = \dfrac{\vec{F}}{\Delta S\hat{n}} = \dfrac{\sigma^2}{2\varepsilon_0}$

$$\boxed{P = \frac{\sigma}{2\varepsilon_0}} \quad \Rightarrow \quad \boxed{P = \frac{1}{2}\varepsilon_0 E_0^2}$$

outwards normal to the surface

Example: Prove that the electrostatic potential in a charge-free space bounded by a conducting enclosure is constant and equal to the potential of the enclosure.

Soln. Any constant potential satisfies Laplace's equation. The constant potential equal to the potential of the enclosure satisfies Laplace's equation and also the boundary conditions of the first kind. Hence, by the first uniqueness theorem, this potential is correct and is the only possible potential for the system under consideration.

Example: Prove that if in an electrostatic system of charged conductors the charge of each conductor is increased n times, the potential (with respect to infinity) at any point of space will also be increased n times.

Soln. Suppose that the potential due to the original charges of the conductors is φ. Being the true potential compatible with the charges of the conductors, φ must satisfies Laplace's equation

$$\nabla^2\varphi = 0$$

and also the boundary conditions of the second kind, which for each conductor are

$$\varphi = \text{constant}, \; -\oint \varepsilon_0 \nabla\varphi \cdot dS = q$$

where, the integral is extended over the surface of the conductor and q is the charge of the conductor. Suppose now that all charges are increased n times. The new potential φ' must satisfy Laplace's equation

$$\nabla^2\varphi' = 0$$

and the new boundary conditions,

$$\varphi' = \text{constant}, \; -\oint \varepsilon_0 \nabla\varphi' \cdot dS = nq$$

But since φ satisfies the first three equations above, $\varphi' = n\varphi$ satisfies the last three. Hence, by the second uniqueness theorem, $\varphi' = n\varphi$ is the correct and the only possible associated with the new charges of the system.

Image Method:

There is a class of electrostatic problems with boundary condition that appear to be difficult to satisfy if the governing Laplace's equation is to solved directly but the condition on the bounding surfaces in these problems can be set up by appropriate image charge and then the potential distribution can be determined in straight forward manner.

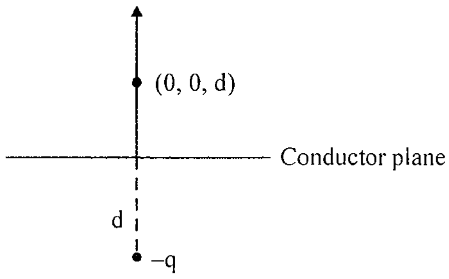
Point charge in front of an grounded conducting plane:

Suppose we have a large grounded conducting plane forming x-y plane and a positive charge placed on the z-axis at $z = d$. The potential will partly due to charge q and partly due to induce charge $-q$. In this process it obey the conditions.

(1) Potential $\phi = 0$ all over the conducting plane at $z = 0$

(2) ϕ must be zero at infinity

(3) ϕ satisfies Laplace's equation in the region $z > 0$ except at the point where the charge q is located.



From the above condition we can write the image charge −q located at $z = -d$.

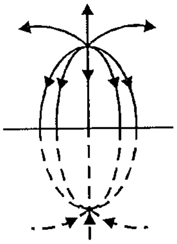
Potential and field at any pont P(x, y, z):

$$\phi(x,y,z)=\frac{1}{4\pi\varepsilon_0}\left(\frac{q}{\sqrt{x^2+y^2+(z-d)^2}}-\frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right) \quad \dots(1)$$

and the field

$$\vec{E}=-\nabla\phi=-\left[\frac{\partial\phi}{\partial x}\hat{i}+\frac{\partial\phi}{\partial y}\hat{j}+\frac{\partial\phi}{\partial z}\hat{k}\right] \quad \dots(2)$$

Line of force:



Force on +q:

$$\vec{F}=\frac{qq'}{4\pi\varepsilon_0(2d)^2}\left(-\hat{k}\right)=-\frac{q^2}{16\pi\varepsilon_0 d^2}\hat{k} \quad \dots(3)$$

Induce surface charge density:

$$\sigma=-\varepsilon_0\left.\frac{\partial\phi}{\partial z}\right|_{z=0}$$

$$=\frac{q}{4\pi}\left[\frac{(z-d)}{\left\{x^2+y^2+(z-d)^2\right\}^{1/2}}+\frac{(z+d)}{\left\{x^2+y^2+(z+d)^2\right\}^{3/2}}\right]_{z=0}$$

$$\therefore \quad \sigma(x,y)=-\frac{q}{2\pi}\frac{d}{\left(x^2+y^2+d^2\right)^{3/2}} \quad \dots(4)$$

Example : A point charge q of mass m is released rest at a distance d from an infinte grounded conducting plate . Find the time taken by the charge to hit plate

ln. When the charge q is at a distance x from the plate the force on it due to the conducting plate is given by the force between q and its image charge $-q$ located at $-x$.

So, the equation of motion is given by

$$m\cdot\frac{d^2x}{dt^2}=-\frac{1}{4\pi\epsilon_0}\cdot\frac{q^2}{(2x)^2}$$

Multiplying both sides by $2\frac{dx}{dt}$ and integrating we get

$$m\left(\frac{dx}{dt}\right)^2=-\frac{q^2}{8\pi\epsilon_0}\int\frac{dx}{x^2}=\frac{q^2}{8\pi\epsilon_0}\cdot\frac{1}{x}+c$$

where, c is a constant.

Now at $x=d,\frac{dx}{dt}=0$. Therefore, $c=-\frac{q^2}{8\pi\epsilon_0}\cdot\frac{1}{d}$ and

$$m\left(\frac{dx}{dt}\right)^2=\frac{q^2}{8\pi\in_0}\left(\frac{1}{x}-\frac{1}{d}\right) \text{ or } dt=-\sqrt{\frac{8\pi\in_0 m}{q^2}}\frac{dx}{\sqrt{\frac{1}{x}-\frac{1}{d}}}$$

Integrating we get the required time as

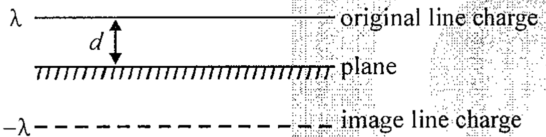
$$T=\int dt=-\sqrt{\frac{8\pi\in_0 m}{q^2}}\int_d^0\frac{dx}{\sqrt{\frac{1}{x}-\frac{1}{d}}}=\sqrt{\frac{8\pi\in_0 md}{q^2}}\int_0^d\frac{\sqrt{x}}{\sqrt{d-x}}dx$$

Substituting $x=d\sin^2\theta$ we find that $\int_0^d\frac{\sqrt{x}}{\sqrt{d-x}}dx=d\frac{\pi}{2}$

$$\therefore \qquad T=\sqrt{\frac{8\pi\in_0 md}{q^2}}\cdot\frac{d\pi}{2}=\frac{\pi}{q}\sqrt{2\pi\in_0 md^3}$$

Exaxmple : Consider uniform line charged λ is placed on an infinite straight wire, a distance d above a grounded conducting plane. The conducting plane is the xy plane. Find the potential in the region above the plane.

Soln.



The potential at P due to the original wire is

$$\phi_1(zy)=\frac{\lambda}{2\pi\varepsilon_0}\ell n\left(\frac{r_0}{r_1}\right)$$

due to image wire

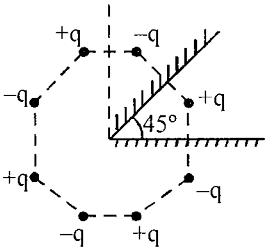
$$\phi_2(yz)=-\frac{\lambda}{2\pi\varepsilon_0}\ell n\left(\frac{r_0}{r_2}\right)$$

Here r_0 is the reference point, where potential is zero.
Therefore, the total potential at P is

$$\phi(y,z)=\frac{\lambda}{2\pi\varepsilon_0}\ell n\left(\frac{r_2}{r_1}\right)=\frac{\lambda}{2\pi\varepsilon_0}\ell n\left\{\frac{y^2+(z+d)^2}{y^2+(z-d)^2}\right\}$$

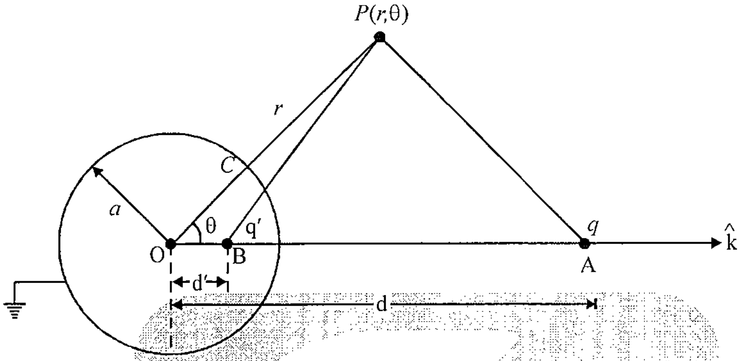
Example: If the angle between to conducting plane 45° then find number of images and draw their position.

Soln. $n=\left(\frac{360°}{45°}-1\right)=(8-1)=7$



A Point charge and a grounded conducting sphere:

Consider a point charge q placed at a distance d from the centre of grounded conducting sphere of radius a. The potential at any external point P will be partly due to q and partly due to induced charges on the sphere. To solve the problem by the method of image we are to replace the induced charges by suitable image charges. The symmetry of the problem suggests that the image charge must be placed symmetrically about the line of symmetry OA. Let us try a single image charge q' located within the sphere at some point B (with $OB = d'$) on the line OA. In this attempt the following conditions must be obeyed.



(i) Potential ϕ is zero all over the surface of the sphere.

(ii) Potential ϕ must be zero at infinity

(iii) Lplace's equation must be satisfied at all possible points excepting at A.

The choice of image charge q' inside the sphere does not affect the conditions, (ii) and (iii).

Now the potential at any point $C(a,\theta)$ on the surface of the sphere must be zero i.e.

$$\frac{1}{4\pi\in_0}\left[\frac{q}{AC}+\frac{q'}{BC}\right]=0 \qquad \dots(5)$$

Or, $$\frac{q}{\sqrt{a^2+d^2-2ad\cos\theta}}+\frac{q'}{\sqrt{a^2+d'^2-2ad'\cos\theta}}=0$$

Or, $$\frac{q/a}{\sqrt{1+\frac{d^2}{a^2}-2\frac{d}{a}\cos\theta}}=\frac{-q'/d'}{\sqrt{1+\frac{a^2}{d'^2}-2\frac{a}{d'}\cos\theta}}$$

To make this equality valid for all values of θ we may choose

$$\frac{q}{a}=-\frac{q'}{d'} \text{ and } \frac{d}{a}=\frac{a}{d'}$$

Therefore, $$\boxed{q'=-\frac{qa}{d}} \text{ and } \boxed{d'=\frac{a^2}{d}} \qquad \dots(6)$$

Potential and field at any point $P(r,\theta)$:

$$\phi(r,\theta)=\frac{q}{4\pi\in_0 AP}+\frac{1}{4\pi\in_0}\frac{q'}{BP}$$

$$\Rightarrow \qquad \phi(r,\theta)=\frac{1}{4\pi\in_0}\left[\frac{q}{\sqrt{r^2+d^2-2rd\cos\theta}}+\frac{q'}{\sqrt{r^2+d'^2-2rd'\cos\theta}}\right] \qquad \dots(7)$$

where q' and d' are given by

Corresponding field is given by

$$\vec{E} = -\vec{\nabla}\phi = -\left[\hat{r}\frac{\partial \phi}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right] \qquad \ldots (8)$$

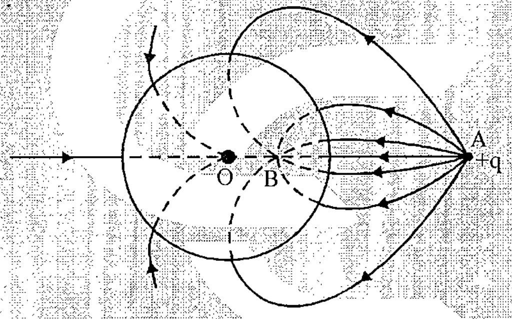
Therefore, the components of $\vec{E}$ are :

$$E_r = -\frac{\partial \phi}{\partial r} = \frac{1}{4\pi \in_0}\left[\frac{q\left(r - d\cos\theta\right)}{\left(r^2 + d^2 - 2rd\cos\theta\right)^{3/2}} + \frac{q'\left(r - d'\cos\theta\right)}{\left(r^2 + d'^2 - 2rd'\cos\theta\right)^{3/2}}\right] \qquad \ldots (9)$$

$$E_\theta = -\frac{1}{r}\frac{\partial \phi}{\partial \theta} = \frac{1}{4\pi \in_0}\left[\frac{qd\sin\theta}{\left(r^2 + d^2 - 2rd\cos\theta\right)^{3/2}} + \frac{q'd'\sin\theta}{\left(r^2 + d'^2 - 2rd'\cos\theta\right)^{3/2}}\right] \qquad \ldots (10)$$

Lines of force:

Lines of force will be the same as that between q at A and $q' = -qa/d$ at B. The lines meet the conducting sphere normally. The lines of force within the sphere are shown by dotted lines because field inside the sphere is zero and they do not exist really.



Force on +q:

Force between +q at A and the sphere will be given by the force between +q at A and the image charge q' placed at B. Therefore, force on +q due to the sphere would be

$$\vec{F} = \frac{1}{4\pi \in_0}\cdot\frac{qq'}{(BA)^2}\hat{k} \quad \text{where } \hat{k} \text{ is a unit vector along OA}$$

$$= -\frac{1}{4\pi \in_0}\cdot\frac{q^2 a/d}{\left(d - a^2/d\right)^2}\hat{k} \quad \left[\because\ BA = OA - OB = d - a^2/d\right]$$

$$\Rightarrow \quad \boxed{\vec{F} = -\frac{q^2 ad}{4\pi \in_0\left(d^2 - a^2\right)^2}\hat{k}} \qquad \ldots (11)$$

The force is attractive in nature.

If the charge q is very close to the sphere, i.e., $d = a + \delta\left(\delta << a\right)$ then

$$\boxed{\vec{F} \to -\frac{1}{4\pi \in_0}\cdot\frac{q^2}{(2\delta)^2}\hat{k}} \qquad \ldots (12)$$

This result is similar to the case of point charge placed in front of a conducting plane.

Induced charges:

The surface density of induced charges at any point $C(a,\theta)$ on the sphere is given by

$$\sigma(\theta) = -\epsilon_0 \left.\frac{\partial\phi}{\partial r}\right|_{r=a} = -\frac{q}{4\pi a}\cdot\frac{d^2-a^2}{\left(a^2+d^2-2ad\cos\theta\right)^{3/2}} \qquad \ldots(13)$$

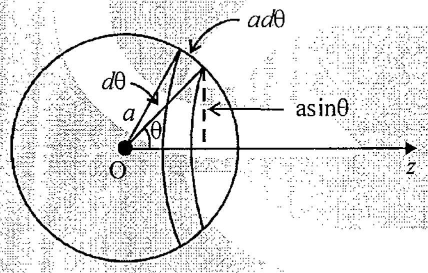
As expected the induced charge density is negative. Note that $\left(a^2+d^2-2ad\cos\theta\right)^{3/2} = AC^3$. Thus, we can say that charge density at a point is inversely proportional to the cube of the distane of the point from q. The charge density is thus maximum at $\theta = 0°$ and minimum at $\theta = 180°$.

$$\boxed{\sigma_{\max} = -\frac{q}{4\pi a}\cdot\frac{d+a}{(d-a)^2}} \text{ and } \boxed{\sigma_{\min} = -\frac{q}{4\pi a}\frac{d-a}{(d+a)^2}} \qquad \ldots(14)$$

Note that σ cannot be zero at any point on the sphere.

Total induced charge on the sphere can be obtained by integrating the charge density over the surface of the sphere. Thus, total induced charge

$$Q = \int_0^{\pi} \sigma(\theta)\cdot 2\pi a\sin\theta\cdot a d\theta = -\frac{q\left(d^2-a^2\right)a}{2}\int_0^{\pi}\frac{\sin\theta d\theta}{\left(a^2+d^2-2ad\cos\theta\right)^{3/2}}$$



Putting $a^2+d^2-2ad\cos\theta = p^2$ we get

$$Q = -\frac{q\left(d^2-a^2\right)}{2d}\int_{d-a}^{d+a}\frac{pdp}{p^3} = \frac{q\left(d^2-a^2\right)}{2d}\left[\frac{1}{p}\right]_{d-a}^{d+a} = -\frac{qa}{d}$$

which is, as expected, equal to the image charge.

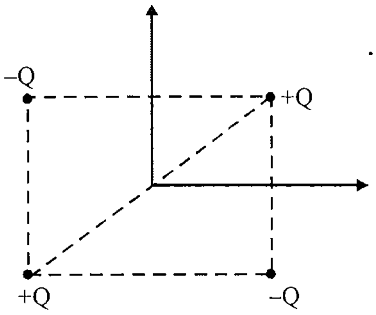
Number of Image:

Suppose a point charge q located at front of two semi infinite coductor plane and the angle between two plane is θ then the number of image charge will be

$$\boxed{n = \left(\frac{360}{\theta}-1\right)} \qquad \ldots(15)$$

Example : for $\theta = 90°$ and $n = 3$

$$n = \left(\frac{360}{90}-1\right) = (4-1) = 3$$



Capacitors:

A capacitor is a device which stores electric charge. Capacitors vary in shape and size, but the basic configurtion is two conductors carrying equal but opposite charges. Capacitors have many important applications in electronics. Some examples are filtering out unwanted frequency signals, forming resonant circuits and making frequency-dependent and independent voltage divides when combined with resistors.

In uncharged state, the charge on the both conductor in the capacitor is zero. During the changing process a charge Q is moved from one conductor to the other one giving one conductor a charge +Q and the other one a charge –Q. A potential difference ΔV is created, with postively charge conductor at higher potential then the negatively charged conductor. Note that wheather charged and unchanced, the net charge on the capacitor as a whole is zero.

Let the amount of charged Q stored in the capacitor and it produce voltage difference ΔV. Thus we may write,

$$Q \propto \Delta V$$

Or, $$Q = C|\Delta V| \Rightarrow \boxed{C = \frac{Q}{|\Delta V|}} \qquad \dots (1)$$

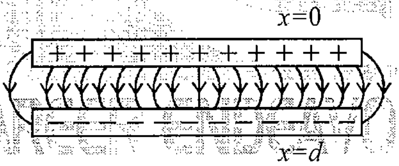
where, C is a positive constant called capacitance.
The S.I. unit of capacitor is the Farad (F).
1F = 1 farad = 1 coulomb/volt

$$1\text{F} = 10^{+12}\,\text{PF} \quad 1\text{F} = 10^{6}\,\mu\text{F}$$

Calculation of Capacitance:

Parallel-plate capacitor:

Consider two metallic plates of equal area A separated by a distance d, as shown in figure. The top plate carries a charge +Q. While the bottom plate carries a charge –Q.



From Gaussian law we can say that the field exist only inside the plate and it is given by

$$\vec{E} = \frac{\sigma}{\varepsilon_0}$$

Therefore, the potential difference between two plate is

$$V = \int_d^0 E \cdot d\ell = +\int_0^d E \cdot d\ell = \frac{\sigma}{\varepsilon_0} d$$

Therefore, the capacitance,

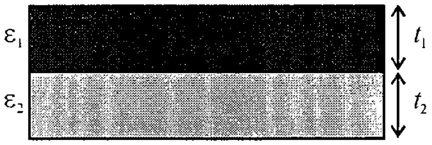
$$C = \frac{Q}{V} = \frac{\sigma A}{V} = \frac{\sigma A \varepsilon_0}{\sigma d} = \frac{A\varepsilon_0}{d}$$

Or, $$\boxed{C = \frac{A\varepsilon_0}{d}} \qquad \dots (2)$$

Special Cases:

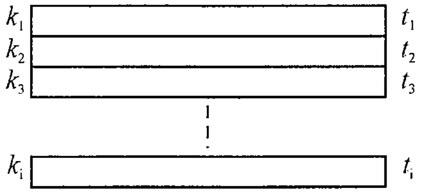
Capacitor with a composite dielectric medium:

Let a capacitor contain two linear dielectric of thickness t_1 and t_2 and permitivities ε_1 and ε_2 respectively. The field inside the two medium is



$$E_1 = \frac{\sigma}{\varepsilon_1} \text{ and } E_2 = \frac{\sigma}{\varepsilon_2}$$

Therefore, potential difference between the plates would be



$$V = -\int_{t_1+t_2}^{0} E \cdot dx = -\int_{t_1+t_2}^{t_1} E_2 dx - \int_{t_1}^{0} E_1 dx$$

$$= \frac{\sigma}{\varepsilon_2}\left[t_1 + t_2 - t_1\right] + \frac{\sigma}{\varepsilon_1} t_1 = \frac{\sigma}{\varepsilon_2} t_2 + \frac{\sigma}{\varepsilon_1} t_1$$

$$\therefore \quad C = \frac{Q}{V} = \frac{\sigma A}{\sigma\left(\frac{t_1}{\varepsilon_1} + \frac{t_2}{\varepsilon_2}\right)} \Rightarrow \boxed{C = \frac{\varepsilon_0 A}{\frac{t_1}{k_1} + \frac{t_2}{k_2}}}$$

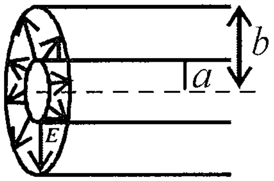
Therefore, the capacitor with several dielectric medium.

$$\boxed{C = \frac{\varepsilon_0 \cdot A}{\sum_i \frac{t_i}{k_i}}} \quad \ldots(3)$$

Cylindrical capacitors:

Consider next a solid cylinderical conductor of radious a surrounded by a coaxial cylindrical shell of inner radius b as shown in figure. The length of both cylinders so that edge effects can be neglected. The capacitor is charged so that the inner cylinder has charge +Q. While the out shell has a charge –Q.

According to Gaussian law,



$$\oint E \cdot dA = \frac{\lambda l}{\varepsilon_0}$$

$$\Rightarrow \quad E \cdot 2\pi r \ell = \frac{\lambda \ell}{\varepsilon_0}$$

$$\Rightarrow \quad \vec{E} = \frac{\lambda}{2\pi r \varepsilon_0}\hat{r} \quad a < r < b$$

$$= 0 \qquad r > b, r < a$$

Therefore, the potential difference,

$$V = -\int_b^a E.dr = \frac{\lambda}{2\pi\varepsilon_0}\int_a^b \frac{dr}{r} = \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{b}{a}\right)$$

Therefore, the capacitance, $c = \frac{s}{V} = \frac{\ell \times 2\pi\varepsilon_0}{\lambda \ln(b/a)} = \frac{2\pi\varepsilon_0 \ell}{\ln\left(\frac{b}{a}\right)}$

$$\boxed{c=\frac{2\pi\varepsilon_0 L}{\ln(b/a)}} \qquad \ldots(5)$$

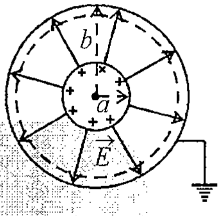
Therefore, capacitor per unit length

$$\boxed{c=\frac{2\pi\varepsilon_0}{\ln(b/a)}} \qquad \ldots(6)$$

(iii) Spherical capacitor :

Let consider a spherical capacitor consist of pair of concentric metal spheres, the space between them being filled with a dielectric and a and b the radius of smaller and larger sphere, the charge of outer is sphere is –Q and inner sphere + Q.

Therefore, the field,



$$\oint E\cdot dS=\frac{Q}{\varepsilon_0}$$

$$\Rightarrow \quad E.4\pi r^2=\frac{Q}{\varepsilon_0}$$

$$\Rightarrow \quad \begin{aligned}\vec{E}&=\frac{1}{4\pi\varepsilon}\frac{Q}{r^2} \qquad a<r<b\\ &=0 \qquad\qquad r>b, r<a\end{aligned}$$

Therefore, the potential difference is

$$V=\int_b^a E\cdot dn=\frac{1}{4\pi\varepsilon_0}Q\left(\frac{1}{a}-\frac{1}{b}\right)$$

Therefore, the capacitor

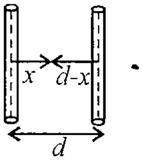
$$c=\frac{Q}{V}$$

$$\boxed{C=\frac{4\pi\varepsilon_0 ab}{(b-a)}}$$

(iv) Capacitor between two long thin parallel wires in air:

Let two wire carries charge $+\lambda$ and $-\lambda$ per unit length and their centre to centre distance d. Let the field at any point (P) inside them is

$$E=\left[\frac{\lambda}{2\pi\varepsilon_0 x}+\frac{\lambda}{2\pi\varepsilon_0(d-x)}\right]\hat{x}$$



$$\therefore \quad V=-\int_{d-a}^{a} E\cdot dx=\frac{\lambda}{2\pi\varepsilon_0}\left[\ell nx-\ell n(d-x)\right]_a^{d-a}$$

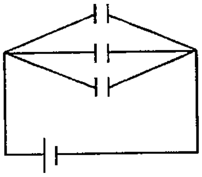
$$=\frac{\lambda}{\pi\varepsilon_0}\ell n\left(\frac{d-a}{a}\right)$$

Therefore, the capacitance, $C=\dfrac{Q}{V}=\dfrac{\lambda L}{\dfrac{\lambda}{\pi\varepsilon_0}\ell n\left(\dfrac{d-a}{a}\right)}$

$$\boxed{C = \frac{L\pi\varepsilon_0}{\ell n\left(\dfrac{d-a}{a}\right)}} \qquad \text{.... (7-b)}$$

Combination of capacitor:

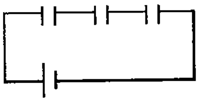
Parallel combination:



$$C_T = c_1 + c_2 + c_3 \ldots\ldots\ldots$$

$$\boxed{C_T = \sum_i c_i} \qquad \text{... (8)}$$

Series combination:



$$\frac{1}{C_T} = \frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3}$$

$$\Rightarrow \qquad \boxed{\frac{1}{C_T} = \sum_i \frac{1}{c_i}} \qquad \text{... (9)}$$

Energy stored in a charged capacitor :

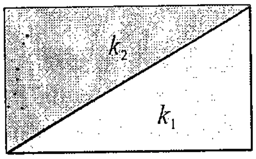
Suppose at any intermediate stage during the capacitor charging the charge on the positive conductor is q and potential difference across the capacitor is $\phi = \dfrac{q}{C}$ work done to increase the charge by an amount dq is

$$dW = \phi dq = \frac{q}{C}dq \Rightarrow W = \int_0^Q \frac{q}{C}dq = \frac{Q^2}{2C} \Rightarrow \boxed{W = \frac{Q^2}{2C}}$$

Also, $Q = CV$

$$\therefore \qquad \boxed{W = \frac{1}{2}CV^2}$$

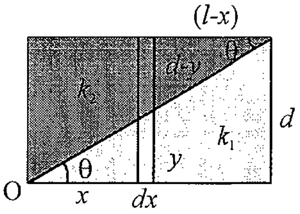
Example : Find the capacitance of a parallel plate capacitor whose upper traingle and lower triangle have dielectric constant k_1 and k_2 respectively as shown in figure.



ln. Let consider element at x distance from left side.
Therefore, the capacitance of the element is

$$dC = \frac{\varepsilon_0 b dx}{\left(\dfrac{y}{k_1} + \dfrac{y-d}{k_2}\right)}$$

$$\therefore \qquad C = \int dC$$



$$= \varepsilon_0 b \int_0^{\ell} \frac{dx}{\frac{x\tan\theta}{k_1}+\frac{(\ell-x)\tan\theta}{k_2}} = \frac{\varepsilon_0 b}{\tan\theta}\int_0^{\ell}\frac{dx}{\frac{x}{k_1}+\frac{(\ell-x)}{k_2}}$$

$$= \frac{\varepsilon_0 b k_1 k_2}{\tan\theta\,(k_2-k_1)}\left[\ell n\left(k_2-k_1\right)x+\ell k_1\right]_0^{\ell}$$

$$= \frac{\varepsilon_0 b\ell}{d}\left(\frac{k_1k_2}{k_2-k_1}\right)\ell n\left(\frac{k_2\ell}{\ell k_1}\right) = \frac{\varepsilon_0 A}{d}\frac{k_1k_2}{k_2-k_1}\,\ell n\frac{k_2}{k_1}$$

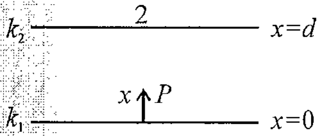
Example: A parallel plate capacitor consist of two plates of area A and plates separated by distance d. If space between the plate is filled with a non-uniform dielectric whose dielectric constant varies linearly from value k_1 at one plate to the value k_2 at other plate. Then find the capacitance of capacitor.

Soln. Let the dielectric constant at P at a distance x from the plate 1 be k and α.

$$\frac{dk}{dx}=\alpha$$

$$dk=\alpha dx$$

$$k=\alpha x+\beta$$



Now, at $x=0$, $k=k_1$ and at $x=d$, $k=k_2$

$$\therefore \qquad k=\frac{(k_2-k_1)x}{d}+k_1$$

Now the the field at P is

$$E=\frac{\sigma}{\varepsilon}=\frac{\sigma}{\varepsilon_0 k}=\frac{\sigma}{\varepsilon_0\left[\frac{(k_2-k_1)x}{d}+k_1\right]}$$

$$\therefore \qquad V=-\int_d^0 E dx=\frac{\sigma}{\varepsilon_0}\int_0^d\frac{dx}{\frac{(k_2-k_1)x}{d}+k_1}$$

$$=\frac{\sigma d}{\varepsilon_0(k_2-k_1)}\ln\left(\frac{k_2-k_1}{d}x+k_1\right)\Bigg|_0^d$$

$$=\frac{\sigma d}{\varepsilon_0(k_2-k_1)}\ln\left(\frac{k_2-k_1+k_1}{k_1}\right)=\frac{Qd}{\varepsilon_0 A(k_2-k_1)}\ln\left(\frac{k_2}{k_1}\right)$$

$$\therefore \qquad \boxed{c=\frac{Q}{V}=\frac{\varepsilon_0 A}{d}\frac{(k_2-k_1)}{\ln(k_2/k_1)}} \qquad \ldots(4)$$

Example: Two capacitors of capacitances C_1 and C_2 carrying charges q_1 and q_2, respectively, are connected in parallel. A spark appears when the connection is made. Find the energy dissipated by this spark if no other energy dissipation is taking place.

Soln. The total electrostatic energy of the two capacitors before connection is,

$$U_{before}=\frac{1}{2}\frac{q_1^2}{C_1}+\frac{1}{2}\frac{q_2^2}{C_2}$$

hence the total capacitance of the two capacitors connected in parallel is equal to to the sum of the individual capacitances and since the total charge are connection is equal to the sum of the original charges, the electrostatic energy after connection is

$$U_{after}=\frac{1}{2}\frac{(q_1+q_2)^2}{C_1+C_2}$$

the principle of the conservation of energy, the energy lost in the spark is then

$$U_{spark}=U_b-U_a=\frac{1}{2}\frac{q_1^2}{C_1}+\frac{1}{2}\frac{q_2^2}{C_2}-\frac{1}{2}\frac{(q_1+q_2)^2}{C_1+C_2}$$

which after simplification becomes,

$$U_{spark}=\frac{1}{2}\frac{(C_1q_2-C_2q_1)^2}{2C_1C_2(C_1+C_2)}$$

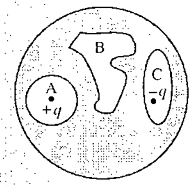
ADDITIONAL SOLVED PROBLEMS

A solid spherical conductor encloses 3 cavities, a cross-section of which are as shown in the figure. A net charge $+q$ resides on the outer surface of the conductor. Cavities A and C contain point charges $+q$ and $-q$, respectively. **[TIFR 2015]**

The net charges on the surfaces of these cavities are

(a) $A=-q, B=q, C=0$ (b) $A=-q, B=0, C=-q$

(c) $A=+q, B=0, C=-q$ (d) $A=-q, B=0, C=+q$



ln. Due to external charge in the cavity the induce charge on the surface is opposite and equal magnitude of external charge.

Therefore, the charge on the surface of the cavity are –q, 0, +q respectively on A, B and C cavity.

The conductors in a 0.75 km long two-wire transmission line are separated by a centre-to-centre distance of 0.2m. If each conductor has a diameter of 4cm, then the capacitance of the line is **[JEST 2013]**

(a) 8.85 μF (b) 88.5 nF (c) 8.85 pF (d) 8.85 nF

ɔln. We capacitance of two line wire is

$$C=\frac{L\pi\varepsilon_0}{\ell n\left(\frac{d-a}{a}\right)}=\frac{0.75\times10^3\times3.14\times8.85\times10^{-12}}{\ell n\left(\frac{0.2-0.02}{0.02}\right)}=\frac{20.84\times10^{-9}}{\ell n9}$$

$$=9.48\times10^{-9}F=9.48nF$$

Correct option is (d)

A coaxial cable of uniform cross-section contains an insulating material of dielectric constant 3.5. The radius of the central wire is 0.01 m and that of the sheath is 0.02 m. The capacitance per kilometer of a cable is :

(a) 280.5 nF (b) 28.05 nF (c) 56.10nF (d) 2.805 nF

oln. Capacitance of coaxial cable is per unit length

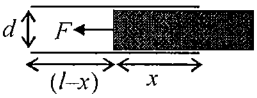
$$C=\frac{2\pi\varepsilon}{\ell n(b/a)}=\frac{2\pi3.5\times8.85\times10^{-12}}{\ell n\left(\frac{0.02}{0.01}\right)}=280.63\times10^{-12}F/m=280.63\,nF/km$$

Correct option is (a)

4. Consider a parallel plate capacitor with plate separation d and each plate having a length l and width a. A dielectric slab of permittivity $\in$, thickness d, length l and width a is partially inserted between the plates. If a constant pd V is maintained between the plates, find the force exerted on the dielectric slab when a length x of the slab remains between the plates.

Soln. The system may be considered as the parallel combination of two capacitors C_1 and C_2, where,

$$C_1 = \frac{\in ax}{d} \text{ and } C_2 = \frac{\in_0 a(\ell - x)}{d}$$



The total capacitance is then

$$C = C_1 + C_2 = \frac{a}{d}\left[(\in - \in_0)x + \in_0 \ell\right]$$

The electrostatic energy of the system is

$$U = \frac{1}{2}CV^2 = \frac{a}{2d}\left[(\in - \in_0)x + \in_0 \ell\right]V^2$$

As V is constant the force experienced by the slab will be

$$\vec{F} = \left.\frac{\partial U}{\partial x}\right|_V \hat{x} = \frac{a}{2d}(\in - \in_0)V^2\hat{x}$$

where $\hat{x}$ is a unit vector in the direction of increasing x. This means that the force is attractive.

5. A parallel plate air capacitor has the plates of area A each.
(a) assuming that the charge Q on it is kept constant find the work done against the electrical forces to increase the plate separation from d_1 to d_2.
(b) Assuming that the pd V across it is maintained constant find the change in energy of the system.

Soln. The energy of the system is

$$U = \frac{Q^2}{2C} = \frac{Q^2 d}{2\in_0 d} \text{ or } U = \frac{1}{2}CV^2 = \frac{\in_0 AV^2}{2d}$$

(a) When the charge is kept constant the work done against the electrical forces is completely spent for increasing the energy of the system. Thus, work done in this case, would be

$$W = \Delta U = \frac{Q^2}{2\in_0 A}(d_2 - d_1)$$

(b) When the pd is maintained constant the change in energy of the system would be

$$\Delta U = \frac{\in_0 AV^2}{2}\left(\frac{1}{d_2} - \frac{1}{d_1}\right)$$

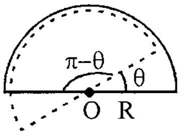
This is negative. To understand it we must consider the work done againstg the electrical forces as well as the energy supplied by the source, which maintained the pd constant.

6. Consider a capacitor consisting of two fixed semicircular plates of radius R separated by a distance d. A semicircular dielectric slab of dielectric constant K almost fills up the space between the plates. The slab is free to rotate about an axis normal to the plates and passing throug the centre. A constant pd V is maintained between the plates. If the slab is now rotated through an angle θ show that the torque tending to pull the slab back inside the capacitor would be $\dfrac{(K-1)\in_0 R^2V^2}{4d}$

Soln. The system may be considered as the parallel combination of two capacitors C_1 and C_2, where C_1 and C_2 are the capacitances of the parts with and without the dielectric. Thus, capacitance

$$C = C_1 + C_2 = \frac{\in \cdot A_1}{d} + \frac{\in_0 \cdot A_2}{d} = \frac{\in}{d}\cdot\frac{1}{2}R^2(\pi-\theta) + \frac{\in_0}{d}\cdot\frac{1}{2}R^2\theta$$

The electrostatic energy of the system is



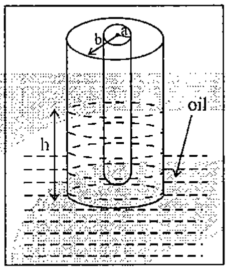
$$U = \frac{1}{2}CV^2$$

As the pd V is constant the torque acting on the system is

$$\tau = \frac{\partial U}{\partial \theta} = \frac{V^2}{2}\cdot\frac{\partial C}{\partial \theta} = \frac{V^2}{2}\cdot\frac{R^2}{2d}(\in_0 - \in) = -(K-1)\in_0 R^2V^2/4d$$

The negative sign indicates that the torque tries to decrease the angle θ.

Two coaxial cylindrical metal tubes (inner radius 'a' and outer radius 'b') stand vertically in a tank of dielectric oil. The succeptibility of the oil is X_e and mass density is ρ. The inner tube is maintanined at potential V, and the outer one is grounded. To what height (h) does the oil rise in the space between the tubes?



ln. Let any time t the liquid rise up certain height x.

The capacitance of empty region

$$c_1 = \frac{2\pi\varepsilon_0(L-x)}{\ell n(b/a)}$$

The capacitance of filled with oil region

$$c_2 = \frac{2\pi\varepsilon_0(1+\chi)x}{\ell n(b/a)}$$

Since these two capacitors connected with same potential i.e. they are in parallel

Therefore, the total capacitance,

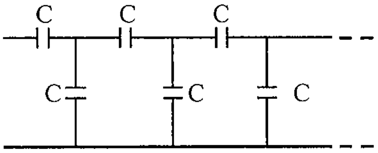
$$c = c_1 + c_2 = \frac{2\pi\varepsilon_0}{\ell n(b/a)}\{(L-x)+(1+x)x\}$$

$$\therefore \quad F = -\frac{dU}{dx} = -\frac{1}{2}V^2\frac{dc}{dx} = \frac{1}{2}V^2\frac{2\pi\varepsilon_0}{\ell n(b/a)}(1-\chi)$$

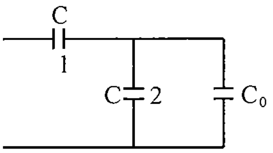
$$\therefore \quad \frac{1}{2}V^2\frac{2\pi\varepsilon_0}{\ell n(b/a)}(1-\chi) = \pi\left(b^2-a^2\right)\rho g h$$

$$\Rightarrow \quad h = \frac{V^2\varepsilon_0(1-\chi)}{\ell n(b/a)\left(b^2-a^2\right)\rho g}$$

Find the capacitance of an infinite circuit formed by the repetation of same link consisting of two indentical capacitor, each with same capacitor.



Soln. The infinite circuit may be reduced to a circuit given below



where C_0 is the net capacitance of the combination. If C' be the equivalent capacitance of C_0 and capacitor 2 of capacity C then

$$C' = C + C_0 \quad \text{(since both inparallel)}$$

Now, C' and capacitor 1 of capacitance C are in series. So,

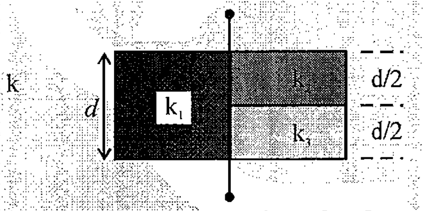
$$\frac{1}{C_0} = \frac{1}{C} + \frac{1}{C + C_0}$$

$$\Rightarrow \qquad \frac{1}{C_0} - \frac{1}{C + C_0} = \frac{1}{C} \quad \Rightarrow C_0^2 + CC_0 - C^2 = 0$$

$$\therefore \qquad C_0 = \frac{-C \pm \sqrt{C^2 + 4C^2}}{2}$$

Taking positive value, $C_0 = \dfrac{-C + \sqrt{5C^2}}{2} \Rightarrow C_0 = \left(\dfrac{\sqrt{5} - 1}{2}\right)C$

9. Parallel plate capacitor is constructed using three different dielectric material as shown in figure. The parallel plates, across which a potential difference is applied are of area A cm² and separated by a distance d cm. Find the capacitance across AB.



Soln. Three region can be treated as three different capacitor C_1, C_2 and C_3.

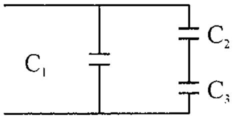
We know that the capacitance of parallel plate capacitor, $C = \dfrac{k\varepsilon_0 A}{d}$

$$\therefore \qquad C_1 = \frac{k_1\varepsilon_0 A/2}{d} = \frac{k_1\varepsilon_0 A}{2d}$$

$$C_2 = \frac{k_2\varepsilon_0 A/2}{d/2} = \frac{k_2\varepsilon_0 A}{d}$$

$$C_3 = \frac{k_3\varepsilon_0 A/2}{d/2} = \frac{k_3\varepsilon_0 A}{d}$$

The equivalent capacitor circuit.



Therefore, the equivalent capacitance.

$$C = C_1 + \frac{C_2 C_3}{C_2 + C_3} = \frac{k_1\varepsilon_0 A}{2d} + \frac{k_1 k_2 \varepsilon_0^2 \; A^2/d^2}{\dfrac{\varepsilon_0 A}{d}(k_1 + k_2)}$$

$$= \frac{\varepsilon_0 A}{d}\left[\frac{k_1}{2} + \frac{k_1 k_2}{k_1 + k_2}\right]$$

. A capacitor of capacitance $C_1 = 1.0\,\mu\text{F}$ withstands the minimum voltage $V_1 = 6.0$ kV while the capacitor of capacitance $C_2 = 2.0\,\mu\text{F}$, the maximum voltage $V_2 = 4.0$ kV. What voltage will the system of these two capacitors withstand if they are connected in series ?

ln. When the two capacitors are connected in series, their equivalent capacitance C is given by

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{1} + \frac{1}{2} = \frac{3}{2} \textbf{ or } C = \frac{2}{3}\mu\text{F} = \frac{2}{3} \times 10^{-6}\text{ F}$$

Now $q_1 = C_1 V_1 = (1 \times 10^{-6}) \times 6000 = 6 \times 10^{-3}$ C

$$q_2 = C_2 V_2 = (2 \times 10^{-6}) \times 4000 = 8 \times 10^{-3}\text{ C}$$

The charge on equivalent capacitor must be less than 6×10^{-3} C because above this charge, the capacitor will break down.

Hence, $V = \dfrac{q}{C} = \dfrac{6 \times 10^{-3}}{(2/3) \times 10^{-6}} = 9 \times 10^{3}\text{ V} = 9\text{kV}$

. A battery of 10 V is connected to a capacitor of capacity 0.1 F. The battery is now removed and this capacitor is connected to a second uncharged capacitor. If the charge distributes equally on these two capacitors, find the total energy stored in the two capacitors. Further, compare this energy with the initial energy stored in the first capacitor.

ln. The initial energy stored in the first capacitor

$$U_0 = \frac{1}{2}CV^2 = \frac{1}{2} \times 0.1 \times (10)^2 = 5.0\text{ J}$$

When this capacitor is connected to the second uncharged capacitor, the charge distributes equally. This shows that the capacitance of the second capacitor is also C. The voltage across each capacitor will be V/2. If U be the energy stored in the two capacitors, then

$$U = \frac{1}{2}C(V/2)^2 + \frac{1}{2}C(V/2)^2 = \frac{1}{4}CV^2 = 2.5\text{ J}$$

$$\frac{U}{U_0} = \frac{2.5}{5.0} = \frac{1}{2}$$

:. Two isolated metallic solid spheres of radii R and $2R$ are charged such that both of these have same charge density σ. The spheres are located far away from each other, and connected by a thin conducting wire. Find the new charge density on the bigger sphere.

ln. Charge on smaller sphere $Q_1 = 4\pi R^2 \cdot \sigma$

Charge on bigger sphere $Q_2 = 4\pi(2R)^2\sigma = 16\pi R^2\sigma$

Therefore, total charge $Q = Q_1 + Q_2 = 20\pi R^2\,\sigma$ ... (1)

Capacitance of two spherical conductors are

$C_1 = 4\pi\varepsilon_0 R$ and $C_2 = 4\pi\varepsilon_0(2R)$

Therefore, total capacitance $C = C_1 + C_2 = 12\pi\varepsilon_0 R$ ... (2)

After connection, the common potential V is given by

$$V = \frac{Q}{C} = \frac{20\pi R^2\sigma}{12\pi\varepsilon_0 R} = \frac{5R\sigma}{3\varepsilon_0}$$

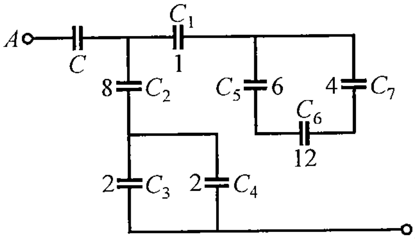
New charge on bigger sphere

$$Q_2' = C_2V = 4\pi\varepsilon_0(2R)\times(5R\sigma/3\varepsilon_0)$$

$$= \frac{40\pi R^2\sigma}{3}$$

Surface density $\sigma_2' = \frac{Q_2'}{\text{surface area}} = \left(\frac{40\pi R^2\sigma}{3}\right)\Big/4\pi(2R)^2 = \frac{5}{6}\sigma.$

13. From the given figure, find the value of the capacitance C if the equivalent capacitance between points A and B is to be 1 μF. All the capacitances are in μF.



Soln. The capacitors C_3 and C_4 are in parallel, therefore their resultant capacity C_8 is 4. The capacitors C_5 and C_6 are in series, therefore, their resultant capacity C_9 is 4. These are shown in figure (a).

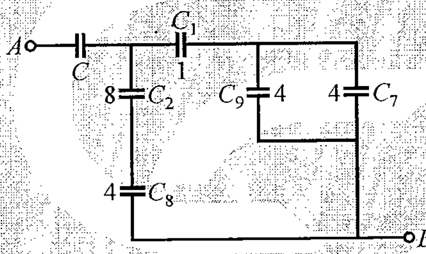


Figure : (a)

Now the capacitor C_2 and C_8 are in series. Their resultant capacity C_{10} is 8/3. Capacitors C_7 and C_9 are in parallel. Their resultant capacity C_{11} is 8. These are shown in figure (b).

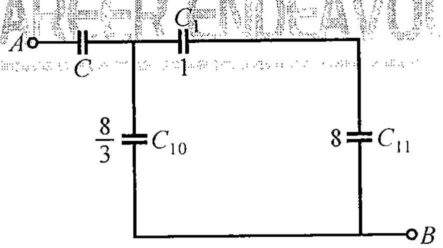


Figure : (b)

C_1 and C_{11} are in series. The equivalent capacitance is 8/9. The parallel combination of 8/3 and 8/9 gives a resultant capacitance 32/9 as shown in figure (c).

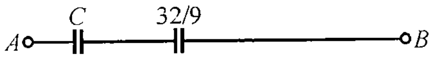
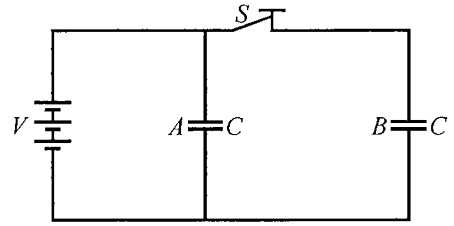


Figure : (c)

From figure (c) we have, $\frac{1}{1} = \frac{1}{C} + \frac{9}{32}$ or $\frac{1}{C} = \frac{23}{32}$

Therefore, $C = \frac{32}{23}$ μF.

Figure shows two identical parallel plate capacitors connected to a battery with switch S closed. The switch is now opened and the free space between the plates of the capacitors is filled with a dielectric of dielectric constant (or relative permittivity) 3. Find the ratio of the total electrostatic energy stored in both capacitors before and after the introduction of the dielectric.



ln. Initially the charge on either capacitor i.e., q_A or q_B is CV coulomb.

When dielectric is introduced, the new capacitance of either capacitor $C_1 = \frac{K_1}{K_2} C = 3C$.

After the opening of switch S, the potential across capacitor A is V volts.

Let the potential across capacitor B is V_1

Therefore, $q_B = CV = C_1V_1$ or $CV = 3CV_1$

Therefore, $V_1 = V/3$ volts

Initial energy of capacitor $A = \frac{1}{2}CV^2$

Energy of capacitor $B = \frac{1}{2}CV^2$

Therefore, total energy $E_i = \frac{1}{2}CV^2 + \frac{1}{2}CV^2 = CV^2$

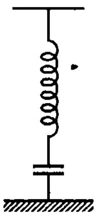
Final energy of capacitor $A = \frac{1}{2}\times(3C)V^2 = \frac{3}{2}CV^2$

Final energy of capacitor $B = \frac{1}{2}\times(3C)\left(\frac{V}{3}\right)^2 = \frac{CV^2}{6}$

Therefore, total final energy $E_f = \frac{3}{2}CV^2 + \frac{CV^2}{6} = \frac{5}{3}CV^2$

Therefore, $\frac{E_i}{E_f} = \frac{CV^2}{(5/3)\,CV^2} = \frac{3}{5}$

A parallel plate capacitor with air as a dielectric is arranged horizontally. The lower plate is fixed and the other connected with a perpendicular spring. The area of each plate is A. In the steady position, the distance between the plates is d_0. When the capacitor is connected with an electric source with the voltage V, a new equilibrium appears, with the distance between the plates as d_1. Mass of the upper plate is m.



(i) Find the spring constant K.

(ii) What is the maximum voltage for a given K in which an equilibrium is possible ?

(iii) What is the angular frequency of the oscillating system around the equilibrium value d_1 (amplitude of the oscillation $<< (d_1)$.

Soln. (i) Let l_0 be the initial extension in the spring.

Therefore, $K l_0 = m g$

After being connected to a voltage source, let the plate separation changes from d_0 to d_1. Hence the extension of the spring is $(d_0 - d_1)$.

In new equilibrium position

$$K\left[l_0 + (d_0 - d_1)\right] = m g + \frac{1}{2}\varepsilon_0 E^2 A$$

or $$K(d_0 - d_1) = \frac{1}{2}\varepsilon_0\left(\frac{V}{d_1}\right)^2 A$$

$$K = \frac{\varepsilon_0 A V^2}{2d_1^2(d_0 - d_1)} \qquad \ldots(1)$$

(ii) From equation (1), $V^2 = \dfrac{2K d_1^2(d_0 - d_1)}{A\varepsilon_0}$

differentiating with respect to d_1, we get $2V\left(\dfrac{dV}{dd_1}\right) = \dfrac{2K}{A\varepsilon_0}\left[2d_1 d_0 - 3d_1^2\right]$

For V to be maximum, $\left(\dfrac{dV}{dd_1}\right) = 0$

Applying this condition, we get $d_1 = \left(\dfrac{2}{3}d_0\right)$

Therefore, $V_{\max}^2 = \dfrac{2K\left(\frac{2}{3}d_0\right)^2\left[d_0 - \frac{2}{3}d_0\right]}{A\varepsilon_0}$

$$\therefore \quad V_{\max} = \sqrt{\frac{K}{A\varepsilon_0}}\left(\frac{2}{3}d_0\right)^{3/2} \qquad \ldots(2)$$

(iii) Let the upper plate be displaced through a small distance x downwards from eequilibrium position. Then the net force on the plate is given by

$$F = -K\left[l_0 + (d_0 - d) + x\right] + m g + \frac{1}{2}\varepsilon_0 A\left[\frac{V^2}{(d_1 - x)^2}\right]$$

$$= -K(d_0 - d) - Kx + \frac{1}{2}\frac{\varepsilon_0 A V^2}{d_1^2}\left[1 - \frac{x}{d_1}\right]^{-2}$$

$$= -K(d_0 - d) - Kx + \frac{1}{2}\frac{\varepsilon_0 A V^2}{d_1^2}\left[1 + \frac{2x}{d_1}\right]$$

$$= -K(d_0 - d) - Kx + K(d_0 - d)\left(1 + \frac{2x}{d_1}\right)$$

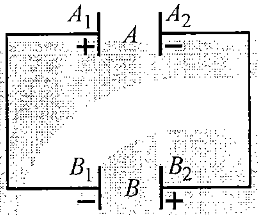
$$= -Kx\left[\frac{3d_1 - 2d_0}{d_1}\right]$$

$$\text{Acceleration} = \frac{F}{m} = -\frac{K}{m}\left(\frac{3d_1 - 2d_0}{d_1}\right)x$$

$$\text{Equation of S.H.M., } \omega = \sqrt{\left[\frac{K}{m}\left(\frac{3d_1 - 2d_0}{d_1}\right)\right]}.$$

Two parallel plate condensers A and B having capacities of 1 μF and 5μF are charged separately to the same potential of 100 V. Now the positive plate of A is connected to the negative plate of B and negative plate of A to the positive plate of B. Find the final charge on each condenser and total loss of electrical energy in the condensers.

ln. The situation is shown in figure :



Charge on condenser $A = 10^{-6} \times 100 = 10^{-4}$ C

Charge on condenser $B = (5 \times 10^{-6}) \times 100 = 5 \times 10^{-4}$ C

Before connecting : Here the plates A_1 and A_2 have the charges $+10^{-4}$ C and -10^{-4} C respectively.

Similarly, the plates B_1 and B_2 have charges -5×10^{-4} C and $+5 \times 10^{-4}$ C respectively.

After connecting : Charge on A_1 and B_1,

$$Q_1 = 10^{-4} + (-5 \times 10^{-4}) = -4 \times 10^{-4}\text{ C}$$

Charge on A_2 and B_2,

$$Q_2 = -10^{-4} + (+5 \times 10^{-4}) = +4 \times 10^{-4}\text{ C}$$

It is obvious that after connection, both A_1 and B_1 have negative charge while A_2 and B_2 have positive charge. This shows that the two condensers are connected in parallel. The capacity of combination is given by

$C = C_A + C_B = 6 \times 10^{-6}$ farad

$$\text{Now common voltage} = \frac{Q_1 \text{ or } Q_2}{C} = \frac{4 \times 10^{-4}}{6 \times 10^{-6}} = \frac{200}{3}\text{ V}$$

So the charge on A and B after connection are

$$q'_A = (10^{-6}) \times \frac{200}{3} = \frac{2}{3} \times 10^{-4}\text{ C} = \mathbf{0.66 \times 10^{-4}\text{ C}}$$

$$q'_B = (5 \times 10^{-6}) \times \frac{200}{3} = \frac{10}{3} \times 10^{-4}\text{ C} = \mathbf{0.33 \times 10^{-4}\text{ C}}$$

$$\text{Original energy} = \frac{1}{2}C_A V_A^2 + \frac{1}{2}C_B V_B^2$$

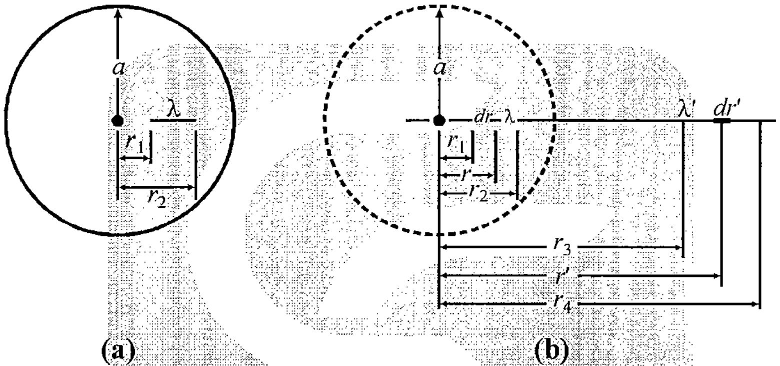
$$=\frac{1}{2}\times 10^{-6}\times(100)^2+\frac{1}{2}(5\times 10^{-6})\times(100)^2$$

$$=0.03\text{ J}$$

Final energy $=\frac{1}{2}CV^2=\frac{1}{2}\times(6\times 10^{-6})\left(\frac{200}{3}\right)^2=\frac{0.04}{3}\text{ J}$

$\therefore$ loss of energy $=0.03-\frac{0.04}{3}=\frac{0.05}{3}\text{ J}=\frac{5}{3}\times 10^{-2}$ joule

17. A line charge charge of uniform density λ is placed radially inside a hallow conducting sphere of radius a, the two ends of the line charge being at the distance r_1 and r_2, respectively, from the centre of the sphere (Figure-a). Find the image charge by which the sphere may be replaced for calculating the potential inside the sphere.



Soln. (a) Line charge λ in a conducting sphere,

(b) The sphere is replaced by an image line charge λ'.

According to the third image system of Table 6-II, the sphere may be replaced by a radial line charge whose ends are at the distance.

$$r_3=\frac{a^2}{r_2}\text{ and }r_4=\frac{a^2}{r_1}$$

from the center of the sphere (Figure-b). The density of the image charge can be found as follows. Consider a differential length element dr of the original line charge. If this element is at a distance r from the center of the sphere, the corresponding element dr' of the image charge is at the distance $r'=a^2/r$ from the center. Differentiating, we obtain for the length of dr'.

$$|dr'|=\left|-\frac{a^2}{r^2}dr\right|=\frac{a^2}{r^2}dr=\frac{r^2}{a^2}dr.$$

PRACTICE SET

. If the separation between two charges is increased the electric potential energy

(a) always decreases (b) may increase or decrease

(c) always increases (d) remains the same.

Energy density of the electric field produced by a point charge varies with the distance r from the point charge as:

(a) $\frac{1}{r^2}$ (b) $\frac{1}{r^4}$ (c) $\frac{1}{r^3}$ (d) $\frac{1}{r^5}$

If two capacitors having the same capacitance C and breakdown voltage V are connected in series, their equivalent capacitance and the breakdown voltage would be

(a) $\frac{C}{2}$ and $\frac{V}{2}$ (b) $2C$ and $2V$ (c) $\frac{C}{2}$ and $2V$ (d) $2C$ and V

If two capacitors having the same capacitance C and breakdown voltage V are connected in parallel, their equivalent capacitance and the breakdown voltage would be

(a) $\frac{C}{2}$ and $\frac{V}{2}$ (b) $2C$ and $2V$ (c) $\frac{C}{2}$ and $2V$ (d) $2C$ and V

A dielectric slab is introduced between the plates of an isolated charged parallel plate air capacitor. Which of the following quantities will remain unchanged?

(a) charge on the capacitor (b) energy of the capacitor

(c) pd across the capacitor (d) electric field inside the capacitor

Capacitance has the dimensions

(a) $\left[M^{-1}L^{-2}T^4I^2\right]$ (b) $\left[MLT^{-3}I^{-1}\right]$ (c) $\left[ML^2T^{-4}I^{-2}\right]$ (d) $\left[M^{-1}L^{-1}T^3I\right]$

A fixed pd V is applied between the plates of a parallel plate capacitor. Now a conducting plate is introduced between the plates such that it divides the space between positive and negative plates in the ratio $d_1 : d_2$. Then the voltage of the conducting plate with respect to the negative plate would be

(a) zero (b) $\frac{d_2}{d_1+d_2}V$ (c) V (d) $\frac{d_1}{d_1+d_2}V$

. The pd across a capacitor is kept constant. If a dielectric slab of dielectric constant K is introduced between the plates, the stored energy will

(a) decrease by a factor K

(b) increase by a factor K

(c) remain constant

(d) increase or decrease depending on the nature of the dielectric

. A parallel plate capacitor of plate area A and plate separation d is charged to a pd of V_0. It is now disconnected from source of pd and then its place separation is increased to 3d. Workd necessary for this is

(a) $3\in_0 AV_0^2/d$ (b) $\in_0 AV_0^2/3d$ (c) $\in_0 AV_0^2/2d$ (d) $\in_0 AV_0^2/d$

0. A metal sphere of radius r_1 carrying a charge q_1 is connected by a thin wire to another metal sphere of radius r_2 carrying a charge q_2. The common potential of the system is

(a) $\frac{1}{4\pi\in_0}\cdot\frac{q_1+q_2}{r_1+r_2}$ (b) $\frac{1}{4\pi\in_0}\cdot\left(\frac{q_1}{r_1}-\frac{q_2}{r_2}\right)$ (c) $\frac{1}{4\pi\in_0}\cdot\left(\frac{q_1}{r_1}+\frac{q_2}{r_2}\right)$ (d) $\frac{1}{4\pi\in_0}\cdot\frac{q_1q_2}{\sqrt{r_1r_2}}$

ANSWER KEY					
Questions	1	2	3	4	5
Option	(b)	(b)	(c)	(d)	(a)
Questions	6	7	8	9	10
Option	(a)	(b)	(b)	(d)	(a)

Chapter 7

Magnetostatics

Electric current: The rate of flow of charges with time is called current.

$$\boxed{I = \frac{dq}{dt}}$$

Current density : The current crossing through the unit area perpendicular to direction of flow is called current density.

$$J = \frac{dI}{ds}$$

This is called volume current density because current flows in volume

$$\boxed{I = \oint \vec{J} \cdot d\vec{s}}$$

Also current density is given,

$$\boxed{\vec{J} = nq\vec{v}}$$

where n is the charge particle per unit volume, q is the charge, and v is the velocity.

If current flows on surface then we define surface current density, $k = \frac{I}{\ell_\perp}$

Equation of Continuity:

Let us consider a volume V enclosed by a surface s. If J is the current density at any surface element ds then total current flowing through s will be

$$I = \oint \vec{J} \cdot d\vec{s}$$

It gives the rate at which charge is flowing out of the volume enclosed by the surface s. According to local conservation charge condition.

$$I = \oint \vec{J} \cdot d\vec{s} = -\frac{dq}{dt} = -\frac{d}{dt}\int \rho dV$$

$$\Rightarrow \qquad \oint \vec{J} \cdot d\vec{s} = -\frac{d}{dt}\int \rho dV$$

$$\Rightarrow \qquad \int \left(\vec{\nabla} \cdot \vec{J}\right) \cdot dV = -\int \frac{d\rho}{dt} \cdot dV$$

$$\Rightarrow \qquad \left(\vec{\nabla} \cdot \vec{J}\right) = -\frac{d\rho}{dt}$$

$\Rightarrow$ $$\boxed{\vec{\nabla}\cdot\vec{J}+\frac{d\rho}{dt}=0}$$

For steady current $\frac{d\rho}{dt}=0$

$\therefore$ $$\boxed{\vec{\nabla}\cdot\vec{J}=0}$$

Biot-Savart's law:

The Biot-Savart law defines the magnetic field produced by a steady current. According to this law the magnetic field at any point due to current element $Id\vec{\ell}$ is given by

$$\vec{B}=\frac{\mu_0}{4\pi}\int\frac{Id\vec{\ell}\times\vec{R}}{R^3}$$

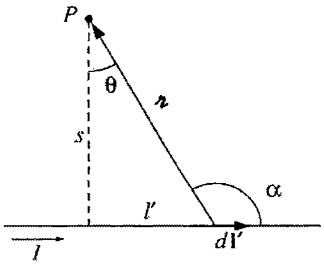
For continuous current density

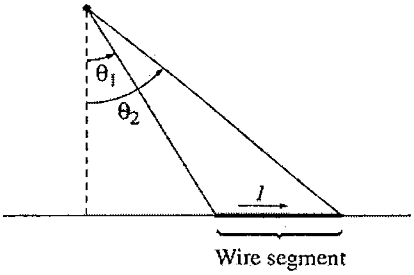
$$B=\frac{\mu_0}{4\pi}\int\frac{\vec{J}\times\vec{R}}{R^3}dV \text{ for volume current density}$$

$$\vec{B}=\frac{\mu_0}{4\pi}\int\frac{\vec{k}\times\vec{R}}{R^3}ds \text{ for surface current density.}$$

Application of Biot-Savart Law:

Find the magnetic field a distance s from a long straight wire carrying a steady current *I*.





In the diagram, $\left(d\vec{\ell}'\times\hat{r}\right)$ points out of the page, and has the magnitude

$$d\ell'\sin\alpha=d\ell'\cos\theta$$

Also, $\ell'=s\tan\theta$, So, $d\ell'=\frac{s}{\cos^2\theta}d\theta$

And $s=r\cos\theta$, So, $\frac{1}{r^2}=\frac{\cos^2\theta}{s^2}$

Thus, $$B=\frac{\mu_0 I}{4\pi}\int_{\theta_1}^{\theta_2}\left(\frac{\cos^2\theta}{s^2}\right)\left(\frac{s}{\cos^2\theta}\right)\cos\theta d\theta$$

$$=\frac{\mu_0 I}{4\pi s}\int_{-\theta_1}^{\theta_2}\cos\theta d\theta$$

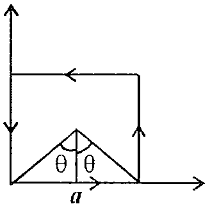
$$\boxed{B=\frac{\mu_0 I}{4\pi s}\left(\sin\theta_2+\sin\theta_1\right)}$$

In the case of infinite wire, $\theta_1 = \frac{\pi}{2}$ and $\theta_2 = \frac{\pi}{2}$, so we obtain

$$B = \frac{\mu_0 I}{2\pi s}$$

The field circles around the wire, following right hand thumb rule.

Example: A current I flows in the anticlockwise direction through a square loop of side a lying in the xoy plane with its center at the origin. Find the magnetic induction at the center of the square loop.



Soln. Magnetic field at distance $\frac{a}{2}$ from finite length wire is given by

$$B_1 = \frac{\mu_0 I}{4\pi(a/2)}(\sin\theta_1 + \sin\theta_2)$$

Since, $\theta_1 = \theta_2$ $\therefore B_1 = \frac{\mu_0 I}{\pi a}\sin\theta = \frac{\mu_0 I}{\pi a}\frac{1}{\sqrt{2}}$

Therefore, the total magnetic field at the centre of square is given by

$$B = B_1 + B_2 + B_3 + B_4 = 4B_1 = \frac{4\mu_0 I}{\pi a}\frac{1}{\sqrt{2}} = \frac{2\sqrt{2}\mu_0 I}{\pi a}$$

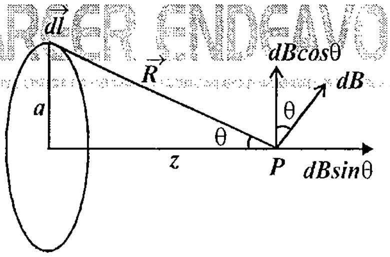
A circular loop of radius carring current *I*. Find the magnetic field on the axis at distance from the centre of the loop.

The field at the point P due to small element d*l* is given by

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{Id\vec{\ell}\times\vec{R}}{R^3}$$

It has two components $dB\sin\theta$ and $dB\cos\theta$

The component $dB\cos\theta$ will be cancelled out due to symmetry.



Therefore, the field at *P* is

$$B = \int dB\sin\theta = \frac{\mu_0}{4\pi}\int\frac{Id\ell\times R}{R^3}\sin\theta$$

$$= \frac{\mu_0 I}{4\pi}\int\frac{d\ell}{R^2}\sin\theta \qquad [\because\ d\ell \perp R]$$

$$= \frac{\mu_0 I}{4\pi}\int\frac{d\ell}{(z^2+a^2)}\frac{a}{\sqrt{z^2+a^2}}$$

$$= \frac{\mu_0 I}{4\pi}\frac{2\pi a^2}{(z^2+a^2)^{3/2}}\hat{z} = \frac{\mu_0 I}{2}\frac{a^2}{(z^2+a^2)^{3/2}}\hat{z}$$

At the centre of circular loop is given by

$$\vec{B} = \frac{\mu_0 I}{2a}\hat{z}$$

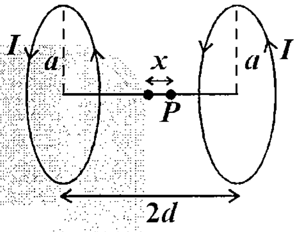
Helmholtz coil:

Let us consider two such coil (1) and (2) each of radius a, same number of turn and same current. Separated by the distance $2d$. Let consider a point P at a small distance x from the mid-point

$$\vec{B} = \frac{\mu_0 IN}{2}\left[\frac{a^2}{\left\{a^2+(d+x)^2\right\}^{3/2}} + \frac{a^2}{\left\{a^2+(d-x)^2\right\}^{3/2}}\right]$$

For B will be uniform near the mid-point $\frac{dB}{dx} = 0$

$$\Rightarrow \qquad \frac{d+x}{\left\{a^2+(d+x)^2\right\}^{5/2}} = \frac{d-x}{\left\{a^2+(d-x)^2\right\}^{5/2}}$$



$$\frac{d+x}{d-x} = \left[\frac{a^2+(d+x)^2}{a^2+(d-x)^2}\right]^{5/2}$$

$$\Rightarrow \qquad \frac{d+x}{d-x} = \frac{1+\frac{5dx}{d^2+a^2}}{1-\frac{5dx}{d^2+a^2}} \quad \Rightarrow \frac{x}{d} = \frac{5dx}{d^2+a^2}$$

$$\Rightarrow \qquad a^2 = 4d^2 \qquad \Rightarrow 2d = a$$

The magnetic field near the mid-point will be uniform if the separation between them equal to the radius. Therefore, the field at mid-point

$$B = \frac{\mu_0 NI}{2}\left[\frac{a^2}{\left\{a^2+(a/2)^2\right\}^{3/2}} + \frac{a^2}{\left\{a^2+(a/2)^2\right\}^{3/2}}\right] = \frac{8\mu_0 NI}{5\sqrt{5}a}$$

Example: An infinite wire carrying current I is bent in the form of a parabola. Find the magnetic field at the focus of the parabola. Take the distance of the focus from the apex as a.

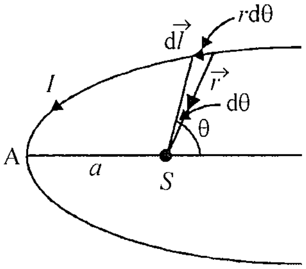
ln. From Biot-Savart law the magnetic field at S is given by

$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{I d\vec{\ell}\times\vec{r}}{r^3}$$

From the figure we note that,

$$\left|d\vec{\ell}\times\vec{r}\right| = \text{area of the parallelogram generated by } d\vec{\ell} \text{ and } \vec{r}$$

$$= 2\times\frac{1}{2}r\cdot r d\theta = r^2 d\theta$$



$$\therefore \qquad \left|\vec{B}\right| = \frac{\mu_0 I}{4a}\int_0^{2\pi}\frac{d\theta}{r}$$

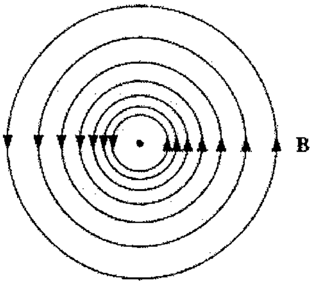
Using $r(1-\cos\theta)=2a$ as the equation to the parabola we get

$$|\vec{B}|=\frac{\mu_0 I}{4a}$$

$\vec{B}$ is directed perpendicular to the plane of the parabola in the usual right-hand sense.

Ampere's law:

The magnetic field of an infinite straight wire.



According to Ampere circuital law, the integral of B around a circular path of radius s, centered at the wire is:

$$\oint \vec{B}.d\vec{\ell}=\oint \frac{\mu_0 I}{2\pi s}d\ell=\frac{\mu_0 I}{2\pi s}\oint d\ell$$

Notice that the answer is independent of s. In fact, it doesn't have to be a circle; any old loop that encloses the wire would give the same answer. For if we use cylindrical coordinates (s, ϕ, z), with the current flowing along the z-axis; we get

$$\vec{B}=\frac{\mu_0 I}{2\pi s}\hat{\phi}$$

And, $d\vec{\ell}=ds\,\hat{s}+s\,d\phi\,\hat{\phi}+dz\,\hat{z}$, So

$$\oint \vec{B}.d\vec{\ell}=\frac{\mu_0 I}{2\pi}\int\frac{1}{s}s d\phi=\frac{\mu_0 I}{2\pi}\int_0^{2\pi}d\phi=\mu_0 I$$

This equation is true not only for a straight wire, but also for any current distribution. This will basically become our Ampere's law if we just replace the current I by enclosed current I_{enc}.

$$\oint \vec{B}.d\vec{\ell}=\mu_0 I_{enclosed}$$

If the flow of charge is represented by a volume current density J, then from the definition of J we get

$$\oint \vec{B}.d\vec{\ell}=\int\left(\vec{\nabla}\times\vec{B}\right).d\vec{a}=\mu_0\int\vec{J}.d\vec{a}$$

Hence, $\vec{\nabla}\times\vec{B}=\mu_0\vec{J}$

That's the curl of $\vec{B}$. Remember the result. It will be needed many times.
Divergence of magnetic field is derivable both from Ampere's law and Biot Savart Law. The result is

$$\boxed{\vec{\nabla}.\vec{B}=0}$$

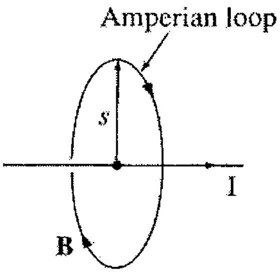
What is the significance of these two results. Well, finite value of curl of magnetic field (proportional to current) simply men that the Magnetic field always curl around the current. And the zero divergence mean the magnetic field, unlike the electric field, does not emerge and solely "diverge" outside from a single point. If magnetic field line has stated from somewhere, it must end somewhere. In other words there is no magnetic monopoles. Notice the different nature of electric and magnetic field.

Like Gauss's law, Ampere's law is always true (for steady currents), but it is not always useful. It is only useful in case of

(1) Infinite straight lines (2) Infinite planes (3) Infinite solenoids (4) Toroids

Application of Ampere's law:

(1) Find the magnetic field a distance s from a long straight wire, carrying a steady current I.



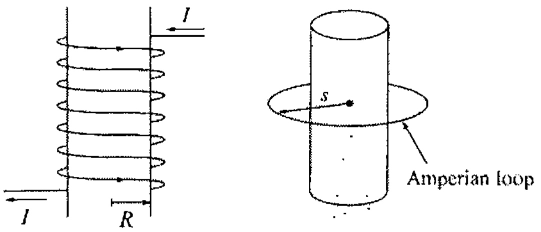
We know the direction of B is "circumferential," circling around the wire as indicated by the right hand thumb rule. By symmetry, the magnitude of **B** is constant around an Amperian loop of radius s, centered on the wire. So Ampere's law gives

$$\oint \vec{B}.d\vec{\ell} = \vec{B}\oint d\vec{\ell} = \vec{B}2\pi s = \mu_0 I_{enclosed} = \mu_0 I$$

So, we get, $\vec{B} = \dfrac{\mu_0 I}{2\pi s}$

We get same answer before.

(2) Find the magnetic field of a very long solenoid, consisting of n closely wound turns per unit length on a cylinder of radius R and carrying a steady current I. The windings are so close that one can take each wind circular.

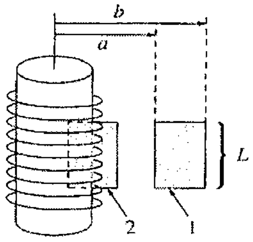


It shall be clear that the magnetic field **B** will not have any radial or circumferential components. If it had radial components than changing the direction of current could alter the direction of field. But changing the direction of current just means the upside downing of the solenoid. So the radial field won't be there.

For B_ϕ would be constant around an amperian loop concentric with the solenoid. Hence

$$\oint \vec{B}.d\vec{\ell} = B_\phi(2\pi s) = \mu_0 I_{enc} = 0$$

So the magnetic field of an infinite, closely wound solenoid runs parallel to the axis. From the right hand rule, we expect that it points upward inside the solenoid and downward outside. Apply Ampere's law to the two rectangular loops in . Loop 1 lies entirely outside the solenoid, with its sides at distances 'a' and 'b' from the axis:



Amperian loops

$$\oint \vec{B}.d\vec{\ell} = \left[B(a) - B(b)\right]L = \mu_0 I_{enc} = 0$$

So, $B(a) = B(b)$.

Evidently the field outside does not depend on the distance from the axis. But we know that it goes to zero for large s. It must therefore be zero everywhere! So outside the solenoid the field is zero.

As for loop 2, which is half inside and half outside, Ampere's law gives

$$\oint \vec{B}.d\vec{\ell} = BL = \mu_0 I_{enc} = \mu_0 n\, I\, L$$

Where **B** is the field inside the solenoid.

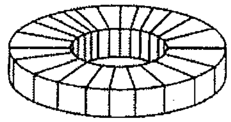
So the field is

$$\vec{B} = \begin{cases} \mu_0 n\, I\, \hat{z}, & \text{inside the solenoid,} \\ 0 & \text{outside the solenoid} \end{cases}$$

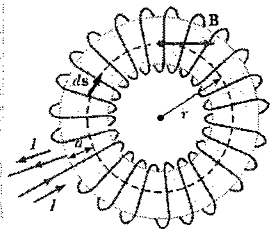
Notice that the field inside is uniform.

(3) Field of a Toroid

A device called a toroid is often used to create an almost uniform magnetic field in some enclosed area. The device consists of a conducting wire wrapped around a ring(a torus) made of a non-conducting material. For a toroid having N closely spaced turns of wire, calculate the magnetic field in the region occupied by the torus, a distance r from the center.



To calculate this field, we must evaluate $\oint \vec{B}.ds$ over the circle of radius r. By symmetry, we see that the magnitude of the field is constant on this circle and tangent to it, so Furthermore, note that



the circular closed path surrounds N loops of wire, each of which carries a current I. Therefore, the right side of Ampere's law is $\mu_0 NI$ in this case.

Ampère's law applied to the circle gives $\oint \vec{B}.d\vec{s} = B\oint ds = B(2\pi r) = \mu_0 NI$

$$\boxed{B = \frac{\mu_0 NI}{2\pi r}}$$

This result shows that B varies as $1/r$ and hence is non-uniform in the region occupied by the torus. However, if r is very large compared with the cross-sectional radius of the torus, then the field is approximately uniform inside the torus.

Infinite sheet of current:

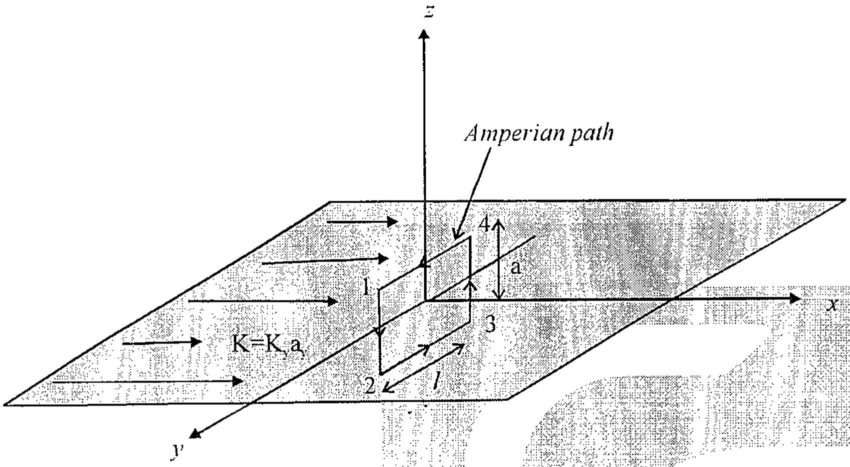
Consider an infinite plane $z = 0$ on which a uniform surface current density $\vec{K} = K\hat{i}$ flows shown in the figure. Dividing the current sheet into pairs of current filaments at +y and –y it is easy to show the resultant magneric field $\vec{B}$ will be directed along $-y$-axis for $z > 0$ and along +y axis for $z < 0$. With this in mind we choose a rectangle PQRS parallel to yz-plane as an Amperian loop. Now, applying Ampere's law we get,

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 K \ell$$

Or, $$\int_{23} \vec{B} \cdot d\vec{\ell} + \int_{34} \vec{B} \cdot d\vec{\ell} + \int_{41} \vec{B} \cdot d\vec{\ell} + \int_{12} \vec{B} \cdot d\vec{\ell} = \mu_0 K \ell$$

Or, $$B\ell + 0 + B\ell + 0 = \mu_0 K \ell \quad \text{or,} \quad B = \frac{\mu_0}{2} K$$

Note that B is independent of the distance from the current sheet.



Vectorially,

$$\vec{B} = \begin{cases} -\dfrac{\mu_0 K}{2} \hat{j} & \text{for } z > 0 \\ +\dfrac{\mu_0 K}{2} \hat{j} & \text{for } z < 0 \end{cases}$$

In general, $$\vec{B} = \frac{\mu_0}{2} \vec{K} \times \hat{n}$$

where $\hat{n}$ is the unit normal from the current sheet to the point of interest.

Example: A long cylindrical kept along z-axis carries a current density $\hat{J} = J_0 r\hat{k}$, where J_0 is a constant and r is the radial distance form the axis of the cylinder. Find the magnetic induction $\hat{B}$ inside the conductor at a distance d from the axis of the cylinder.

ln. We know that, Ampere law,

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 \int \vec{J} \cdot d\vec{s}$$

$$\Rightarrow \quad B \cdot 2\pi d = \mu_0 J_0 \int_0^d \int_0^{2\pi} r dr d\theta$$

$$\Rightarrow \quad B \cdot 2\pi d = \mu_0 J_0 \frac{d^3}{3} 2\pi$$

$$\Rightarrow \quad B = \frac{\mu_0 J_0 d^2}{3} \hat{\phi}$$

Magnetic dipole

You remember what is an electric dipole, right? When two opposite charge placed a small distance apart, it called an electric dipole. Similarly, when a wire carrying a current forms a small closed loop, then it is called magnetic dipole.

If **m** is the magnetic moment of magnetic dipole then

$$\boxed{m \equiv I\int da = Ia}$$

Here **a** is the "vector area" of the loop. If the loop is flat, **a** is the ordinary area enclosed, with the direction assigned by the usual right hand rule (fingers in the direction of the current).

Magnetic vector potential of a magnetic dipole with magnetic moment **m** is:

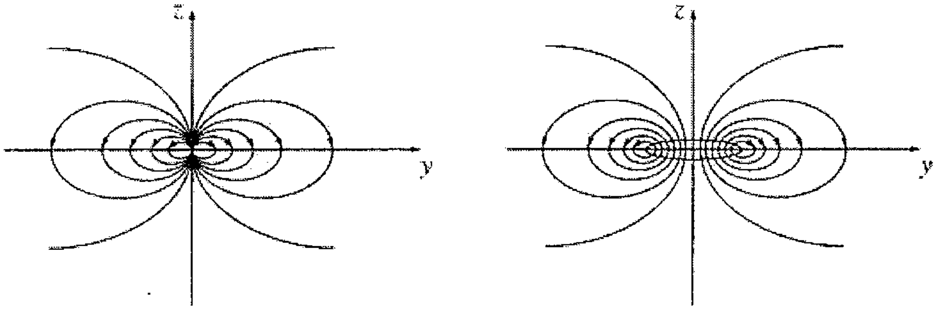
$$\boxed{\vec{A}_{dip}(r) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2}}$$

In polar coordinate A becomes

$$\vec{A}_{dip}(r) = \frac{\mu_0}{4\pi}\frac{\vec{m}\sin\theta}{r^2}\hat{\phi}$$

Hene the magnetic field is:

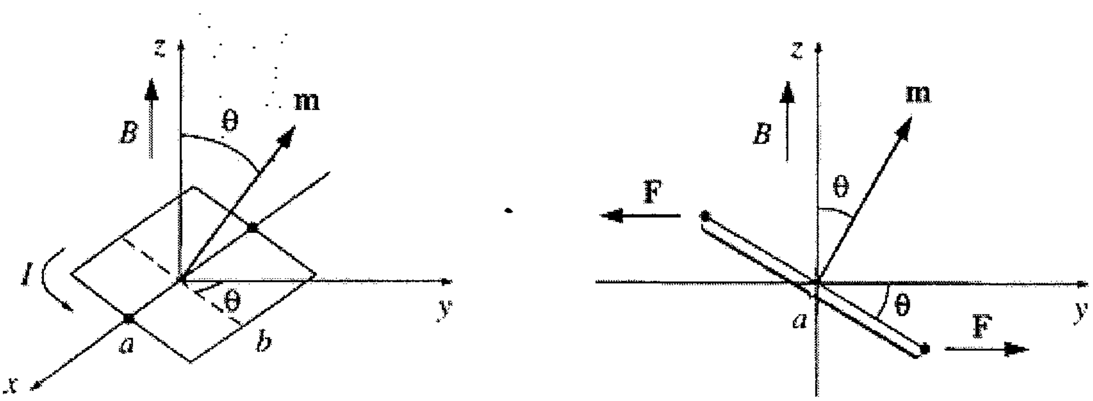
$$\vec{B}_{dip}(r) = \vec{\nabla}\times\vec{A} = \frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right)$$



(a) Field of a "pure" dipole (a) Field of a "physical" dipole

Torque on a magnetic dipole

A magnetic dipole experiences a torque in a magnetic field, just as an electric dipole does in an electric field. Let's calculate the torque on a rectangular current loop in a uniform field B.



(a) (b)

Take the center of loop at the origin, and tilt it an angle θ from the z axis towards the y-axis. Let B point in the z-direction. The forces on the two sloping sides cancel as they tend to stretch the loop, but they don't rotate it. The forces on the "horizontal" sides are likewise equal and opposite (so the net force on the loop is zero), but they do generate a torque:

$$\vec{N} = aF\sin\theta\,\hat{x}$$

The magnitude of the force on each of these segments is

$$\vec{F} = I\,b\,B$$

And hence the torque is

$$\vec{N} = I\,ab\,B\sin\theta\,\hat{x} = mB\sin\theta\,\hat{x}$$

Or, $$\boxed{\vec{N} = \vec{m}\times\vec{B}}$$

where, $m = Iab$ is the magnetic dipole moment of the loop. The presence of a uniform field; in a nonuniform field it is the exact torque (about the center) for a perfect dipole of infinitesimal size.

Energy of a magnetic dipole is

$$U = -\vec{m}.\vec{B}$$

The force on a magnetic dipole for an infinitesimal loop, with dipole moment ***m***, in a field $\vec{B}$, is

$$\boxed{\vec{F} = \vec{\nabla}\left(\vec{m}.\vec{B}\right)}$$

Magnetic vector potential

We have seen that

$$\vec{\nabla}.\vec{B} = 0$$

This implies that there exist a vector function **A** such that

$$\boxed{\vec{B} = \vec{\nabla}\times\vec{A}} \qquad \text{(From vector calculus)}$$

This A is called the magnetic vector potential.

Magnetic vector potential due to a wire is given as

$$\vec{A} = \frac{\mu_0}{4\pi}\int\frac{I d\vec{\ell}}{r}$$

Example: A current distribution gives rise to the magnetic vector potential

$$\vec{A}\left(x,y,z\right) = x^2y\hat{i} + y^2x\hat{j} - xyz\hat{k} \qquad T.m.$$

Find the corresponding magnetic field $\vec{B}$ at $\left(-1,2,5\right)$

oln. By definition,

$$\vec{B}\left(x,y,z\right) = \vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & y^2x & -xyz \end{vmatrix} = -xz\hat{i} + yz\hat{j} + \left(y^2 - x^2\right)\hat{k}$$

$$\therefore \qquad \vec{B}\left(-1,2,5\right) = 5\hat{i} + 10\hat{j} + 3\hat{k}\ T$$

Calculation of $\vec{A}$ in a few simple cases:

(1) A straight current carrying conductor:

Let us consider a long straight wire *AB* carrying a steady current *I*. We are to calculate the magnetic vector potential $\vec{A}$ at any point *P*. We choose cylindrical coordinates (r,θ,z) with z-axis along the wire and the foot *O* of the perpendicular from *P* on the wire as the origin. Let us consider an element $d\vec{\ell}=\hat{z}dz$ at a distance *z* from *O*. Now by definition

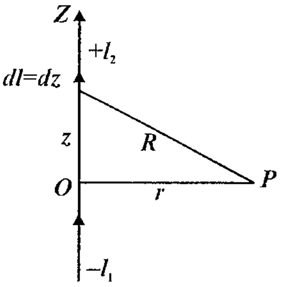
$$\vec{A}=\frac{\mu_0}{4\pi}\int\frac{Id\vec{\ell}}{R}=\hat{z}\frac{\mu_0 I}{4\pi}\int_{-\ell_1}^{+\ell_2}\frac{dz}{\sqrt{r^2+z^2}}=\hat{z}\frac{\mu_0 I}{4\pi}\left[\ell n\left(z+\sqrt{r^2+z^2}\right)\right]_{-\ell_1}^{+\ell_2}$$

$$=\hat{z}\frac{\mu_0 I}{4\pi}\ell n\left[\frac{\ell_2+\sqrt{r^2+\ell_2^2}}{-\ell_1+\sqrt{r^2+\ell_1^2}}\right]$$

If the wire is of length 2L and the point P is just above the centre of the wire then $\ell_1=\ell_2=L$ and we can write

$$\vec{A}=\hat{z}\frac{\mu_0 I}{4\pi}\ell n\left[\frac{L+\sqrt{r^2+L^2}}{-L+\sqrt{r^2+L^2}}\right]$$

Now if $r/L<<1$ then we can approximate



$$\vec{A}=\hat{z}\frac{\mu_0 I}{4\pi}\ell n\left[\frac{1+\left(1+r^2/L^2\right)^{1/2}}{-1+\left(1+r^2/L^2\right)^{1/2}}\right]\approx\hat{z}\frac{\mu_0 I}{4\pi}\ell n\left[\frac{2+\dfrac{r^2}{2L^2}}{\dfrac{r^2}{2L^2}}\right]$$

$$=\hat{z}\frac{\mu_0 I}{4\pi}\ell n\left(\frac{4L^2}{r^2}+1\right)\approx\hat{z}\frac{\mu_0 I}{2\pi}\ell n\left(\frac{2L}{r}\right)$$

Using equation (27), we can calculate the magnetic field due to a long straight wire as follows:

$$\vec{B}=\vec{\nabla}\times\vec{A}=\frac{1}{r}\begin{vmatrix}\hat{r} & r\hat{\theta} & \hat{z}\\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial z}\\ 0 & 0 & A_z\end{vmatrix}=-\hat{\theta}\frac{\partial A_z}{\partial r}$$

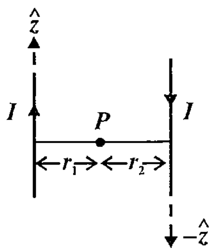
$$=-\hat{\theta}\frac{\partial}{\partial r}\left[\frac{\mu_0 I}{2\pi}\ell n\left(\frac{2L}{r}\right)\right]=\hat{\theta}\frac{\mu_0 I}{2\pi r}$$

Two long straight parallel wires carrying same current:

Suppose we have two long parallel wires each of length 2L carrying current I in opposite directions. The magnetic vector potential $\vec{A}$ at any point *P* due to this system can be calculated by using the result (27). Thus,

$$\vec{A}=\vec{A}_1+\vec{A}_2=\hat{z}\frac{\mu_0 I}{2\pi}\ell n\left(\frac{2L}{r_1}\right)-\hat{z}\frac{\mu_0 I}{r_2}=\hat{z}\frac{\mu_0 I}{2\pi}\ell n\left(\frac{r_2}{r_1}\right)$$

where r_1 and r_2 are the distances of the point of observation *P* from the wires and $\hat{z}$ is a unit vector parallel to the wires as shown in figure below.



Infinite sheet of current:

Consider which shows an infinite current sheet. From equation (17.b) we can write

$$\vec{B} = \vec{\nabla}\times\vec{A} = \pm\frac{\mu_0 K}{2}\hat{j}$$

where + and – signs apply to $z < 0$ and $z > 0$ regions, respectively. As the current is along x-axis we expect $\vec{A}$ to be also x-axis. Thus

$$\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ A_x & 0 & 0\end{vmatrix} = \pm\frac{\mu_0 K}{\partial z} = \pm\frac{\mu_0 K}{2}$$

Integrating we get,

$$A_x = \pm\frac{\mu_0 K z}{2} + \text{constant}$$

where the constant may be used to define the position at which $A = 0$

Long solenoid:

Consider an infinite solenoid with n turns per unit length, radius a and carrying current I. Symmetry of the problem suggests that $\vec{A}$ should have the direction of the current, i.e. the circumferential direction $(\hat{\theta})$. The magnetic field inside is along the axis (z-axis) of the solenoid and is given by

$$\vec{B} = \mu_0 n I\hat{z}$$

Now, considering a circular Amperian loop of radius $r(r < a)$ inside the solenoid in xy-plane we can write.

$$\oint_C \vec{A}\cdot d\vec{\ell} = \int_S \vec{\nabla}\times\vec{A}\cdot d\vec{S} = \int_S \vec{B}\cdot d\vec{S}$$

Or, $\quad A\cdot 2\pi r = \mu_0 n I\cdot\pi r^2$

Or, $\quad \vec{A} = \dfrac{\mu_0 n I}{2} r\hat{\theta}$, for $r < a$

For a circular Amperian loop of radius $r(r > a)$ outside the solenoid,

$$\int_S \vec{B}\cdot d\vec{S} = \mu_0 n I\cdot\pi a^2$$

Since the field extends only up to $r = a$. Thus,

$$A\cdot 2\pi r = \mu_0 n I\cdot\pi a^2 \text{ or } \vec{A} = \frac{\mu_0 n I a^2}{2r}\hat{\theta} \text{ for } r > a$$

It is easy to show that $\vec{A}$ as given by the equation (29) or (30) satifies the relations $\vec{\nabla}\times\vec{A}=\vec{B}$ and $\vec{\nabla}\cdot\vec{A}=0$

$$\vec{A}=\frac{\mu_0 nI}{2}r\left(-\sin\theta\hat{i}+\cos\theta\hat{j}\right)=\frac{\mu_0 nI}{2}\left(-y\hat{i}+x\hat{j}\right)$$

Now,
$$\vec{\nabla}\times\vec{A}=\frac{\mu_0 nI}{2}\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial y}\\ -y & x & 0\end{vmatrix}=\hat{k}\mu_0 nI=\vec{B} \qquad \left(\because\ \hat{z}=\hat{k}\right)$$

And
$$\vec{\nabla}\cdot\vec{A}=\frac{\mu_0 nI}{2}\left[\frac{\partial}{\partial x}(-y)+\frac{\partial}{\partial y}(x)\right]=0$$

Motion of charged particle:

The study of the motion of charged particles in specified field is important since it provides a good physical insight for understanding some of the dynamical processes in plasmas. It also permits to obtain information on some macroscopic phenomena which are due to the collective behaviour of large number of particles. Not all of the components of the detailed microscopic particle motion contribute to macroscopic effect but it is possible to isolate the components of the individual motion that contribute to the collective plasma behaviour.

Lorentz force:

We know that when a charged particle moving in presence of electric field it feels a force $q\vec{E}$. Similarly, in the presence of a magnetic field $\vec{B}$, a charge q experices an additional force.

$$\boxed{\vec{F}_m=q\vec{v}\times\vec{B}} \quad \text{(where } v \text{ is the velocity)}$$

Note

(1) There is no force on a charge at rest.

(2) A force is exerted on the charge only if there is component of the magnetic field perepndicular to the direction of the velocity.

(3) $v.F_m=0$, which shows that the magnetic force does not do any work.

In case where both the electric and magnetic field are present ,the force on the charge particle is given by

$$\boxed{\vec{F}=q\left(\vec{E}+\vec{v}\times\vec{B}\right)}$$

This is called Lorentz force

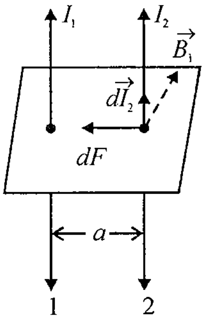
When a current carrying wire is placed in magnetic field force on wire due to field is given by

$$\vec{F}=\int Id\vec{\ell}\times\vec{B}$$

Force between two long straight parallel conductors carrying current:

Let us consider two long parallel wires (1 and 2) carrying currents I_1 and I_2, separated by a distance a shown in figure. The conductor 1 produces a magnetic field

$$B_1 = \frac{\mu_0}{4\pi}\cdot\frac{2I_1}{a}$$



at the position of conductor 2. It is directed perpendicular to the conductor 2 as shown in above figure.

The $d\vec{F}$ acting on the element $d\vec{\ell}_2$ of the conductor 2 due to $\vec{B}_1$ is

$$d\vec{F} = I_2\left(d\vec{\ell}_2 \times \vec{B}_1\right) \text{ or } \left|d\vec{F}\right| = dF = \frac{\mu_0}{4\pi}\cdot\frac{2I_1I_2}{a}d\ell_2$$

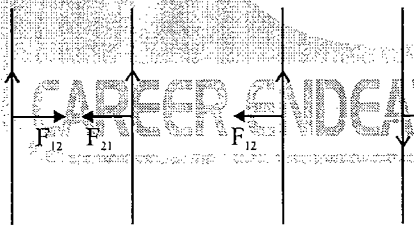
This force is directed towards the conductor 1, i.e. attractive if I_1 and I_2 flow in the same direction. On the other hand, the force is directed away from the conductor 1 if I_1 and I_2 flow in opposite directions.

The force per unit length of each wire is of magnitude

$$\left|\vec{F}\right| = \frac{\mu_0}{4\pi}\cdot\frac{2I_1I_2}{a}$$

and directed perpendicularly to their lengths.

(i) If current flow same direction then the force between two line is attractive in nature.
(ii) If the current flow opposite direction with each other then the force is repulsive in nature.



Motion of charge particle in uniform magnetic field :

Let us suppose a particle of mass m and charge q is thrown into a uniform field $\vec{B} = B\hat{k}$ with initial velocity $\vec{v} = v_\perp\hat{i} + v_\parallel\hat{k}$.

Let the position of particle at initial moment $t = 0$ to be (0, 0, 0).

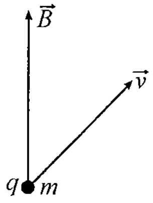
Force on the particle at initial moment is $\vec{F}_i = q\vec{v}\times\vec{B} = q(v_\perp\hat{i} + v_\parallel\hat{k})\times B\hat{k} = qv_\perp B(-j)$.

Due to this force particle will deflect in y-direction. Therefore, at later instant of time velocity of particle can be written as $\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$.

Equation of motion of particle,

$$\vec{F} = m\vec{a}$$

$$q\vec{v}\times\vec{B} = m\vec{a}$$



$$q(v_x\hat{i}+v_y\hat{j}+v_z\hat{k})\times B\hat{k}=m\vec{a}$$

$$qB(-v_x\hat{j}+v_y\hat{i})=m(a_x\hat{i}+a_y\hat{j}+a_z\hat{k})$$

$$qB(-v_x\hat{j}+v_y\hat{i})=m\left(\frac{dv_x}{dt}\hat{i}+\frac{dv_y}{dt}\hat{j}+\frac{dv_z}{dt}\hat{k}\right)$$

$$\therefore \qquad m\frac{dv_x}{dt}=qBv_y \qquad \text{... (i)}$$

$$m\frac{dv_y}{dt}=-qBv_x \qquad \text{... (ii)}$$

$$m\frac{dv_z}{dt}=0 \Rightarrow v_z=\text{constant}=\text{initial value}=v_{\parallel} \qquad \text{... (iii)}$$

Differentiate equation (i) with respect to time,

$$m\frac{d^2v_x}{dt^2}=qB\frac{dv_y}{dt}$$

Use equation (ii),

$$\frac{d^2v_x}{dt^2}=-\left(\frac{qB}{m}\right)^2 v_x$$

This is 2nd order ordinary differential equation, its solution can be written as

$$v_x=c_1\cos\left(\frac{qB}{m}t\right)+c_2\sin\left(\frac{qB}{m}t\right)$$

At $t=0,\ v_x=v_\perp \Rightarrow c_1=v_\perp$, and at $t=0, \frac{dv_x}{dt}=0$ (from (i))

$\Rightarrow c_2=0$

$$\therefore \qquad v_x=v_\perp\cos\left(\frac{qB}{m}t\right) \qquad \text{... (iv)}$$

Put in equation (i), to get

$$v_y=-v_\perp\sin\left(\frac{qB}{m}t\right) \qquad \text{... (v)}$$

If we square and add equations (iii), (iv), and (v), we get

$$v_x^2=v_y^2+v_z^2=v_\parallel^2+v_\perp^2$$

This implies that of particle doesn't change in magnetic field.
From equation (iii),

$$v_z=v_\parallel$$

$$\therefore \qquad \frac{dz}{dt}=v_\parallel \Rightarrow z=v_\parallel t+c_3$$

Since, at $t=0,\ z=0 \Rightarrow c_3=0$

$$\therefore \qquad \boxed{z = v_{\|}t} \qquad \text{... (vi)}$$

From equation (iv),

$$v_x = v_\perp \cos\left(\frac{qB}{m}t\right)$$

$$\frac{dx}{dt} = v_\perp \cos\left(\frac{qB}{m}t\right)$$

At $t = 0,\ x = 0 \Rightarrow c_4 = 0$

$$\therefore \qquad x = \frac{v_\perp m}{qB}\sin\left(\frac{qB}{m}t\right) \qquad \text{... (vii)}$$

From equation (v),

$$v_y = -v_\perp \sin\left(\frac{qB}{m}t\right)$$

$$\frac{dy}{dt} = -v_\perp \sin\left(\frac{qB}{m}t\right)$$

$$\therefore \qquad y = \frac{v_\perp m}{qB}\cos\left(\frac{qB}{m}\right) + c_5$$

At $t = 0,\ y = 0 \Rightarrow c_5 = 0$

$$\therefore \qquad y = \frac{v_\perp m}{qB}\cos\left(\frac{qB}{m}t\right) \qquad \text{... (viii)}$$

Square and add equation (vii) and (viii) to get

$$x^2 + y^2 = \left(\frac{v_\perp m}{qB}\right)^2 \qquad \text{... (ix)}$$

This is equation of circle of radius

$$\boxed{r = \frac{v_\perp m}{qB}} \qquad \text{... (x)}$$

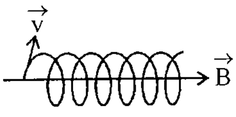
From equation (vii) and (viii) we also see that values of x and y repeat after time

$$\boxed{T = \frac{2\pi m}{qB}}$$

This is time of revolution of particle.

Conclusion :

- From equation (vi) and (x) we conclude that if charge particle has initial velocity at some angle with $\vec{B}$ then particle revolves as well as advances in direction of $\vec{B}$. That is path is helix.



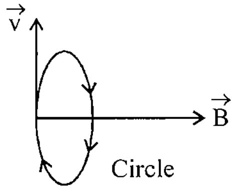
Helix

- Distance moved by particle in direction of magnetic field during one revolution is

$$\boxed{z = v_{\parallel} T = v_{\parallel} \frac{2\pi m}{qB}}$$

This is called pitch.

- If $v_{\parallel} = 0$ that is initial velocity is perpendicular to $\vec{B}$ then particle revolves in xy-plane only. So its path is circle.



Circle

- If $v_{\perp} = 0$, then particle moves in straight line.

Straight line

Motion of charge particle in uniform electric field :

Suppose a charge particle of mass m and charge q moving in presence of uniform electric field $\vec{E}$,then we can write

$$m\frac{d^2\vec{r}}{d^2t} = q\vec{E} \qquad \text{...(1.f)}$$

$$\Rightarrow \qquad \frac{d\vec{r}}{dt} = \frac{q\vec{E}t}{m} + \vec{C}$$

Let the particle has initial velocity v_0 and position r_0.

$$\Rightarrow \qquad \frac{d\vec{r}}{dt} = \frac{q\vec{E}t}{m} + \vec{v}_0$$

$$\Rightarrow \qquad \boxed{\vec{r}(t) = \frac{1}{2}\left(\frac{q\vec{E}}{m}\right)t^2 + \vec{v}_0 t + \vec{r}_0} \qquad \text{...(1.g)}$$

Therefore, the charge particle moving with a constant acceleration $\left(\frac{qE}{m}\right)$, in the direction of E if $a > 0$ and in the opposite direction if $a < 0$. In perpendicular direction there is no motion.

Motion of charge particle in uniform electric and magnetic field:

Let a uniform magnetic field $\vec{B} = B\hat{k}$ and uniform electric field $\vec{E} = E_x\hat{i} + E_y\hat{j} + E_z\hat{k}$ exist in space.

Suppose a particle of mass m and charge q is thrown into the field at $t = 0$ from point (0, 0, 0) with initial velocity $v_{0x}\hat{i} + v_{0y}\hat{j} + v_{0z}\hat{k}$.

Let at some instant of time velocity of particle becomes $\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$.

Therefore equation of motion at some instant will be

$$m\frac{d\vec{v}}{dt} = \vec{F} = q\vec{E} + q\,\vec{v} \times \vec{B}$$

or $$m\left(\frac{dv_x}{dt}\hat{i}+\frac{dv_y}{dt}\hat{j}+\frac{dv_z}{dt}\hat{k}\right)=q\left[E_x\hat{i}+E_y\hat{j}+E_z\hat{k}+\left(v_x\hat{i}+v_y\hat{j}+v_z\hat{k}\right)\times B\hat{k}\right]$$

$$\therefore\quad \frac{dv_x}{dt}=\frac{q}{m}\left(E_x+v_yB\right) \qquad \dots(1)$$

$$\frac{dv_y}{dt}=\frac{q}{m}\left(E_y-v_xB\right) \qquad \dots(2)$$

$$\frac{dv_z}{dt}=\frac{q}{m}E_z \qquad \dots(3)$$

Differentiate equation (1) with respect to t and use equation of (2) to get

$$\frac{d^2v_x}{dt^2}=-\left(\frac{qB}{m}\right)^2\left(v_x-\frac{E_y}{B}\right)$$

put $v_x-\dfrac{E_y}{B}=v_x'$

$$\frac{d^2v_x'}{dt^2}=-\omega^2v_x', \qquad \omega=\frac{qB}{m}\ \text{(cyclotron frequecny)}$$

Solution of this equation is

$$v_x'=c_1\cos\omega t+c_2\sin\omega t$$

$$\therefore\quad v_x=\frac{E_y}{B}+c_1\cos\omega t+c_2\sin\omega t$$

using inital conditions $\left[t=0,\ v_x=v_{0x},\ \dfrac{dv_x}{dt}=\dfrac{q}{m}\left(E_x+v_{0y}B\right)\right]$

we get, $c_1=v_{0x}-\dfrac{E_y}{B},\quad c_2=\left(\dfrac{E_x}{B}+v_{0y}\right)$

$$\therefore\quad v_x=\frac{E_y}{B}+\left(v_{0x}-\frac{E_y}{B}\right)\cos\omega t+\left(\frac{E_x}{B}+v_{0y}\right)\sin\omega t \qquad \dots(4)$$

$$\therefore\quad x=\frac{E_y}{B}t+\frac{1}{\omega}\left(V_{0x}-\frac{E_y}{B}\right)\sin\omega t-\frac{1}{\omega}\left(\frac{E_x}{B}+V_{0y}\right)\cos\omega t+\frac{1}{\omega}\left(\frac{E_x}{B}+v_{0y}\right) \qquad \dots(5)$$

put v_x in equation (2) to get,

$$v_y=-\frac{E_x}{B}+\left(\frac{E_x}{B}+v_{0y}\right)\cos\omega t+\left(\frac{E_y}{B}-v_{0x}\right)\sin\omega t \qquad \dots(6)$$

$$y=-\frac{E_x}{B}t+\frac{1}{\omega}\left(\frac{E_x}{B}+v_{0y}\right)\sin\omega t+\frac{1}{\omega}\left(v_{0x}-\frac{E_y}{B}\right)\cos\omega t-\frac{1}{\omega}\left(v_{0x}-\frac{E_y}{B}\right) \qquad \dots(7)$$

Integrate equation (3) to get

$$v_z=\frac{q}{m}E_zt+v_{0z} \qquad \dots(8)$$

$$\therefore\quad z=\frac{qE_z}{2m}t^2+v_{0z}t \qquad \dots(9)$$

Special Cases :

- **Case I:** $\vec{E} = E_z\hat{k}$, i.e. $\vec{E} // \vec{B}$

  Here, $E_x = 0$, $E_y = 0$

$$\therefore \quad x = \frac{v_{0x}}{\omega}\sin\omega t - \frac{v_{0y}}{\omega}\cos\omega t + \frac{v_{0y}}{\omega}$$

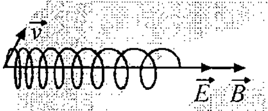
and $y = \dfrac{v_{0y}}{\omega}\sin\omega t + \dfrac{v_{0x}}{\omega}\cos\omega t - \dfrac{v_{0x}}{\omega}$

From above two equation we get

$$\left(x - \frac{v_{0y}}{\omega}\right)^2 + \left(y + \frac{v_{0x}}{\omega}\right) = \frac{v_{0x^2} + v_{0y^2}}{\omega^2} \qquad \ldots (10)$$

This is equation of circle in $x - y$ plane

using equation (9) and (10) we conclude that particle will move in helical path of constant radius but varying pitch.

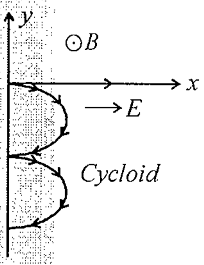


Case II : $\vec{E} \perp \vec{B}$

Let $\vec{E} = E_x\hat{i}$, $E_y = 0$, $E_z = 0$, i.e.

and initially particle is at rest.

Therefore from equation (5) and (7) weget



$$x = -\frac{E}{B\omega}\cos\omega t + \frac{E}{B\omega} \quad \text{or} \quad x - \frac{E}{B\omega} = \frac{E}{B\omega}\cos\omega t$$

$$y = -\frac{Et}{B} + \frac{E}{B\omega}\sin\omega t \quad \text{or} \quad y + \frac{Et}{B} = \frac{E}{B\omega}\sin\omega t$$

This is equation of circle whose center is moving in y direction with velocity $-\dfrac{E}{B}$.

This is also called cycloid.

Example: A beam of protons with a velocity 4 × 10^{-5} m/sec enters a uniform magnetic field of 0.3 tesla at an angle of 60° to the magnetic field. Find the radius of the helical path taken by the proton beam. Also find the pitch of the helix (which is the distance travelled by a proton in the beam parallel to the magnetic field during one period of revolution).

Soln. The parallel component of velocity $v_{||}$ to the magnetic field B is given by

$$v_{||} = v\cos 60° = 4\times 10^5 \times \tfrac{1}{2} = 2\times 10^5 \text{ m/s.}$$

The perpendicular component of velocity $v_\perp$ to the magnetic field is given by

$$v_\perp = v\sin 60° = 4\times 10^5 \times \tfrac{\sqrt{3}}{2} = 3.464\times 10^5 \text{ m/s.}$$

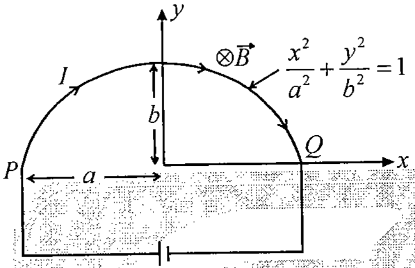
The path of the proton is a helix because it moves in a circular path under $v_\perp$ and advances along B under $v_{||}$. The radius of the helix is given by

$$r = \frac{mv_{\perp}}{qB} = \frac{(1.67\times10^{-27}\ \text{kg})\,(3.464\times10^{5}\ \text{m/s})}{(1.6\times10^{-19}\ \text{C})\,(0.3\ \text{tesla})} = 12\times10^{-3}\ \text{m} = 12\ \text{mm}$$

The period of rotation is $T = \dfrac{2\pi m}{qB} = \dfrac{2\times3.14\times1.67\times10^{-27}}{1.6\times10^{-19}\times0.3} = 21.85\times10^{-8}$ sec.

Therefore, pitch of the helix is $p = v_{\parallel}\times T = (2\times10^{5}\text{m/s})\,(21.85\times10^{-8}\ \text{s}) = 43.7\times10^{-3}\ \text{m} = 43.7\ \text{mm}$

Example: In figure shown there exists a uniform magnetic field $\vec{B}$ in the region. Calculate net force on the wire PQ due to the magnetic field.



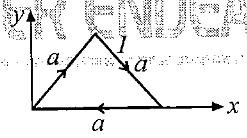
ln. Force on the wire is $\vec{F} = \int I\,d\vec{l}\times\vec{B}$

$$= I\int\left(dx\hat{i}+dy\hat{j}\right)\times B\left(-\hat{k}\right) = I\,B\int\left(dx\hat{j}-dy\hat{i}\right)$$

$$= IB\left[\hat{j}\int_{-a}^{+a}dx - \hat{i}\int_{0}^{0}dy\right] = 2IBa\hat{j}$$

Example: In the figure shown there exists a non uniform magnetic field $\vec{B} = x\hat{k}$. claculate force on the circular loop shown in figure.

ln. Magnetic force on the loop is given as



$$\vec{F} = \oint I\,d\vec{l}\times\vec{B}$$

$$= I\oint\left(dx\hat{i}+dy\hat{j}\right)\times kx\hat{k}$$

$$= Ik\left[\int xdx\left(-\hat{j}\right)+\int xdy\left(\hat{i}\right)\right] = Ik\left[-\hat{j}\int_{0}^{0}xdx+\hat{i}\int_{0}^{0}xdy\right]$$

$$= Ik\left\{-\hat{j}\left[\frac{x^2}{2}\right]_0^0+\hat{i}\left[\int_0^{a/2}\frac{y}{\sqrt{3}}dy+\int_{a/2}^{0}\left(a-\frac{y}{\sqrt{3}}\right)dy+\int_a^0 0\,dy\right]\right\}$$

$$= Ik\left\{0+\hat{i}\left[\frac{a^2}{8\sqrt{3}}-\frac{a^2}{2}+\frac{a^2}{8\sqrt{3}}\right]\right. = \frac{-Ika^2}{4}\left[2-\frac{1}{\sqrt{3}}\right]\hat{i}$$

Example: Find the value of the magnetic field required to maintain non-relativistic protons of energy 1 MeV in a circular orbit of radius 100 mm.

(Given: $m_p = 1.67\times10^{-27}kg, e = 1.6\times10^{-19}C$

Soln. $\frac{1}{2}mv^2 = 1\times10^6 \times 1.6\times10^{-19} J$

$$\Rightarrow \qquad v = \sqrt{\frac{3.2\times10^{-13}}{m}} = \sqrt{\frac{3.2\times10^{-13}}{1.67\times10^{-27}}} = \sqrt{1.91677\times10^{14}} = 1.384474\times10^7 \ m/\text{sec}$$

Now, for circular orbit

$$q\,vB = \frac{mv^2}{r}$$

$$B = \frac{mv}{qr} = \frac{1.67\times10^{-27}\times1.384474\times10^7}{1.6\times10^{-19}\times10^{-1}}$$

$$= \frac{1.67\times1.384474}{1.6} = 1.4450 \text{ Tesla}$$

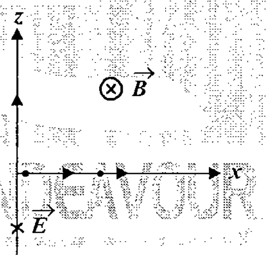
Example: A particle of mass 1×10^{-26} kg and charge $+1.6\times10^{-19}$ C travelling with a velocity 1.28×10^6 m/s in the +x-direction enters a region in which a uniform electric field E and a uniform magnetic field of induction B are present such that $E_x = E_y = 0,\ E_z = 102.4$ kV/m and $B_x = B_z = 0,\ B_y = 8\times10^{-2}$ Wb m^{-2}. The particle enters this region at the origin at time $t = 0$. Determine the location x, y and z coordinates of the particle at $t = 5\times10^{-6}$ s. If the electric field is switched off at this instant (with magnetic field still present), what will be the position of the particle at $t = 7.45\times10^{-6}$ s?

Soln. Velocity of particle in +x-direction $= 1.28\times10^6$ m/sec.

$E_x = E_y = 0,\ E_z = 102.4$ kV/m

$B_x = B_z = 0,\ B_y = 8\times10^{-2}$ Wb m^{-2}

These are shown in figure



Electric force $\vec{F}_e$ on the charge is given by

$|\vec{F}_e| = q\,\vec{E} = (1.6\times10^{-19})\,(102.4\times10^3)$

$\quad = 163.84\times10^{-16}$ along –z-direction

Magnetic force $\vec{F}_m$ on the charge is given

$|\vec{F}_m| = q\,v\,B$

$\quad = (1.6\times10^{-19})\,(1.28\times10^6)\,(8\times10^{-2})$

$\quad = 163.84\times10^{-16}$ along +z-direction

Thus there are two equal, opposite and collinear forces acting on the particle along z-axis. Hence the resultant force on particle is zero. The particle moves thus along the x-axis without deflection.

(a) At time $t = 5\times10^{-6}$ s, the distance x travelled by the particle.

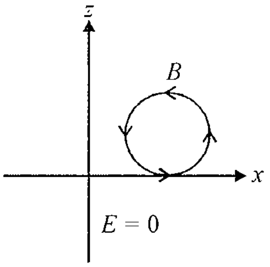
$x = vt = (1.28\times10^6)\,(5\times10^{-6}) = 6.4$ m

Therefore, coordinates of the particle = (6.4, 0, 0).

(b) When the electric field is switched off, there will be a force 163.84×10^{-16} along +z-axis acting on the particle moves in uniform magnetic field and describes a circular path in xz-plane as shown in figure. We know that when a particle of mass m and charge q is subjected to a magnetic field B acting perpendicular to it, the particle moves with velocity v in a circular trajectory whose radius r is given by

$$Bqv = mv^2/r \text{ or } r = \frac{mv}{Bq}$$

$$\therefore \quad r = \frac{(1\times10^{-26})(1.28\times10^{6})}{(8\times10^{-2})(1.6\times10^{-19})} = 1 \text{ m}$$



The time period t of revolution of particle due to magnetic field is given by

$$t = \frac{2\pi r}{v} = \frac{2\times3.14\times1}{1.28\times10^{6}} = 4.9\times10^{-6} \text{ sec.}$$

The time during which the magnetic field acts

$$= 7.45\times10^{-6} - 5\times10^{-6} = 2.45\times10^{-6} \text{ sec}$$

As the time period is 4.9×10^{-6} sec, and the magnetic field acts only for a time 2.45×10^{-6} sec i.e. for half the time period hence the particle covers a distance i.e. 2 metre along z-direction.
The coordinates of particle are (6.4 m, 0, 2 m).

ADDITIONAL SOLVED PROBLEMS

A magnetic field $\vec{\mathrm{B}} = \mathrm{B}_0\left(\hat{\mathrm{i}}+2\hat{\mathrm{j}}-4\hat{\mathrm{k}}\right)$ exists at a point. If a test charge moving with a velocity $\vec{\upsilon} = \upsilon_0\left(3\hat{\mathrm{i}}-\hat{\mathrm{j}}+2\hat{\mathrm{k}}\right)$ experiences no force at a certain point. The electric field at that point in SI units is

(a) $\vec{\mathrm{E}} = -\upsilon_0\mathrm{B}_0\left(3\hat{\mathrm{i}}-2\hat{\mathrm{j}}-4\hat{\mathrm{k}}\right)$ (b) $\vec{\mathrm{E}} = -\upsilon_0\mathrm{B}_0\left(\hat{\mathrm{i}}+\hat{\mathrm{j}}+7\hat{\mathrm{k}}\right)$ **[JEST 2012]**

(c) $\vec{\mathrm{E}} = \upsilon_0\mathrm{B}_0\left(14\hat{\mathrm{j}}+7\hat{\mathrm{k}}\right)$ (d) $\vec{\mathrm{E}} = -\upsilon_0\mathrm{B}_0\left(14\hat{\mathrm{j}}+7\hat{\mathrm{k}}\right)$

oln. $\vec{B} = B_0\left(\hat{i}+2\hat{j}-4\hat{k}\right) \Rightarrow \vec{v} = v_0\left(3\hat{i}-\hat{j}+2\hat{k}\right)$

Lorentz force, $\vec{F} = q\left(\vec{E}+\vec{v}\times\vec{B}\right) = 0$

$$\Rightarrow \quad q\vec{E} = -q\left(\vec{v}\times\vec{B}\right) \Rightarrow \quad \vec{E} = \vec{B}\times\vec{v}$$

$$= vB_0\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -4 \\ 3 & -1 & 2 \end{vmatrix} = \hat{i}(0)+\hat{j}(-14)+\hat{k}(-7) = -v_0B_0\left(14\hat{j}+7\hat{k}\right)$$

Correct option is (d)

A proton moves with a speed of 300 m/s in a circular orbit in the xy-plane in a magnetic field 1 tesla along the positive z-direction. When an electric field of 1 V/m is applied along the positive y-direction, the centre of the circular orbit

(a) remains stationary
(b) moves at 1 m/s along the negative x-direction
(c) moves at 1 m/s along the positive z-direction
(d) moves at 1 m/s along the positive x-direction

oln. The drift velocity of guiding centre is given by

$$\vec{V}_E = \frac{\vec{E}_\perp\times\vec{B}}{B^2} = \frac{\hat{y}\times\hat{z}}{1} = 1\,\hat{i}$$

Therefore, the particle will moves at 1 m/sec along positive x-direction.
Correct option is (d)

3. Four partices-electron, proton, He^+ and Li^+ are projected in a circular orbit of same radius perpendicular to a given magnetic field. Then the velocity will be smallest for

(a) electron (b) He^+ (c) proton (d) Li^+

Soln. We know that radius of the circle

$$r_L = \frac{mv}{qB}$$

Since, q, B and r are fixed

$$\therefore \quad v \propto \frac{1}{m}$$

Therefore, the velocity of He^+ will be smallest.

Correct option is (c)

4. A positively charged particle, with a charge q, enters a region in which there is a uniform electric field $\vec{E}$ and a uniform magnetic field $\vec{B}$, both directd parallel to the positive y-axis. At $t = 0$, the particle is at the origin and has a speed v_0 directed along the positive x-axis. The orbit of the particle, projected on the x-z plane, is a circle. Let T be the time taken to complete one revolution of this circle. The y-coordinate of the particle at $t = T$ is given by

(a) $\frac{\pi^2 mE}{2qB^2}$ (b) $\frac{2\pi^2 mE}{qB^2}$ (c) $\frac{\pi^2 mE}{qB^2} + \frac{v_0 \pi m}{qB}$ (d) $\frac{2\pi m v_0}{qB}$

Soln. Force along y direction $q\vec{E}$. Therefore, acceleration along y-axis, $f = \frac{q\vec{E}}{m}$

For circular motion we can write

$$\frac{mv^2}{r} = qvB \quad \Rightarrow \frac{v}{r} = \left(\frac{qB}{m}\right)$$

Time taken for complete one revolution.

$$T = \frac{2\pi r}{v} = \left(2\pi \frac{m}{qB}\right)$$

Therefore, the y-coordinate at a time t = T is given by

$$y = \frac{1}{2} f\, T^2 = \frac{1}{2}\frac{qE}{m}\frac{4\pi^2 m^2}{q^2B^2} = \frac{2\pi^2 mE}{qB^2}$$

Correct option is (b)

5. Consider an infinitely long straight cylindrical conductor of radius R with its axis along the z-direction, which carries a current of 1 A uniformly distributed over its cross section. Which of the following statements is correct?

(a) $\vec{\nabla} \times \vec{B} = 0$ everywhere

(b) $\vec{\nabla} \times \vec{B} = \frac{\mu_0}{\pi R^2}\hat{z}$ everywhere

(c) $\vec{\nabla} \times \vec{B} = 0$ for $r > R$

(d) $\vec{\nabla} \times \vec{B} = \frac{\mu_0}{\pi R^2}\hat{z}$ for $r > R$

Soln. We can write from Ampere law,

$$B = \frac{\mu_0 I}{2\pi r}\hat{\theta} \qquad r > R$$

$$= \frac{\mu_0 I r}{2\pi R^2}\hat{\theta} \; r < R$$

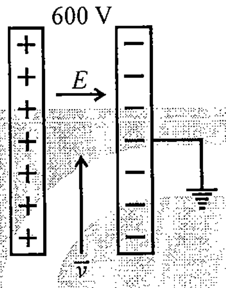
$$\therefore \quad \vec{\nabla}\times\vec{B} = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\mu_0 I}{2\pi}\right) = 0 \qquad r > R$$

$$\vec{\nabla}\times\vec{B} = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\mu_0 I r^2}{2\pi R}\right)\hat{z}$$

$$= \frac{\mu_0 I}{\pi R^2}\hat{z} \neq 0 \qquad r < R$$

Correct option is (c)

A potential difference of 600 volts is applied across the plates of a parallel plate condenser. The separation between the plates is 3 mm. An electron projected vertically parallel to the plates with a velocity of 2×10^6 m/sec moves undeflected between the plates. Find the magnitude and direction of the magnetic field in the region between the condenser plates (neglect the edge effects). Given : charge of the electron $= -16 \times 10^{-19}$ coulomb.



ln. We know that, electric intensity $E = \frac{V}{d} = \frac{600}{3\times10^{-3}} = 2\times10^5$ volt per metre.

Since the electron moves undeflected between the plates, the force due to magnetic field gets balanced by the force to electric field.

Therefore, $B\,e\,v = e\,E$, when B is magnetic induction and e is the charge on the electron.

Now, $B = \frac{E}{v} = \frac{2\times10^5}{2\times10^6} = 0.1\frac{\text{Weber}}{\text{m}^2}$

The direction of the field is perpendicular to the plane of the paper.

Which one of the following current densities, $\vec{J}$, can generate the magnetic vector potential $\vec{A} = \left(y^2\hat{i} + x^2\hat{j}\right)$?

(a) $\frac{2}{\mu_0}\left(x\hat{i} + y\hat{j}\right)$ (b) $-\frac{2}{\mu_0}\left(\hat{i} + \hat{j}\right)$ (c) $\frac{2}{\mu_0}\left(\hat{i} - \hat{j}\right)$ (d) $\frac{2}{\mu_0}\left(x\hat{i} - y\hat{j}\right)$

oln. We know that. $\vec{\nabla}\times\vec{B} = \mu_0 J$

$$\vec{\nabla}\times\vec{A} = B$$

$$\therefore \quad J = \frac{1}{\mu_0}\left(\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right)\right) = \frac{1}{\mu_0}\left(\vec{\nabla}\times(2x-2y)\hat{k}\right)$$

$$= \frac{1}{\mu_0}2\left(-\hat{i}-\hat{j}\right) = -\frac{2}{\mu_0}\left(\hat{i}+\hat{j}\right)$$

Correct option is (b)

8. The vector potential in a region is given as $\vec{A}(x,y,z) = -y\hat{i} + 2x\hat{j}$. The associated magnetic induction is $\vec{B}$ is

(a) $\hat{i}+\hat{k}$ (b) $3\hat{k}$ (c) $-\hat{i}+2\hat{j}$ (d) $-\hat{i}+\hat{j}+\hat{k}$

Soln. $\vec{A} = -y\hat{i} + 2x\hat{j}$

$$\vec{B} = \vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y & 2x & 0 \end{vmatrix} = (2+1)\hat{k} = 3\hat{k}$$

Correct option is (b)

9. The value of $\oint_c \vec{A}.d\vec{\ell}$ along a square loop of side L in a uniform field $\vec{A}$ is :

(a) 0 (b) 2LA (c) 4LA (d) L^2A

Soln. $\int \vec{A}\cdot d\vec{\ell} = \int(\vec{\nabla}\times\vec{A})\cdot d\vec{s} = 0\,(\vec{A}\text{ is uniform})$

Correct option is (a)

10. Which of the following expressions for a vector potential $\vec{A}$ **DOES NOT** represent a uniform magnetic field of magnitude B_0 along the z-direction?

(a) $\vec{A} = (0, B_0x, 0)$ (b) $\vec{A} = (-B_0y, 0, 0)$ (c) $\vec{A} = \left(\dfrac{B_0x}{2}, \dfrac{B_0y}{2}, 0\right)$ (d) $\vec{A} = \left(-\dfrac{B_0y}{2}, \dfrac{B_0x}{2}, 0\right)$

Soln. We know that, $\vec{\nabla}\times\vec{A} = \vec{B}$

$\therefore$ $\vec{\nabla}\times(B_0x\hat{j}) = B_0\hat{k}$ is uniform field.

$\vec{\nabla}\times(-B_0y\hat{j}) = -B_0\hat{k}$ is uniform field.

$$\vec{\nabla}\times\left(\frac{B_0x}{2}\hat{i} + \frac{B_0y}{2}\hat{j}\right) = 0$$

$\vec{\nabla}\times\left(-\dfrac{B_0y}{2}\hat{i} + \dfrac{B_0x}{2}\hat{j}\right) = \dfrac{B_0}{2}(\hat{k})$ is uniform field.

Correct option is (c)

Statement for linked Answer Q.13 and Q.14:

An infinitely long hollow cylinder of radius R carrying a surface charge density s is rotated about its cylinderical axis with a constant angular speed w

11. The magnitude of the surface current is :

(a) $\sigma R\omega$ (b) $2\sigma R\omega$ (c) $\pi\sigma R\omega$ (d) $2\pi\sigma R\omega$

Soln. We know that current,

$$I = \frac{Q}{T} = \frac{2\pi R\sigma}{T} = \omega R\sigma$$

Correct option is (a)

2. The magnitude of vector potential inside the cylinder at a distance from its axis is :

(a) $2\mu_0\sigma R\omega r$ (b) $\mu_0\sigma R\omega r$ (c) $\frac{1}{2}\mu_0\sigma R\omega r$ (d) $\frac{1}{4}\mu_0\sigma R\omega r$

oln. We know that,

$$\oint \vec{A}\cdot d\vec{\ell} = \int \vec{B}\cdot d\vec{s} \qquad \Rightarrow A\cdot 2\pi r = \mu_0\sigma\omega R\cdot \pi r^2$$

$$\Rightarrow \qquad A = \frac{1}{2}\mu_0\sigma\omega Rr$$

Correct option is (c)

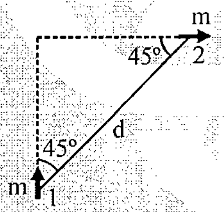
3. A toroidal coil has N closely-wound turns. Assume the current through the coil to be I and the toroid is filled with a magnetic material of relative permittivity μ_r. The magnitude of magnetic induction $\vec{B}$ inside the toroid, at a radial distance r from the axis, is given by

(a) $\mu_r\mu_0 NIr$ (b) $\frac{\mu_r\mu_0 NI}{r}$ (c) $\frac{\mu_r\mu_0 NI}{2\pi r}$ (d) $2\pi\mu_r\mu_0 NIr$

oln. $\oint \vec{B}\cdot d\vec{\ell} = \mu_r\mu_0 IN \Rightarrow B = \frac{\mu_r\mu_0 IN}{2\pi r}$

Correct option is (c)

4. Two magnetic dipoles of magnitude m each are placed in a plane as shown



The energy of interaction is given by

(a) Zero (b) $\frac{\mu_0}{4\pi}\frac{m^2}{d^3}$ (c) $\frac{3\mu_0}{2\pi}\frac{m^2}{d^3}$ (d) $\frac{3\mu_0}{8\pi}\frac{m^2}{d^3}$

oln. The interaction energy between two magnetic dipole moment is given by

$$w_{12} = \frac{\mu_0}{4\pi}\left(3\frac{(\vec{m}_1\cdot\vec{r}_{12})}{r_{12}^5} - \frac{\vec{m}_1}{r_{12}^3}\right)\cdot\vec{m}_2$$

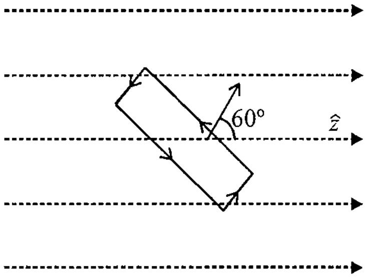
$$= \frac{\mu_0}{4\pi}\left\{3\frac{(m_1\cdot r_{12})(m_2\cdot r_{12})}{r_{12}^5} - \frac{\vec{m}_1\cdot\vec{m}_2}{r_{12}^3}\right\}$$

$$= \frac{\mu_0}{4\pi}\left[\frac{3m^2\cos^2 45^\circ}{d^3} - 0\right] \qquad (\because\ m_1 \perp m_2)$$

$$= \frac{\mu_0}{4\pi}\frac{3m^2}{d^3}\frac{1}{2} = \frac{\mu_0}{8\pi}\frac{3m^2}{d^3}$$

Correct option is (d)

15. In a constant magnetic field of 0.6 Tesla along the z direction, find the value of the path integral $\oint \vec{A}.\vec{dl}$ in the units of (Tesla m^2) on a square loop of side length $(1/\sqrt{2})$ meters. The normal to the loop makes an angle of 60° to the z-axis, as shown in he figure. The answer should be up to two decimal places. ____



Soln. $\oint \vec{A}\cdot d\vec{\ell} = \oint \left(\vec{\nabla}\times\vec{A}\right)\cdot d\vec{S} = \oint \vec{B}\cdot d\vec{S} = BA\cos\theta = 0.6\times\frac{1}{2}\cdot\cos 60° = 0.6\times\frac{1}{2}\times\frac{1}{2} = 0.15$

16. If the vector potential, $\vec{A} = \alpha x\hat{x} + 2y\hat{y} - 3z\hat{z}$
satisfies the Coulomb gauge, the value of the constant α is ____________

Soln. The condition of Coulomb gauge

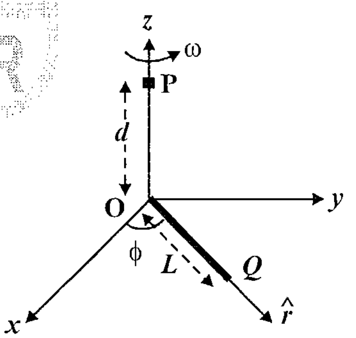
$$\nabla\cdot A = 0$$

$$\Rightarrow \quad \frac{\partial}{\partial x}(\alpha x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(-3z) = 0$$

$$\Rightarrow \quad \alpha + 2 - 3 = 0$$

$$\Rightarrow \quad \alpha = 1$$

17. A rod of length L with uniform charge density λ per unit length is in the xy-plane and rotating about z-axis passing through one of its edge with an angular velocity $\vec{\omega}$ as shown in the figure below. $\left(\hat{r}, \hat{\phi}, \hat{z}\right)$ refer to the unit vectors at Q, $\vec{A}$ is the vector potential at a distance d from the origin O along z-axis for $d >> L$ and $\vec{J}$ is the current density due to the motion of the rod.
Which one of the following statements is correct?



(a) $\vec{J}$ along $\hat{r}$; $\vec{A}$ along $\hat{z}$; $\left|\vec{A}\right| \propto \frac{1}{d}$

(b) $\vec{J}$ along $\hat{\phi}$; $\vec{A}$ along $\hat{\phi}$; $\left|\vec{A}\right| \propto \frac{1}{d^2}$

(c) $\vec{J}$ along $\hat{r}$; $\vec{A}$ along $\hat{z}$; $\left|\vec{A}\right| \propto \frac{1}{d^2}$

(d) $\vec{J}$ along $\hat{\phi}$; $\vec{A}$ along $\hat{\phi}$; $\left|\vec{A}\right| \propto \frac{1}{d}$

Soln. Since rod is rotated in the xy-plane the direction of J and A will be along $\hat{\phi}$

Since, $d >> L$, so we can consider rod as magnetic dipole, the vector potential due to magnetic dipole as

$$A = \frac{\mu_0}{4\pi}\frac{\vec{M}\times\vec{d}}{d^3} \Rightarrow A \propto \frac{1}{d^2}$$

Correct option is (b)

. The magnetic field (in A m^{-1}) inside a long solid cylindrical conductor of radius a = 0.1 *m* is, $\vec{H}=\frac{10^4}{r}\left[\frac{1}{\alpha^2}\sin(\alpha r)-\frac{r}{\alpha}\cos(\alpha r)\right]\hat{\phi}$, where $\alpha=\frac{\pi}{2a}$. What is the total current (in A) in the conductor?

(a) $\frac{\pi}{2a}$ (b) $\frac{800}{\pi}$ (c) $\frac{400}{\pi}$ (d) $\frac{300}{\pi}$

ln. $\vec{J}=\left(\vec{\nabla}\times\vec{H}\right)$

$$=\frac{1}{r}\frac{\partial}{\partial r}\left[10^4\left(\frac{1}{\alpha^2}\sin(\alpha r)-\frac{r}{\alpha}\cos(\alpha r)\right)\right]\hat{k}$$

$$=\frac{10^4}{r}\left[\frac{1}{\alpha}\cos\alpha r-\frac{1}{\alpha}\cos(\alpha r)+r\sin(\alpha r)\right]\hat{k}=10^4\sin(\alpha r)\hat{k}$$

$$I=\int J\cdot ds=10^4\int_0^a \sin\alpha r\,2\pi r dr=10^4 2\pi\int_0^a \sin(\alpha r)rdr$$

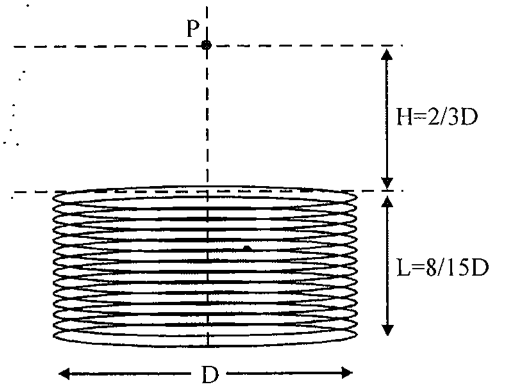
$$=2\pi\times10^4\left[-\frac{(\cos(\alpha r))\cdot r}{\alpha}+\frac{\sin(\alpha r)}{\alpha^2}\right]_0^a=2\pi\times10^4\left[-\frac{\cos\alpha a}{\alpha}\cdot a+\frac{\sin\alpha a}{\alpha^2}\right]$$

$$=2\pi\times10^4\left[-\frac{\cos\frac{\pi}{2}}{\alpha}\cdot a+\frac{\sin\frac{\pi}{2}}{\left(\frac{\pi^2}{4a^2}\right)}\right]=\frac{8a^2}{\pi}\times10^{+4}=\frac{800}{\pi}$$

Correct option is (b)

. A short solenoid with n turns per unit length has diameter D and length L = 8D/15, as shown in the figure, and it carries a constant current I. The magnetic field B at a point P on the axis of the solenoid at a distance H = 2D/3 from its near end (see figure) is **[TIFR 2014]**

[use $\int dx\left(1+x^2\right)^{-3/2}=x\left(1+x^2\right)^{-1/2}$]



(a) $\frac{4}{65}\mu_0 nI$ (b) $\frac{4}{13}\mu_0 nI$ (c) $\frac{24}{15}\mu_0 nI$ (d) $\frac{112}{65}\mu_0 nI$

oln. Let consider an elementary ring of thickness *dx* at distance *x*.

$$dB = \frac{\mu_0 Indxa^2}{2\left(a^2 + x^2\right)^{3/2}}$$

$$B = \frac{\mu_0 In}{2}\int_H^{H+L} \frac{a^2 dx}{\left(a^2 + x^2\right)^{3/2}} = \frac{\mu_0 In}{2}\int_{2/3D}^{6/5D} \frac{\left(D/2\right)^2 dx}{\left(\left(D/2\right)^2 + x^2\right)^{3/2}}$$

$$= \frac{\mu_0 In}{2}\int_{4/3}^{12/5} \frac{dz}{\left(1+z^2\right)^{3/2}} = \frac{\mu_0 In}{2}\left[\frac{z}{\sqrt{1+z^2}}\right]_{4/3}^{12/5}$$

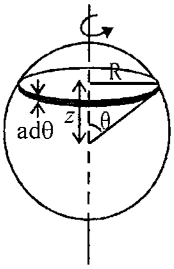
$$= \frac{\mu_0 In}{2}\left[\frac{12/5}{\sqrt{1+\frac{144}{25}}} - \frac{4/3}{\sqrt{1+\frac{10}{9}}}\right] = \frac{\mu_0 In}{2}\left[\frac{12/5}{13/5} - \frac{4/3}{5/3}\right]$$

$$= \frac{\mu_0 In}{2}\left(\frac{12}{13} - \frac{4}{5}\right) = \frac{\mu_0 In}{2}\left(\frac{60-52}{65}\right) = \frac{4}{65}\mu_0 In$$

Correct option is (a)

20. A spherical shell of radius a, carrying a uniform surface charge density σ, is set spinning with a angular velocity ω about a diamter. Find the magnetic field at the centre of the shell. Also find the equivalent magnetic dipole moment.

Soln. Let us consider the sphere is rotating about z-axis. Take an elementary ring Therefore, the current associated with this ring is



$$dI = kd\ell = kRd\theta = \sigma vRd\theta = \sigma\omega a\sin\theta ad\theta = \omega\sigma a^2 \sin\theta d\theta$$

Therefore, dipole moment of the current ring is

$$d\vec{m} = \text{current} \times \text{area} = \omega\sigma a^2 \sin\theta d\theta \pi a^2 \sin^2\theta$$

Therefore, total magnetic moment,

$$\vec{m} = \hat{z}\pi\omega a^4\sigma\int_0^{\pi}\sin^3\theta d\theta = \frac{4}{3}\pi\sigma\omega a^4\hat{z}$$

The magnetic field at O due to the current ring is

$$d\vec{B} = \hat{z}\frac{\mu_0}{4\pi}\frac{2\pi(dI)(R)^2}{\left\{(R)^2+(z)^2\right\}^{3/2}} = \hat{z}\frac{\mu_0}{2}\frac{\omega\sigma a^2\sin\theta d\theta a^2\sin^2\theta}{a^3} = \frac{\mu_0}{2}\omega\sigma a\sin^3\theta d\theta$$

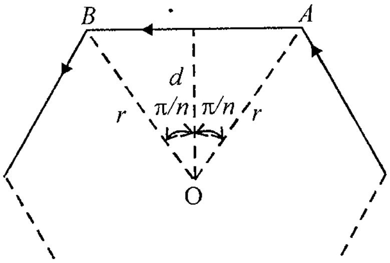
Total magnetic fielda at O is

$$\vec{B} = \frac{\mu_0}{2}\omega\sigma a\int_0^{\pi}\sin^3\theta d\theta = \frac{2\mu_0\omega\sigma a}{3}\hat{z}$$

21. A wire carrying current I is bent into the form of a regular n-sided polygon. The distance of any vertex from the centre of the polygon is r. Find the magnetic field at its centre. Discuss the case $n \to \infty$.

Soln. An arm of a polygon of n sides makes an angle of $\frac{2\pi}{n}$ at the centre (O). Now the field at O due the entire wire will be equal to n times the field due to a single side. Thus,

$$B = n\cdot\frac{\mu_0 I}{4\pi d}\cdot\left(\sin\frac{\pi}{n} + \sin\frac{\pi}{n}\right) = \frac{\mu_0 nI}{4\pi}\cdot\frac{2\sin\frac{\pi}{n}}{r\cos\frac{\pi}{n}} \text{ or } B = \frac{\mu_0 nI}{2\pi r}\tan\frac{\pi}{n}$$



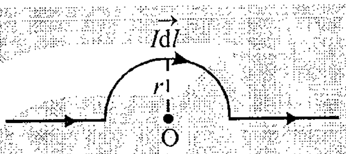
In the limit $n \to \infty$, $\tan\frac{\pi}{n} \to \frac{\pi}{n}$ and then $B = \frac{\mu_0 I}{2r}$

which is equal to the magnetic field at the center of a circular loop of radius r carrying current I.

. An infinitely long wire carries a current I. It is bent so as to have a semicircular detour around the origin, with radius r. Calculate the magnetic field at the origin.

ln. The straight parts of the wire do not contribute to the field at O since for them $Id\vec{\ell} \times \vec{r} = 0$. Considering the semicircular part we get for the field at O as

$$\vec{B} = \frac{\mu_0}{4\pi r^2}\int \frac{Id\vec{\ell} \times \vec{r}}{r^3}$$



Obviously, $d\vec{\ell} \perp \vec{r}$ and $\vec{B}$ points into the page. The magnitude of $\vec{B}$ as

$$\vec{B} = \frac{\mu_0 I}{4\pi r^2}\int d\ell = \frac{\mu_0 I}{4\pi r^2} \cdot \pi r = \frac{\mu_0 I}{4r}$$

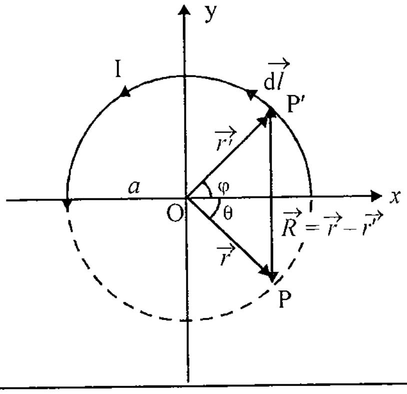
. A semicircular wire carries a steady current I. Show that the magnetic field at the point P on the other half of the circle is given by $\vec{B} = \vec{k}\frac{\mu_0 I}{8\pi a}\ell n\left[\frac{\tan\frac{\theta+\pi}{4}}{\tan\frac{\theta}{4}}\right]$

where 'a' is the radius to the circle, $\hat{k}$ unit outward normal to the plane of the circle and θ is the angle which the line joining the centre of the circle and P makes with the diameter of the semicircle.

oln. We can write for the position vectors of the source point P' as $\vec{r}\,' = a\cos\varphi\hat{i} + a\sin\varphi\hat{j}$ and that of the field point P as $\vec{r} = \cos\theta\hat{i} - a\sin\theta\hat{j}$.

Therefore, $\vec{R} = \vec{r} - \vec{r}\,' = a(\cos\theta - \cos\varphi)\hat{i} - a(\sin\theta + \sin\varphi)\hat{j}$

And $d\vec{\ell} = -a\sin\varphi d\varphi\hat{i} + a\cos\varphi d\varphi\hat{j}$



$$\therefore\quad d\vec{\ell}\times\vec{R}=a^2 d\varphi\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -\sin\varphi & \cos\varphi & 0\\ \cos\theta-\cos\varphi & -(\sin\theta+\sin\varphi) & 0\end{vmatrix}=\hat{k}a^2 d\varphi\{1-\cos(\theta+\varphi)\}$$

Now, $$\vec{B}=\frac{\mu_0 I}{4\pi}\int\frac{d\vec{\ell}\times\vec{R}}{R^3}=\hat{k}\frac{\mu_0 I a^2}{4\pi}\int_0^{\pi}\frac{\{1-\cos(\theta+\varphi)\}d\varphi}{\left\{a^2(\cos\theta-\cos\varphi)^2+a^2(\sin\theta+\sin\varphi)\right\}^{3/2}}$$

Or, $$\vec{B}=\vec{k}\frac{\mu_0 I}{4\pi a}\int_0^{\pi}\frac{\{1-\cos(\theta+\varphi)\}d\varphi}{\{2-2\cos(\theta+\varphi)\}^{3/2}}=\vec{k}\frac{\mu_0 I}{4\pi a\cdot 2^{3/2}}\int_0^{\pi}\frac{d\varphi}{\sqrt{1-\cos(\theta+\varphi)}}$$

$$=\vec{k}\frac{\mu_0 I}{4\pi a\cdot 2^{3/2}}\int_0^{\pi}\frac{d\varphi}{\sqrt{2}\cdot\sin\dfrac{\theta+\varphi}{2}}=\vec{k}\frac{\mu_0 I}{16\pi a}\cdot\left[2\ell n\left|\tan\frac{\theta+\varphi}{4}\right|\right]_0^{\pi}$$

$$=\vec{k}\frac{\mu_0 I}{8\pi a}\ell n\left[\frac{\tan\dfrac{\theta+\pi}{4}}{\tan\dfrac{\theta}{4}}\right]$$

24. Consider a long straight wire carrying current I current along +z-direction. Assuming the ion for $\vec{B}$ due to it and the relation $\vec{B}=\vec{\nabla}\times\vec{A}$, determine the vector potential $\vec{A}$ at any outside point. Check that $\vec{\nabla}\cdot\vec{A}=0$ and $\vec{\nabla}\times\vec{A}=\vec{B}$. Also, find $\vec{A}$ at any inside point. Assume that the current is uniformly distributed over the cross-section fo the wire of radius a.

Soln. Using cylinderical coordinates the magnetic field at any outside point (r,θ,z) can be written in the form

$$\vec{B}=\frac{\mu_0}{4\pi}\cdot\frac{2I}{r}\hat{\theta}$$

As the current is along +z-direction $\vec{A}$ can be written in the form $\vec{A}=A\hat{z}$. Now,

$$\vec{B}=\vec{\nabla}\times\vec{A}\text{ or }\frac{\mu_0 I}{2\pi i}\hat{\theta}=\frac{1}{r}\begin{vmatrix}\hat{r} & r\hat{\theta} & \hat{z}\\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial z}\\ 0 & 0 & A\end{vmatrix}$$

Or, $$\frac{\mu_0 I}{2\pi r}=-\frac{\partial A}{\partial r}$$

Or, $$A=-\frac{\mu_0 I}{2\pi}\int\frac{dr}{r}=-\frac{\mu_0}{2\pi}I\ell nr+\text{constant}$$

$\vec{A}$ points in the direction of the current and hence we can write

$$\vec{A}=-\hat{z}\frac{\mu_0 I}{2\pi}\ell n\frac{r}{b}$$

where, b is an arbitrary constant.

$$\vec{\nabla}\cdot\vec{A}=\frac{\partial A_z}{\partial z}=0,\ \vec{\nabla}\times\vec{A}=-\hat{\theta}\frac{\partial A_z}{\partial r}=\hat{\theta}\frac{\mu_0 I}{2\pi r}=\hat{\theta}\frac{\mu_0}{4\pi}\cdot\frac{2I}{r}$$

The magnetic field at any inside point is given by Ampere's law as

$$\oint\vec{B}\cdot d\vec{\ell}=\mu_0\cdot J\cdot\pi r^2 \text{ or } B\cdot 2\pi r=\mu_0 J\pi r^2$$

Or, $$B=\frac{\mu_0}{2}Jr=\frac{\mu_0 I}{2\pi a^2}\cdot r$$

Vectorially, $\vec{B}=\dfrac{\mu_0 I}{2\pi a^2}r\hat{\theta}$

As before,

$$\frac{\mu_0 I}{2\pi a^2}r=-\frac{\partial A}{\partial r} \text{ or } A=-\frac{\mu_0 I}{4\pi a^2}r^2+\text{constant}$$

∴ $$\vec{A}=-\hat{z}\frac{\mu_0 I}{4\pi a^2}r^2+\text{constant}$$

5. If $\vec{B}$ is uniform show that $\vec{A}=-\dfrac{1}{2}(\vec{r}\times\vec{B})$, where $\vec{r}$ is the position vector of the point in question. Check that $\vec{\nabla}\cdot\vec{A}=0$ and $\vec{\nabla}\times\vec{A}=\vec{B}$

oln. Let $\vec{B}=B\hat{k}$ and $\vec{C}=\vec{r}\times\vec{B}$

∴ $$\vec{\nabla}\times\vec{C}=\vec{\nabla}\times(\vec{r}\times\vec{B})=(\vec{B}\cdot\vec{\nabla})\vec{B}+\vec{r}(\vec{\nabla}\cdot\vec{B})-\vec{B}(\vec{\nabla}\cdot\vec{r})$$

Now, $$(\vec{B}\cdot\vec{\nabla})\vec{r}=B\frac{\partial}{\partial z}(x\hat{i}+y\hat{j}+z\hat{k})=B\hat{k}=\vec{B}$$

$$(\vec{r}\cdot\vec{\nabla})\vec{B}=0 \text{ as } \vec{B} \text{ is uniform}$$

$$\vec{\nabla}\cdot\vec{B}=0$$

And $$\vec{\nabla}\cdot\vec{r}=3$$

Therefore, $$\vec{\nabla}\times\vec{C}=\vec{B}-0+0-3\vec{B}=-2\vec{B}$$

Or, $$\vec{B}=-\frac{1}{2}\vec{\nabla}\times\vec{C}=\vec{\nabla}\times\left(-\frac{1}{2}\vec{C}\right)=\vec{\nabla}\times\vec{A}$$

∴ $$\vec{A}=-\frac{1}{2}\vec{C}=-\frac{1}{2}(\vec{r}\times\vec{B})$$

$$\vec{\nabla}\cdot\vec{A}=\frac{1}{2}\vec{\nabla}\cdot(\vec{r}\times\vec{B})=-\frac{1}{2}\left[\vec{B}\cdot(\vec{\nabla}\times\vec{r})-\vec{r}\cdot(\vec{\nabla}\cdot\vec{B})\right]$$

Now, $\vec{\nabla}\cdot\vec{A}=0; \vec{\nabla}\times\vec{B}=0$ as $\vec{B}$ is uniform.

Therefore, $\vec{\nabla}\cdot\vec{A}=0$

$$\vec{\nabla}\times\vec{A}=-\frac{1}{2}\vec{\nabla}\times(\vec{r}\times\vec{B})=-\frac{1}{2}\left[\vec{r}(\vec{\nabla}\cdot\vec{B})-\vec{B}(\vec{\nabla}\cdot\vec{r})+(\vec{B}\cdot\vec{\nabla})\vec{r}-(\vec{r}\cdot\vec{\nabla})\vec{B}\right]$$

Now, $\vec{\nabla}\cdot\vec{A}=0$ and $(\vec{r}\cdot\vec{\nabla})\vec{B}=0$ because $\vec{B}$ is uniform, $\vec{\nabla}\cdot\vec{r}=3$ and

$$\left(\vec{B}\cdot\vec{\nabla}\right)\vec{r} = \left(B_x\frac{\partial}{\partial x}+B_y\frac{\partial}{\partial y}+B_z\frac{\partial}{\partial z}\right)\left(x\hat{i}+y\hat{j}+z\hat{k}\right)$$

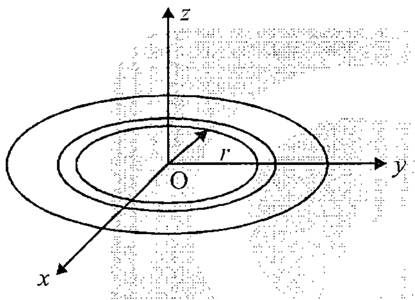
$$= B_x\hat{i}+B_y\hat{j}+B_z\hat{k} = \vec{B}$$

Thus, $\vec{\nabla}\times\vec{A} = -\frac{1}{2}\left[-3\vec{B}+\vec{B}\right] = \vec{B}$

26. A thin disk of radius 'a' carrying uniform surface charge density σ is rotating with constant angular velocity ω about its axis (z-axis). Suppose there is a uniform magnetic field $\vec{B} = \vec{B}\hat{j}$. Show that the torque acting on the disk is of magnitude $\frac{1}{4}\pi\sigma\omega Ba^4$.

Soln. Consider a ring of radius 'r' and radial thickness dr. The charge on the ring is $\sigma\cdot 2\pi r dr$. It constitutes a current loop of current.

$$dI = kdr = \sigma vdr = \sigma\omega rdr = \sigma\omega rdr$$



The magnetic moment of this ring is

$$d\vec{m} = \text{current}\times\text{area of the loop} = dI\cdot\pi r^2\hat{k} = \pi\sigma\omega r^2 dr\hat{k}$$

Therefore, torque on the ring is $d\vec{\tau} = d\vec{m}\times\vec{B} = -\hat{i}\,\pi\sigma\omega Br^3 dr$

Hence, total torque on the disk is

$$\vec{\tau} = -\hat{i}\,\pi\sigma\omega B\int_0^a r^3 dr = -\hat{i}\frac{1}{4}\pi\sigma\omega Ba^4$$

27. A circular coil of radius a carrying current I is lying in the xy-plane with its centre at the origin. There is an external uniform field $\vec{B} = \left(\frac{1}{\sqrt{2}}\hat{i}+\frac{1}{\sqrt{2}}\hat{j}\right)B_0$. Find the force and the torque acting on the coil.

Soln. Let the current is flowing in such a direction that the magnetic dipole moment of the coil becomes $\vec{m} = I\pi a^2\hat{k}$. Therefore, force on the coil

$$\vec{F} = \left(\vec{m}\cdot\vec{\nabla}\right)\vec{B} = I\pi a^2\frac{\partial\vec{B}}{\partial z} = 0$$

Torque on the coil is

$$\vec{\tau} = \vec{m}\times\vec{B} = I\pi a^2\cdot\frac{B_0}{\sqrt{2}}\left(\hat{k}\times\hat{i}+\hat{k}\times\hat{j}\right) = \frac{\pi a^2 IB_0}{\sqrt{2}}\left(\hat{j}-\hat{i}\right)$$

. Find the force between two ideal magnetic dipoles of moments $\vec{m}_1$ and $\vec{m}_2$ separated by a distance r. Assume that $\vec{m}_1$ and $\vec{m}_2$ point in the direction of the vector joining them.

ln. The force on the dipole $\vec{m}_2$ due to $\vec{m}_1$ is given by

$$\vec{F}=\left(\vec{m}_2\cdot\vec{\nabla}\right)\vec{B}_1 \qquad \text{where, } \vec{B}_1=\frac{\mu_0}{4\pi}\left[-\frac{\vec{m}_1}{r^3}+\frac{3\left(\vec{m}_1\cdot\vec{r}\right)\vec{r}}{r^5}\right]$$

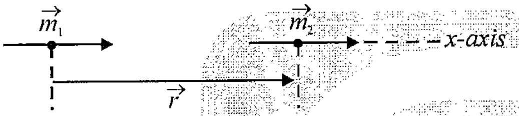
is the magnetic field at the position of $\vec{m}_2$ due to $\vec{m}_1$. Now from figure below.

$$\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}=r\hat{i},\ \vec{m}_1=m_1\hat{i},\quad \vec{m}_2=m_2\hat{i}$$

$$\therefore \quad \vec{F}=\left(m_2\frac{\partial}{\partial x}\right)\left[\frac{\mu_0}{4\pi r^3}\left(-m_1+3m_1\right)\right]\hat{i}=\hat{i}\,\frac{\mu_0 m_1 m_2}{2}\frac{\partial}{\partial x}\left(\frac{1}{x^3}\right)$$

$$=-\hat{i}\,\frac{3\mu_0}{2\pi}\frac{m_1 m_2}{x^4}-=-\hat{i}\,\frac{\mu_0}{4\pi}\cdot\frac{6m_1 m_2}{x^4}$$

a force of attraction along the line joining the dipoles.



). Find the force between two parallel ideal coplanar magnetic dipoles of moments $\vec{m}_1$ and $\vec{m}_2$ separated by a distance r. Assume that the vector joining them is perpendicular to the directions of the dipole, which point in the same direction.

ln. As in the solved problem (31), we can write

$$\vec{F}=\left(\vec{m}_2\cdot\vec{\nabla}\right)\vec{B}_1=\frac{\mu_0}{4\pi}\left(\vec{m}_2\cdot\vec{\nabla}\right)\left[-\frac{\vec{m}_1}{r^3}+\frac{3\left(\vec{m}_1\cdot\vec{r}\right)\vec{r}}{r^3}\right]$$

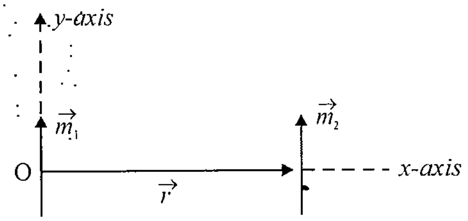
Here, $\vec{m}_1=m_1\hat{j},\vec{m}_2=m_2\hat{j}$, $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}=r\hat{i}$

$$\therefore \quad \vec{F}=\frac{\mu_0}{4\pi}m_2\frac{\partial}{\partial y}\left[-\frac{m_1}{x^3}\hat{j}+\frac{3\left(m_1 y\right)x\hat{i}}{x^5}\right]$$

[Note that $\vec{m}_1\cdot\vec{r}=0$ but $\vec{\nabla}\left(\vec{m}_1\cdot\vec{r}\right)\neq 0$]

$$=\frac{\mu_0 m_2}{4\pi}\cdot\frac{3m_1}{x^4}\hat{i}=\frac{\mu_0}{4\pi}\cdot\frac{3m_1 m_2}{r^4}\hat{i}$$

A force of repulsion along the line joining the dipoles.



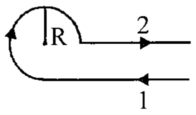
0. Find the magnetic field at the point O if a current carrying wire has the shape shown in figure. The radius of the current part of the wire is R. The linear parts assume to be very long.

oln. Magnetic field at the centre due to linear part 1

$$B_1=\frac{\mu_0 I}{4\pi R}$$

Magnetic field due to the curve,

$$B_2 = \frac{3}{4}\frac{\mu_0 I}{2R} = \frac{\mu_0 I}{4\pi}\frac{3\pi}{2R}$$



Magnetic field due to linear part 2 is

$$B_3 = 0$$

Therefore, total field

$$B = B_1 + B_2 + B_3 = \frac{\mu_0}{4\pi}\frac{I}{R} + \frac{\mu_0 I}{4\pi}\frac{3\pi}{R} + 0 \quad = \frac{\mu_0}{4\pi}\frac{I}{R}\left[1 + \frac{3\pi}{R}\right]$$

31. A steady current I flows down a long cylindrical conductor of radius a. The current density at a distance r from the axis of the conductor is proportional to r. Calculate the magnetic field both inside and outside the wire as a function of r.

Soln. Let us consider an Amperian loop in the form of a circle of radius $r\left(r < a\right)$ with its centre on the axis of the cylinder. Symmetry of the problem indicates that $\vec{B}$ is tangential to the Amperian loop everywhere and also of constant magnitude all over it.

So, by applying Ampere's law we get

$$\oint \vec{B}\cdot d\vec{\ell} = \mu_0 I_{enc}$$

Or, $$B\cdot 2\pi r = \mu_0\int_0^r 2\pi r dr\cdot J\left(r\right) = 2\pi\mu_0 K\int_0^r r^2 dr = 2\pi\mu_0 K\frac{r^3}{3}$$

where we assume that $J\left(r\right) = Kr$, K being a proportionality constant.

$\therefore$ $$B = \mu_0 K\frac{r^3}{3} \text{ for } r \le a$$

For any external point $r > a, I_{enc} = I$ and then

$$B\cdot 2\pi r = \mu_0 I \text{ or } B = \frac{\mu_0 I}{2\pi r} \text{ for } r \ge a$$

Total current,

$$I = \int_0^a 2\pi r dr\cdot J\left(r\right) = 2\pi K\int_0^a r^2 dr = 2\pi K\frac{a^3}{3} \text{ or } K = \frac{3I}{2\pi a^3}$$

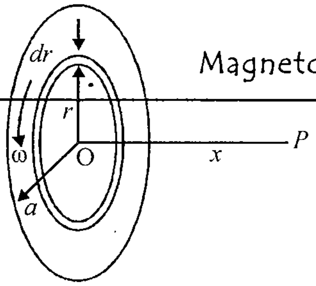
Thus, $$B = \frac{\mu_0 I r^2}{2\pi a^3} \text{ for } r \le a$$

32. A thin disk of radius a carrying uniform surface charge density σ, is rotating at constant angular velocity ω about its own axis. Find the magnetic field at any axial point. Also, find the magnetic field at the centre.

Soln. Consider a ring of radius r and radial thickness dr. The speed of any point on this ring is $v = \omega\cdot r$. The ring constitutes a current loop of current $dI = \sigma\cdot\omega r dr$. The magnetic field due to this ring at an axial point P at a distance x from the centre is

$$dB = \frac{\mu_0}{2}\cdot\frac{dI\cdot r^2}{\left(r^2 + x^2\right)^{3/2}} = \frac{\mu_0\sigma\omega}{2}\cdot\frac{r^3 dr}{\left(r^2 + x^2\right)^{3/2}}$$

Therefore, total field at P will be



$$B=\frac{\mu_0\sigma\omega}{2}\int_0^a\frac{r^3dr}{\left(r^2+x^2\right)^{3/2}}$$

Substituting $r^2+x^2=z^2$ we get

$$B=\frac{\mu_0\sigma\omega}{2}\int_{|x|}^{\sqrt{a^2+x^2}}\left(1-\frac{x^2}{z^2}\right)dz=\frac{\mu_0\sigma\omega}{2}\left[\frac{a^2+2x^2}{\sqrt{a^2+x^2}}-2|x|\right]$$

If σ is positive then $\vec{B}$ will be directed along $\overline{OP}$. At the centre, $x=0$ and

$$B=\frac{\mu_0\sigma\omega a}{2}$$

3. The space between two long coaxial conducting cylinders of radius a and b ($a<b$) and length L is filled with a slightly conducting material of conducting σ. A potential difference is maintained between the cylinders such that a cylinderically symmetric current flows between them. Find the resistance against such a current flow?

oln. Consider a cylindrical shell of radius r and $r+dr$. The area of the shell through which current flows is $A(r)=2\pi rL$ and length of current flow is dr.

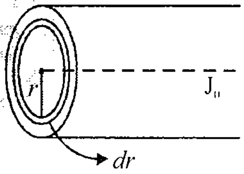
Therefore, resistance of this shell is $dR=\frac{1}{\sigma}\frac{dr}{2\pi rL}$

Total resistance, $R=\int dR=\int_a^b\frac{1}{\sigma}\frac{dr}{2\pi rL}=\frac{\ell n(b/a)}{2\pi L\sigma}$

4. In a conducting wire of radius a current density varies with radial distance according to the law $\vec{J}=\left(\frac{r}{a}\right)\vec{J}_0$ is parallel to the axis. Calculate the total current through the wire.

oln.

So, current through this ring,



$$I=\int\vec{J}\cdot d\vec{s}=\int Jrdrd\phi=\int_0^a J_0\cdot\frac{r}{a}rdr\int_0^{2\pi}d\phi=\frac{r}{a}J_0 2\pi rdr$$

Total current,

$$I=\frac{2\pi J_0}{a}\int_0^a r^2dr=\frac{2\pi J_0}{a}\frac{a^3}{3}=\frac{2\pi}{3}a^2J_0$$

5. The magnetic scalar potential in a certain region is given by

$$\phi\cdot(x,y,t)=\frac{B_0}{\mu_0}z\left(1+\frac{x}{a}\right)$$

where, B_0 and a are constant. Find the static magnetic field.

oln. $\vec{B}=-\mu_0\vec{\nabla}\phi$

$$=-B_0\left\{\frac{z}{a}\hat{i}+\hat{k}\left(1+\frac{x}{a}\right)\right\}$$

36. Find the steady current density that can give rise to a magnetic field

$$\vec{B} = k\left(\hat{i}y - \hat{j}x\right), \text{ where } k \text{ is a constant.}$$

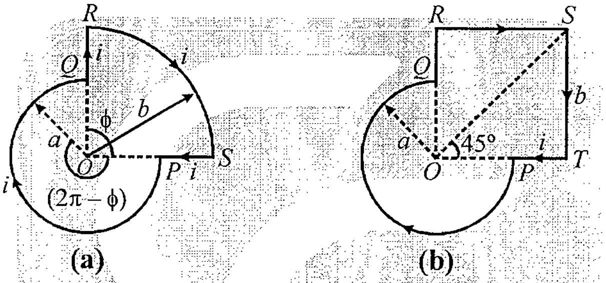
Soln. Since, we know that

$$\mu_0 \vec{J} = \vec{\nabla} \times \vec{B}$$

$$\Rightarrow \qquad \vec{J} = \frac{k}{\mu_0}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = \frac{k}{\mu_0}[-1-1]\hat{k} = -\frac{2k}{\mu_0}\hat{k}$$

37. Find the magnetic induction of the field at the point O of a loop with current i, whose shape is illustrated
(a) In figure (a), the radii a and b, as well as the angle ϕ are known,
(b) In figure (b), the radius a and the side b are known

Soln. (a) Magnetic field at O due to segment PQ



$$B_1 = \frac{\mu_0}{4\pi} \times \frac{i}{a} \times (2\pi - \phi)$$

Magnetic field at O due to parts QR and PS is zero because $d\vec{\ell}$ and $\vec{r}$ are parallel

$$\left[\because \; d\vec{B} = \frac{\mu_0}{4\pi} \times i \times \frac{(d\vec{\ell} \times \vec{r})}{r^3}\right]$$

Magnetic field at O due arc RS which subtend an angle ϕ is given by

$$B_2 = \frac{\mu_0}{4\pi} \times \frac{i}{b} \times \phi$$

Therefore, $B = B_1 + B_2 = \frac{\mu_0}{4\pi} \times i \times \left[\frac{2\pi - \phi}{a} + \frac{\phi}{b}\right]$

(b) Field at O due to arc PQ

$$B_1 = \frac{\mu_0}{4\pi} \times \frac{i}{a} \times \frac{3\pi}{2}$$

Field at O due to parts QR and PT is zero

Field at O due to part MN $\quad B_2 = \frac{\mu_0}{4\pi} \times \frac{i}{b} \times \sin 45° = \frac{\mu_0}{4\pi} \times \frac{i}{b} \times \frac{1}{\sqrt{2}}$

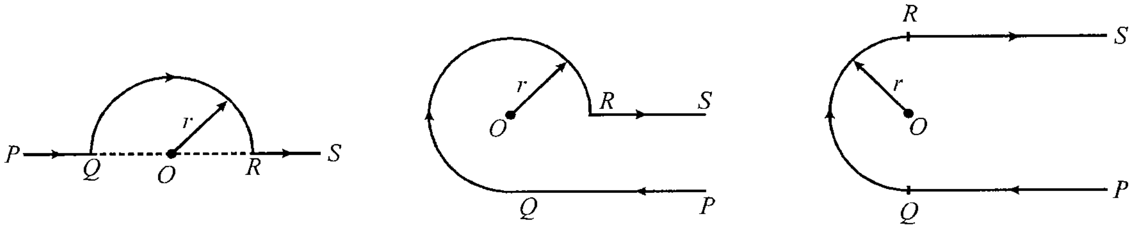
Field at O due to part ST

$$B_3 = \frac{\mu_0}{4\pi} \times \frac{i}{b} \times \left(\sin 0° + \sin 45°\right) = \frac{\mu_0}{4\pi} \times \frac{i}{b} \times \frac{1}{\sqrt{2}}$$

Now, $B = B_1 + B_2 + B_3$

Therefore, $B = \frac{\mu_0}{4\pi} i \left[\frac{3\pi}{2a} + \frac{1}{\sqrt{2}b} + \frac{1}{\sqrt{2}b}\right] = \frac{\mu_0 i}{4\pi}\left[\frac{3\pi}{2a} + \frac{\sqrt{2}}{b}\right].$

3. Find the magnetic induction of the field at the point O if a current carrying wire has the shape shown in figure (a), (b) and (c). The radius of the curved part of the wire is r, the linear parts are assumed to be very long.



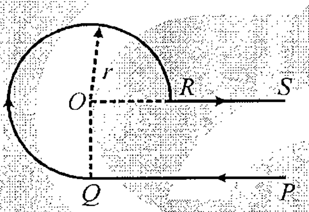
(a) (b) (c)

oln. (a) Magnetic field at O due to straight wires [PQ and RS] is zero.
The magnetic field at O due to circular part QR is given by

$$B_1 = \frac{\mu_0}{4\pi}\times\frac{i}{r}\times\pi$$

$$\therefore \quad B = 0+\frac{\mu_0}{4\pi}\times\frac{i\pi}{r}+0=\frac{\mu_0 i}{4r}$$

(b) See below figure :



(i) The magnetic induction at O due to wire RS is zero because O lies on the line RS itself when extended backward.

(ii) Magnetic induction at O due to circular portion QR of the wire is given by $B_1 = \frac{3}{4}\cdot\frac{\mu_0 i}{2r}$

(iii) Magnetic induction at O due to infinitely long straight wire LK is given by

$$B_2 = \frac{\mu_0}{4\pi}\times\frac{i}{r}[\sin\theta_1+\sin\theta_2]$$

Here $\theta_1 = 0$ and $\theta_2 = \frac{\pi}{2}$. Therefore, $B_2 = \frac{\mu_0 i}{4\pi r}$

Now, $$B = B_1 + B_2 = \frac{\mu_0 i}{4\pi r}\times\frac{3\pi}{2}\times\frac{\mu_0 i}{4\pi r}=\frac{\mu_0 i}{4\pi r}\left[1+\frac{3\pi}{2}\right]$$

(c) In this case, the magnetic field will be

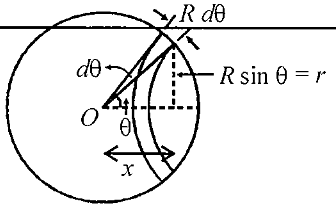
$$B = \frac{\mu_0}{4\pi}\frac{i}{r}+\frac{\mu_0}{4\pi}\frac{i}{r}\pi+\frac{\mu_0}{4\pi}\frac{i}{r} \text{ or } B=\frac{\mu_0}{4\pi}\frac{i}{r}(2+\pi).$$

9. A non-conducting sphere of radius $R = 50$ mm charged uniformly with surface density $\sigma = 10.0\ \mu\text{C/m}^2$ rotates with an angular velocity $\omega = 70$ rad/s about the axis passing through its centre. Find the magnetic induction at the centre of the sphere.

oln. **See figure :** The sphere is non-conducting and hence the charge will remain uniform on the outer surface. Consider a ring as shown in figure.

$$di = k\,d\ell = \sigma v R\,d\theta$$
$$di = \sigma\omega R\sin\theta\,R\,d\theta$$

Magnetic induction at the centre O due to the ring $= \dfrac{\mu_0 dir^2}{\left(r^2+x^2\right)^{3/2}}$

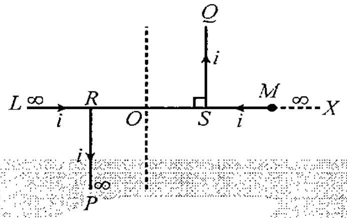


$$dB = \frac{\mu_0}{2}\sigma\omega R\sin^3\theta\, d\theta$$

$$\therefore \quad B = \frac{\mu_0}{2}\sigma\omega R\int_0^{\pi/2}\sin^3\theta\, d\theta = \frac{2}{2}\mu_0\sigma\omega R$$

Substituting the values, we get $B = 26\ pT$

40. A pair of stationary and infinitely long bent wires are placed in the XY-plane as shown in figure.



The wires carry current of $i = 10$ amperes each as shown. The segments L and M are along the X-axis. The segments P and Q are parallel to the Y-axis such that $OS = OR = 0.2$ m. Find the magnitude and direction of the magnetic induction at the origin O.

Soln. Here we consider the magnetic induction $\vec{B}$ at O due to each segment separately.

(i) $\vec{B}$ at O due to segment L is zero because current element $d\vec{\ell}$ and $\vec{r}$ are parallel $(d\vec{\ell}\times\vec{r} = 0)$.

(ii) $\vec{B}$ at O due to segment M is also zero.

(iii) $\vec{B}$ at O due to segments Q

$$B_1 = \frac{1}{2}\left(\frac{\mu_0}{2\pi}\frac{i}{r}\right)$$

$$\therefore \quad B_1 = \frac{1}{2}\left(\frac{4\pi\times10^{-7}}{2\pi}\times\frac{10}{10.02}\right)\text{weber/m}^2$$

$$= 0.5\times10^{-4}\ \text{Weber/m}^2\ \text{(Out of page)}$$

(iv) $\vec{B}$ at O due to segments P

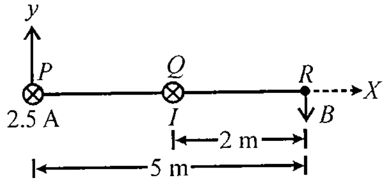
Magnetic field, $B_2 = 0.5\times10^{-4}$ Weber/m^2 (Out of page)

$$\text{Now, } \vec{B} = \vec{B}_1 + \vec{B}_2$$

$$= (0.5\times10^{-4} + 0.5\times10^{-4})$$

$$= 10^{-4}\ \text{Weber/m}^2$$

Two long parallel wires carrying currents 2.5 amperes and I amperes in the same direction (directed into the plane of the paper) are held at P and Q respectively such that they are perpendicular to the plane of paper. The points P and Q are located at a distance of 5 metres and 2 metres respectively from a collinear point R (See below figure).



(i) An electron moving with a velocity of 4×10^5 m/s along the positive x-direction experiences a force of magnitude 3.2×10^{-20} N at the point R. Find the value of I.

(ii) Find all the positions at which a third long parallel wire carrying a current of magnitude 2.5 amperes may be placed so that the magnetic induction at R is zero.

ln. The magnetic field induction at point R due to current in wires P and Q

$$B = B_1 + B_2 = \left(\frac{\mu_0 I_1}{2\pi r_1} + \frac{\mu_0 I_2}{2\pi r_2}\right) \text{ along } Y\text{-axis}$$

or $$\vec{B} = -2 \times 10^{-7}\left[\frac{2.5}{5} + \frac{I}{2}\right]\hat{j}$$

(i) We know that $\vec{F} = q\vec{v}\,\vec{B}$

Therefore, $$\vec{F} = \left[(-1.6 \times 10^{-19}) \times (4 \times 10^5\,\hat{i}) \times \{10^{-7}(1+I)\,\hat{j}\}\right]$$

or $$\vec{F} = 6.4 \times 10^{-21}(1+I)\,\hat{k}$$

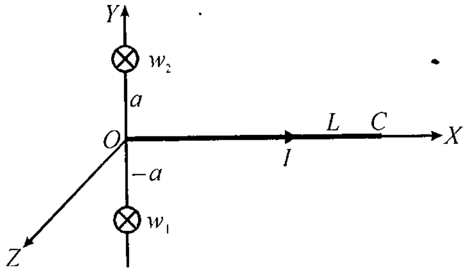
or $$\vec{F} = 6.4 \times 10^{-21}(1+I) = 3.2 \times 10^{-20} \text{ Newton}$$

(ii) Let the third wire be placed at a distance x on the left. Suppose the wire carries a current upwards perpendicular to the plane of paper.

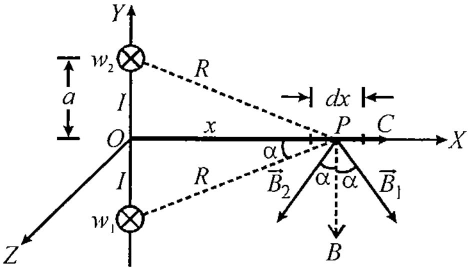
Now, $$B_1 + B_2 + B_3 = \frac{\mu_0}{2\pi}\left(\frac{I_1}{r_1} + \frac{I_2}{r_2} - \frac{I_3}{x}\right) = 0 \Rightarrow \frac{2.5}{5} + \frac{4}{2} - \frac{2.5}{x} = 0 \textbf{ or } x = \pm 1$$

at a distance of ± 1 metre from R along x-axis.

2. A straight segment OC (of length L metre) of a circuit carrying a current I amp. is placed along the X-axis (figure). Two infinitely long straight wires w_1 and w_2, each extending from $Z = -\infty$ to $+\infty$, are fixed at $y = -a$ metre and $y = +a$ metre respectively as shown in the figure. If the wires w_1 and w_2 each carry a current I amp. into the plane of the paper, obtain the expression for the force acting on the segment OC.



Soln. Consider a small element P of length dx at a distance x from the end O as shown in figure.



The field at P due to w_1 and w_2 are as shown in figure

$$B_1 = \left(\frac{\mu_0 I}{2\pi}\right)\frac{1}{R}$$

Similarly, magnetic field $\vec{B}_2$ produced at $P(x, 0, 0)$ due to wire w_2

$$B_2 = \left(\frac{\mu_0 I}{2\pi}\right)\frac{1}{R}$$

The components of $\mathbf{B}_1$ and $\mathbf{B}_2$ along X-axis cancel while those along Y-axis are added up. Hence the total field at P is given by

$$B = 2\left(\frac{\mu_0 I}{2\pi}\right)\frac{1}{R}\cos\alpha = 2\left(\frac{\mu_0 I}{2\pi}\right)\frac{1}{R}\cdot\left(\frac{x}{R}\right)$$

$$= \frac{\mu_0 I}{\pi}\frac{x}{(x^2+a^2)} \text{ along } Y\text{-axis} \qquad \left\{\because R^2 = (x^2+a^2)\right\}$$

The force $d\vec{F}$ acting on the current element due to field is

$$d\vec{F} = I\, d\vec{\ell} \times \vec{B} = -I\, dx\,(\hat{i}\times\hat{j})B$$

$$= \left(\frac{\mu_0 I^2}{\pi}\right)(-\hat{k})\frac{x\,dx}{(x^2+a^2)}$$

Force **F** on the whole conductor

$$\vec{F} = \left(\frac{\mu_0 I^2}{\pi}\right)(-\hat{k})\int_0^L \frac{x\,dx}{(x^2+a^2)}$$

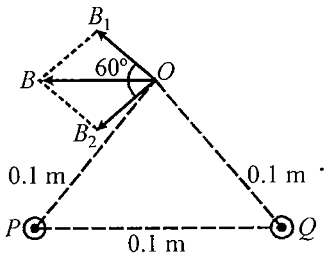
$$= \left(\frac{\mu_0 I^2}{\pi}\right)(-\hat{k})\frac{1}{2}\int_0^L \frac{2x\,dx}{(x^2+a^2)}$$

$$= \left(\frac{\mu_0 I^2}{2\pi}\right)(-\hat{k})\left[\log(x^2+a^2)\right]_0^L = \left(\frac{\mu_0 I^2}{2\pi}\right)(-\hat{k})\left[\log(L^2+a^2) - \log a^2\right]$$

$$= \left(\frac{\mu_0 I^2}{2\pi}\right)(-\hat{k})\log_e\left(\frac{L^2+a^2}{a^2}\right) = \frac{\mu_0 I^2}{2\pi}\log_e\left(\frac{L^2+a^2}{a^2}\right) \text{acting along } (-\hat{k}) \text{ direction.}$$

3. Two straight infinitely long and thin parallel wires are spaced 0.1 m apart and carry a current of 10 ampere each. Find the magnetic field at a point distant 0.1 m from both wires in the two cases when the currents are in the (i) same and (ii) opposite directions.

oln. (i) When the currents are in the same direction. The situation is shown in figure :



Let the currents in both the wires are perpendicular to the plane of the paper and directed upward.

The magnetic field B_1 at point O due to current carrying wire P is

$$B_1 = \frac{\mu_0}{2\pi}\frac{i}{r} = 2\times 10^{-7}\left(\frac{10}{0.1}\right) = 2\times 10^{-5} \text{ tesla.}$$

The direction of B_1 at point O will be perpendicular to OP and directed towards left

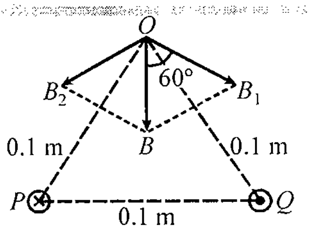
Similarly, the magnetic field B_2 at point O due to current carrying wire Q is $B_2 = 2\times 10^{-5}$ tesla.

The direction of B_2 will be perpendicular to OQ and directed towards left.

The resultant magnetic field B at O is

$$\begin{aligned} B &= B_1\cos 30^\circ + B_2\cos 30^\circ \quad (\because \angle B_1 O B_2 = 60^\circ) \\ &= (2\times 10^{-5})\cos 30^\circ + (2\times 10^{-5})\cos 30^\circ \\ &= 2\times(2\times 10^{-5})\cos 30^\circ = 2\times(2\times 10^{-5})\times(\sqrt{3}/2) \\ &= 3.46\times 10^{-5} \text{ Tesla (parallel to QP)} \end{aligned}$$

(ii) When the currents are in opposite directions.
The situation is shown in figure :



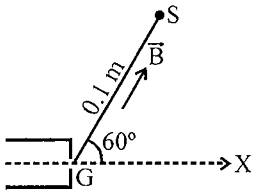
Let the current in wire P be directed into the paper while in wire Q be directed upward.

Here $B_1 = 2\times 10^{-5}$ tesla and $B_2 = 2\times 10^{-5}$ tesla

The direction of B_1 will be perpendicular to OP towards right and the direction of B_2 will be perpendicular to OQ towards left. The resultant magnetic field at O is given by

$$\begin{aligned} B &= B_1\cos 60 + B_2\cos 60 \\ &= 2\times(2\times 10^{-5})\times\tfrac{1}{2} \\ &= 2\times 10^{-5} \text{ Tesla (perpendicular to PQ)} \end{aligned}$$

44. An electron gun G emits electron of energy 2 keV travelling in the positive X-direction. The electron are required to hit the spot S where $GS = 0.1$ m, and the line GS makes an angle 60° with the X-axis as shown in figure.



A uniform magnetic field B parallel to GS exists in the region outside the electron gun. Find the minimum value of B needed to make the electrons hit S.

Soln. Given that the energy of electrons is 2 keV.

Thus, $\frac{1}{2}mv^2 = 2\text{ keV}$

$$= 2\times10^3\times(1.6\times10^{-19})\text{ J}$$

or $\quad m^2v^2 = 2m\times2\times10^3\times(1.6\times10^{-19})$

or $\quad mv = [2(9.1\times10^{-31})\times2\times10^3\times(1.6\times10^{-19})]^{1/2} \quad \ldots(1)$

$$= 7.63\times10^{-25}$$

Electrons will move in spiral path. Therefore, electrons will hit the target only if

$GS = \text{N.pitch}$
$N = \text{integer}$

$$GS = N\cdot\frac{2\pi m}{qB}V_{11}$$

$$B = N\cdot\frac{2\pi m}{q(GS)}V_{11} \text{ for } B \text{ to be minimum } N = 1$$

$$B_{\min} = \frac{2\pi m}{q(GS)}V_{11} = \frac{2\pi m}{q(GS)}V\cos60^\circ = \frac{\pi mV}{q(GS)} = 1.5\times10^{-4}\text{ Tesla}$$

45. An electron beam passes through a perndicular magnetic field 2×10^{-3} weber/m^2 and an electric field 3.4×10^4 volts/m both acting simultaneously. If the path of electrons remains undeviated, calculate the speed of the electrons. If the electric field is removed, what will be the radius of the electron path ?

Soln. For the undeviated path, therefore Net force is zero $qE = qvB$ or $v = \frac{E}{B} = \frac{3.4\times10^4}{2\times10^{-3}} = 1.7\times10^7\ m/s$

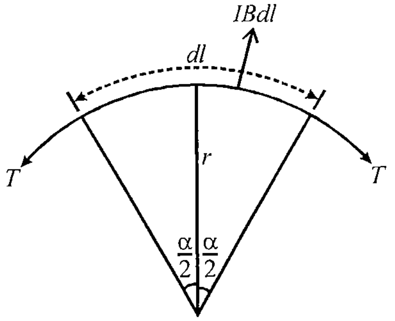
When the electric field is removed, then $Bev = \frac{mv^2}{r}$

or $\quad r = \frac{mv}{Be} = \frac{9\times10^{-31}\times(1.7\times10^7)}{(2\times10^{-3})\times(1.6\times10^{-19})} \qquad (\because m = 9\times10^{-31}\text{ kg})$

$$= 4.78\times10^{-2}\ m$$

i. A loop of flexible conducting wire of length 0.5 m lies in a magnetic field of 1.0 tesla perpendicular to the plane of the loop. Show that when a current is passed through the loop, it opens into a circle. Also calculate the tension developed in the wire if the current is of 1.57 amp.

ln. Consider an element of loop is shown figure :



Force on element $= i\,B\,dl$. This is perpendicular to the element. Hence the loop opens into a circle.

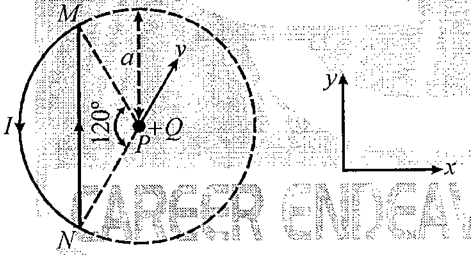
For equilibrium of element, $2T\sin\frac{\alpha}{2} = i\,B\,dl$

$$T\alpha = i\,B\,dl \qquad \left(\because \sin\frac{\alpha}{2} = \frac{\alpha}{2}\right)$$

$$\text{or}\quad T = \frac{i\,B\,dl}{\alpha} = i\,B\,r \qquad \left(\because \frac{dl}{r} = \alpha\right)$$

$$= \frac{i\,Bl}{2\pi} = \frac{1.57\times 1.0\times 0.5}{2\times 3.14} = 0.125 \text{ Newton.}$$

7. A wire loop carrying a current I is placed in the xy-plane as shown in below figure :



(a) If a particle with charge $+Q$ and mass m is placed at the centre P and given a velocity v along NP (see the above figure), find its instantaneous acceleration.

(b) If an external uniform magnetic induction field $\vec{B} = B\hat{i}$ is applied, find the force and the torque acting on the loop due to this field.

oln. **(a)** Magnetic field at P is due to circular are subtending an angle 120° and the straight wire MN.

The magnetic field B_1 due to circular arc is given by

The magnetic field due to straight wire

$$B_2 = \frac{\mu_0\, nI}{4\pi R}(\sin\theta_1 + \sin\theta_2), \text{ where } \theta_1 = \theta_2 = 60°.$$

The distance of the wire from $P = R = a\cos 60° = \frac{a}{2}$

$$\text{Therefore, } B_2 = \frac{\mu_0\, nI}{4\pi(a/2)}\left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}\,\mu_0 I}{2\pi a} \text{ along } (-Z) \text{ axis}$$

Net magnetic field, $B = B_1 - B_2 = \frac{\mu_0 I}{6a} - \frac{\sqrt{3}}{2}\frac{\mu_0 I}{\pi a} = \frac{\mu_0 I}{a}\left[\frac{1}{6} - \frac{\sqrt{3}}{2\pi}\right]$ along Z-axis

$$\vec{B} = \frac{\mu_0 I}{a}\left[\frac{1}{6} - \frac{\sqrt{3}}{2\pi}\right]\hat{k}$$

The velocity v makes an angle of 60° with X-axis.

Therefore, $\vec{\mathbf{v}} = \hat{i}\, v\cos 60 + \hat{j}\, v\sin 60 = \hat{i}\frac{v}{2} + \hat{j}\frac{\sqrt{3}}{2}v = \frac{v}{2}(\hat{i} + \sqrt{3}\hat{j})$

The magnetic force on charge q is given by $\vec{F}_m = Q(\vec{v} \times \vec{B})$

Therefore, acceleration $\vec{a} = \frac{\vec{F}_m}{m} = \frac{Q}{m}\vec{v} \times \vec{B}$

or $\vec{a} = \frac{Q}{m}\left[\frac{v}{2}\left(\hat{i} + \sqrt{3}\hat{j}\right) \times \frac{\mu_0 I}{a}\left(\frac{1}{6} - \frac{\sqrt{3}}{2\pi}\right)\hat{k}\right]$

Solve this expression and obtained the magnitude of a.

$$a = \frac{0.109\, \mu_0 I Q v}{m}$$

(b) Since, $\vec{B}$ is uniform net force on loop is is

Torque $= I(\vec{A} \times \vec{B})$

Area of loop = Area of sector – area of triangle zero

$$= \frac{1}{3}\pi a^2 - \frac{1}{2}\text{base} \times \text{height} = \frac{1}{3}\pi a^2 - \frac{1}{2}\left(\frac{a}{2}\right)(\sqrt{3}a) = \frac{\pi a^2}{3} - \frac{\sqrt{3}a^2}{4} = (0.6136a^2)$$

Therefore, Torque $= I(0.6136a^2\,\hat{k}) \times (B\,\hat{i}) = 0.6136a^2\, IB\hat{j}$

48. Three infinitely long thin wires, each carrying current i in the same direction are in the XY-plane of a gravity free space. The central wire is along the Y-axis while the other two are along $x = \pm d$.

(i) Find the locus of the points for which magnetic field B is zero.

(ii) If the central wire is displaced along the Z-direction by a small amount and released, show that it will execute simple harmonic motion. If the linear density of the wire is λ. Find the frequency of oscillation.

Soln. **(i)** The situation is shown in below figure :

The magnetic field at any point is given by $B = \mu_0 i / 2\pi x$. The resultant magnetic field due to all the three wires will be zero only at the position of the middle wire itself. Thus locus of the points for which the magnetic field **B** is zero is $x = 0$

(ii) When the middle wire is slightly displaced, the field on the position of wire is

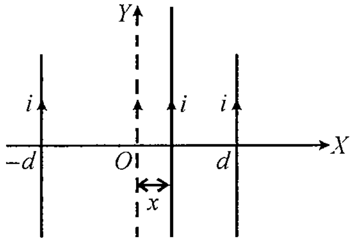
$$B = \frac{\mu_0}{2\pi}\left[\frac{i}{(d-x)} - \frac{i}{(d+x)}\right] \approx \frac{\mu_0}{\pi} i \frac{x}{d^2}$$

Restoring force per unit length of the middle wire is

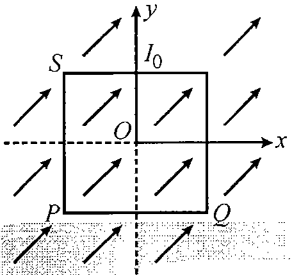
$$\frac{F}{L} = -\frac{\mu_0 i^2}{\pi d^2}x = \lambda \times \text{acceleration.}$$

Therefore, acceleration $= -\dfrac{\mu_0 i^2 x}{\pi d^2 \lambda} \propto$ displacement

Hence, $T = 2\pi\sqrt{\left(\dfrac{\pi d^2 \lambda}{\mu_0 i^2}\right)}$ or $f = \dfrac{1}{2\pi}\sqrt{\left(\dfrac{\mu_0 i^2}{\pi d^2 \lambda}\right)}$.



9. A uniform; constant magnetic field B is directed at an angle of 45° to the x-axis in xy-plane. $PQRS$ is a rigid square wire frame carrying a steady current I_0, with its centre at the origin O. At time $t = 0$, the frame is at rest in the position shown in the figure, with its side parallel to x and y axes. Each side of the frame is of mass M and length L.



(a) What is the torque $\vec{\tau}$ about O acting on the frame due to magnetic field ?

(b) Find the angle by which the frame rotates under the action of this torque in a short interval of time Δt, and the axis about which this rotation occurs. (Δt is so short that any variation in the torque during this interval may be neglected).

Given : the moment of inertia of the frame about an axis through its centre perpendicular to its plane is $\frac{4}{3}mL^2$.

oln. (a) $\vec{B} = B(\cos 45\,\hat{i} + \sin 45\,\hat{j})$ and $\vec{A} = L^2\hat{k}$

Now, $\vec{\tau} = I_0(\vec{A}\times\vec{B}) = I_0\left[L^2\hat{k}\times B(\cos 45\,\hat{i} + \sin 45\,\hat{j})\right]$

$$= I_0 L^2 B\left[\cos 45°\,\hat{j} + \sin 45°(-\hat{i})\right] = \frac{I_0 L^2 B}{\sqrt{2}}(-\hat{i}+\hat{j})$$

(b) Axis of rotation $= \dfrac{(-\hat{i}+\hat{j})}{\sqrt{2}}$ i.e., the frame will experience a torque about axis SQ in its plane and magnitude of torque $\tau = I_0 L^2 B$

Applying the theorem of perpendicular axes, the moment of inertia about the axis of rotation SQ

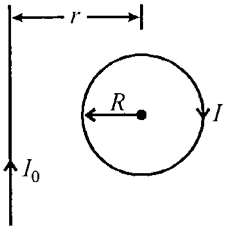
$$= \frac{1}{2}\left[\frac{4}{3}ML^2\right] = \frac{2}{3}ML^2$$

We know that $\tau = I\alpha$

Therefore, angular acceleration $\alpha = \dfrac{\tau}{I}$ or $\alpha = \dfrac{I_0 L^2 B}{\frac{2}{3}ML^2} = \dfrac{3I_0 B}{2M}$

Further, angular rotation $= \dfrac{1}{2}\alpha(\Delta t)^2 = \dfrac{1}{2}\times\dfrac{3I_0 B}{2M}\times\Delta t^2 = \dfrac{3I_0 B}{4M}\times(\Delta t)^2$.

50. A long straight wire is coplaner with a current carrying circular loop of radius R as shown in figure. Current flowing through wire and the loop is I_0 and I respectively. If distance between centre of loop and wire is $r = 2R$, calculate force of attraction between the wire and the loop.

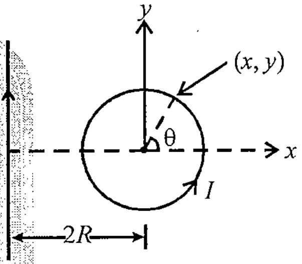


Soln. $\vec{F} = \int I d\vec{\ell} \times \vec{B} = \int \left(Idx\hat{i} + dy\hat{j}\right) \times \frac{\mu_0 I}{2\pi(2R+x)}\left(-\hat{k}\right) = \frac{\mu_0 I I_0}{2\pi}\int \frac{dx\hat{j} - dy\hat{i}}{(2R+x)}$

Change into polar coordinate

$x = R\cos\theta,\ y = R\sin\theta$

$dx = -R\sin\theta d\theta,\ dy = R\cos\theta d\theta$



$$\vec{F} = \frac{\mu_0 I I_0}{2\pi}\left[\int_0^{2\pi} \frac{-R\sin\theta d\theta \hat{j}}{2R + R\cos\theta} - \int_0^{2\pi} \frac{R\cos\theta d\theta \hat{i}}{2R + R\cos\theta}\right]$$

$$= \frac{\mu_0 I I_0}{2\pi}\left[\hat{j}\left\{\ln(2+\cos\theta)\right\}_0^{2\pi} - \hat{i}\int_0^{2\pi} \frac{\cos\theta d\theta}{2+\cos\theta}\right]$$

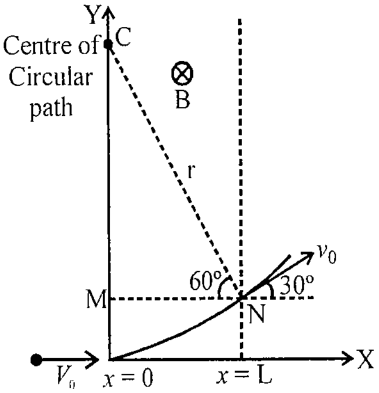
$$= \frac{\mu_0 I I_0}{2\pi}\left[\hat{j}\times 0 + \hat{i}\cdot\frac{2\left(2-\sqrt{3}\right)\pi}{\sqrt{3}}\right] \Rightarrow \vec{F} = \mu_0 I I_0\left(\frac{2-\sqrt{3}}{\sqrt{3}}\right)\hat{i} = \mu_0 I_0 I\left(\frac{2-\sqrt{3}}{\sqrt{3}}\right).$$

51. The region between $x = 0$ and $x = L$ is filled with uniform, steady magnetic field $B_0\hat{k}$. A particle of mass m positive charge q and velocity $v_0\hat{i}$ travels along X-axis and enters the region of magnetic field. Neglect the gravity throughout the question.

(a) Find the value of L if the particle emerges from the region of magnetic field with its final velocity at an angle 30° to its initial velocity.

(b) Find the final velocity of the particle and the time spent by it is the magnetic field, if the magnetic field now extends upto $2.1\,L$.

Soln. (a) See below figure :



Here the velocity of particle $\vec{v}$ is perpendicular to magnetic field $\vec{B}$ and hence the particle will move along circular arc. The radius r of the arc is given by

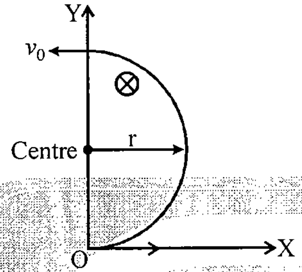
$$r = \frac{m v_0}{qB} \qquad \dots (1)$$

Let the particle leaves the magnetic field at N.
From triangle CMN

$\cos 60° = \frac{L}{r}$ or $L = r\cos 60°$

$$\therefore \quad L = \frac{m v_0}{qB} \times \frac{1}{2} = \frac{m v_0}{2qB} \qquad \dots (2)$$

(b) In this case, the magnetic field extends upto 2.1 L i.e., $L > 2r$. So, the particle completes half circle before leaving the magnetic field. This is shown is figure.

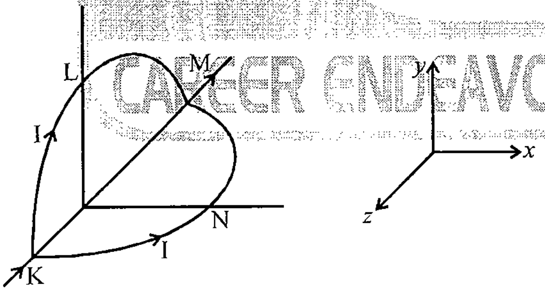


We know that magnetic force is always perpendicular to velocity vector. Therefore, the magnitude of velocity will remain same.

Therefore, final velocity $= v_0(-\hat{i})$

Time spent $= \frac{\pi r}{v_0} = \frac{\pi m}{2qB}$.

2. A circular loop of radius R is bent along a diameter and given a shape as shown in the figure. One of the semicircles (KNM) lies in the xz-plane and the other one (KLM) in the yz-plane with their centres at the origin. Current I is flowing through each of the semicircles as shown in figure.



(a) A particle of charge q is released at the origin with a velocity $v = v_0\,\hat{\mathbf{i}}$. Find the instantaneous force **f** on the particle. Assume that space is gravity free.

(b) If an external uniform magnetic field $B\,\hat{\mathbf{j}}$ is applied, determined the forces F_1 and F_2 on the semicircles KLM and KNM due to this field and the net force F on the loop.

oln. (a) Magnetic field $\vec{B}_O$ at the origin is given by

$$\vec{B}_O = \vec{B}_{KNM} + \vec{B}_{KLM}$$

$$= \left(\frac{\mu_0 I}{4\pi R}\right)\pi\hat{k} + \left(\frac{\mu_0 I}{4\pi R}\right)\pi(-\hat{i}) = \frac{\mu_0 I}{4R}\left[(-\hat{i} + \hat{k})\right]$$

Force $\vec{F}$ on charge q at origin

$$\vec{F} = q(\vec{v}\times\vec{B})$$

$$= q\left[(-v_0\,\hat{i})\left(\frac{\mu_0 I}{4R}\right)(-\hat{i}+\hat{k})\right] \quad = -\frac{\mu_0 I q v_0}{4R}\hat{k}$$

(b) In a uniform magnetic field, the force on a wire is independent of its shape and depends on the vector joining the beginning and ending of the current carrying wire. Thus,

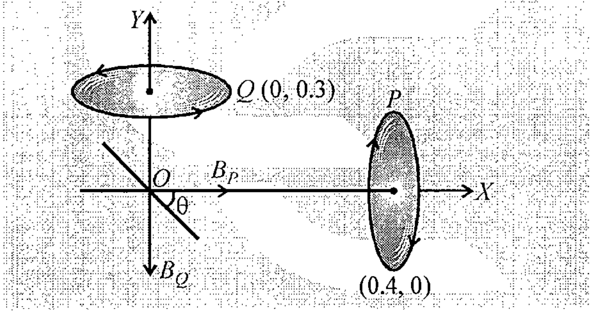
Force on KLM, $\vec{F}_1 = I(\overrightarrow{KM}\times\vec{B})$ $\qquad [\because\ F = I(l\times B)]$

or $\quad \vec{F}_1 = I\left[(-2R\,\hat{k})\times B\,\hat{i}\right] = 2IRB\,\hat{i}$

Similarly, force on KNM, $\vec{F}_2 = 2IRB\,\hat{i}$

Net force on loop $= \vec{F}_1 + \vec{F}_2 = 4IRB\,\hat{i}$

53. Centres of two similar coils P and Q having same number of turns are located at the coordinates (0.4, 0) and (0, 0.3) such that the plane of coils are perpendicular to X and Y axes respectively. The areas of cross section of coils P and Q are in the ratio 4 : 3. Coil P has 16 A current in clockwise direction and coil Q has $9\sqrt{3}$ A current in anti-clockwise direction as seen from the origin. A small compass needle is placed at the origin. Find the deflection of the needle, assuming the earth's magnetic field negligible and the radii of the coils very small compared to their distances from the origins.



Soln. See above figure :

We know that the magnetic field on the axis of a current carrying coil of radius r and number of turns N at a distance x from the centre of coil is given by

$$B = \frac{\mu_0 N i r^2}{2(r^2+x^2)^{3/2}} = \frac{\mu_0 N i(\pi r^2)}{2\pi(r^2+x^2)^{3/2}} = \frac{\mu_0 N i A}{2\pi(r^2+x^2)^{3/2}},$$

where A = area of the coil. When $r \ll x$, then $B = \dfrac{\mu_0 N i A}{2\pi x^3}$

Now, magnetic field at O due to coil P is $B_P = \dfrac{\mu_0 N i_P A_P}{2\pi x^3}$ (along OX)

Similarly, $\quad B_Q = \dfrac{\mu_0 N i_Q A_Q}{2\pi y^3}$ (along YO)

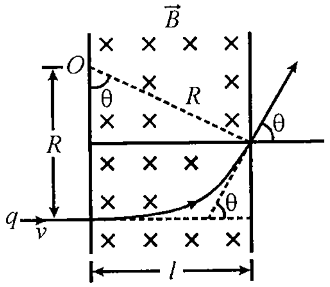
Suppose, the compass needle makes an angle θ with X-axis, then

$$\tan\theta = \frac{B_Q}{B_P} = \left(\frac{i_Q}{i_P}\right)\left(\frac{A_Q}{A_P}\right)\left(\frac{x^3}{y^3}\right) = \left(\frac{9\sqrt{3}}{16}\right)\left(\frac{3}{4}\right)\left(\frac{0.4}{0.3}\right) = \sqrt{3}$$

Therefore, $\qquad \theta = \tan^{-1}\sqrt{3} = 60^\circ$

. An α-particle is accelerated by a potential difference of 10^4 V. Find the change in its direction of motion, if it enters normally in a region of thickness 0.1 m having transverse magnetic induction of 0.1 Tesla. (Given : mass of α-particle 6.4×10^{-27} kg).

ln. The situation is shown in figure :



When a charged particle with charge q is accelerated through a potential difference V volts, then

$$\frac{1}{2}mv^2 = qV$$

$$\text{or} \quad v = \sqrt{\left(\frac{2qV}{m}\right)} \qquad \dots(1)$$

α-particle in magnetic field moves in a circle of radius R given by

$$\frac{mv^2}{R} = qvB \quad \text{or} \quad R = \frac{mv^2}{qvB}$$

$$\text{or} \quad R = \frac{1}{B}\sqrt{\left(\frac{2mV}{q}\right)} \qquad \dots(2)$$

The change in direction of α-particle (θ) from figure is given by

$$\sin\theta = \frac{l}{R} = lB\sqrt{\left(\frac{q}{2mV}\right)}$$

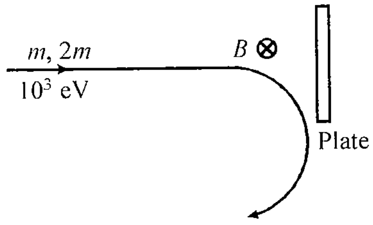
Here $l = 0.1$ m, $B = 0.1$ tesla, $V = 10^4$ volt

$q = 2e = 2 \times 1.6 \times 10^{-19} = 3.2 \times 10^{-19}$ coul and $m = 6.4 \times 10^{-27}$ kg

Therefore, $\sin\theta = 0.1 \times 0.1 \times \sqrt{\left(\dfrac{3.2 \times 10^{-19}}{2 \times 6.4 \times 10^{-27} \times 10^4}\right)} = \dfrac{1}{2}$ or $\theta = 30°$.

5. A beam of charged particles having kinetic energy 10^3 eV, contains masses 8×10^{-27} kg and 1.6×10^{-26} kg emerge from the end of an acceleration tube. There is a plate at a distance 10^{-2} m from the end of the tube and placed perpendicular to the beam. Calculate the magnitude of the smallest magnetic field which can prevent the beam from striking the plate.

oln. See below figure :



We know that $Bev = \frac{mv^2}{r}$ or $Be = \frac{mv}{r}$ ... (1)

If K be the kinetic energy of the particle, then

$$\frac{1}{2}mv^2 = K \text{ or } mv = \sqrt{2mK} \qquad \text{... (2)}$$

$$\therefore \quad Be = \frac{\sqrt{2mK}}{r} \text{ or } B^2 = \frac{2mK}{r^2e^2} \qquad \text{... (3)}$$

Substituting the values, we get

$$B^2 = \frac{2\times(8\times10^{-27})\times(10^3\times1.6\times10^{-19})}{10^4\times(1.6\times10^{-19})^2}$$

$$B = 1 \text{ Tesla} \qquad \text{... (4)}$$

From $2m$ mass particle, we have

$$B = \sqrt{2} \text{ Tesla} \qquad \text{... (5)}$$

From equation (3), $B = \frac{\sqrt{2mK}}{re}$

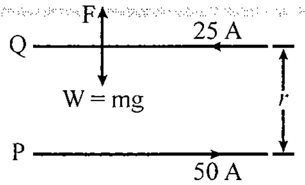
The minimum value of B corresponds to a radius 0.01 m.

Further, $r \propto \frac{1}{B}$

We want $r_{\max} = 0.01$ m. Hence B is minimum.

$$B_{\text{smallest}} = \frac{\sqrt{4mK}}{re} = \sqrt{2} \text{ Tesla}$$

56. A long horizontal wire P carries a current of 50 A. It is rigidly fixed. Another find wire Q is placed directly above and parallel to P. The weight of the wire Q is 0.075 N/m and it carries a current of 25 A. Find the position of the wire Q from the wire P so that Q remains suspended due to the magnetic repulsion. Also indicate the direction of current in Q with respect to P.



Soln. The magnetic force per unit length on the wire Q due to the current in the wire P is

$$F = \frac{\mu_0}{2\pi}\left(\frac{i_1 i_2}{r}\right) \text{ N/m}$$

The currents in P and Q must be *opposite* because than Q will experience a repulsive force which would balance the weight of Q.

$$\therefore \qquad F = mg = \frac{\mu_0}{2\pi}\left(\frac{i_1 i_2}{r}\right)$$

or $$r = \frac{\mu_0}{2\pi}\times\left(\frac{i_1 i_2}{mg}\right) = (2\times10^{-7})\frac{50\times25}{0.075} = 3.33\times10^{-3} \text{ m} = 3.33 \ mm$$

7. A coil in the shape of an equilateral triangle of side 0.02 m is suspended from a vertex such that it is hanging in a vertical plane between the pole pieces of a permanent magnet producing a horizontal magnetic field of 5×10^{-2} tesla. Find the couple acting on the coil when a current of 0.1 ampere is passed through it and the magnetic parallel to its plane.

oln. The couple acting on a closed loop is given by $\tau = N\,i\,A\,B\sin\theta$, where, N = number of loops, A = area of loops, B = magnetic field and θ = angle between magnetic field and normal to the surface of the loop.

Here $N = 1$, $i = 0.1$ amp, $B = 5\times10^{-2}$ tesla, $\theta = 90°$ and $A = \frac{1}{2}$(Base × height)

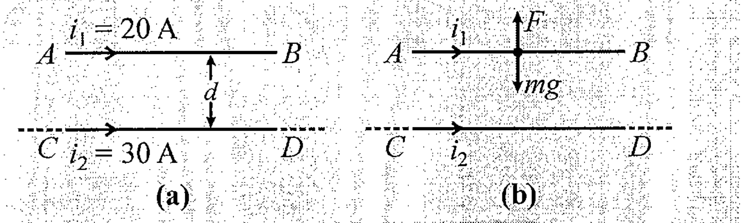
$$= \tfrac{1}{2}(0.02 \times 0.01732) = 1.732\times10^{-4}\ \text{m}^2$$

$$\therefore \quad \tau = 1\times 0.1 \times (1.732\times10^{-4}) \times (5\times10^{-2}) \times 1$$

$$= 8.66\times10^{-7}\ \text{newton/metre}$$

8. A long horizontal wire AB which is free to move in a vertical plane and carries a steady current of 20 A, is in equilibrium at a height of 0.01 m over another parallel long wire CD, which is fixed in a horizontal plane and carries a steady current of 30 A, as shown in figure. Show that when AB is slightly depressed, it executes simple hormonic motion. Find the period of oscillation.

oln. The situation is shown in below figure :



Let m be the mass per unit length of wire AB. The weight per unit length of wire $AB = mg$. Let i_1 and i_2 be the currents in AB and CD respectively. The magnetic force F per unit length of AB, due to current carrying conductor CD is given by

$$F = \frac{\mu_0 i_1 i_2}{2\pi d},$$

where d is the separation between two wires.

$$\text{For equilibrium, } mg = \frac{\mu_0 i_1 i_2}{2\pi d} \qquad \dots(1)$$

Now let AB be depressed by a small distance x towards CD. Then force per unit length

$$= \frac{\mu_0 i_1 i_2}{2\pi(d-x)}$$

Therefore, resultant force per unit length of $AB = \dfrac{\mu_0 i_1 i_2}{2\pi(d-x)} - mg$ ... (2) (directed upwards)

So, resultant force per unit length

$$= \frac{\mu_0 i_1 i_2}{2\pi(d-x)} - \frac{\mu_0 i_1 i_2}{2\pi d} = \frac{\mu_0 i_1 i_2}{2\pi}\left[\frac{1}{(d-x)} - \frac{1}{d}\right]$$

$$= \frac{\mu_0 i_1 i_2}{2\pi} \times \frac{x}{d(d-x)} \text{ direct upward} \approx \frac{\mu_0 i_1 i_2 x}{2\pi d^2} \text{ if } d \gg x$$

So, $F \propto x$. Hence AB will executes S.H.M.

Now, $F = -\frac{\mu_0 i_1 i_2}{2\pi d} \times \frac{x}{d} = -\frac{mg}{d} x = -k\,x$, where $k = \frac{mg}{d}$

We know that the time period of S.H.M. is given by

$$T = 2\pi\sqrt{\left(\frac{m}{k}\right)} = 2m\sqrt{\left(\frac{m}{(mg/d)}\right)}$$

$$= 2\pi\sqrt{\left(\frac{d}{g}\right)} = 2\pi\sqrt{\left(\frac{0.01}{9.8}\right)} = 0.2 \text{ sec}$$

59. A small coil of radius 0.002 m is placed on the axis of a magnet of magnetic moment 10^5 joule/tesla and length 0.1 m at a distance of 0.15 m from the centre of the magnet. The plane of the coil is perpendicular to the axis of the magnet. Find the force on the coil when the current of 2.0 amp. is passed through it.

Soln. The magnetic field on the axis of a coil at a distance x is given by

$$\frac{\mu_0}{4\pi} \times \frac{2\pi a^2}{(x^2 + a^2)^{3/2}} = \frac{\mu_0}{4\pi} \times \frac{2M'}{x^3} \quad \text{(from a small coil)}$$

where $M' = i\pi a^2 = 2 \times 3.14 \times (0.002)^2 = 25.12 \times 10^6$ amp. m^2

$$\text{Force} = \frac{\mu_0}{4\pi} \times 2M'm\left[\frac{1}{(d-l)^3} - \frac{1}{(d+l)^3}\right] = \frac{\mu_0}{4\pi}\frac{2MM'(3d^2 + l^2)}{(d^2 - l^2)^3} \text{ where } M = 2\,ml$$

Here $\frac{\mu_0}{4\pi} = 10^7$, $M = 10^5$ joule/tesla, $M' = 25.12 \times 10^{-6}$ amp. m^2, $d = 0.15$ m and $l = 0.1$ m.

Putting we get $\vec{F} = 43.96 \times 10^{-4}$ newton

PRACTICE SET

A dipole of magnetic moment $\vec{m}$ is placed in an inhormogeneous magnetic field $\vec{B}$. The force experienced by the dipole is :

(a) $\vec{F}=\vec{m}\times\left(\vec{\nabla}\times\vec{B}\right)$ (b) $\vec{F}=\left(\vec{m}.\vec{\nabla}\right)\vec{B}$ (c) $\vec{F}=\left(\vec{\nabla}.\vec{m}\right)\vec{B}$ (d) $\vec{F}=\vec{\nabla}\times\left(\vec{m}\times\vec{B}\right)$

A very long solenoid with n turns per unit length carries a current I. The magnetic field at a point, which is on its axis and its end face, is :

(a) $\mu_0 nI$ (b) $(2/3)\mu_0 nI$ (c) $(1/3)\mu_0 nI$ (d) $(1/2)\mu_0 nI$

If in interior of a unit sphere we have $\vec{\nabla}.\vec{J}=a$ positive constant, where $\vec{J}$ is the current density then we may conclude that;

(a) According to Gauss' theorem, the charge contained in the unit sphere is constant in time.
(b) According to Gauss' theorem, charge is flowing into the unit sphere.
(c) According to the continuity equation, the charge density within the unit sphere must necessarily be uniform.
(d) According to the continuity equation, the change density inside the unit sphere diminishes with time.

Choose the correct statement from the following:

(a) The magnetic field $\vec{B}$ is a vector and not a pseudovector.
(b) $\vec{E}.\vec{B}$ is a scalar and not a pseudoscalar.
(c) $\vec{E}\times\vec{B}$ is a pseudovector.
(d) The magnetic vector potential $\vec{A}$ is a pseudovector.

A wire of circular cross section of radius R carries a volume current density $\vec{J}=kr^2\hat{z}$ along the length of the wire. The magnetic field at a position r < R is :

(a) $\frac{\mu_0}{2}kr^3\hat{z}$ (b) $\frac{\mu_0}{4}k\,r^3\hat{\phi}$ (c) $\frac{\mu_0}{4\pi}k\,r^3\hat{r}$ (d) $\frac{\mu_0}{4\pi}k\,r^2\hat{\phi}$

Consider an infinite plane at z = 0 with uniform surface current density $\vec{K}=K\hat{i}$. The direction of the magnetic field $\vec{B}$ and the vector potential $\vec{A}$ of this current distribution at a distance d along the positive z-axis are :

(a) $\vec{B}$ along $\hat{j}$ and $\vec{A}$ along $\hat{k}$ (b) $\vec{B}$ along $\hat{k}$ and $\vec{A}$ along $\hat{j}$

(c) $\vec{B}$ along $-\hat{j}$ and $\vec{A}$ along $\hat{i}$ (d) $\vec{B}$ along $\hat{i}$ and $\vec{A}$ along $-\hat{j}$

The magnitude of vector potential at a point $r\,(r<R)$ in an infinite solenoid with N-turns per unit length, radius R and carrying a current I, is given by

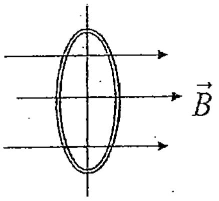
(a) $A=\mu_0 N\,I\,r^2$ (b) $A=\frac{\mu_0 N\,I\,r^2}{2}$ (c) $A=\mu_0 N\,I\,r$ (d) $A=\frac{\mu_0 N\,I\,r}{2}$

Magnetic induction $|B|$ due to current I in a coil of radius R placed in y-z plane at a distance X along the axis of the coil is given by $\frac{\mu_0 I}{2}\frac{R^2}{\left(R^2+X^2\right)^{3/2}}$. Another similar coil of radius R, carrying equal current I is placed in the y-z plane at a distance 2R from the first coil along the same axis. The value of $|B|$ at a point midway between the two coils and on the x-axis is:

(a) $\frac{\mu_0 I}{2\sqrt{2}\,R}$ (b) $\frac{\mu_0 I\,R}{2\sqrt{2}}$ (c) 0 (d) $\frac{\mu_0 I}{4\sqrt{2}\,R}$

9. The expression $\vec{A} = \frac{B}{2}\left(-\hat{i}y + \hat{j}x\right)$ for the vector potential represents
(a) a uniform magnetic field in the z-direction (b) a non- uniform magnetic field in the z-direction
(c) a uniform magnetic field in xy-plane (d) a non-uniform magnetic field in xy-plane

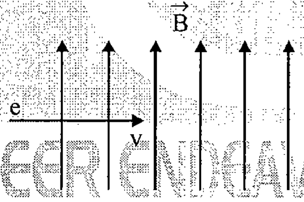
10. A loop of wire of radius a and carrying a current I is suspended vertically in a uniform horizontal magnetic field B as shown. Calculate the work done in rotating the loop around the vertical axis through an angle π



(a) $\frac{\pi^2 I^2}{2a}$ (b) $2\pi a^2 I$ (c) $\frac{\pi a^2 I}{2}$ (d) 0

11. A charge q is revolving with angular velocity ω in a plane perpendicular to a uniform magnetic field B. If the field is reduced to B/2.
(a) The linear velocity is doubled (b) The linear velocity is halved
(c) The radius is halved (d) The radius is doubled.

12. A stream of electrons eneters, from left to right, a uniform magnetic field (B) which is pointing upwards as shown in the figure. The field deflects the stream of electrons.



(a) Downward (in the plane of the paper) (b) Upward (in the plane of the paper)
(c) Into the paper (d) Out of the Paper.

13. The ratio of cyclotron frequencies of an α-particle and a proton moving in the same magnetic field $\vec{B}$ is:
(a) 1/2 (b) 2 (c) 4 (d) 1/4

14. An electron moves with velocity $\hat{v} = \hat{j}.v_0$ under the combined influence of $\vec{E} = \hat{i}E_0$ and $\vec{B} = -\hat{k}B_0$ fields. The force on the electron is in the
(a) Y direction (b) Z direction (c) YZ plane (d) X direction

15. A charged particle is projected with certain velocity parallel to a magnetic field. It will
(a) trace a circular path
(b) trace a helical path
(c) come to rest instantly
(d) continue its motion along a straight the line without any change

16. Time taken by a changed particle to move once around a circular orbit in a magnetic field is independent of
(a) mass (b) strength of magnetic field
(c) charge (d) speed

7. Four partices-electron, proton, He$^+$ and Li$^+$ are projected with the same velocity perpendicular to a given magnetic field. Then the circle described will be smallest for

(a) electron (b) He$^+$ (c) proton (d) Li$^+$

8. A uniformly charged spherical metal ball of mass M, radius R and charge q is rotating with an angular velocity ω about one of its dimeter. Then the ratio of its magnetic dipole moment to the mechanical moment (angular momentum) is

(a) $\frac{q}{2MR}$ (b) $\frac{q}{M}$ (c) $\frac{q}{2M}$ (d) $\frac{q}{M\omega}$

ANSWER KEY					
Questions	1	2	3	4	5
Option	(b)	(d)	(d)	(b)	(b)
Questions	6	7	8	9	10
Option	(c)	(d)	(a)	(a)	(b)
Questions	11	12	13	14	15
Option	(d)	(c)	(a)	(d)	(d)
Questions	16	17	18		
Option	(d)	(d)	(c)		

Chapter 8

Magnetization & Magnetic Materials

Introduction :

So far we have read the magnetic field $\vec{B}$ produced by the current in vacuum. In this chapter we shall discuss the effect of magnetic field inside the medium.

Magnetization

Any current loop has a magnetic field and thus has a magnetic dipole moment, including the atomic-level current loops described in some models of the atom. Thus, the magnetic moments in a magnetized substance may be described as arising from these atomic-level current loops. For the Bohr model of the atom, these curren9+t loops are associated with the movement of electrons around the nucleus in circular orbits. In addition, a magnetic moment is intrinsic to electrons, protons, neutrons, and other particles; it arises from a property called spin.

Ordinarily, the current loops inside an object cancel each other out because of the random orientation of the atoms. But when a magnetic field is applied, a net alignment of these magnetic dipoles occurs, and the medium becomes magnetically polarized, or magnetized.

Let's define,

M ≡ magnetic dipole moment per unit volume

$$\therefore \quad \vec{M} = \lim_{\Delta v\to 0}\frac{1}{\Delta v}\sum_{i=1}^{N} m_i$$

M is called the magnetization; it plays a role analogous to the polarization **P** in electrostatics. Rightnow the equation is how the magnetization got there it could be paramagnetism, diamagnetism, or even ferromagnetism. We shall take **M** as given, and calculate the field this magnetization itself produces.

Volume and surface current

In electric field we have shown the occurrence of bound charges (volume and surface bound charges). While calculating the field of a magnetized object the bound currents come. There are bound surface current and volume current. We won't derive the expression –

Field of a magnetized object is

$$\bar{A}(r) = \frac{\mu_0}{4\pi}\int\frac{1}{r}\left[\nabla'\times M(r')\right]d\tau' + \frac{\mu_0}{4\pi}\oint\frac{1}{r}\left[\bar{M}(r')\times d\bar{a}'\right]$$

The first term looks just like the potential of a volume current,

$$\boxed{\bar{J}_b = \bar{\nabla}\times\bar{M}} \qquad \ldots (3)$$

while the second looks like the potential of a surface current,

$$\boxed{\bar{K}_b = \bar{M}\times\hat{n}} \qquad \ldots (4)$$

where $\hat{n}$ is the normal unit vector. With these definitions,

$$A(r)=\frac{\mu_0}{4\pi}\int_v \frac{J_b(r')}{r}d\tau'+\frac{\mu_0}{4\pi}\int_s \frac{K_b(r')}{r}da' \quad \dots(5)$$

Auxalury field H

As we have seen, there are both free and bound current. So the total current **J** could be written as

$$J=J_b+J_f \quad \dots(6)$$

Where $\mathbf{J}_b$ is bound and $\mathbf{J}_f$ is the free current.
We know the Ampere's Law tells us

$$\oint \vec{B}.d\vec{\ell}=\mu_0 I_{enc.}$$

And, $\vec{\nabla}\times\vec{B}=\mu_0\vec{J}$

So from there we replace the value of **J** and use

$$\boxed{\vec{J}_b=\vec{\nabla}\times\vec{M}}$$

$$\frac{1}{\mu_0}\left(\vec{\nabla}\times\vec{B}\right)=\vec{J}=\vec{J}_f+\vec{J}_b=\vec{J}_f+\left(\vec{\nabla}\times\vec{M}\right)$$

Or, $$\vec{\nabla}\times\left(\frac{1}{\mu_0}\vec{B}-\vec{M}\right)=\vec{J}_f \quad \dots(7)$$

Now, define the quantity inside the curl as **H**

$$\boxed{\vec{H}\equiv\frac{1}{\mu_0}\vec{B}-\vec{M}} \quad \dots(8)$$

So now the equation becomes

$$\boxed{\vec{\nabla}\times\vec{H}=\vec{J}_f} \quad \dots(9)$$

Or in integral terms

$$\oint \vec{H}.d\vec{\ell}=I_{fenc} \quad \dots(10)$$

Where I_{fenc} is the total free current passing through the Amperian loop. These are the forms of Ampere's law inside matter. **H** plays a role in magnetostatics analogous to **D** in electrostatics.

Linear and non-linear medium

For linear medium **M** is proportional to **H**

$$\boxed{\vec{M}=\chi_m H} \quad \dots(11)$$

The constant of proportionality X_m is called the magnetic susceptibility; it is a dimensionless quantity that varies from one substance to another—positive for paramagnets and negative for diamagnets. Typical values are around 10^{-5}.

From $\left[\vec{H}\equiv\frac{1}{\mu_0}\vec{B}-\vec{M}\right]$, we get

$$\vec{B}=\mu_0\left(\vec{H}+\vec{M}\right)=\mu_0\left(1+\chi_m\right)\vec{H} \quad \dots(12)$$

Or defining, $$\mu\equiv\mu_0\left(1+\chi_m\right) \quad \dots(13)$$

We get, $$\vec{B}=\mu\vec{H} \quad \dots(14)$$

μ is called the permeability of the material. In a vacuum, where there is no matter to magnetize, the susceptibility χ_m vanishes, and the permeability is μ_0. That's why μ_0 is called the permeability of free space.

Magnetic moment of atoms

Electrons not only spin; they also revolve around the nucleus—for simplicity, let's assume the orbit is a circle of radius R (Fig. 6.1). Although technically this orbital motion does not constitute a steady current, in practice the period $T = 2\pi R / v$ is so short that unless you blink awfully fast, it's going to look like a steady current:

$$I = \frac{e}{T} = \frac{ev}{2\pi R}$$

The orbital dipole moment is: $m = IA = I\pi R^2$

$$m = -\frac{1}{2} ev\, R\, \hat{z}$$

The minus sign accounts for the negative charge of the electron. Like any other magnetic dipole, this one is subject to a torque **($\mathbf{m} \times \mathbf{B}$)** when the atom is placed in a magnetic field.

Diamagnet – paramagnet - feromagnet

Unlike electric polarization, which is almost always in the same direction as **E**, some materials acquire a magnetization parallel to **B** (paramagnets) and some opposite to **B** (diamagnets). A few substances (called ferromagnets, in deference to the most common example, iron) retain their magnetization even after the external field has been removed.

For these (ferromagnets) the magnetization is not determined by the present field but by the whole magnetic "history" of the object. Permanent magnets made of iron are the most familiar examples of magnetism, though from a theoretical point of view they are the most complicated. Let's begin with paramagnetism and diamagnetism.

For details study about Diamagnetic, Paramagnetic and Feromagnetic material please read any solid state book.

Magnetostatic Boundary Condition

Like the electrostatic case we have magnetostatic boundary conditions too. But the conditions are not same. For magnetostatics the boundary conditions are

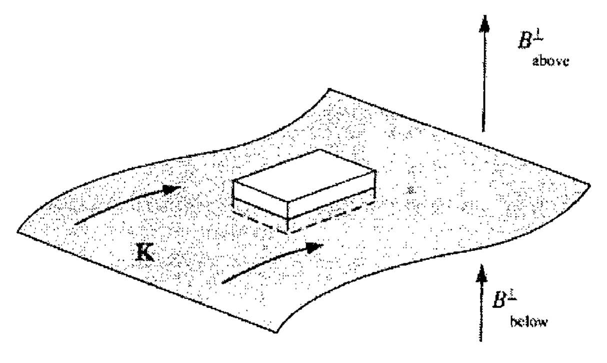


Fig. 5.24

$$\vec{B}^{\perp}_{above} = \vec{B}^{\perp}_{below} \qquad \text{...(15)}$$

$$B^{||}_{above} - B^{||}_{below} = \mu_0 K \qquad \text{...(16)}$$

Where K is the surface current density.

Combined form of equation (15) and (16) is given as

$$\vec{B}_{above} - \vec{B}_{below} = \mu_0 \left(K \times \hat{n}\right) \qquad \text{...(17)}$$

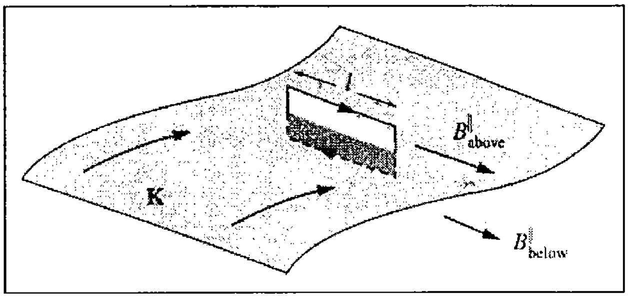
Where $\hat{n}$ is a unit vector perpendicular to the surface, pointing "upward".

Like the scalar potential of electrostatics the magnetic vector potential is also continuous

$$A_{above} = A_{below} \quad \text{... (18)}$$

But the derivative of vector potential has a discontinuity

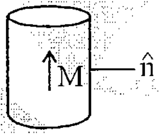
$$\frac{\partial A_{above}}{\partial n} - \frac{\partial A_{below}}{\partial n} = -\mu_0 K \quad \text{... (19)}$$



ADDITIONAL SOLVED PROBLEMS

. An infinitely long circular cylinder carries a uniform magnetization M parallel to its axis. Find the magnetic field (due to M) inside and outside the cylinder.

oln. $J_b = \nabla \times M = 0;\ K_b = M \times \hat{n} = M\hat{\phi}$



The field is that of a surface current $K_b = M\hat{\phi}$ but that's just a solenoid, so the field **outside is zero**, and inside $B = \mu_0 K_b = \mu_0 M$. Moreover, it points upward (in the drawing).

. A long copper rod of radius R carries a uniformly distributed free current I (Fig. 6.4). Find H inside and outside the rod

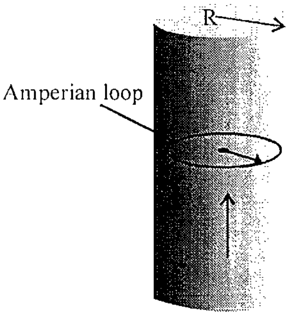
oln. Take an Amperian loop of radius s < R. Apply

$$\oint H.dl = I_{fenc.}$$

We get (as enclosed current is not I, but fraction of I enclosed by the loop)

$$H(2\pi s) = I_{fenc} = I\frac{\pi s^2}{\pi R^2}$$

$$H = \frac{I}{2\pi R^2} s\hat{\phi} \quad (s \le R)$$



Outside of the wire $(s \ge R)$ the situation is different (Enclosed current = I)

$$H = \frac{I}{2\pi s}\hat{\phi} \qquad (s \ge R)$$

Outside, (as always, in empty space) M = 0, so

$$B = \mu_0 H = \frac{\mu_0 I}{2\pi s}\hat{\phi} \qquad (s \ge R)$$

As there is no way right now to determine M inside the wire,we can't determine B inside the wire now.

3. Consider a plane interface of a magnetic medium of permeability μ and vacuum. If the magnetic field $\vec{B}$ in the medium in the vicinity of a point on the interface makes an angle θ with the normal to the interface, find the magnetic field $\vec{B}_0$ in vacuum in the vicinity of the above point.

Soln. Continuity of the normal component of $\vec{B}$ indicates that $B_{0n} = B\cos\theta$.

From the continuity of the tangential component of $\vec{H}$ we get

$$H_{0t} = H_t \text{ or } \frac{B_{0t}}{\mu_0} = \frac{B_t}{\mu} \text{ or } B_{0t} = \frac{\mu_0 B_t}{\mu} = \mu_0 B\sin\theta/\mu$$

Therefore, the magnitude of $\vec{B}$-field in vacuum would be

$$B_0 = \sqrt{B_{0n}^2 + B_{0t}^2} = B^2\sqrt{\cos^2\theta + \mu_0^2\sin^2\theta/\mu^2}$$

The angle θ_0 made by $\vec{B}_0$ with the normal to the interface is given by

$$\tan\theta_0 = \frac{B_{0t}}{B_{0n}} = \frac{\mu_0}{\mu}\tan\theta$$

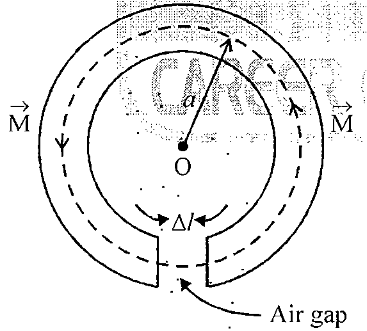
4. Consider a thin permanently magnetized toroid of mean radius a having a small air gap $\Delta\ell$ between its pole faces. If it has a magnetization $\vec{M}$ of constant magnitude in circumferential direction find $\vec{B}$ and $\vec{H}$ both in the air gap and within the material.

Soln. Applying Ampere's circuital law to the dotted path in figure.

$$\oint \vec{H}\cdot d\vec{\ell} = 0 \text{ or } H_i(2\pi a - \Delta\ell) + H_0\cdot\Delta\ell = 0$$

Now we note that the magnetic flux (BA) or flux density (B) may be assumed to be the same throughout. Magnetization inside the air gap is zero. Therefore, we can write,

$$H_0 = \frac{B}{\mu_0} \text{ and } H_i = \frac{B}{\mu_0} - M$$



Therefore,

$$\left(\frac{B}{\mu_0} - M\right)(2\pi a - \Delta\ell) + \frac{B}{\mu_0}\cdot\Delta\ell = 0 \text{ or } B = \frac{\mu_0 M(2\pi a - \Delta\ell)}{2\pi a}$$

both in the material and in the air gap. Now the magnetic intensity in air gap is $H_0 = \dfrac{B}{\mu_0}$ and in the material,

$$H_i = \frac{B}{\mu_0} - M = -\frac{M\cdot\Delta\ell}{2\pi a}$$

PRACTICE SET

The intrinsic magnetic moment of the atoms of a material is not zero. The material
(a) msut be paramagnetic (b) must be diamagnetic
(c) must be ferromagnetic (d) may be paramagnetic or ferromagnetic

The relative permeability of a material is 0.98. The material must be
(a) paramagnetic (b) ferromagnetic (c) diamagnetic (d) none of these

Susceptibility χ of a paramagnetic gas varies with temperature as
(a) $\chi \propto T$ (b) $\chi \propto T^{-1}$ (c) $\chi \propto T^{-2}$ (d) $\chi \propto T^{0}$

Which of the following statements is not applicable to ferromagnetic materials?
(a) susceptibility $\chi_m >> 1$
(b) it shows the phenomenon of hysteresis
(c) its permeability has a fixe value
(d) it can show spontaneous magnetization below certain temperature

When a ferromagnetic material is taken through a hysteresis loop its susceptibility (χ) defined through $\vec{M} = \chi\vec{H}$ may not take up a value which is
(a) zero (b) negative (c) infinity (d) constant.x

ANSWER KEY					
Questions	1	2	3	4	5
Option	(d)	(c)	(b)	(c)	(d)

Chapter 9

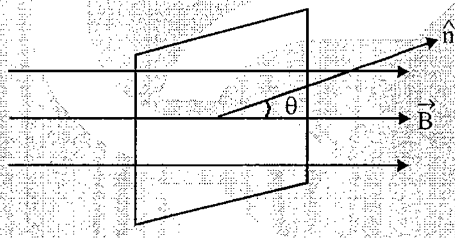
Faraday's law & Electromagnetic Induction

Introduction :

In 1931 Michael Faraday performed experiments to check whether current is produced in a closed wire when placed in a magnetic field. He found no current in that case but he realized that a time-varing current in the loop is obtained while the magnet is being moved towards or away from it. The law he formulated is known as Faraday's law of electromagnetic induction. It is most important law of electromagnetism. Without it there would be no electricity from rotating generators, no telephone, no radio and television.

Magnetic flux:

Consider a uniform magnetic field passing through a surface *s*, as shown in figure below.



Let the area vector be $\vec{A} = A\hat{n}$, where A is the area of the surface and $\hat{n}$ its unit normal.
The magnetic flux through the surface is given by

$$\boxed{\phi_B = \vec{B}\cdot\vec{A} = BA\cos\theta} \quad \text{... (1.a)}$$

Where, θ is the angle between A and B.

For non-uniform magnetic field , ϕ is given by

$$\boxed{\phi = \iint_S \vec{B}\cdot d\vec{A}} \quad \text{... (1.b)}$$

Faraday's Laws of Electromagnetic Induction:

(i) Induced emf in a circuit is proportional to the rate of change of magnetic flux linked with the circuit.
(ii) The direction of induced emf is such that it tries to oppose the cause of its generation, i.e., the variation of magnetic flux producing it.
If Φ be the magnetic flux linked with a circuit at any time 't' then the laws of electromagnetic induction can be expressed mathematically as

$$\varepsilon = -\frac{d\Phi}{dt} \quad \text{... (1.c)}$$

where ε is the induced emf.

Self-Induction:

When a current flows in a circuit it produces a magnetic field in the surrounding space and hence there will be a magnetic flux linked with the circuit. If the current through the circuit varies with time the flux linked with the circuit also varies with time and gives rise to an induced emf in the circuit. This phenomenon is known as self-inductance. According to Lenz's law the induced emf opposes the change in current in the circuit. For this reason it is called a back emf. Self-inductance plays somewhat the same role in electrical circuits that mass plays in mechanical system. If the circuit is in a non-ferromagnetic is a measure of its inertia to changes of current. If the circuit is in a non-ferromagnetic medium the flux Φ linked with it will be proportional to the current I in the circuit. Thus,

$$\Phi = LI \qquad \ldots (2)$$

where L is a constant, which depends on the geometry of the circuit, its dimensions and the permeability of the surrounding medium. The quantity L is called the coefficient of self-inductance or simply the self-inductance of the circuit. Numerically self-inductance may be defined as the total flux linked with the circuit when the unit current flows through it.

The self induced emf in the circuit is given by

$$\varepsilon = -\frac{d\Phi}{dt} = -\frac{d}{dt}(LI)$$

If the medium is not ferromagnetic and the circuit is rigid then L is a constant and we can write.

$$\varepsilon = -L\frac{dI}{dt} \qquad \ldots (3)$$

Thus, self-inductance may also be defined as numerically equal to the emf induced in the circuit for a unit rate of change of current in it.

In case of ferromagnetic material Φ is not linearly related to I. Then for a rigid stationary circuit where flux change is caused by change in current only, we can write

$$\varepsilon = -\frac{d\Phi}{dt} = -\frac{d\Phi}{dI}\cdot\frac{dI}{dt} = -L\frac{dI}{dt} \qquad \ldots (4)$$

where $L = \frac{d\Phi}{dI}$ is called incremental self-inductance.

Calculation of self-inductance in simple cases:

(1) A solenoid :

Suppose we consider an air-cored long solenoid of length l containing N number of turns. If a current I flows through it the axial magnetic field at any inside point is given by

$$B = \mu_0 nI = \frac{\mu_0 NI}{\ell}$$

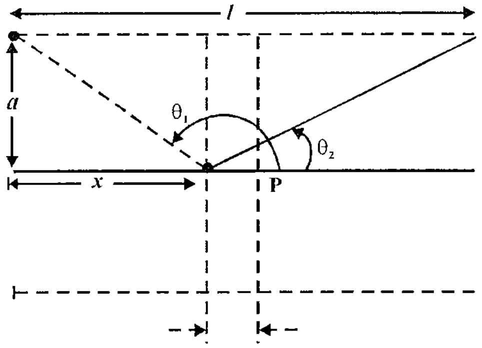
If A be the area of cross-section of the solenoid and the magnetic field B is assumed to have the same value over the cross-section then total magnetic flux linking the N-turns of the solenoid would be

$$\Phi = N\cdot B\cdot A = \frac{\mu_0 N^2 A}{\ell} I$$

Now the self-inductance is L is defined a the flux linkage per unit current. Therefore, the self-inductance of the solenoid is

$$L = \frac{\Phi}{I} = \frac{\mu_0 N^2 A}{\ell} \qquad \ldots (5)$$

$$L = \frac{\mu N^2 A}{\ell}$$ (for material having constant permeability μ)



If the solenoid is not very long then the axial magnetic field at any axial point P is given by

$$L = \frac{\mu_0 NI}{2}(\cos\theta_2 - \cos\theta_2) = \frac{\mu_0 Nl}{2l}\left[\frac{l-x}{\sqrt{(l-x)^2+a^2}} + \frac{x}{\sqrt{x^2+a^2}}\right]$$

In a length dx about p there are (N/l)dx number of turns and hence and the flux linking these turns is

$$d\phi = \left(\frac{N}{l}dx\right)BA$$

In a length dx about P there are $\left(\frac{N}{\ell}\right)dx$ number of turns and hence the flux linking these turns is

$$d\Phi = \left(\frac{N}{\ell}dx\right)BA$$

where, we assume that B is the uniform over the entire cross-section.
Therefore, total magnetic flux linking the solenoid is

$$\Phi = \int d\Phi = \int_0^\ell \frac{BAN}{\ell}dx = \frac{\mu_0 N^2 AI}{2\ell^2}\int_0^\ell \left[\frac{\ell - x}{\sqrt{(\ell-x)^2+a^2}} + \frac{x}{\sqrt{x^2+a^2}}\right]$$

Or, $$\Phi = \frac{\mu_0 N^2 AI}{2\ell^2}\left[-\sqrt{(\ell-x)^2+a^2} + \sqrt{x^2+a^2}\right]_0^\ell$$

$$= \frac{\mu_0 N^2 A}{\ell^2}\left[\sqrt{\ell^2+a^2} - a\right]\cdot I$$

Therefore, $$L = \frac{\mu_0 N^2 A}{\ell^2}\left[\sqrt{\ell^2+a^2} - a\right]$$

$$= \frac{\mu_0 N^2 A}{\ell}\left[\sqrt{1+(a/\ell)^2} - a/\ell\right] \qquad \text{...(6)}$$

(2) A toroidal coil:

The mganetic field inside a toroidal coil having N number of turns, mean length l, cross-sectional area A and carrying current I is given by the equation ( ) as

$$B = \mu_0 nI = \mu_0 \cdot \frac{N}{\ell} \cdot I$$

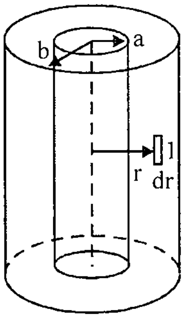
Neglecting any variation of B over the cross-sectional area we can write for the flux linking the N turns of the toroid as

$$\Phi = N \cdot B \cdot A = \frac{\mu_0 N^2 A}{\ell} I$$

Therefore, the self-inductance is

$$L = \frac{\Phi}{I} = \frac{\mu_0 N^2 A}{\ell} \qquad \text{... (7)}$$

(3) Long coaxial cable:



Consider a long coaxial cable consisting of two concentric cylinders. The inner one has radius a and carries a current I while the outer one has radius b and carries a return current I. Applying Ampere's circuital law it is easy to show that the magnetic field outside the cable is zero and at an internal point at a distance r from the axis $(a < r < b)$ the field is given by

$$B = \frac{\mu_0 I}{2\pi r}$$

The lines of $\vec{B}$ are circles about the axis of the cable. If we consider an elemental area of radial width dr and axial length unity then flux through it would be

$$d\Phi = B \cdot dr \times 1 = \frac{\mu_0 I}{2\pi r} dr$$

Therefore, total flux per unit axial length is

$$\Phi = \int d\Phi = \frac{\mu_0 I}{2\pi} \int_a^b \frac{dr}{r} = \frac{\mu_0 I}{2\pi} \ell n \frac{b}{a}$$

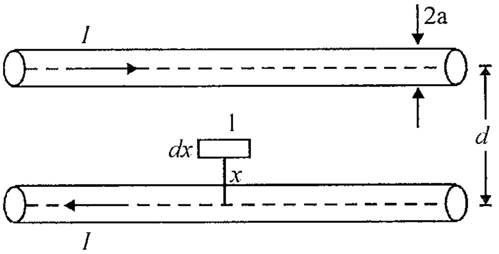
So, the self-inductance per unit length is

$$L = \frac{\Phi}{I} = \frac{\mu_0}{2\pi} \ell n \frac{b}{a} \qquad \text{... (8)}$$

In the above discussion we have neglected the flux within the materials of the two cylinders. This is justified when $b >> a$.

(4) Two-wire transmission line:

Let us consider two long parallel wires separated by a distance d in air. The wires carrying the same current I in opposite directions. We shall assume that the radius a of each wire is much less than d, so that the flux inside the material of the wires may



be neglected. The flux lines will be mainly concentrated in the region between the wires. Total magnetic field at any point at a distance x from one wire is

$$B = \frac{\mu_0 I}{2\pi}\left[\frac{1}{x} + \frac{1}{d-x}\right]$$

The flux through an elemental area of width dx and length unity is

$$d\Phi = B \cdot dx \times 1 = \frac{\mu_0 I}{2\pi}\left[\frac{1}{x} + \frac{1}{d-x}\right]dx$$

Hence, the total flux through the entire area between the two wires of unit length is

$$\Phi = \int d\Phi = \frac{\mu_0 I}{2\pi}\int_a^{d-a}\left[\frac{1}{x} + \frac{1}{d-x}\right]dx$$

$$= \frac{\mu_0 I}{2\pi}\left[\ell n\frac{d-a}{a} - \ell n\frac{a}{d-a}\right] = \frac{\mu_0 I}{\pi}\ell n\frac{d-a}{a}$$

Therefore, self-inductance per unit length is

$$L = \frac{\Phi}{I} = \frac{\mu_0}{\pi}\ell n\frac{d-a}{a} \qquad \text{...(10)}$$

Mutual Induction:

Suppose two coils are placed near each other, as shown in the figure below.

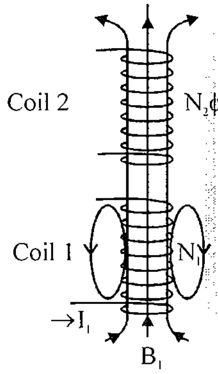


Figure-1

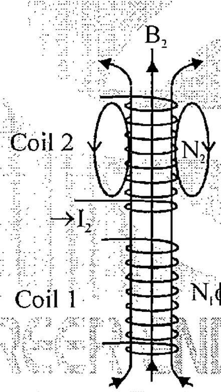


Figure-2

The first coil has N_1 turn and carries a current I_1 which gives rise to a magnetic field $\vec{B}_1$. Since the two coils are close to each other, some of the magnetic field lines through coil 1 will also pass through coil 2. Let ϕ_{21} denote the magnetic flux through one turn of coil 2 due to I_1. Now, by varying I_1 with time, there will be an induced emf associated with the changing magnetic flux in second coil.

$$\varepsilon_{21} = -N_2\frac{d\phi_{21}}{dt} = -\frac{d}{dt}\iint \vec{B}_1 \cdot d\vec{A}_2$$

The time rate of change of magnetic flux ϕ_{21} in coil 2 is proportional to the time rate of change of the current in coil 1.

$$N_2\frac{d\phi_{21}}{dt} = M_{21}\frac{dI_1}{dt}$$

where the proportionality constant M_{21} is called the mutual induction. It can be writen as

$$M_{21} = \frac{N_2\phi_{21}}{I_1}$$

In similar manner suppose instead there is a current I_2 in the second coil and it is varying with time. Then the induced emf in coil 1 becomes

$$\varepsilon_{12} = -N_1\frac{d\phi_{12}}{dt} = -\frac{d}{dt}\iint \vec{B}_2\cdot d\vec{A}_1$$

and a current is induced in coil 1.

This changing flux in coil 1 is proportional to the changing current in coil 2.

$$N_1\frac{d\phi_{12}}{dt} = M_{12}\frac{dI_2}{dt}$$

$$\Rightarrow \qquad M_{12} = \frac{N_1\phi_{12}}{I_2}$$

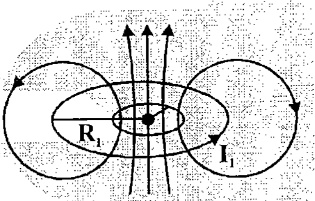
However, using the reciprocity theorem which combine Ampere's law ans Biot-Savart law, one may show that the constant are equal

$$M_{12} = M_{21} = M$$

Calculation of Mutual inductances in simple cases:

(1) Consider two single-turn co-planar, concentric coils of radii R_1 and R_2 with $R_1 >> R_2$ as shown in figure. The outer ring caries current I_1. So, the magnetic field at the centre is

$$B_1 = \frac{\mu_0 I}{2R_1}$$



Since, $R_1 >> R_2$, we appropriate the magnetic field through the entire inner coil by B_1. Hence, the flux through the second coil is

$$\phi_{21} = B_1A_2 = \frac{\mu_0 I_1}{2R_1}\pi R_2^2 = \frac{\mu_0\pi I_1 R_2^2}{2R_1}$$

Thus, the mutual inductance is given by

$$M = \frac{\phi_{21}}{I_1} = \frac{\mu_0\pi R_2^2}{2R_1}$$

The result shows that M depends only on the geometrical factors, R_1 and R_2, and is independent of the current I_1 in the coil.

Energy in the magnetic field

The work done on a unit charge, against the back emf, in one trip around the circuit is -å (the minus sign records the fact that this is the work done by you against the emf, not the work done by the emf). The amount of charge per unit time passing down the wire is I. So the total work done per unit time is

$$\frac{dW}{dt} = -\varepsilon I = LI\frac{dI}{dt}$$

Hence, integrating we get

$$\boxed{W = \frac{1}{2}LI^2} \qquad \ldots(11)$$

We can prove that

$$\boxed{W = \frac{1}{2\mu_0}\int_{\text{all space}} B^2 d\tau} \qquad \text{... (12)}$$

(Here "$d\tau$" is the volume element "dv")
So, summarizing energy stored in electric and magnetic field are

$$\left.\begin{aligned} W_{elec} &= \frac{1}{2}\int (V\rho)d\tau = \frac{\epsilon_0}{2}\int E^2 d\tau \\ W_{mag} &= \frac{1}{2}\int (\bar{A}.\bar{J})d\tau = \frac{1}{2\mu_0}\int B^2 d\tau \end{aligned}\right\} \qquad \text{... (13)}$$

ADDITIONAL SOLVED PROBLEMS

1. A thin conducting wire is bent into a circular loop of radius r and placed in a time dependent magnetic field of magnetic induction.

$$\vec{B}(r) = B_0 e^{-\alpha t}\hat{e}_z,\ (B_0 > 0 \text{ and } \alpha > 0),$$

such that, the plane of the loop is perpendicular to $\vec{B}(t)$. Then the induced emf in the loop is :

(a) $\pi r^2 \alpha B_0\, e^{-\alpha t}$ (b) $\pi r^2 B_0\, e^{-\alpha t}$ (c) $-\pi r^2 \alpha B_0\, e^{-\alpha t}$ (d) $-\pi r^2\, B_0\, e^{-\alpha t}$

Soln. The flux through the circular disc is

$$\phi = \pi r^2 B(t) = \pi r^2 B_0 e^{-\alpha t}$$

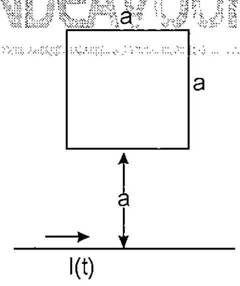
Therefore, the induce emf $= -\frac{d\phi}{dt} = \pi r^2 \alpha B_0 e^{-\alpha t}$

Correct option is (a)

2. An infinitely long wire carrying a current $I(t) = I_0 \cos(wt)$ is placed at a distance a from a square loop of side a as shown in the figure. If the resistance of the loop is R, then the amplitude of the induced current in the loop is :

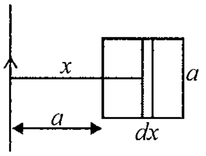
(a) $\frac{\mu_0}{2\pi}\frac{aI_0\omega}{R}\ln 2$ (b) $\frac{\mu_0}{\pi}\frac{aI_0\omega}{R}\ln 2$

(c) $\frac{2\mu_0}{\pi}\frac{aI_0\omega}{R}\ln 2$ (d) $\frac{\mu_0}{2\pi}\frac{aI_0\omega}{R}$



Soln. $B = \frac{\mu_0 I}{2\pi x}$

$$d\phi = Badx = \frac{\mu_0 I}{2\pi x}adx$$



$$= \frac{\mu_0 Ia}{2\pi}\int_a^{2a}\frac{dx}{a} = \frac{\mu_0 Ia}{2\pi}\ell n\left(\frac{2a}{a}\right) = \frac{\mu_0 Ia}{2\pi}\ell n2$$

$$\varepsilon = -\frac{\partial\phi}{\partial t} = -\frac{d}{dt}\left(\frac{\mu_0 aI_0\cos(\omega t)}{2\pi}\ell n2\right) = \frac{\omega\mu_0 aI_0}{2\pi}\ell n2 = \frac{\mu_0}{2\pi}a\omega I_0 \ln 2$$

The induce current $I = \frac{\varepsilon}{R} = \frac{\mu_0}{2\pi}\frac{a\omega I_0}{R}\ln 2$

Correct option is (a)

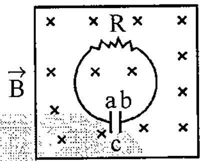
Consider a circular loop of wire of radius r. lying in the xy-plane as shown in figure below. The loop contain a resistor R and a capacitor C, and is placed in a uniform magnetic field which point into the page and decreases at a rate $\frac{dB}{dt} = -\alpha$, with $\alpha > 0$

(i) Find the maximum amount of charge on the capacitor

(ii) Which plate, a or b, has a higher potential? What causes charges to separate.

oln. According to Faraday's law,

$$\varepsilon = -\frac{\partial\phi}{\partial t} = -\frac{\partial B}{\partial t}\pi r^2 = \alpha\pi r^2$$



So, we can say capacitor will be charged by the battery of emf= $\varepsilon = \alpha\pi r^2$ through the resistance

$\therefore \quad q = q_0\left(1 - e^{-t/RC}\right)$

$\therefore \quad q_{\max} = q_0 = \alpha\pi r^2 C$

According to Lenz's law, the direction of induced emf is such that it tries to oppose the cause of its generation. So, the induce current will flow in clockwise direction.

Therefore, electron will flow in anti-clockwise direction.

Therefore, plate b has a lower potential and a plate has a higher potential. Because DC current can not flow through the capacitor.

. A small magnet is dropped down a long vertical copper tube in a uniform gravitational field. After a long time, the magnet **[JEST 2012]**

(a) Attains a constant velocity (b) moves with a constant acceleration

(c) moves with a constant deceleration (d) executes simple harmonic motion.

oln. When magnet bar fall down through a long copper tube in a uniform gravitational field, due to Lorentz force an additional force acting on it opposite to its motion due to this it attains a constant velocity.

. A circular conducting ring of radius 'R' rotates with constant angular velocity ω about its diameter placed along the x-axis. A uniform magnetic field B is applied along the y -axis. If at time t =0 the ring is entirely in the xy- plane, the emf induced in the ring at time t > 0 is **[JEST 2012]**

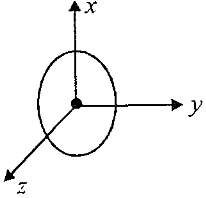
(a) $B\omega^2\pi R^2 t$ (b) $B\omega\pi R^2\tan(\omega t)$ (c) $B\omega\pi R^2\sin(\omega t)$ (d) $B\omega\pi R^2\cos(\omega t)$

oln. Initially the normal of the plate is perpendicular to the xy plane at any time t the normal make the angle ωt

$$\phi = B\cdot ds = B\pi R^2\sin(\omega t)$$

$$\varepsilon = -\frac{\partial\phi}{\partial t} = \omega\pi R^2 B\cos(\omega t)$$

$$\boxed{|\varepsilon| = \omega\pi R^2 B\cos(\omega t)}$$



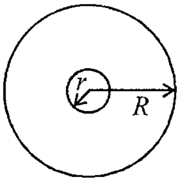
Correct option is (d)

6. A large circular coil of N turns and radius R carries a time varying current $i = i_0 \sin(wt)$. A small circular coil of n turns and radius r (r <<R) is placed at the center of the large coil such that the coils are concentric and coplanar. The induced emf in the small coil
   (a) Leads the current in the large coil by $\pi/2$
   (b) Lags the current in the large coil by π
   (c) Is in phase with the current in the large coil
   (d) lags the current in the large coil by $\pi/2$

Soln. We have, $I = I_0 \sin(\omega t)$

The magnetic field at the centre of large circular surface is

$$B = \frac{\mu_0 IN}{2R} = \frac{\mu_0 N I_0}{2R}\sin\omega t$$

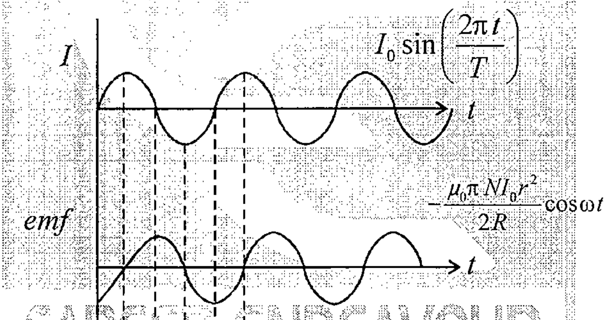


Therefore, the magnetic flux passing through the smaller circular surface.

$$\varphi = \frac{\mu_0 N I_0}{2R}\pi r^2 \sin\omega t$$

$$\therefore \quad emf = -\frac{d\varphi}{dt} = -\frac{\mu_0 N I_0}{2R}\pi r^2 \omega\cos\omega t$$

$$emf = -\frac{\mu_0 \pi N I r^2}{2R}\cos\omega t$$



Therefore, induce emf in the small coil lags behind the current in the large coil by $\frac{\pi}{2}$

Correct option is (d)

7. A conducting loop L of surface area S is moving with a velocity $\vec{\upsilon}$ in a magnetic field $\vec{B}(\vec{r}, t) = \vec{B}_0 t^2$, B_0 is a positive constant of suitable dimensions. The emf induced, V_{emf}, in the loop is given by :

(a) $-\int_S \frac{\partial \vec{B}}{\partial t}.d\vec{S}$ (b) $\oint_L (\vec{\upsilon}\times\vec{B}).d\vec{L}$ (c) $-\int_S \frac{\partial \vec{B}}{dt}.d\vec{S} - \oint_L (\vec{\upsilon}\times\vec{B}).d\vec{L}$ (d) $-\int_S \frac{\partial \vec{B}}{dt}.d\vec{S} + \oint_L (\vec{\upsilon}\times\vec{B}).d\vec{L}$

Soln. Let consider an elementary surface is

Therefore, the magnetic flux pass through the elementary area dS is

$$d\varphi = \phi\, B\cdot dS$$

$$\therefore \quad \text{Induce emf} = -\frac{\partial\varphi}{dt} = -\int\frac{\partial \vec{B}}{\partial t}dS$$

Correct option is (a)

(a) A coil of N turns and cross-sectional area A is rotated at a constant angular velocity ω about a diameter in a magnetic field $B = B_0 \sin\omega t$ directed normal to the axis of rotation. Find the induced emf in the coil. Assume at $t = 0$ the coil axis coincides with the direction of the magnetic field.

(b) A uniform magnetic field $\vec{B}(t)$ pointing along the z-axis, fills up a circular region centred at the origin and lying on the xy-plane. Show that the induced electric field at a distance r from the origin is given by

$$\vec{E} = -\frac{r}{2}\frac{dB}{dt}\hat{\theta}$$

where, $\hat{\theta}$ is the unit cross-radial vector.

oln. (a) At any instant t the coil axis makes an angle ωt with the direction of magnetic field. Hence, flux linked with the coil at any instant t is

$$\Phi = NAB\cos\omega t = NAB\sin\omega t\cos\omega t = \frac{1}{2}NAB_0\sin 2\omega t$$

Now, from Faraday's law, the induced emf is

$$\varepsilon = -\frac{d\Phi}{dt} = -NAB_0\omega\cos 2\omega t$$

(b) Considering an Amperian loop or radius r we can write from Faraday's law, induced emf.

$$\varepsilon = -\frac{d\Phi}{dt} \text{ or } \oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\left[\pi r^2 B(t)\right]$$

Or, $$E\cdot 2\pi r = -\pi r^2\frac{dB}{dt} \qquad \left[\because \vec{E} \text{ points in the circumferential direction}\right]$$

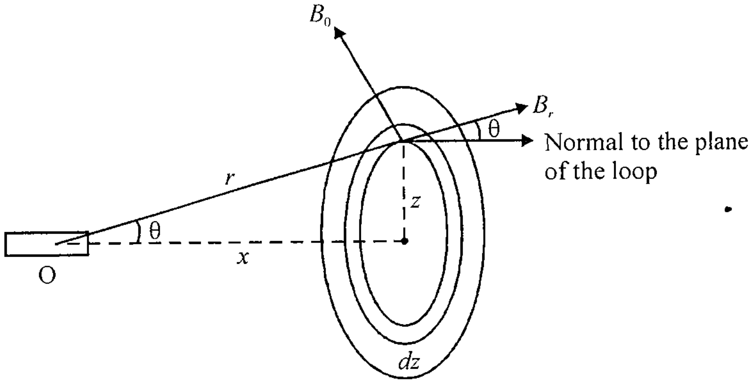
Therefore, $$\vec{E} = -\frac{r}{2}\cdot\frac{dB}{dt}\hat{\theta}$$

A small magnet of moment M is approaching toward a circular loop of radius a along its axis with a constant speed v. Show that when the magnet is at a distance x from the coil the current induced in the loop is given by

$$i = \frac{3\mu_0 Ma^2vx}{\left(a^2+x^2\right)^{5/2}}$$

where R is the resistance of the loop.

oln. The magnetic field at a point z from the centre of the loop as shown in the figure



$$\vec{B} = \frac{\mu_0}{4\pi}\left[-\frac{\vec{M}}{r^3} + \frac{3\left(\vec{M}\cdot\vec{r}\right)\vec{r}}{r^5}\right]$$

Its radial and transverse components are

$$B_r = \frac{\mu_0}{4\pi}\cdot\frac{2M\cos\theta}{r^3} \text{ and } B_\theta = \frac{\mu_0}{4\pi}\cdot\frac{M\sin\theta}{r^3}$$

So, the component of $\vec{B}$ normal to the plane of the loop is

$$B_n = B_r\cos\theta + B_\theta\cos(90°+\theta) = \frac{\mu_0}{4\pi}\cdot\frac{M}{r^3}\left(3\cos^2\theta - 1\right)$$

Now, considering an elementary ring of radii z and $z+dz$ in the plane of the loop we find that the magnetic flux through this ring is $B_n\cdot 2\pi z dz$. Therefore, total flux through the loop is

$$\Phi = \frac{\mu_0}{4\pi}.\int_0^a \frac{M}{r^3}\left(3\cos^2\theta - 1\right)\cdot 2\pi z dz = \frac{\mu_0 M}{2}\int_0^a \frac{\left(2x^2 - z^2\right)}{\left(x^2+z^2\right)^{5/2}}dz$$

where, we use $\cos\theta = x/\sqrt{x^2+z^2}$

Substituting $x^2+z^2 = y^2$ and integrating we get

$$\Phi = \frac{\mu_0 M}{2}\cdot\frac{a^2}{\left(a^2+x^2\right)^{3/2}}$$

Now, the induced emf $\varepsilon = -\frac{d\Phi}{dt}$ and induced current.

$$i = \frac{\varepsilon}{R} = -\frac{1}{R}\frac{d\Phi}{dt} = -\frac{1}{R}\frac{d\Phi}{dt}.\frac{dx}{dt} = -\frac{v}{R}\frac{d\Phi}{dx} = \frac{3\mu_0 M v a^2 x}{2R\left(a^2+x^2\right)^{5/2}}$$

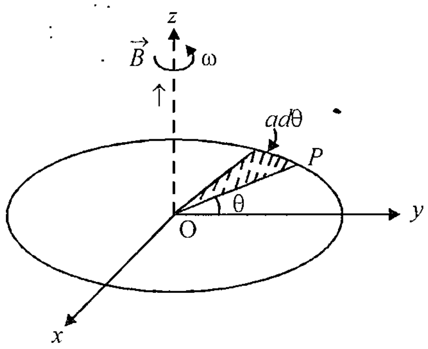
10. A conducting disc of radius a is rotating about its own axis with a constant angular velocity ω in a uniform magnetic field $\vec{B}$ acting perpendicular to the plane of the disc. Find the value of emf induced between the rim and the centre of the disc.

Soln. Suppose at any time instant t any radius OP of the

$$d\Phi = B\cdot\frac{1}{2}a^2 d\theta$$

Hence the value of the induced between the centre and the rim would be the

$$\varepsilon = \left|\frac{d\Phi}{dt}\right| = \frac{1}{2}Ba^2\frac{d\theta}{dt} = \frac{1}{2}Ba^2\omega$$



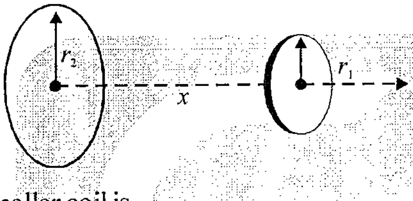
Consider two coplanar concentric flat circular coils of average radii r_1 and r_2 and consisting of N_1 and N_2 turns respectively $(r_1 << r_2)$. If the smaller coil is moved along the common axis with a cosntant speed v keeping its plane parallel to the plane of the larger coil show that the induced emf in it will be maximum when it is at a distance $r_2 / 2$ from the larger coil.

oln. The magnetic field at the centre of the smaller coil due to a current I_2 in larger coil is

$$B = \frac{\mu_0}{4\pi} \cdot \frac{2\pi N_2 I_2 r_2^2}{\left(r_2^2 + x^2\right)^{3/2}}$$

where x is the distance between the coils. Assuming B to be uniform over the area of the smaller coil we can write for the flux linked with the smaller coil as

$$\Phi = B \cdot \pi r_1^2 \cdot N_1 = \frac{\mu_0 \pi N_1 N_2 r_1^2 r_2^2 I_2}{2\left(r_2^2 + x^2\right)^{3/2}}$$



Therefore, induced emf in the smaller coil is

$$\varepsilon = -\frac{d\Phi}{dt} = -\frac{d\Phi}{dx}\frac{dx}{dt} = -v\frac{d\Phi}{dx} = \frac{3\mu_0 \pi N_1 N_2 r_1^2 r_2^2 I_2 v \cdot x}{2\left(r_2^2 + x^2\right)^{5/2}}$$

Now for ε to be maximum,

$$\frac{d\varepsilon}{dx} = 0 \text{ or } \frac{d}{dx}\left[\frac{x}{\left(r_2^2 + x^2\right)^{5/2}}\right] = 0$$

This requires that $x = \frac{r_2}{2}$.

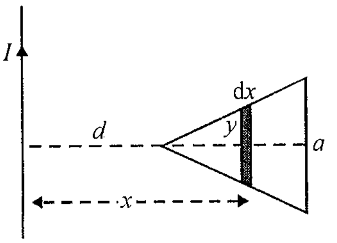
2. Consider a wire loop in the form an equilateral triangle of side a. A long straight wire carrying I is placed in the plane of the loop at a distance d from the nearest vertex of the triangle and parallel to the farthest side of the triangle. Find the total magnetic flux through the loop.

oln. Consider an element of thickness dx and length y in the loop. The magnetic field at this element due to the straight wire is

$$B = \frac{\mu_0}{4\pi} \cdot \frac{2I}{x}$$

Therefore, the flux through the element is

$$d\Phi = B \cdot y dx$$



From above figure,

$$\frac{y}{x-d}=\frac{a}{h}$$

where, $h=\frac{\sqrt{3}}{2}a$ is the height of the loop

Therefore, $d\Phi=\frac{\mu_0 I}{2\pi x}\cdot\frac{a(x-d)}{h}\cdot dx=\frac{\mu_0 Ia}{2\pi h}\cdot\frac{x-d}{x}dx$

So, the total flux through the loop is

$$\Phi=\frac{\mu_0 Ia}{2\pi h}\int_d^{d+h}\frac{x-d}{x}dx=\frac{\mu_0 Ia}{2\pi h}\cdot\left[x-d\ell nx\right]_x^{d+h}$$

$$=\frac{\mu_0 Ia}{2\pi h}\left[h-d\ell n\frac{d+h}{d}\right]=\frac{\mu_0 Ia}{2\pi}\left[1-\frac{d}{h}\ell n\left(1+\frac{h}{d}\right)\right]$$

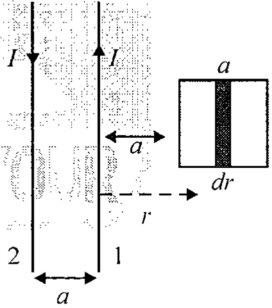
13. Two long fixed straight parallel wires separated by a distance a carry the same current I in opposite directions with I increasing at the rate $\frac{dI}{dt}$. A square loop of side a is placed in the plane of the wires with its near side at a distance a from one of the wires. Find the induced emf in the square loop. What is the direction of the induced current?

Soln. Magnetic field produced by the straight wire 1. at a distance *r* from it is

$$B_1=\frac{\mu_0}{4\pi}\cdot\frac{2I}{r}$$

So, the flux through an elementary area *adr* is

$$d\Phi_1=B_1\cdot adr=\frac{\mu_0 Ia}{2\pi r}dr$$



Total magnetic flux through the loop due to wire 1 is

$$\Phi_1=\frac{\mu_0 Ia}{2\pi}\int_a^{2a}\frac{dr}{r}=\frac{\mu_0 Ia}{2\pi}\ell n2$$

Similarly, the flux due to wire 2 is

$$\Phi_2=\frac{\mu_0 Ia}{2\pi}\int_{2a}^{3a}\frac{dr}{r}=\frac{\mu_0 Ia}{2\pi}\ell n\frac{3}{2}$$

As the two fluxes are in opposite directions the net flux through the loop is

$$\Phi=\Phi_1-\Phi_2=\frac{\mu_0 Ia}{2\pi}\ell n\frac{4}{3}$$

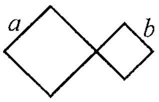
pointing the towards the page.

Therefore, the induced emf in the loop is

$$\varepsilon=-\frac{d\Phi}{dt}=-\frac{\mu_0 a}{2\pi}\ell n\frac{4}{3}\cdot\frac{dI}{dt}$$

According to Lenz's law the direction of induced current will be such as to oppose the change in flux, i.e., it must produce a flux pointing out from the page. So, in figure above the induced current will be anticlockwise as seen from the above.

4. A plane loop shown in figure is shaped as two squares with sides a = 20 cm and b = 10 cm and is introduced into a uniform magnetic field at right angles to the loop's plane. The magnetic induction varies with time as $B_0 = B_0 \sin \omega t$, where $B_0 = 10\ mT$ and $\omega = 100\ s^{-1}$. Find the amplitude of the current induced in the loop if its resistance per unit length is equal to $\rho = 50\ m\Omega/m$



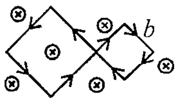
oln. Loops are connected in such a way that if current flow in clockwise in one and in other, it will flow clocwise. Hence, induced emf in b is opposite in the loop a.

Therefore, emf in the loop with side a is

$$\frac{d}{dt}\left(a^2 B\right) = a^2 \frac{d}{dt}\left(B_0 \sin \omega t\right) = a^2 \omega \cos \omega t B_0$$

and emf in the loop with side b is

$$b\frac{d}{dt}\left(B_0 \sin \omega t\right) = b^2 \omega B_0 \cos \omega t$$



Hence, net emf in the circuit $= \left(a^2 - b^2\right) B_0 \omega \cos \omega t$

Resistance of the circuit $= 4(a+b)\rho$

Therefore, the amplitude of the current,

$$= \frac{\left(a^2 - b^2\right) B_0 \omega}{4(a+b)\rho} = \frac{(a-b) B_0 \omega}{4\rho} = \frac{(20-10)\times 10\times 10^5 \times 100}{50\times 4\times 10^{-3}} = \frac{10^{+2}}{200} = \frac{100}{200} = 0.5A$$

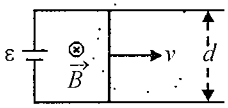
5. Consider two long parallel rails of negligible resistance separated by a distance d and connected to a cell of emf of ε at one end. A jumper of mass m and resistance R slides without friction along the rails. An external uniform magnetic field B is aplied normal to the plane of the rails. Find the velocity of the jumper as a function of time. Find its terminal velocity and also current through it when it attains the terminal velocity.

oln. Let at any instant of time t the jumper is moving with velocity v. The emf induced in it is Bvd. So, from Ohm's law current in the circuit is

$$i = \frac{\varepsilon - Bvd}{R} \quad \ldots (i)$$

As the force on the conductor is Bid, its equation of motion is

$$m\frac{dv}{dt} = Bid = \frac{Bd}{R}\left(\varepsilon - Bvd\right) \text{ or } \frac{dv}{v - \frac{\varepsilon}{Bd}} = -\frac{B^2 d^2}{mR} dt$$



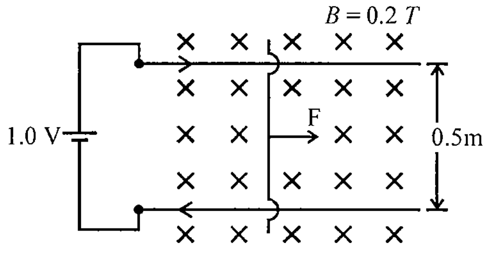
Integrating,

$$\int_0^v \frac{dv}{v - \frac{\varepsilon}{Bd}} = -\frac{B^2 d^2}{mR}\int_0^t dt \text{ or } \ell n \frac{v - \varepsilon / Bd}{-\varepsilon / Bd} = -\frac{B^2 d^2}{mR} t \text{ or } V = \frac{\varepsilon}{Bd}\left(1 - e^{-B^2 d^2 t / mR}\right)$$

It gives the velocity v as a function of time t. As $t \to \infty$, v tends to a constant value $v_T = \frac{\varepsilon}{Bd}$, which is called the terminal velocity. Equation (1) shows that at $v = v_T = \frac{\varepsilon}{Bd}$ the current $i = 0$. Note that the current due to ε tries to move the jumper away with an acceleration whereas the current due to induced emf tries to oppose this motion. When the induced emf equals the cell emf net current in the jumper becomes zero and the jumper

16. Two long parallel wires of zero resistance are connected to each other by a battery of 0.1 volt. The separation between the wires is 0.5 m. A metallic bar, which is perpendicular to the wires and of resistance 10 Ω moves on these wires when magnetic field of 0.2 Tesla is acting perpendicular to the plane containing the bar and the wires. Find the steady state velocity of the bar. If the mass of the bar is 0.002 kg, find its velocity as a function of time.

Soln. See below figure :



The current i in the bar is given by $i = \dfrac{1.0 \text{ volt}}{10\,\Omega} = 0.1$ amp.

The current carrying bar is placed in the magnetic field $\vec{B} = 0.2$ T. $\vec{B}$ is perpendicular to the plane of paper and directed downward. The magnetic force on the bar is given by, $F = i\,B\,l = 0.1 \times 0.02 \times 0.5 = 1 \times 10^{-3}$ newton.

The induced e.m.f. in the bar

$e = B\,v\,l = 0.02 \times v \times 0.5 = 1 \times 0.01\,v$ volt, where v = instantaneous velocity of the bar.

By Lent'z law, e will oppose the motion of the bar which will ultimately attain a steady velocity. In this state, the induced e.m.f. e will be equal to the applied e.m.f. (1.0 volt).

Therefore, $0.01\,v = 1.0$ or $v = 100\ m/s$

If m be the mass of the bar, its acceleration is given by

$$\frac{dv}{dt} = \frac{F}{m}$$

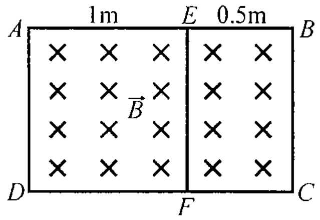
Integrating, we get, $v = \dfrac{F}{m}t + C$

Where C = integrating constant.

At $t = 0$, $v = 0$ and hence $C = 0$

Therefore, $v = \dfrac{F}{m}t = \dfrac{1 \times 10^{-3}}{0.002}t$ or $v = 0.5\,t$.

17. A rectangle frame $ABCD$ made of uniform metal wire has a straight connection between E and F made of the same wire as shown in figure. $AEFD$ is a square of side 1 m and $EB = FC = 0.5$ m. The entire circuit is placed in a steadily increasing, uniform magnetic field directed into the plane of the paper and normal to it. The rate of change of magnetic field is 1 T/s. The resistance per unit length of the wire is 1 Ω/m. Find the magnitudes and directions of the currents in the segments AE, BE and EF.



oln. Induced e.m.f. $= -\dfrac{d\phi}{dt} = -\dfrac{d}{dt}(BA) = -A\dfrac{dB}{dt}$

Therefore, induced e.m.f. in $AEFD = 1 \times 1 = 1\text{ V}$

Induced e.m.f. in $EBCF = \dfrac{1}{2} \times 1 = 0.5\text{ V}$ (area $= \frac{1}{2}\text{m}^2$)

Total induced e.m.f. = 1 + 0.5 = 1.5 V

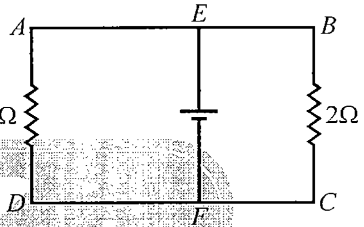
Given that resistance per unit length of the wire is 1 Ω/m. Hence the equivalent circuit takes the form as shown in figure. The resistance 3Ω and 2Ω are in parallel. Hence their equivalent resistance would

$$\frac{(3\times 2)}{(3+2)} = \left(\frac{6}{5}\right)$$

Therefore, current from $EF = \dfrac{1.5}{(6/5)} = \dfrac{5}{4}$amp

Current in $AE = \dfrac{5}{4} \times \dfrac{2}{5} = \dfrac{1}{2}$amp

Current in $BE = \dfrac{5}{4} \times \dfrac{3}{5} = \dfrac{3}{4}$amp



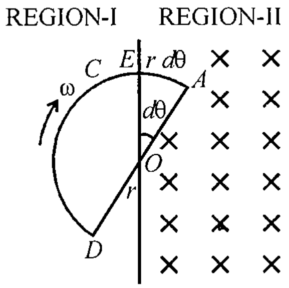
8. Space is divided by the line AD into two regions. Region I is field free and the region II has a uniform magnetic field B directed into the plane of paper. ACD is semicircular conducting loop of radius r with centre at O (figure), the plane of the loop being in the plane of the paper. The loop is now made to rotate with a constant angular velocity ω about an axis passing through O and perpendicular to the plane of the paper. The effective resistance of the loop is R.
   (i) Obtain an expression for the magnitude of the induced current in the loop.
   (ii) Show the direction of the current when the loop is entering into the region II.
   (iii) Plot a graph between the induced e.m.f. and time of rotation for two periods of rotation.

oln. (i) When the loop is in region I, the magnetic field linked with the loop is zero. When the loop enters in magnetic field in region II (See figure), the magnetic flux linked with it, is given by $\phi = BA$.

Therefore, e.m.f. induced $e = -\dfrac{d\phi}{dt} = -\dfrac{d(BA)}{dt} = -B\dfrac{dA}{dt} = B\dfrac{dA}{dt}$ (numerically)

Let $d\theta$ be the angle rotated by the loop in time dt, then

dA = Area of the triangle $OEA = (1/2)r \cdot r\,d\theta$



Therefore, $e = B\cdot\dfrac{1}{2}\cdot\dfrac{r\cdot r\,d\theta}{dt} = \dfrac{1}{2}Br^2\dfrac{d\theta}{dt} = \dfrac{1}{2}Br^2\omega$

Induced current $= \dfrac{\text{e.m.f.}}{R} = \dfrac{1}{2}\dfrac{Br^2\omega}{R}$

dA can also be calculated in the following way :

The area corresponding 2π (angle) is πr^2

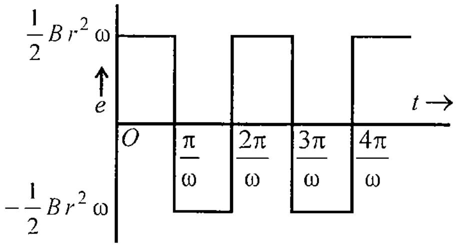
Therefore, area corresponding to unit angle $= \dfrac{\pi r^2}{2\pi}$

Area corresponding to angle $d\theta = \dfrac{\pi r^2}{2\pi} \times d\theta$

Therefore, $dA = \dfrac{\pi r^2}{2\pi} \times d\theta = \dfrac{1}{2} r^2 \, d\theta$.

(ii) According to Lenz's law, the direction of current induced is to oppose the change in magnetic flux. So the current induced must be upward. In this way, the direction of current must be *anticlockwise*.

(iii) The graph is shown in figure. When the loop enters the magnetic field, the magnetic flux linked increases and e.m.f. $e = (1/2) B r^2 \omega$ in one direction. When the loop enters comes out of the field, the decreases and e.m.f. is induced in opposite sense.



19. A long solenoid having 200 turns per cm carries a current of 1.5 amp. At the centre of it is placed a 100 turn of cross-sectional area 3.14×10^{-4} m^2 having its axis parallel to the field produced by the solenoid. When the direction of current in the solenoid is reversed within 0.05 sec, what is the e.m.f. induced in the coil ?

Soln. We know that at the centre of solenoid, the magnetic field B is given by

$$B = \mu_0 n i$$

$$= \left(4\pi \times 10^{-7} \frac{\text{water}}{\text{amp. metre}}\right) \times \left(200 \times 10^2 \frac{\text{turns}}{\text{metre}}\right) \times (1.5 \text{ amp.})$$

$$= 3.8 \times 10^{-2} \frac{\text{weber}}{\text{metre}^2}$$

The flux through each turn of the coil of area $A = 3 \times 10^{-4}$ metre^2

Therefore, $\phi = BA = (3.8 \times 10^{-2}) \times (3.14 \times 10^{-4})$

$$= 1.2 \times 10^{-5} \text{ weber}$$

When the current in the solenoid is reversed, the change in flux

$$= 2 \times (1.2 \times 10^{-5}) = 2.4 \times 10^{-5} \text{ weber}$$

Induced e.m.f. $e = N \dfrac{d\phi}{dt} = 100 \times \dfrac{2.4 \times 10^{-5}}{0.05} = 0.048$ volt

20. A long cylindrical coil of inductance L_1 is wound on a bobbin of diameter d_1. The magnetic iinduction in the coil connected to a current source is B_1. After rewinding the coil on a bobbin of diameter d_2 its inductance become L_2. Find the magnetic induction B_2 of the field in the new coil connected to the same current source asssuming that the length of wire is much larger than that of coil.

Soln. The cross-sectional area of coils are $A_1 = \dfrac{\pi d_1^2}{4}$ and $A_2 = \dfrac{\pi d_2^2}{4}$.

Let ϕ be the magnetic flux linked with each turn due to a current i and N be the number of turns in the coil. The total flux linked with the coil will be $N\phi = N B A$. If L be the self inductance of the coil, then

$$NBA = Li \text{ or } B = \left(\frac{Li}{NA}\right)$$

$$\therefore \quad \frac{B_2}{B_1} = \frac{L_2 i_2 N_1 A_1}{L_1 i_1 N_2 A_2} = \frac{L_2 N_1 A_1}{L_1 N_2 A_2}$$

(Since, $i_1 = i_2$, because wire and current source remain unchanged).

Further $\frac{N_1}{N_2} = \frac{d_2}{d_1}$

Hence, $\frac{B_2}{B_1} = \frac{L_2}{L_1} \times \frac{d_2}{d_1} \times \frac{\pi d_1^2}{\pi d_2^2} = \frac{L_2 d_1}{L_1 d_2}$

$$B_2 = B_1\left(\frac{L_2 d_1}{L_1 d_2}\right).$$

A long solenoid having 1000 turns per cm carries an alternating current of peak value 1 ampere. A search coil having a cross-section area of 1×10^{-4} m^2 and 20 turns is kept in the solenoid so that its plane is perpendicular to the axis of the solenoid. The search coil registers a peak voltage of 2.5×10^{-2} volt. Find the frequency of the current in the solenoid.

oln. We know that the magnetic field of the solenoid is given by

$B = \mu i N,$ ... (1)

where i is the current and N is the number of turns per matre.

The magnetic flux-linked with the search coil is given by

$\phi = B A N_s$ ... (2)

where A = area of search coil and N_s is the total number of turns in the search coil.

Substituting the value of B from equation (1) and (2), we get $\phi = (\mu_0 i N) A N_s$

$$\text{Now, } \frac{d\phi}{dt} = \mu_0 A N N_s \frac{di}{dt}$$

$$= \mu_0 A N N_s (i_0 \omega \cos \omega t) \quad \text{... (3)}$$

The peak value is given by

$$\left(\frac{d\phi}{dt}\right)_{\text{peak}} = \mu_0 A N N_s i_0 \omega \quad \text{... (4)}$$

Substituting the given values, we have

$$2.5 \times 10^{-2} = (4\pi \times 10^{-7}) \times (1 \times 10^{-4}) \times (10^5) \times (20) \times 1 \times \omega$$

Solving for ω, we get $\omega = 99.47$ per sec.

$$\therefore \quad f = \frac{\omega}{2\pi} = \frac{99.47}{2 \times 3.14} = 15.8 \text{ per sec}$$

22. Two parallel vertical metallic rails AB and CD are separated by 1 m. They are connected at the two ends by resistances R_1 and R_2 as shown in figure. A horizontal metallic bar L of mass 0.2 kg slides without friction, vertically down the rails under the action of gravity. There is a uniform horizontal magnetic field of 0.6 T perpendicular to the plane of the rails. It is observed that when the terminal velocity is attained, the powers dissipated in R_1 and R_2 are 0.76 W and 1.2 W respectively. Find the terminal velocity of the bar L and the values of R_1 and R_2.

Soln. See figure. The forces acting on the bar are

(i) Weight mg acting vertically downward.

(ii) Magnetic force $B\,i\,l$ acting vertically upwards.

When the two forces become numerically equal, the bar acquires terminal velocity v_T.

Therefore, $mg = Bil$ or $i = \frac{mg}{Bl}$ or $i = \frac{0.2\times 9.8}{0.6\times 1} = \frac{49}{15}$ amp.

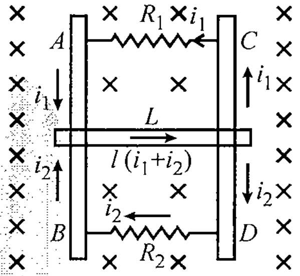
Let i_1 and i_2 be the currents in R_1 and R_2 respectively. Then,

$i = i_1 + i_2$ or $\frac{49}{15} = i_1 + i_2$ ... (1)

Now $e = i_1 R_1 = i_2 R_2$ ... (2) (R_1 and R_2 are in parallel)

Further $P_1 = e\,i_1$ or $0.76 = e\,i_1$ ... (3)

and $P_2 = e\,i_2$ or $1.2 = e\,i_2$ ... (4)



From Eqs. (3) and (4),

$\frac{i_1}{i_2} = \left(\frac{0.76}{1.2}\right)$ or $i_1 = \frac{19}{30} i_2$ ... (5)

Solving Eqs. (1) and (5), we get

$i_2 = 2$ amp. and $i_1 = \left(\frac{19}{15}\right)$ amp.

From equation (4),

$1.2 = e\times 2$ or $e = 0.6$ volt ... (6)

Now $e = B v_T l$ or $v_T = \left(\frac{e}{Bl}\right)$

Therefore, $v_T = \frac{0.6}{0.6\times 1} = 1$ m/s

Again, $R_1 = \frac{e}{i_1} = \frac{0.6\times 15}{19} = 0.47\,\Omega$ and $R_2 = \frac{e}{i_2} = \frac{0.6}{2} = 0.3\,\Omega$

23. A very small circular loop of area 5×10^{-4} m^2, resistance 2 Ohm and negligible inductance is initially coplanar and concentric with a much larger fixed circular loop of radius 0.1 m. A constant current of 1 ampere is passed in the bigger loop and the smaller loop is rotated with angular velocity ω rad/sec about a diameter.

Calculate : (i) the flux linked with the smaller loop,

(ii) induced e.m.f. and

(iii) induced current in the smaller loop, as a function of time.

Soln. (i) Magnetic field produced by the bigger loop at the centre $= \mu_0 n i/2r$.

Therefore, $B = \frac{(4\pi\times 10^{-7})\,(1)\,(1)}{2\times 0.1} = 2\pi\times 10^{-6}$ weber/m^2

The field is perpendicular to the plane of the loop.

Charge of flux linked up with the coil during one rotation $= 2\,B\,A$

$\phi = 2\times(2\pi\times 10^{-6})\times(5\times 10^{-4}) = 20\pi\times 10^{-10}$

Therefore, rate of change of flux $= \dfrac{20\pi\times10^{-10}}{\text{time}} = \dfrac{20\pi\times10^{-10}}{(2\pi/\omega)} = 10^{-9}\omega =$ induced e.m.f.

So, induced e.m.f. $= 10^{-9}\ \omega$

Now induced current $= \dfrac{\text{induced e.m.f.}}{\text{Resistance}} = \dfrac{10^{-9}\omega}{2}$

This is maximum current.

Current at any instant $i = i_0 \sin \omega t$

Therefore, $i = \dfrac{10^{-9}\omega}{2}\sin\omega t$ ampere.

4. A long solenoid of diameter 0.1 m has 2×10^4 turns per metre. At the centre of the solenoid a 100 turn coil of radius 0.01 m is placed with its axis coinciding with the solenoid axis. The current in the solenoid is decreased at a constant rate from +2 A to –2 A in 0.05 sec. Find the e.m.f. induced in the coil. Also find the total charge flowing through the coil during this time when the resistance of the coil is $10\pi^2\,\Omega$.

oln. Magnetic induction due to the solenoid at its centre is given by $B = \mu_0 ni$ weber/m^2.

Flux linked with each turn of the coil $\phi = BA$ weber $= \mu_0 ni A$ weber

Now, induced e.m.f. in N turn coil is given by

$$e = -N\frac{d\phi}{dt} = -N\frac{d}{dt}(\mu_0 ni A) = -N\mu_0 nA\frac{di}{dt}$$

$$\frac{di}{dt} = -\frac{4}{0.05} = -80$$

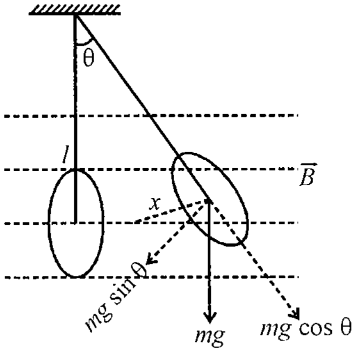
Therefore, $e = -100\times(4\pi\times10^{-7})\times(2\times10^4)\times(\pi\times10^{-4})\times(-80) = 6.4\,\pi^2\times10^{-3}$ volt

Now, $q = i\times t = \dfrac{e}{R}\times t$

$$= \frac{(6.4\,\pi^2\times10^{-3})\,(0.05)}{10\,\pi^2} = 3.2\times10^{-5}\ \text{C} = 32\ \text{micro-coulomb}$$

5. A wire frame of area 3.92×10^{-4} m^2 and resistance 20 Ω is suspended freely from a 0.392 m long thread. There is a uniform magnetic field of 0.784 tesla and the plane of wire-frame is perpendicular to the magnetic field. The frame is made to oscillate under gravity by displacing it through 2×10^{-2} m from its initial position along the direction of magnetic field. The plane of the frame is always along the direction of thread and does not rotate about it. What is the induced e.m.f. in wire frame as a function of time ? Also find the maximum current is the frame ?

oln. The situation is shown in below figure:



The instantaneous flux through the frame when displaced through an angle θ is given by $\phi = BA\cos\theta$.
Instantaneous induced e.m.f.

$$e = -\frac{d\phi}{dt} = BA\sin\theta\frac{d\theta}{dt} = BA\theta\frac{d\theta}{dt} \qquad (\because\ \sin\theta = \theta) \qquad \ldots(1)$$

Applying Newton's law $m\frac{d^2x}{dt^2} = -mg\sin\theta$ or $\frac{d^2x}{dt^2} = -g\sin\theta$

From figure, $\sin\theta = \theta = \frac{x}{l}$ or $x = l\theta$

Therefore, $\frac{d^2x}{dt^2} = -\frac{g}{l}\theta$ or $\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta$

Putting $\omega = \sqrt{(g/l)}$, we get, $\frac{d^2\theta}{dt^2} + \omega^2\omega = 0$ ... (2)

This is the equation of S.H.M.

Solution of equation (2) is given by $\theta = \theta_0 \sin\omega t$ ... (3)

Substituting the value of θ in equation (1), we get

$$e = BA(\theta_0\sin\omega t)\frac{d}{dt}(\theta_0\sin\omega t)$$

$$= BA\ \theta_0\sin\omega t\ \theta_0\ \omega\cos\omega t$$

$$= \frac{1}{2}BA\omega\theta_0^2\sin 2\omega t$$

Here, $\omega = \sqrt{\left(\frac{g}{l}\right)} = \sqrt{\left(\frac{9.8}{0.392}\right)} = 5\ \text{sec}^{-1}$ and $\theta_0 = \frac{x_0}{l} = \frac{2\times10^{-2}}{0.392}$

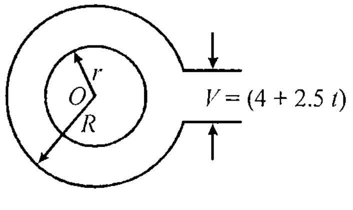
Substituting the value, we get

$$e = \frac{1}{2}\times(0.784)\times(3.92\times10^{-4})\times(5)\times\left(\frac{2\times10^{-2}}{0.392}\right)^2\sin 10t$$

$$= 2\times10^{-6}\sin 10t \text{ or } e_{\text{max.}} = 2\times10^{-6}\text{ volt and } i_{\text{max.}} = \frac{e_{\text{max.}}}{R} = \frac{2\times10^{-6}\text{ volt}}{20\,\Omega} = 10^{-7}\text{ amp}$$

26. Two concentric coplanar circular loops made of wire with resistance per unit length $10^{-4}\,\Omega\,\text{m}^{-1}$, have diameters 0.2 m and 2 m. A time varying potential difference $(4 + 2.5t)$ volts is applied to the larger loop. Calculate the current in the smaller loop.

Soln. The situation is shown in figure :



The field at O, due to the current in large loop is given by $B_0 = \frac{\mu_0 i}{2R}$

where $i = \dfrac{\text{Potential difference}}{\text{Resistance}} = \dfrac{(4+2.5t)}{(2\pi R)\rho}$

Therefore, $B_0 = \dfrac{\mu_0}{2R}\left[\dfrac{(4+2.5t)}{(2\pi R)\rho}\right]$

Here we assume that the central loop is very small i.e., $r \ll R$, so the field may be regarded as constant within the loop. Thus within the loop the field is B_0. The flux in smaller loop

$$\phi = B_0 \times (\pi r^2) = \frac{\mu_0}{2R}\left[\frac{(4+2.5t)}{(2\pi R)\rho}\right]\cdot \pi r^2$$

Induced e.m.f. $= -\dfrac{d\phi}{dt} = \dfrac{\mu_0 r^2}{4R^2\rho} \times (2.5)$

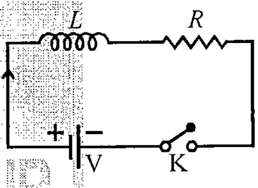
Thus current i in the smaller loop

$$i = \frac{\text{e.m.f.}}{\text{Resistance}} = \frac{\mu_0 r^2}{4R^2\rho} \times 2.5 \times \left(\frac{1}{2\pi r \rho}\right) = \frac{2.5\mu_0 r}{8\pi R^2 \rho^2}$$

$$i = \frac{2.5 \times (4\pi \times 10^{-7})(0.1)}{8\pi \times 1 \times 10^{-8}} = 1.25 \text{ amp}$$

7. A solenoid has an inductance of 10 henry and a resistance of 2 ohm. It is connected to a 10 volt battery. How long will it take for the magnetic energy to each 1/4 of its maximum value ?

oln. The L-R circuit is shown in figure. Let at an instant t, after shorting of switch, current through the circuit is i.



Then, $L\dfrac{di}{dt} + Ri = V$ or $\dfrac{di}{V - Ri} = \dfrac{dt}{L}$

Integrating this expression and keeping $i = 0$ when $t = 0$, we get

$$i = \frac{V}{R}(1 - e^{-Rt/L}) = i_0(1 - e^{-Rt/L}) \quad \dots (1)$$

where $i_0 = \dfrac{V}{R}$ = maximum current.

Magnetic energy stored $= \dfrac{1}{2}Li^2$

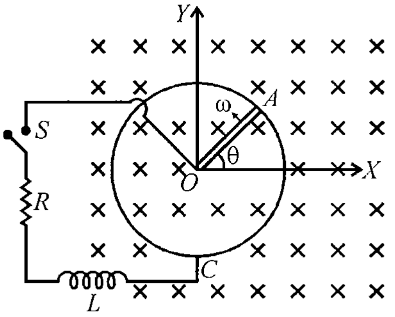
Here $\dfrac{1}{2}Li^2 = \dfrac{1}{4}\left(\dfrac{1}{2}Li_0^2\right)$ or $i = \dfrac{i_0}{2} = \dfrac{V}{2}R$

Substituting the values in equation (1), we get

$$\frac{V}{2R} = \frac{V}{R}(1 - e^{-2t/10}) \text{ or } 0.5 = 1 - e^{-0.2t}$$

or $\quad e^{-0.2t} = 0.5$ *which gives* $t = 3.478$ sec

28. A metal rod OA of mass m and length l is kept rotating with a constant angular speed ω in a vertical plane about a horizontal axis at the end O. The free end A is arranged to slide without friction along a fixed conducting circular ring in the same plane as that of rotation. A uniform and constant magnetic field induction B is applied perpendicular and into the plane of rotation as shown in figure. An inductor and an external resistance R are connected a switch S between the point O and a point C on the ring to form an electrical circuit. Neglect the resistance of the ring and the rod. Initially, the switch is open.



(a) What is the induced e.m.f. across the terminals of the switch ?

(b) The switch S is closed at time $t = 0$

  (i) Obtain an expression for the current as a function of time.

  (ii) In the steady state, obtain the time dependence of the torque required to maintain the constant angular speed, given that the rod OA was along the positive X-axis at $t = 0$.

Soln. (a) The flux covered by enclosed area of sector OAC at the given instant of time

$$\Phi_B = B\left(\frac{1}{2}\cdot r\,r\,\theta\right) = \frac{1}{2}Br^2\theta$$

Therefore, the induced e.m.f. developed between two ends O and A of the rod, according to Faraday's law

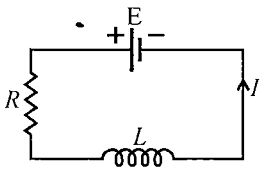
$$e = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\left(\frac{1}{2}Br^2\theta\right) = -\frac{1}{2}Br^2\frac{d\phi}{dt}$$

$$= -\frac{1}{2}Br^2\omega$$

The negative sign indicates the direction of e.m.f. which would be from A towards O so as to resist the motion in anticlockwise direction.

(b) (i) Here we can replace the rod by a battery of e.m.f. $= \frac{1}{2}Br^2\omega$ with positive terminal pointing towards O. Now switch, resistance and inductance are connected in series as shown in figure. At $t = 0$, the switch is closed.

Let a current I flows through the circuit. When the current increases in the circuit, a self induced e.m.f. $\left[-L\frac{dI}{dt}\right]$ is developed in the inductance.



By Kirchoff's law, we have $E = IR - L\frac{dI}{dt} = 0$

or $L\frac{dI}{dt} = E - IR$ or $\frac{dI}{E - IR} = \frac{dt}{L}$

or $\int_0^I \frac{dI}{E - IR} = \int_0^t \frac{dt}{L}$ ($\because$ At $t = 0$, $I = 0$ and at time t, current is I)

$$-R\frac{dI}{dZ}=1 \text{ or } dI=-\frac{dZ}{R}$$

$$\therefore \quad \int_{E}^{E-RI}\frac{-dZ}{R\cdot Z}=\int_{0}^{t}\frac{dt}{L}$$

$$\text{or} \quad -\frac{1}{R}[\log_e Z]_{E}^{E-RI}=\frac{t}{L} \qquad \text{or} \quad -\frac{1}{R}[\log_e(E-RI)-\log_e E]=\frac{t}{L}$$

$$\text{or} \quad -\frac{1}{R}\log_e\left\{\frac{(E-RI)}{E}\right\}=\frac{t}{L}$$

$$\text{or} \quad \frac{(E-RI)}{E}=e^{-Rt/L} \text{ or } E-RI=E\,e^{-Rt/L}$$

$$\text{or} \quad RI=E\left[1-e^{-Rt/L}\right] \text{ or } I=\frac{E}{R}\left[1-e^{-Rt/L}\right]$$

$$\therefore \quad I=\frac{Br^2\omega}{2R}(1-e^{-Rt/L})$$

(ii) The force acting on rod due to induced current I is given by

$$\mathbf{F}=I(\mathbf{I}\times\mathbf{B})=\frac{Br^2\omega}{2R}(1-e^{-Rt/L})\,\mathbf{r}\,\mathbf{B}$$

$$\text{or} \quad F=\frac{B^2r^3\omega}{2R}(1-e^{-Rt/L})$$

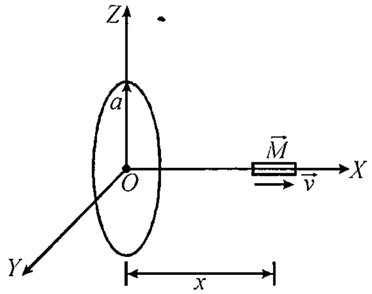
This force may be regarded as acting on the middle point of rod OA. The torque produced is given by

$$\tau=F\cdot\left(\frac{r}{2}\right)=\frac{B^2r^4\omega}{4R}(1-e^{-Rt/L})$$

The torque acts in clockwise direction. To maintain a constant angular speed ω, an external torque τ is needed to be applied in anticlockwise direction with time dependence.

29. An infinitesimally small bar magnet of dipole moment M is pointing and moving with the speed v is the X-direction. A small closed circular conducting loop of radius a and negligible self-inductance lies is the YZ-plane with its centre at $x=0$, and its axis coinciding with the X-axis. Find the force oppositing the motion of the magnet, if the resistance of the loop is R. Assume that the distance x of the magnet from the centre of the loop is much greater than a.

Soln. The varying magnetic field due to magnetic dipole will produce induced current in the coil. Finally, the circular conducting loop will also behave like a small bar magnet. This bar magnet will produce a force oppositing the motion of the magnet.



The magnetic field at the centre O of the coil due to bar magnet, $B = \frac{\mu_0}{4\pi} \times \frac{2M}{x^3}$

The magnetic flux through the loop due to bar magnet.

$$\phi = \frac{\mu_0}{4\pi} \times \frac{2M}{x^3} \times (\pi a^2) \qquad (\because\ \phi = BA)$$

Induced e.m.f. on circular loop

$$e = -\frac{d\phi}{dt} = -\frac{d}{dt}\left[\frac{\mu_0}{4\pi} \times \frac{2M}{x^3} \times (\pi a^2)\right] = -\frac{\mu_0 M a^2}{2}\left(\frac{-3}{x^4} \times \frac{dx}{dt}\right)$$

$$= \frac{3\mu_0 M a^2 v}{2x^4}, \text{ where } v = \frac{dx}{dt}$$

Therefore, induced current $i = \frac{e}{R} = \frac{3\mu_0 M a^2 v}{2x^4 R}$

Let M_{coil} be the magnetic moment of the coil. Then,

$$M_{\text{coil}} = i \times A = \frac{3\mu_0 M a^2 v}{2x^4 R} \times \pi a^2$$

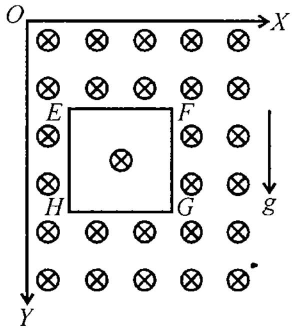
$$= \frac{3\pi \mu_0 M a^4 v}{2x^4 R}$$

Now, oppositing force is given by

$$F = \frac{\mu_0}{4\pi} \times \frac{6MM_{\text{coil}}}{x^4}$$

$$= \frac{\mu_0}{4\pi}\frac{6M}{x^4}\left[\frac{3\pi\mu_0 M a^4 v}{2x^4 R}\right] = \frac{9\mu_0^2 M^2 a^4 v}{4Rx^8}.$$

30. A magnetic field $B = (B_0 y/a)k$ is into the paper in the +Z-direction, B_0 and a are positive constants. A square loop $EFGH$ of side a and mass m and resistance R, in XY-plane, start falling under the influence of gravity. Note the directions of X and Y axes in figure.



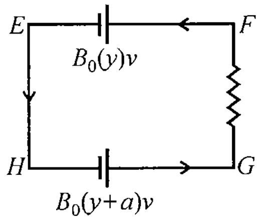
Find,

(a) the induced current in the loop and indicate its direction.

(b) the total lorentz force acting on the loop and indicate its direction, and

(c) an expression for the speed of the loop, $v(t)$ and its terminal velocity.

Soln. (a) Let v be the velocity of the loop at any time t. Induced e.m.f. in HG is given by



$$e_{HG} = \frac{B_0\,(y+a)}{a} va = B_0\,(y+a)\,v \qquad \text{... (1)}$$

The potential of G will be higher than the potential of H

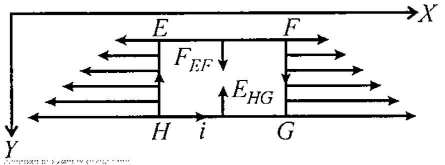
Similarly, $e_{EF} = \frac{B_0\,y}{a} v\,a = B_0\,y\,v \qquad \text{... (2)}$

Here F is the higher potential, e_{EH} and $e_{FG} = 0$.

So, the current in the circuit $i = \frac{B_0\,(y+a)\,v - B_0\,y\,v}{R} = \frac{B_0\,a\,v}{R}$

The current will be in anticlockwise direction as shown in figure.

(b) The lorentz forces on EF and HF are given as



$$F_{EF} = \frac{B_0\,y}{a} \times i \times a \;\; \text{(along } +Y\text{-axis)}$$

$$F_{HG} = \frac{B_0\,(y+a)}{a} \times i \times a \;\; \text{(along } -Y\text{-axis)}$$

Different elements will experience different forces as shown in figure.

Net Lorentz force $F = B_0\,(y+a)\,i - B_0\,a\,i$ (along $-Y$-axis)

Therefore, $F = \frac{B_0^2\,a^2 v}{R}\,(-j)$

(c) Net force acting on frame $F = mg - \frac{B_0^2\,a^2\,v}{R}$

or $\quad m\frac{dv}{dt} = mg - \frac{B_0^2\,a^2 v}{R}$

or $\quad \int \frac{dv}{\left[mg - \frac{B_0^2\,a^2\,v}{R}\right]} = \int \frac{dt}{m}$

$$\frac{\log\left(mg - \frac{B_0^2\,a^2\,v}{R}\right)}{-B_0^2\,a^2/R} = \frac{t}{m} + C$$

Where C is constant of integration. To obtain the value of C, we apply the following condition at $t = 0$, $v = 0$.

Hence, $\frac{C = \log\,(mg)}{\left[(-B_0^2\,a^2/R)\right]}$

$$\therefore \quad \log\left(mg - \frac{B_0^2\,a^2\,v}{R}\right) = -\frac{B_0^2\,a^2}{R} \times \frac{t}{m} + \frac{B_0^2\,a^2}{R}\,\frac{\log\,(mg)}{(B_0^2\,a^2/R)}$$

or $\log\left[mg - \frac{B_0^2 a^2 v}{R}\right] = -\frac{B_0^2 a^2}{R} \times \frac{t}{m} + \log mg$

$\therefore \quad v = \frac{mg\,R}{B_0^2 a^2}\left[1 - e^{-B_0^2 a^2 t/mR}\right]$

and terminal velocity $v_{\text{terminal}} = \frac{mg\,R}{B_0^2 a^2}$ (at $t = \infty$).

31. A rectangle conducting loop in the vertical XZ-plane has length L, width W, mass M and resistance R. It is dropped length wise from rest. At $t = 0$ the bottom of the loop is at a height h above the horizontal X-axis. There is a uniform magnetic field B perpendicular to XZ-plane, below the X-axis. The bottom and top of the loop cross this axis at $t = t_1$ and $t = t_2$ respectively. Obtain the expression for the velocity of the loop for the time $t_1 \le t \le t_2$.

Soln. For $0 \le t \le t_1$, there is free fall. Hence, $v = gt$. Therefore, $v_1 = g t_1 = \sqrt{2gh}$.

For $t_1 \le t \le t_2$, the induced e.m.f. in the loop is given by $e = -\frac{d\phi}{dt} = -BvW$ (where W = width).

Therefore, induced current $= -\frac{BvW}{R}$ in clockwise direction.

So, magnetic force $F = W/B$. Therefore, $F = -\frac{B^2 v W^2}{R}$ upwards.

Net force $= mg - \frac{B^2 v W^2}{R}$

Hence, $m\frac{dv}{dt} = mg - \frac{B^2 v W^2}{R}$

or $dv = \left(g - \frac{B^2 v W^2}{mR}\right) dt$

or $\int_{v_1}^{v} \frac{dv}{v_1\left[\left(g - \frac{B^2 W^2 v}{mR}\right)\right]} = \int_{t_1}^{t} dt$

Integrating $\frac{mR}{B^2 W^2}\left[\log\left(g - \frac{B^2 W^2 v}{mR}\right)\right]_{v_1}^{v} = -(t - t_1)$

or $\left[\frac{g - (B^2 W^2 v/mR}{g - (B^2 W^2 v_1/mR}\right] = e^{-(t-t_1)B^2 W^2/mR}$

or $v = \frac{mRg}{B^2 W^2} + \left(v_1 - \frac{mgR}{B^2 W^2}\right) \times e^{-(t-t_1)B^2 W^2/mR}$.

2. A thermocole vessel contains 0.5 kg of distilled water at 30°C. A metal coil of area 5×10^{-3} m^2, number of turns 100, mass 0.06 kg and resistance 1.6 Ω is lying horizontally at the bottom of the vessel. A uniform, time varying magnetic field is set up to pass vertically through the coil at time $t = 0$. The field is first increased from zero to 0.8 T at a constant rate between 0 and 0.2 s and then decreased to zero at the same rate between 0.2 and 0.4 s. This cycle is repeated 12000 times. Make sketch of the current through the coil and the power dissipated in the coil as functions of time for the first two cycles. Clearly indicate the magnitudes of the quantities on the axes. Assume that no heat is lost to the vessel or the surroundings. Determine the final temperature of the water under thermal equilibrium. Specific heat of the metal = 500 J kg^{-1} K^{-1} and the specific heat of water = 4200 J kg^{-1} K^{-1}. Neglect the inductance of the coil.

oln. Induced e.m.f. $e = -N\dfrac{d\phi}{dt} = -NA\dfrac{dB}{dt}$

Induced current $i = \dfrac{e}{R} = -\dfrac{NA}{R}\dfrac{dB}{dt}$

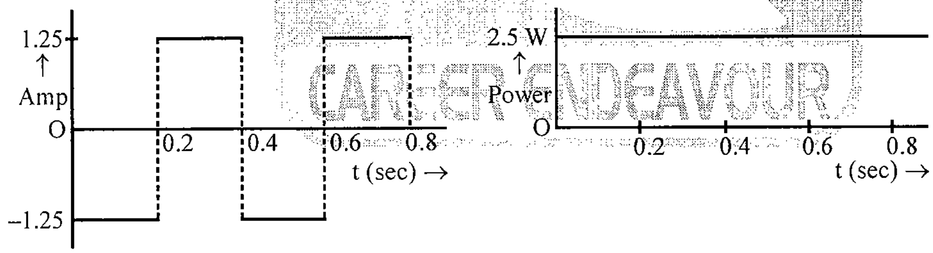
Given that $\left(\dfrac{dB}{dt}\right) = 0.4$ Tesla/sec. for time interval 0 to 0.2 sec. Hence, current for this interval

$$(i)_{\min} = -\frac{100 \times (5 \times 10^{-3})}{1.6} \times 0.4 = -1.25 \text{ amp.}$$

Further, $\left(\dfrac{dB}{dt}\right) = -0.4$ Tesla/sec. for time interval 0.2 to 0.4 sec. Hence current in this interval

$$(i)_{\max} = \frac{100 \times (5 \times 10^{-3})}{1.6} \times 0.4 = +1.25 \text{ amp.}$$

Power $P = \dfrac{dW}{dt} = I^2 R = (1.25)^2 \times (1.6) = 2.5W$ (for all time)



(a) (b)

The sketches of the current through the coil and power dissipated in the coil as functions of time for the first two cycles are shown in above figure respectively.

Time for which the magnetic field varying = 0.4 × 1200 = 4800 seconds.

Therefore, amount of heat generated in the coil $= I^2 Rt = (1.25)^2 \times 1.6 \times 4800 = 12000$ joule.

Let T be the final temperature, then

$m_w \cdot s_w \cdot \Delta T + m_m \cdot s_m \cdot \Delta T = 12000$ or $(0.5)(4200)(T-30) + (0.06)(500)(T-30) = 12000$

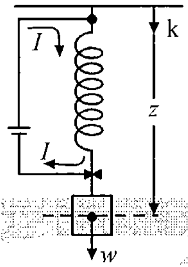
Solving we get, $T = 35.6\,°C$

33. A long, flexible, cylindrical coil of negligible weight, radius a, and length l has n turns of wire. The coil is suspended from its upper end, and to its lower end a weight w is attached. Neglecting end effects of the coil, find the current I that should be sent through the coil in order that the coil will support the weight w without stretching or contracting.

Soln. Let z be the distance from the point from which the upper end of the coil is suspended to the center of gravity of w. Since the end effects of the coil may be neglected, we may assume that the field of the coil is just

$$H = \frac{nI}{l}$$

If a current of certain magnitude is sent through the coil, the coil lifts the weight. A coil of this type is known as "Roget's spiral."



and is confined to the interior of the coil. The energy of this field is, by

$$U = \frac{\mu_0}{2}\int H^2 dv = \frac{\mu_0}{2} H^2 \pi a^2 l = \frac{\mu_0}{2}\frac{n^2 I^2}{l}\pi a^2$$

The force exerted by the coil on the weight w is, by

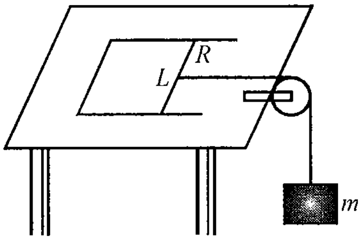
$$F = \frac{\partial U}{\partial z} k \bigg|_{I=\text{constant}},$$

and since $\Delta z = \Delta l$,

$$F = \frac{\partial U}{\partial l} k \bigg|_{I=\text{constant}} = -\frac{\mu_0}{2}\frac{n^2 I^2}{l^2}\pi a^2 k$$

for equilibrium we must have $F = -wk$, so that $I = \frac{l}{na}\sqrt{\frac{2w}{\mu_0 \pi}}$

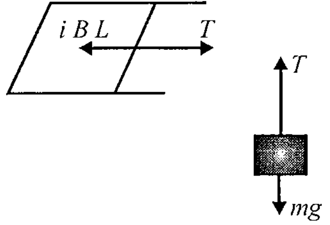
34. A pair of parallel horizontal conducting rails of negligible resistance shorted at one end is fixed on a table. The distance between the rails is L. A conducting massless rod of resistance R can slide on the rails frictionlessly. The rod is tied to a massless string which passes over a pulley fixed to the edge of the table. A mass m, tied to the other end of the string, hangs vertically. A constant magnetic field B exists perpendicular to the table. If the system is released from rest, calculate



(i) the terminal velocity achieved by the rod, and

(ii) the acceleration of the mass at the instant when the velocity of rod is half the terminal velocity.

Soln. (i) See figure :



When terminal velocity is achieved

$$T = i\,B\,L = mg \text{ or } \frac{B^2 L^2 v_T}{R} = mg$$

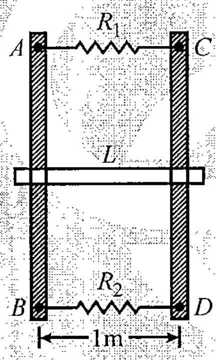
Therefore, $v_T = \dfrac{mgR}{B^2 L^2}$ ... (1)

(ii) For the block, $a = \dfrac{mg - T}{m} = \dfrac{mg - i\,B\,L}{m} = \dfrac{mg - (B^2 L^2 v/R)}{m}$

When $v = \dfrac{v_T}{2}$, then $a = \dfrac{mg - B^2 L^2 \left(\frac{v_T}{2} R\right)}{m} = g - \dfrac{B^2 L^2 (mg\,R/B^2\,L^2\,R)}{2m} = g - \dfrac{g}{2} = \dfrac{g}{2}.$

5. Two parallel vertical metallic rails AB and CD are separated by 1 m. They are connected at the two ends by resistances R_1 and R_2 as shown in figure. A horizontal metallic bar L of mass 0.2 kg slides without friction, vertically down the rails under the action of gravity. There is a uniform horizontal magnetic field of 0.6 T perpendicular to the plane of the rails. It is observed that when the terminal velocity is attained, the powers dissipated in R_1 and R_2 are 0.76 W respectively. Find the terminal velocity of the bar L and the values of R_1 and R_2.

Soln. The situation is shown in figure :



Due to the movement of metallic bar, there will be a change in magnetic flux linked with the close circuits and an induced e.m.f. is produced.

The forces acting on the bar are

(i) weight mg acting vertically downward,

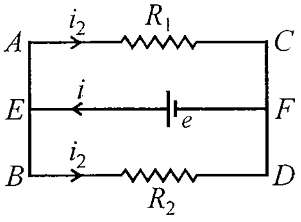
(ii) magnetic force $B\,i\,L$ acting vertically upward.

When the above two forces become numerically equal, the bar will be in dynamical equilibrium and attains terminal velocity (say) v.

Thus, $mg = B\,i\,L$ or $i = \frac{mg}{B\,L}$.

Therefore, $i = \dfrac{0.2 \times 9.8}{0.6 \times 1} = 3.267$ amp. ... (1)

Let i_1 and i_2 be the currents in R_1 and R_2 respectively as shown in figure :



Now, $i = i_1 + i_2 = 3.267$ ... (2)

As R_1 and R_2 are in parallel with the bar, then

$e = R_1\,i_1 = R_2\,i_2$ ... (3)

where e is the induced e.m.f. in the bar.

The power in R_1,

$P_1 = e i_1 = 0.76$ watt ... (4)

The power in R_2,

$P_2 = e i_2 = 1.2$ watt ... (5)

Dividing equation (4) by equation (5), we get

$$\frac{i_1}{i_2} = \frac{0.76}{1.2} = \frac{19}{30} \text{ or } i_1 = \frac{19}{30} i_2 \quad \text{... (6)}$$

From equation (2), $\frac{19}{30} i_2 + i_2 = 3.267$

Solving it, we get, $i_2 = 2$ amp.

From equation (6), $i_1 = \frac{19}{30} \times 2 = \frac{19}{15} = 1.267$ amp.

From equation (4), $e \times 1.267 = 0.76$

Therefore, $e = 0.6$ V

Further, $e = BvL$ or $v = \frac{e}{BL}$.

Therefore, $v = \frac{0.6}{0.6 \times 1} = 1\ m/s$

From equation (3), $R_1 = \frac{e}{i_1} = \frac{0.6}{1.267} = 0.474\ \Omega$ and $R_2 = \frac{e}{i_2} = \frac{0.6}{2} = 0.3\ \Omega$

PRACTICE SET

The electric field produced by a uniformly moving charge is

(a) conservative (b) irrotational (c) non-coservative (d) none of these

A conducting rod of length l rotates with a constant angular velocity ω about its perpendicular bisector. An external uniform magnetic field $\vec{B}$ is applied parallel to the axis of rotation. The pd between the two ends of the rod is

(a) $\frac{1}{2}B\omega\ell^2$ (b) $\frac{1}{4}B\omega\ell^2$ (c) $B\omega\ell^2$ (d) zero

A small bar magnet is allowed to fall freely from rest along the axis of a very long vertical conducting tube. After sometime the magnet will be found to

(a) come to rest (b) oscillate

(c) move with an acceleation g, (d) move with a constant speed

The differential form of Faraday's law of electromagnetic induction is

(a) $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ (b) $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$ (c) $\vec{\nabla}\times\vec{B} = -\frac{\partial\vec{E}}{\partial t}$ (d) $\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}$

Lenz's law is a consequence of the law of conservartion of

(a) charge (b) energy (c) momentum (d) mass

What are the dimensions of $\nabla\Phi / R$, where $\nabla\Phi$ is the change in magnetic flux and R is the resistance?

(a) $[IT]$ (b) $[MLT^{-3}I^{-1}]$ (c) $[IT^{-1}]$ (d) $[ML^2T^{-2}I^{-1}]$

A circular loop of radius r is lying in the xy-plane with its centre at the origin. If there exists a time varying magnetic field $\vec{B}(t) = B_0 e^{-\alpha t}\hat{z}$ $(B_0 > 0, \alpha > 0)$ then the induced emf in the loop is

(a) $-\pi r^2 \alpha B_0 e^{-\alpha t}$ (b) $-\pi r^2 B_0 e^{-\alpha t}$ (c) $\pi r^2 B_0 e^{-\alpha t}$ (d) $\pi r^2 \alpha B_0 e^{-\alpha t}$

The current passing through a choke coil of 1 H is decreasing at the rate of 2A/s. Then, the emf developed in the coil is

(a) 2V (b) 0.5V (c) –2V (d) –0.5V

A copper ring is held horizontally and a bar magnet is dropped through the ring with its length along the axis of the ring. Then, the acceleration of the falling magnetic is

(a) equal to g, (b) greater than g (c) less than g (d) zero

0. The equation $\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}$ is the differential form of

(a) faraday's law (b) Gauss's law (c) Ampere's law (d) Coulomb's law

ANSWER KEY					
Questions	1	2	3	4	5
Option	(c)	(d)	(d)	(a)	(b)
Questions	6	7	8	9	10
Option	(a)	(d)	(a)	(c)	(c)

Chapter 10

Maxwell's Equations, Poynting vector, EM wave

Displacement current:

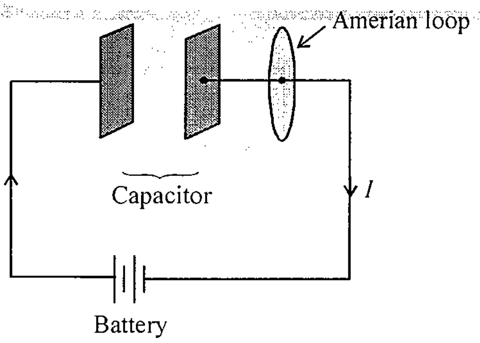
Till now we have come across four basic laws about electromagnetism. These are essentially:

$$\left.\begin{array}{ll} \text{(i) } \nabla.E = \frac{1}{\epsilon_0}\rho & (\text{Gauss's law}), \\ \text{(ii) } \nabla.B = 0 & (\text{no name}), \\ \text{(iii) } \nabla \times E = -\frac{\partial B}{\partial t} & (\text{Faraday's law}) \\ \text{(iv) } \nabla \times B = \mu_0 J & (\text{Ampere's law}) \end{array}\right\} \quad \text{... (1)}$$

Faraday's law (iii) says that changing magnetic field creates a electric field. Does changing electric field produces magnetic field too? Answer is YES. To understand this, consider the following case.
Consider the fig 8.1. There is a capacitor connected with a battery and it is charging. The Ampere's law is:

$$\oint B.d\ell = \mu_0 I_{enc}$$

If we apply ampere's law in the deep shadowed circular loop right side of right plate of capacitor. The wire, carrying current **I,** cutting through the loop. So enclosed current by the loop $\mathbf{I}_{\mathbf{enc}}$ is **I.** But if we apply the Ampere's law in the balloon shaped loop (lightly shadowed) then no current passing through the loop. So $\mathbf{I}_{\mathbf{enc}}$ = Zero! So there must be going something wrong there.



Maxwell fixed the problem by adding a displacement current in the ampere's law as

$$J_d \equiv \epsilon_0 \frac{\partial E}{\partial t} \qquad \text{... (2)}$$

Or,
$$I_d \equiv \epsilon_0 \frac{\partial \Phi_E}{dt} \qquad \text{... (3)}$$

Here Φ_E is Electric flux (flux created by electric field).

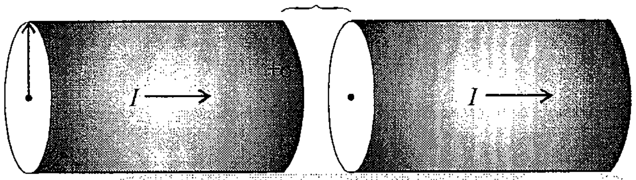
So the integral form of Ampere's law becomes

$$\boxed{\int B.d\ell = \mu_0 I_{enclosed} + \mu_0 \in_0 \int\left(\frac{\partial E}{\partial t}\right)\cdot da} \qquad \ldots (4)$$

And in the curl form

$$\boxed{\nabla \times B = \mu_0 J + \mu_0 \in_0 \frac{\partial E}{\partial t}} \qquad \ldots (5)$$

Example: A fat wire, radius a, carries a constant current I, uniformly distributed over its cross-section. A narrow gap in the wire, of width w << a, forms a parallel-plate capacitor, as shown in Figure. Find the magnetic field in the gap, at a distance s < a from the axis.



oln. The displacement current density is

$$J_d = \in_0 \frac{\partial E}{\partial t} = \frac{I}{A} = \frac{I}{\pi a^2}\hat{z}$$

Drawing an "amperian loop" at radius s

$$\oint B.d\ell = B.2\pi s = \mu_0 I_{d_{enc}} = \mu_0 \frac{I}{\pi a^2}.\pi s^2 = \mu_0 I \frac{s^2}{a^2} \quad \Rightarrow \quad B = \frac{\mu_0 I s^2}{2\pi s a^2}$$

$$B = \frac{\mu_0 I s}{2\pi a^2}\hat{\phi}$$

Example: An alternating current $I = I_0 \cos(\omega t)$ flows down a long straight wire, and returns along a co-axial conducting tube of radius a.

(a) What is the Electric field induced? In what direction does the induced electric field point (radial, circumferential, or longitudinal)?

oln. $E(s,t) = \frac{\mu_0 I_0 \omega}{2\pi}\sin(\omega t)\,\ell n\left(\frac{a}{s}\right)\hat{z}$

Maxwell's equations:

So, the Maxwell's equations now after addition of displacement current are

(i) $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\in_0}$ (Gauss's law in electrostatic) ... (6)

(ii) $\vec{\nabla}\cdot\vec{B} = 0$ (Gauss's law in Magnetostatic) ... (7)

(iii) $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (Faraday's law) ... (8)

(iv) $\vec{\nabla}\times\vec{B} = \left.\begin{matrix} \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial \vec{E}}{\partial t} \\ \mu_0 J + \mu_0\frac{\partial \vec{D}}{\partial t}\end{matrix}\right\}$ (Ampere's law with Maxwell's correction) ... (9)

Properties of Maxwell's equation:

(i) Maxwell's equations are linear

(ii) Maxwell's equations obey the law of superposition

(iii) Maxwell's equations are relativistic invariants

(iv) Maxwell's equations are not symmetry with respect to electric and magnetic fields. Because electric charge exist in nature but magnetic charge do not exist. Maxwell's equations become symmetry when $\rho = 0$ and $J = 0$

(v) From Maxwell's equations we can derive the equations of continuity.

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\Rightarrow \quad \vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{H}\right) = 0 \text{ (divergence of curl of any vector is zero)}$$

$$\Rightarrow \quad \vec{\nabla} \cdot \vec{J} + \frac{\partial}{\partial t}\left(\vec{\nabla} \cdot \vec{D}\right) = 0$$

$$\Rightarrow \quad \boxed{\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0} \qquad \ldots (10)$$

Poynting Theorem

We know that the electric energy stored in electromagnetic field is

$$W_e = \frac{\epsilon_0}{2} \int E^2 d\tau$$

And the magnetic energy is

$$W_m = \frac{1}{2\mu_0} \int B^2 d\tau$$

So the total energy stored in the electromagnetic field is

$$U_{em} = \frac{1}{2} \int \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) d\tau \qquad \ldots (11)$$

Suppose we have some charge and current configuration which, at time 't' produces fields E and B. In the next instant, dt, the charges move around a bit. Question: How much work dW, is done by the electromagnetic forces acting on these charges in the interval dt? According to the Lorentz law, the work done on a charge q is

$$F.d\ell = q\left(E + v \times B\right).vdt = qE.vdt$$

Now, $q = \rho d\tau$ and $\rho v = J$, so the rate at which work is done on all the charges in a volume v is:

$$\frac{dW}{dt} = \int_v \left(E.J\right) d\tau$$

This quantity (dw/dt) can be proved to be equals to

$$\frac{dW}{dt} = -\frac{d}{dt} \int_v \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) d\tau - \frac{1}{\mu_0} \oint_s \left(E \times B\right).da \qquad \ldots (12)$$

where 'S' is the surface bounding v. This is Poynting's theorem; it is the "work-energy theorem" of electrodynamics. The first integral on the right is the total energy stored in the fields, U_{em}. The second term evidently represents the rate at which energy is carried out of v, across its boundary surface, by the electromagnetic fields. Poyntings theorem says, then, that the work done on the charges by the electromagnetic force is equal to the decrease in energy stored in the field, less the energy that flowed out through the surface.

The energy per unit time, per unit area, transported by the fields is called the Poynting vector:

$$\bar{S} \equiv \frac{1}{\mu_0}\left(\bar{E} \times \bar{B}\right) \qquad \ldots (13)$$

By the definition of S, dw/dt can be written as

$$\frac{dW}{dt} = -\frac{dU_{em}}{dt} - \oint_S \bar{S}.d\bar{a} \qquad \ldots (14)$$

Of course, the work W done on the charges will increase their mechanical energy (kinetic, potential, or whatever). If we let $\mathbf{U}_{\mathbf{mech}}$ denote the mechanical energy density, so that

$$\frac{dW}{dt} = \frac{d}{dt}\int_V u_{mech} d\tau$$

And use U_{em} for the energy density of the fields,

$$u_{em} = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)$$

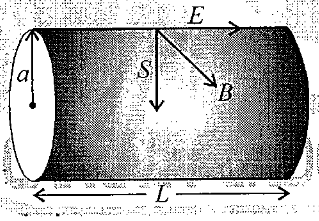
Then the equation becomes

$$\frac{d}{dt}\int_v \left(u_{mech} + u_{em}\right) d\tau = -\oint_S S.da = -\int_v \left(\bar{\nabla}.\bar{S}\right) d\tau$$

So, in more compact form

$$\frac{\partial}{\partial t}\left(u_{mech} + u_{em}\right) = -\bar{\nabla}.\bar{S} \qquad \ldots (15)$$

When current flows down a wire, work is done, which shows up as Joule heating of the wire. Though there are certainly easier ways to do it, the energy per unit time delivered to the wire can be calculated using the Poynting vector. Assuming it's uniform, the electric field parallel to the wire is



$$E = \frac{V}{L}$$

where V is the potential difference between the ends and L is the length of the wire (Fig. 8. 3). The magnetic field is "circumferential"; at the surface (radius a) it has the value

$$B = \frac{\mu_0 I}{2\pi a}$$

The magnitude of the Poynting vector is

$$S = -\frac{1}{\mu_0}\frac{V}{L}\frac{\mu_0 I}{2\pi a} = \frac{V I}{2\pi a L}$$

And it points radially inward. The energy per unit time passing in through the surface of the wire is therefore.

$$\int S.da = S\left(2\pi a L\right) = V\ I$$

Electromagnetic wave in free space:

In free space, $\rho = 0$ and $J = 0$

The Maxwell's equations take the following form.

$$\vec{\nabla}\cdot\vec{E} = 0 \quad \text{... (16)}$$

$$\vec{\nabla}\cdot\vec{H} = 0 \quad \text{... (17)}$$

$$\vec{\nabla}\times\vec{E} = -\mu_0\frac{\partial\vec{H}}{\partial t} \quad \text{... (18)}$$

$$\vec{\nabla}\times\vec{H} = \varepsilon_0\frac{\partial\vec{E}}{\partial t} \quad \text{... (819)}$$

Taking curl of the equation (8.4-4), we get,

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{H}\right) = \varepsilon_0\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{E}\right)$$

$$\Rightarrow \quad \vec{\nabla}\left(\vec{\nabla}\cdot\vec{H}\right) - \vec{\nabla}^2 H = -\mu_0\varepsilon_0\frac{\partial^2 H}{\partial t^2} \qquad \text{[Using equation (18 and 17)}$$

$$\Rightarrow \quad \boxed{\vec{\nabla}^2\vec{H} - \mu_0\varepsilon_0\frac{\partial^2\vec{H}}{\partial t^2} = 0} \quad \text{... (20)}$$

Similarly taking used of equation (18) and using equation (16) and (19), we get

$$\boxed{\vec{\nabla}^2\vec{E} - \mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2} = 0} \quad \text{... (21)}$$

Now both $\vec{E}$ and $\vec{H}$ satisfy the wave equation,

$$\boxed{\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0} \quad \text{... (22)}$$

So, from equations (17), (21) and (20) we can write the speed of electromagnetic wave propagation in free space is

$$c = \frac{1}{\sqrt{\varepsilon_0\mu_0}} \simeq 3\times10^8\ m/\sec$$

which is speed of light in free space. This says that the light is an electromagnetic wave.

Plane Wave solutions:

The solution of the equations (5) and (6) are plane wave.

$$\left.\begin{aligned}\vec{E}(r,t) &= \vec{E}_0\, e^{i\left(\vec{k}\cdot\vec{r}-\omega t\right)}\\ \vec{H}(r,t) &= \vec{H}_0\, e^{i\left(\vec{k}\cdot\vec{r}-\omega t\right)}\end{aligned}\right\} \quad \text{... (23)}$$

$\vec{E}_0$, $\vec{H}_0$ are the complex amplitude, and $\vec{k}$ is the propagating constant. $\vec{k}$ is defined as

$$k = \frac{2\pi}{\lambda}\hat{k} = \frac{\omega}{c}\hat{k} \qquad (\hat{k} \text{ is unit vector along propagation direction})$$

Relative direction of $\vec{E},\vec{H},\vec{k}$

We have solution of wave equation,

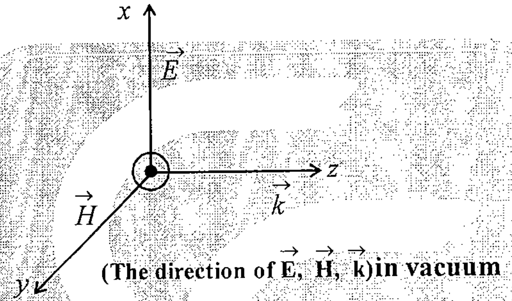
$$\vec{E}(r,t)=\vec{E}_0\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$\vec{H}(r,t)=\vec{H}_0\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

Now,

$$\left.\begin{array}{lll}\vec{\nabla}\cdot\vec{E}=0 & \Rightarrow i\vec{k}\cdot\vec{E}=0 & \therefore \vec{k}\perp\vec{E} \\ \vec{\nabla}\cdot\vec{H}=0 & \Rightarrow i\vec{k}\cdot\vec{H}=0 & \therefore \vec{k}\perp\vec{H} \\ \vec{\nabla}\times\vec{E}=-\dfrac{\partial\vec{B}}{\partial t} & \Rightarrow i\vec{k}\times\vec{E}=i\omega\mu_0\vec{H} & \therefore \vec{k}\times\vec{E}=\omega\mu_0\vec{H} \\ \vec{\nabla}\times\vec{H}=\varepsilon_0\dfrac{\partial\vec{E}}{\partial t} & \Rightarrow i\vec{k}\times\vec{H}=-i\omega\varepsilon_0\vec{E} & \therefore \vec{k}\times\vec{H}=\omega\varepsilon_0\vec{E}\end{array}\right\}\quad \ldots.(24)$$

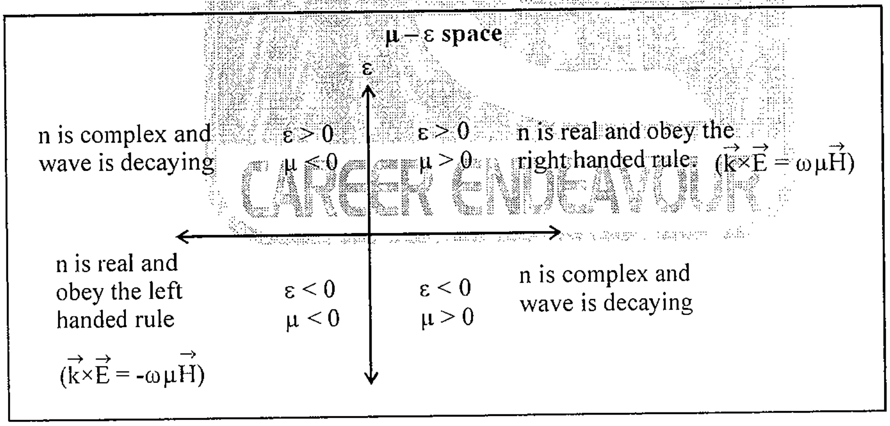
From equation we can say $\vec{k},\vec{E},\vec{H}$ are mutually perpendicular to each other.



(The direction of $\vec{E}$, $\vec{H}$, $\vec{k}$) in vacuum

Special Note :

Right handed and Left handed systems :



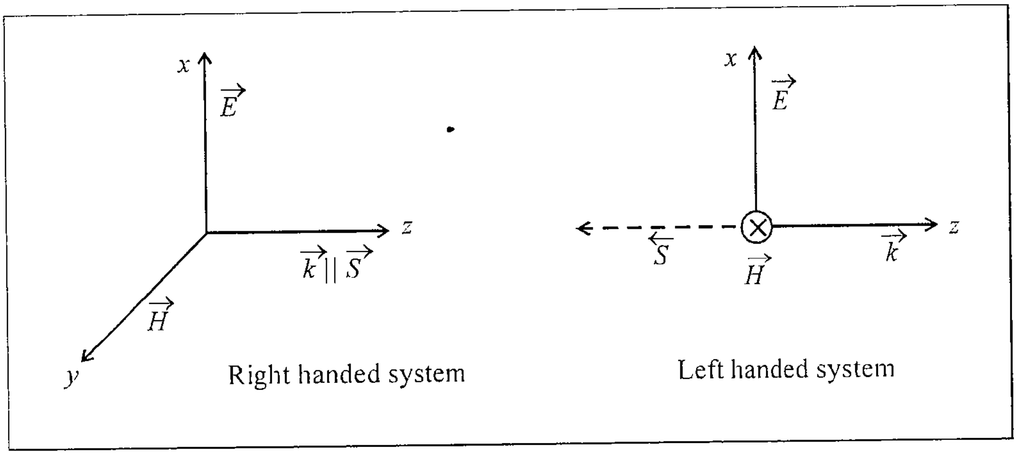
μ – ε space

n is complex and wave is decaying — $\varepsilon > 0$, $\mu < 0$

$\varepsilon > 0$, $\mu > 0$ — n is real and obey the right handed rule. $(\vec{k}\times\vec{E}=\omega\mu\vec{H})$

n is real and obey the left handed rule $(\vec{k}\times\vec{E}=-\omega\mu\vec{H})$ — $\varepsilon < 0$, $\mu < 0$

$\varepsilon < 0$, $\mu > 0$ — n is complex and wave is decaying



Right handed system

Left handed system

Relative phase of $\vec{E}$ and $\vec{H}$:

We have the wave equation $\nabla^2 E - \varepsilon_0\mu_0 \frac{\partial^2 E}{\partial t^2} = 0$ and the solution, $\vec{E} = \vec{E}_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

So, we can write,

$$k^2 - \varepsilon_0\mu_0\omega^2 = 0$$

$$k^2 = \varepsilon_0\mu_0\omega^2 \qquad \text{... (25)}$$

Therefore, k is real quantity in free space and $\vec{E}$ and $\vec{H}$ are in phase.

Wave Impedance :

Let z_0 is a quantity of the ratio of the magnitude of $\vec{E}$ and $\vec{H}$ is

$$[z_0] = \frac{[VL^{-1}]}{[IL^{-1}]} = \frac{[V]}{[I]} = [R]$$

$$z_0 = \frac{|\vec{E}|}{\vec{H}} = \frac{\mu_0\omega}{k} = \sqrt{\frac{\mu_0}{\epsilon_0}} \qquad \text{[Using equation (24)]}$$

$\mu_0 = 4\pi\times10^{-7}$ *S.I.* unit

$\varepsilon_0 = 8.85\times10^{-12}$ *S.I.* unit

$z_0 = 376.6\,\Omega$

z_0 is called the empidence of wave in free space.

Poynting vector :

The poynting's vector for the plane electromagnetic wave in free space is

$$\vec{s} = \vec{E}\times\vec{H} = \frac{1}{\mu_0\omega}\vec{E}\times(\vec{k}\times\vec{E}) \qquad \text{[Using equation (24)]}$$

$$= \frac{1}{\mu_0\omega}\left[\vec{k}(\vec{E}\cdot\vec{E}) - \vec{E}(\vec{E}\cdot\vec{k})\right] = \frac{E^2}{\mu_0\omega}\vec{k} \qquad \text{... (26)}$$

Thus, the energy flow is in the direction of wave propagation. Since $\vec{E}$ is normal to $\vec{k}$, from equation (9), we can write an terms of magnitude,

$$kE = \mu_0\omega H \quad or \quad \sqrt{\epsilon_0}E = \sqrt{\mu_0}H \quad or \quad \frac{1}{2}\epsilon_0 E^2 = \frac{1}{2}\mu_0 H^2 \qquad \text{... (28)}$$

This shows that in case of electromagnetic waves in free space electromagnetic energy is equally shared between electric and magnetic fields.

Total electromagnetic energy density is

$$u = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2}\mu_0 H^2 = \epsilon_0 E^2 \qquad \text{... (29.a)}$$

So, equation (11) can also be written as

$$\boxed{\vec{s} = uc\hat{n}} \qquad \text{... (29.b)}$$

where, $c = \frac{1}{\sqrt{\mu_0\epsilon_0}} = \frac{\omega}{k}$ is the speed of the wave.

Time average Poynting's vector:

$$\langle \vec{s} \rangle = \frac{1}{2}\text{Re}\left[\vec{E}^{*} \times \vec{H}\right]$$

The magnitude of the time average of the Poynting's vector is called the intensity of radiation (I). Thus, the intensity.

$$I = \left|\langle s \rangle\right| = \frac{1}{2}E_0 H_0 = \frac{E_0}{\sqrt{2}}\frac{H_0}{\sqrt{2}} = E_{rms} \times H_{rms}$$

We also know, $\langle \mu \rangle = \left\langle \varepsilon_0 E_0^2 \right\rangle = \frac{1}{2}\varepsilon_0 E_0^2 = \varepsilon_0 E_{rms}^2$

Since, Maxwell's equation is symmetric of electric field and magnetic field.
So, we can write,

$$\sqrt{\varepsilon_0}\, E_{rms} = \sqrt{\mu_0}\, H_{rms}$$

$$\therefore \quad \frac{I}{\langle u \rangle} = \frac{1}{\varepsilon_0} \cdot \frac{H_{rms}}{E_{rms}} = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = c$$

$$\text{or} \quad \boxed{I = \langle u \rangle c} \qquad \ldots(30)$$

Boundary Conditions :

When electromagnetic wave exist in more than one medium the field vectors must satisfy certain conditions at the interface of two media. These conditions are known as boundary conditions.
These conditions are

$$\left.\begin{aligned}
&(1)\, D_2 \cdot \hat{n} - D_1 \cdot \hat{n} = \sigma \\
&(2)\, B_2 \cdot \hat{n} = B_1 \cdot \hat{n} \\
&(3)\, \hat{n} \times \vec{H}_2 - \hat{n} \times \vec{H}_1 = \vec{k} \\
&(4)\, \vec{E}_2 \cdot \hat{\tau}_2 = \vec{E}_1 \cdot \hat{\tau}_1
\end{aligned}\right\} \qquad \ldots(31)$$

Special Cases:

(1) Dielectric-dielectric interface : In this case, there is no free charge or free current at the interface and we have

$$\left.\begin{aligned}
&\varepsilon_2 E_{2n} = \omega_1 E_{1n},\ B_{2n} = B_{1n} \\
&\frac{1}{\mu_2} B_{2t} = \frac{1}{\mu_1} B_{1t},\ E_{2t} = E_{1t}
\end{aligned}\right\} \quad \ldots(32) \quad \begin{aligned}&(n \text{ stands for normal component}) \\ &(t \text{ stands for tangential component})\end{aligned}$$

(2) Conductor-dielectric interface : For conducting medium inside field

$$\vec{E}_1 = 0$$

Therefore, $\vec{H}_1$ is also zero. So, both the tangential and normal component of $\vec{H}_1$ and $\vec{E}_1$ are also zero.

$$E_{1n} = E_{1t} = 0 \text{ and } H_{1n} = H_{1t} = 0$$

So, in this case, the boundary conditions become

$$\left.\begin{array}{l}\varepsilon_2 E_{2n} = \sigma,\ B_{2n} = B_{1n} = 0 \\ \dfrac{1}{\mu_2}\hat{n}\times\vec{B}_2 = \vec{K},\ E_{2t} = E_{1t} = 0\end{array}\right\} \qquad \ldots (33)$$

Ordinary laws of reflection and refraction:

Suppose that the xy-plane forms the interface ($z = 0$) between two linear homogeneous dielectric (non-conducting) media characterized by permitivities ε_1 and ε_2, and permeabilities μ_1 and μ_2, respectively as shonw in figure. Let a plane electromagnetic wave be incident on the interface at the point O. The wave is partily reflected and partly refracted. Let us represent the electric fields associated with these waves by

$$\vec{E} = \vec{E}_0 e^{j(\vec{k}\cdot\vec{r}-\omega t)}$$

$$\vec{E}_1 = \vec{E}_{10} e^{j(\vec{k}_1\cdot\vec{r}-\omega_1 t)}$$

/And $$\vec{E}_2 = \vec{E}_{20} e^{j(\vec{k}_2\cdot\vec{r}-\omega_2 t)}$$

where, $\vec{E}_0$, $\vec{E}_{10}$ and $\vec{E}_{20}$ are the amplitudes, $\vec{k}$, $\vec{k}_1$ and $\vec{k}_2$ are the propagation vectors, ω, ω_1 and ω_2 are the frequencies of the incident, reflected and refracted waves respectively.

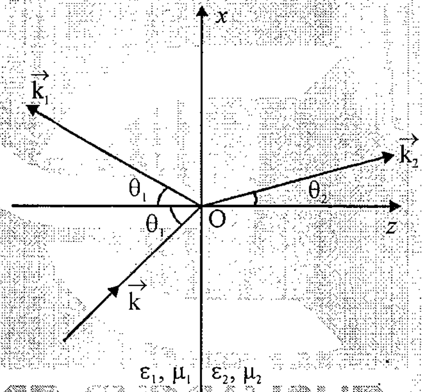


Figure : Reflection and refraction of EM waves

Let us now apply the boundary condition that the tangential components of electric field are continuous across the interface $z = 0$. Thus,

$$\left(\vec{E}_0\right)_t \cdot e^{j(\vec{k}\cdot\vec{r}-\omega t)} + \left(\vec{E}_{10}\right)_t \cdot e^{j(\vec{k}_1\cdot\vec{r}-\omega_1 t)} = \left(\vec{E}_{20}\right)_t \cdot e^{j(\vec{k}_2\cdot\vec{r}-\omega_2 t)}$$

where the subscript t is used to denote tangential components. The condition must be valid for all values of time and it immediately follows that

$$\omega = \omega_1 = \omega_2$$

Thus, we find that the frequency of electromagnetic wave does not change on reflection and refraction.

The condition must also hold at all points on the interface ($z = 0$). So, we must have

$$\vec{k}\cdot\vec{r} = \vec{k}_1\cdot\vec{r} = \vec{k}_2\cdot\vec{r} \text{ at } z = 0$$

Therefore, $$k_x = k_{1x} = k_{2x} \text{ and } k_y = k_{1y} = k_{2y}$$

If the incident beam is assumed to be in the xz-plane then $k_y = 0$. Consequently $k_{1y} = k_{2y} = 0$ and both $\vec{k}_1$ and $\vec{k}_2$ lie in the xz-plane.

Since z-axis is normal to the interface we may conclude that the incident, reflected, refracted waves and normal to the interface at the point of incidence all lie in the the same plane

From above figure,

$$k_x = k\sin\theta,\ k_{1x} = k_1\sin\theta_1 \text{ and } k_{2x} = k_2\sin\theta_2$$

Therefore, from the condition, $k_x = k_{1x}$ we get

$$k\sin\theta = k_1\sin\theta_1$$

Since the vectors $\vec{k}$ and $\vec{k}_1$ lie in the same medium we can write for the phase velocity in medium 1 as

$$\frac{\omega}{k} = \frac{\omega_1}{k_1} = \frac{1}{\sqrt{\mu_1 \in_1}}$$

As $\omega = \omega_1$, therefore, $k = k_1$ and from equation ( ), we get

$$\theta = \theta_1$$

That is, the angle of incidence is equal to the angle of reflection.

$$k_{1x} = k_{2x}$$

Or, $$k_1\sin\theta_1 = k_2\sin\theta_2$$

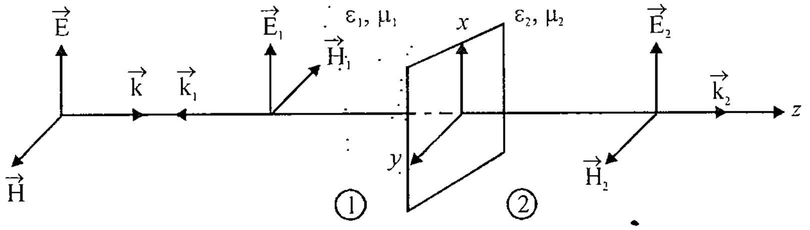
Or, $$\frac{\sin\theta_1}{\sin\theta_2} = \frac{k_2}{k_1} = \frac{\omega/k_1}{\omega/k_2} = \frac{v_1}{v_2} = \frac{c/v_2}{c/v_1} = \frac{n_2}{n_1}$$

where v_1 and v_2 are the phase velocity of the wave in the medium 1 and 2 respectively and n_1 and n_2 are the reflective index of two medium.

Therefore, $\boxed{n_1\sin\theta_1 = n_2\sin\theta_2}$ is called Snell's law of refraction.

Reflection and Transmission for Normal Incidence:

Suppose xy-plane forms an interface ($z = 0$) between two dielectric medium characterized by the permitivity ε_1 and ε_2 and permiability μ_1 and μ_2. Let a plane electromagnetic wave of frequency ω polarized along x-direction and travelling along z-direction be incident normally on the interface. $\vec{E}\ \vec{H}$ describe the incident wave $\vec{E}_1$ and $\vec{H}_1$ describe the reflected and $\vec{E}_2\ \vec{H}_2$ are transmitted wave.



Incident wave : $\vec{E} = \hat{i}E_0 e^{i(kz-\omega t)}$ $\vec{H} = \hat{J}\,H_0\,e^{i(kz-\omega t)}$

Reflected wave : $\vec{E}_1 = \hat{i}E_{10}e^{-i(k_1 z+\omega t)}$ $\vec{H}_1 = -\hat{J}\,H_{10}\,e^{-i(k_1 z+\omega t)}$

Transmitted wave : $\vec{E}_2 = \hat{i}E_{20}e^{i(k_2 z-\omega t)}$ $\vec{H}_2 = \hat{J}\,H_{20}\,e^{i(k_2 z-\omega t)}$

we know that,

$$\vec{k}\times\vec{E} = \mu\omega\vec{H}$$

$\Rightarrow \qquad \vec{H} = \frac{1}{\mu}\frac{1}{v}\hat{k} \times \vec{E}$

$\Rightarrow \qquad \left|\vec{H}\right| = \sqrt{\frac{\varepsilon}{\mu_0}}\left|\vec{E}\right| \qquad$ ($\mu = \mu_0$ for non-magnetic medium)

$\therefore \qquad H_0 = \sqrt{\frac{\varepsilon_1}{\mu_1}}E_0 \quad H_{10} = \sqrt{\frac{\varepsilon_1}{\mu_1}}E_{10}$ and $H_{20} = \sqrt{\frac{\varepsilon_2}{\mu_2}}E_{20}$

Therefore, according to Boundary condition the tangential component of E is continuous.

$\therefore \qquad E_0 + E_{10} = E_{20} \qquad \ldots (34)$

Similarly, the continuity of $\vec{H}$ field across the boundary gives,

$$H_0 + H_{10} = H_{20}$$

$\Rightarrow \qquad \sqrt{\frac{\varepsilon_1}{\mu_1}}\left(E_0 - E_{10}\right) = \sqrt{\frac{\varepsilon_2}{\mu_2}}E_{20} \qquad \ldots (35)$

For non-magnetic medium, refractive index,

$$n = \frac{c}{v} = \sqrt{\frac{\varepsilon}{\varepsilon_0}} \qquad \ldots (36)$$

Therefore, we can write the equation (35) as

$$n_1\left(E_0 - E_{10}\right) = n_2 E_{20} \qquad \ldots (37)$$

Now, multiplying by n_1, to the equation (34) and adding with equation (37) we get

$$2n_1\left(E_0\right) = \left(n_1 + n_2\right)E_{20}$$

$\Rightarrow \qquad \frac{E_{20}}{E_0} = \frac{2n_1}{n_1 + n_2}$

$\Rightarrow \qquad \boxed{t_{12} = \frac{2n_1}{n_1 + n_2}}$ it is called amplitude transmission coefficient.

Again, multiplying by n_2, to the equation (34) and substracting equation (37) we get

$$E_0\left(n_2 - n_1\right) = -\left(n_1 + n_2\right)E_p$$

$\Rightarrow \qquad \boxed{r_{12} = \frac{E_{10}}{E_0} = \frac{n_1 - n_2}{n_1 + n_2}}$ amplitude reflection coefficient.

ADDITIONAL SOLVED EXAMPLES

. The electric field E(r, t) at a point r at time t in a metal due to the passage of electrons can be described by the equation

$$\nabla^2\vec{E}(\vec{r},t)=\frac{1}{c^2}\left[\frac{\partial^2\vec{E}(\vec{r},t)}{\partial t^2}+\omega'^2\vec{E}(\vec{r},t)\right]$$

Where ω' is a characteristic associated with the metal and c is the speed of light in vacuum. The dispersion relation corresponding to the plane wave solutions of the form $\exp\left[i\left(\vec{k}.\vec{r}-\omega t\right)\right]$ is given by :

(a) $\omega^2=c^2k^2-\omega'^2$ (b) $\omega^2=c^2k^2+\omega'^2$ (c) $\omega^2=ck-\omega'$ (d) $\omega=ck+\omega'$

oln. Let $E(r,t)=E_0e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

We have, $\nabla^2E(r,t)=\frac{1}{c^2}\left[\frac{\partial^2E(r,t)}{\partial t^2}+\omega'^2E(r,t)\right]$

Replacing $\vec{\nabla}$ by ik and $\frac{\partial}{\partial t}=-i\omega$ we get

$$-k^2=\frac{1}{c^2}\left[-\omega^2+\omega'^2\right] \qquad (\text{since, } E_0e^{i(\vec{k}\cdot\vec{r}-\omega t)}\neq 0]$$

$\Rightarrow$ $k^2c^2=\omega^2-\omega'^2$

$\Rightarrow$ $\omega^2=c^2k^2+\omega'^2$

Correct option is (b)

. An electromagnetic wave is propagating in free space in the z-direction. If the electric field is given by $E=\cos(\omega t-kz)i$, where $\omega t=ck$, then the magnetic field is given by :

(a) $B=(1/c)\cos(\omega t-kz)j$ (b) $B=(1/c)\sin(\omega t-kz)j$

(c) $B=(1/c)\cos(\omega t-kz)i$ (d) $B=(1/c)\cos(\omega t-kz)ji$

oln. We know the relation between $\vec{E},\vec{k}$ and $\vec{B}$ is

$$\vec{k}\times\vec{E}=\omega\vec{B}$$

$\Rightarrow$ $\vec{B}=\frac{1}{\omega}\left(\vec{k}\times\vec{E}\right)=\frac{k}{\omega}\left(\vec{k}\times\vec{E}\right)=\frac{1}{c}\left(\vec{k}\times\vec{E}\right)$

We have, $E=\cos(\omega t-kt)i$

$\therefore$ $\hat{k}=\hat{z}$

$\therefore$ $\vec{B}=\frac{1}{c}\left[\hat{z}\times\hat{i}\cos(\omega t-kt)\right]=\frac{1}{c}\cos(\omega t-kt)\hat{j}$

Correct option is (a)

. Which one of the following Maxwell's equations implies the absence of magnetic monopoles?

(a) $\vec{\nabla}.\vec{E}=\pi/\varepsilon_0$ (b) $\vec{\nabla}.\vec{B}=0$

(c) $\vec{\nabla}\times\vec{E}=-\partial\vec{B}/\partial t$ (d) $\vec{\nabla}\times\vec{B}=\left(1/c^2\right)\partial\vec{B}/\partial t+\mu_0\vec{J}$

Soln. $\vec{\nabla}\cdot\vec{B}=0$

$\Rightarrow \quad \int\vec{\nabla}\cdot\vec{B}dv=0$

$\Rightarrow \quad \oint\vec{B}\cdot d\vec{S}=0$

$\Rightarrow \quad \varphi=0$

That say that monopole does not exist.

Correct option is (b)

Common Data for Q. 4 and Q. 5

Let $\tilde{E}=\hat{x}E_0\exp\left[i\hat{k}\cdot\vec{r}-\omega t\right]$, where $\vec{k}=\hat{z}\left(k\cos\phi+ik\sin\phi\right)$, $k=1\vec{k}$ and $\hat{x}$, $\hat{y}$ and $\hat{z}$ are cartesian unit vectors, represent an electric field of a plane electro magnetic wave of frequency ω.

4. Which one of the following statements is TRUE?

(a) The magnitude of the electric field is attenuated as the wave propagates

(b) The energy of the e.m. wave flows along the x-direction

(c) The magnitude of the electric field of the wave is a constant

(d) The speed of the wave is the same as c (speed of light in free space)

Soln. $\vec{E}=\hat{x}E_0\,e^{i\left(\vec{k}\cdot\vec{r}-\omega t\right)}$

$\vec{k}=\vec{z}\left(k\cos\varphi+ik\sin\phi\right)$

$E=\hat{x}\,E_0\,e^{i}\left[\hat{z}\left(k\cos\varphi+ik\sin\varphi\right)\cdot\vec{r}-\omega t\right]$

$=\hat{x}E_0\,e^{-k\sin\varphi z}e^{i}\left(k\cos\varphi z-\omega t\right)$

This says that the amplitude decay as the propagation.

Correct option is (a)

5. The magnetic field $\tilde{B}$ of the wave is :

(a) $\hat{y}\frac{k}{\omega}E_0\exp\left(-zk\sin\phi\right)\exp\left[i\left(zk\cos\phi-\omega t\right)\right]$

(b) $\hat{y}\frac{k}{\omega}E_0\exp\left(-zk\sin\phi\right)\exp\left[i\left(zk\cos\phi-\omega t+\phi\right)\right]$

(c) $\hat{y}\frac{k}{\omega}E_0\exp\left[i\left(zk\cos\phi-\omega t+\phi\right)\right]$

(d) $\hat{y}\frac{k}{\omega}E_0\exp\left(-zk\cos\phi\right)\exp\left[i\left(zk\sin\phi-\omega t\right)\right]$

Soln. We know,

$\vec{k}\times\vec{E}=\omega\vec{B}$

$\Rightarrow \quad \vec{B}=\frac{k}{\omega}\left(\hat{k}\times\vec{E}\right)$

We have,

$$\vec{E} = \hat{x}E_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} = \hat{x}E_0\, e^{-kt\sin\varphi}\, e^{i(kt\cos\varphi-\omega t)} \text{ and } \hat{k} = \hat{z}$$

$$\vec{B} = \frac{k}{\omega}E_0\, e^{-kt\sin\varphi}\, e^{i(kt\cos\varphi-\omega t)}\left[\hat{z}\times\hat{x}\right]$$

$$= \hat{y}\frac{k}{\omega}E_0\, e^{-kt\sin\varphi}\, e^{i(kt\cos\varphi-\omega t)}$$

Correct option is (b)

5. For this to represent a standing wave in the empty region between the plates :

(a) $k = \pi/L$ and $\alpha = 0$ (b) $k = 2\pi/L$ and $\alpha = \pi/2$

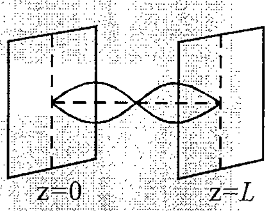
(c) $k = \pi/(2L)$ and $\alpha = \pi/2$ (d) $k = \pi/2L$ and $\alpha = 0$

Soln. We have,

$$A(r,t) = A_0\hat{i}\cos(kz+\alpha)\cos(kct),\ \varphi(r,t) = 0$$

$$\therefore\quad \vec{E} = -\frac{\partial\vec{A}}{\partial t} - \nabla\vec{\varphi} = -\frac{\partial A}{\partial t} + 0 = +A_0kc\cos(kz+\alpha)\sin(kct)$$

For standing wave the boundary condition is



$$E(0,t) = E(L,t) = 0$$

$$\therefore\quad \cos\alpha = 0;\qquad \alpha = \frac{\pi}{2}$$

And $\cos(kL+\alpha) = 0$

$$(kL+\alpha) = (2m+1)\times\frac{\pi}{2}\qquad (m = 0, 1, 2, 3, ...)$$

$$kL + \frac{\pi}{2} = 2\pi + \frac{\pi}{2} = \frac{5\pi}{2}\quad \Rightarrow k = \left(\frac{2\pi}{L}\right)$$

$$\therefore\quad k = \frac{2\pi}{L} \text{ and } \alpha = \frac{\pi}{2}$$

Correct option is (b)

. In a non-conducting medium characterized by $\varepsilon = \varepsilon_0, \mu = \mu_0$ and conductivity $\sigma = 0$, the electric field (in $\mathrm{Vm^{-1}}$) is given by $\vec{E} = 20\sin\left[10^8 t - kz\right]\hat{j}$. The magnetic field, $\vec{H}$(in $\mathrm{A\ m^{-1}}$), is given by :

(a) $20k\cos\left[10^8 t - kz\right]\hat{i}$ (b) $\frac{20k}{10^8\mu_0}\sin\left[10^8 t - kz\right]\hat{j}$

(c) $-\frac{20k}{10^8\mu_0}\sin\left[10^8 t - kz\right]\hat{i}$ (d) $-20k\cos\left[10^8 t - kz\right]\hat{j}$

Soln. We know the relation between $\vec{E}, \vec{k}, \vec{H}$ is

$$\vec{k}\times\vec{E} = \omega\mu_0\vec{H}$$

$$\vec{H} = \frac{k}{\omega\mu_0}\left(\vec{k}\times\vec{E}\right)$$

We have,

$$\vec{E} = 20\sin\left(10^8 t - kt\right)\hat{j}$$

$$\hat{k} = \hat{z}, \ \omega = 10^8$$

$$\therefore \quad \vec{H} = \frac{20k}{\omega\mu_0}\left(\hat{z}\times\hat{j}\right)\sin\left(10^8 t - kt\right) = -\frac{20k}{10^8\mu_0}\sin\left(10^8 t - kt\right)\hat{i}$$

Correct option is (c)

8. Which one of the following sets of Maxwell's equations for time - independent charge density ρ and current density $\hat{J}$ is correct?

(a) $\vec{\nabla}.\vec{E} = \rho/\varepsilon_0$, $\vec{\nabla}.\vec{B} = 0$, $\vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t}$, $\vec{\nabla}\times\vec{B} = \mu_0\varepsilon_0\dfrac{\partial\vec{E}}{\partial t}$

(b) $\vec{\nabla}.\vec{E} = \rho/\varepsilon_0$, $\vec{\nabla}.\vec{B} = 0$, $\vec{\nabla}\times\vec{E} = 0$, $\vec{\nabla}\times\vec{B} = \mu_0\hat{J}$

(c) $\vec{\nabla}.\vec{E} = 0$, $\vec{\nabla}.\vec{B} = 0$, $\vec{\nabla}\times\vec{E} = 0$, $\vec{\nabla}\times\vec{B} = \mu_0\hat{J}$

(d) $\vec{\nabla}.\vec{E} = \rho/\varepsilon_0$, $\vec{\nabla}.\vec{B} = \mu_0\hat{J}$, $\vec{\nabla}\times\vec{E} = 0$, $\vec{\nabla}\times\vec{B} = \mu_0\varepsilon_0\dfrac{\partial\vec{E}}{\partial t}$

Soln. The Maxwell's equations are

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}, \ \vec{\nabla}\cdot\vec{B} = 0, \ \vec{\nabla}\times\vec{E} = -\frac{\partial B}{\partial t}$$

$$\vec{\nabla}\times\vec{B} = \mu_0 J + \varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t} = \mu_0 J + \mu_0\frac{\partial\vec{B}}{\partial t}$$

For steady state circuit flow $\dfrac{\partial B}{\partial t} = 0$

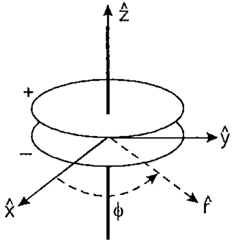
$\therefore \quad \nabla\times B = \mu_0 J$ and since J is time independent $\therefore \dfrac{\partial B}{\partial t} = 0$

Therefore, the Maxwell's equation for time independent charge density and current density are

$$\vec{\nabla}.\vec{E} = \rho/\varepsilon_0, \ \vec{\nabla}.\vec{B} = 0, \ \vec{\nabla}\times\vec{E} = 0, \ \vec{\nabla}\times\vec{B} = \mu_0\hat{J}$$

Correct option is (b)

9. A parallel plate capacitor is being discharged. What is the direction of the energy flow in terms of the poynting vector in the space betwen the plates?



(a) Along the wire in the positive z axis
(b) Radially inward $(-\hat{r})$
(c) Radially outward $(\hat{r})$
(d) Circumferential (ϕ)

Soln. The electric field inside the two parallel plate is along $-\hat{z}$ and from right handed rule we can say the magnetic field along $\hat{\theta}$.

Therefore, poynting vector, $\vec{s} = \vec{E} \times \vec{H}^*$

Therefpre. direction of poynting $\vec{s} = -\hat{z} \times \hat{\theta}$

Velocity $= \hat{r}$

Correct option is (c)

Statement for Linked Answer Q.10 and Q.11.

Consider the propagation of electromagnetic waves in a linear, homogenous and isotropic material medium with electric permitivity ε, and magnetic permeability μ.

1. For a plane wave of angular frequency ω and propagation vector $\vec{k}$ propagating in the medium Maxwell's equations reduce to

(a) $\vec{k}.\vec{E} = 0; \vec{k}.\vec{H} = 0; \vec{k} \times \vec{E} = \omega\varepsilon\vec{H}; \vec{k} \times \vec{H} = -\omega\mu\vec{E}$

(b) $\vec{k}.\vec{E} = 0; \vec{k}.\vec{H} = 0; \vec{k} \times \vec{E} = -\omega\varepsilon\vec{H}; \vec{k} \times \vec{H} = \omega\mu\vec{E}$

(c) $\vec{k}.\vec{E} = 0; \vec{k}.\vec{H} = 0; \vec{k} \times \vec{E} = -\omega\mu\vec{H}; \vec{k} \times \vec{H} = \omega\varepsilon\vec{E}$

(d) $\vec{k}.\vec{E} = 0; \vec{k}.\vec{H} = 0; \vec{k} \times \vec{E} = \omega\mu\vec{H}; \vec{k} \times \vec{H} = -\omega\varepsilon\vec{E}$

oln. Let $\vec{E} = \vec{E}_0\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$

We know that Maxwell's equations,

$$\vec{\nabla} \cdot \vec{E} = 0,\ \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t},\ \vec{\nabla} \times \vec{B} = \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

Replacing $\vec{\nabla}$ by ik and $\frac{\partial}{\partial t}$ by $-i\omega$, we get

$$i\vec{k} \cdot \vec{E} = 0 \Rightarrow \vec{k} \cdot \vec{E} = 0$$

$$i\vec{k} \cdot \vec{B} = 0 \Rightarrow \vec{k} \cdot \vec{B} = 0$$

$$i\vec{k} \times \vec{E} = i\omega\mu\vec{H} \Rightarrow \vec{k} \times \vec{E} = \mu\omega\vec{H}$$

$$i\vec{k} \times \vec{H} = i\omega\varepsilon\vec{E} \Rightarrow \vec{k} \times \vec{H} = -\varepsilon\omega\vec{E}$$

Correct option is (d)

1. If ε and μ assume negative values in a certain frequency range, then the directions of the propagation vector $\vec{k}$ and the Poynting vector $\vec{S}$ in that frequency range are related as

(a) $\vec{k}$ and $\vec{S}$ are parallel

(b) $\vec{k}$ and $\vec{S}$ are anti-parallel.

(c) $\vec{k}$ and $\vec{S}$ are perpendicular to each other

(d) $\vec{k}$ and $\vec{S}$ make an angle that depends on the magnitude of $|\varepsilon|$ and $|\mu|$.

Soln. $\vec{S} = \vec{E} \times \vec{H}^* = \vec{E} \times \left(\vec{k} \times \vec{E}^*\right) \frac{1}{\mu\omega}$

$$= \frac{1}{\mu\omega}\left[\vec{k}\left(\vec{E}.\vec{E}^*\right) - \left(k \cdot E\right)\vec{E}^*\right] = \frac{1}{\mu\omega}\vec{k}|E|^2 = \frac{k}{\mu\omega}|E|^2\,\hat{k}$$

Since μ is negative.

$$\vec{S} = -\frac{k}{\mu\omega}|E|^2\,\hat{k}$$

Therefore, k and S are anti-parallel.

Correct option is (b)

Statement for Linked Answer Q. 12 and Q. 13:

A plane electromagnetic wave has the magnetic field given by $\vec{B}(x,y,z,t) = B_0 \sin\left[(x+y)\frac{k}{\sqrt{2}} + \omega t\right]\hat{k}$

Where k is the wave number and $\hat{i}, \hat{j}$ and $\hat{k}$ are the Cartesian unit vectors is x, y and z directions, respectively.

12. The electric field $\vec{E}(x,y,z,t)$ corresponding to the above wave is given by

(a) $cB_0 \sin\left[(x+y)\frac{k}{\sqrt{2}} + \omega t\right]\frac{(\hat{j}-\hat{i})}{\sqrt{2}}$
(b) $cB_0 \sin\left[(x+y)\frac{k}{\sqrt{2}} + \omega t\right]\frac{(\hat{i}+\hat{j})}{\sqrt{2}}$

(c) $cB_0 \sin\left[(x+y)\frac{k}{\sqrt{2}} + \omega t\right]\hat{i}$
(d) $cB_0 \sin\left[(x+y)\frac{k}{\sqrt{2}} + \omega t\right]\hat{j}$

Soln. $\vec{B} = B_0 \sin\left[(x+y)\frac{k}{\sqrt{2}} + \omega t\right]\hat{k}$

We know the solution,

$$\vec{E} = -\frac{1}{\omega\varepsilon_0\mu_0}\left[\vec{k}\times\vec{B}\right] = -\frac{k}{\omega}c^2\left[\hat{k}\times\vec{B}\right]$$

$$= -cB_0 \sin\left[(x+y)\frac{k}{\sqrt{2}} + \omega t\right]\left[(\hat{i}+\hat{j})\frac{1}{\sqrt{2}}\times\hat{k}\right]$$

$$= cB_0 \sin\left[(x+y)\frac{k}{\sqrt{2}} + \omega t\right](\hat{j}-\hat{i})\frac{1}{\sqrt{2}}$$

$$= cB_0\frac{(\hat{j}-\hat{i})}{\sqrt{2}}\sin\left[(x+y)\frac{k}{\sqrt{2}} + \omega t\right]$$

Correct option is (a)

13. The average Poynting vector is given by

(a) $\frac{cB_0^2}{2\mu_0}\frac{(\hat{i}-\hat{j})}{\sqrt{2}}$
(b) $-\frac{cB_0^2}{2\mu_0}\frac{(\hat{i}-\hat{j})}{\sqrt{2}}$
(c) $\frac{cB_0^2}{2\mu_0}\frac{(\hat{i}+\hat{j})}{\sqrt{2}}$
(d) $-\frac{cB_0^2}{2\mu_0}\frac{(\hat{i}+\hat{j})}{\sqrt{2}}$

Soln. $\langle S\rangle = \left\langle\left(E\times H^*\right)\right\rangle = \frac{cB_0^2}{T\mu_0}\int_0^T \sin^2\left[(x+y)\frac{k}{\sqrt{2}} + \omega t\right]\frac{(\hat{i}+\hat{j})}{\sqrt{2}}dt$

$$= \frac{cB_0^2}{2\sqrt{2}\mu_0}(\hat{i}+\hat{j})$$

Correct option is (c)

4. The space-time dependence of the electric field of a linearly polarized light in free space is given by $\hat{x}E_0\cos(\omega t - kz)$ where E_0 ω and k are the amplitude, the angular frequency and the wavevector, respectively. The time averaged energy density associated with the electric field is

(a) $\frac{1}{4}\varepsilon_0 E_0^2$ (b) $\frac{1}{2}\varepsilon_0 E_0^2$ (c) $\varepsilon_0 E_0^2$ (d) $2\varepsilon_0 E_0^2$

Soln. We have,

$\vec{E} = E_0\cos(\omega t - kt)\hat{x}$

Therefore, the time average energy density associated with electric field

$$\langle u\rangle = \frac{\frac{1}{2}\varepsilon_0}{T}\int E^2 dt = \frac{1}{2}\frac{\varepsilon_0 E_0^2}{T}\int_0^T \cos^2(\omega t - kt)\,dt = \frac{1}{2}\varepsilon_0 E_0^2\frac{1}{2} = \frac{1}{4}\varepsilon_0 E_0^2$$

Correct option is (a)

5. Interference fringes are seen at an observation plane $z = 0$, by the superposition of two plane waves $A_1\exp[i(\vec{k}_1\cdot\vec{r} - \omega t)]$ and $A_2\exp[i(\vec{k}_2\cdot\vec{r} - \omega t)]$, where A_1 and A_2 are real amplitudes. The condition for interference maximum is

(a) $(\vec{k}_1 - \vec{k}_2)\cdot\vec{r} = (2m+1)\pi$ (b) $(\vec{k}_1 - \vec{k}_2)\cdot\vec{r} = 2m\pi$

(c) $(\vec{k}_1 + \vec{k}_2)\cdot\vec{r} = (2m+1)\pi$ (d) $(\vec{k}_1 + \vec{k}_2)\cdot\vec{r} = 2m\pi$

Soln. The phase difference between two wave is

$\varphi = (\vec{k}_1\cdot\vec{r} - \vec{k}_2\cdot\vec{r}) = (\vec{k}_1 - \vec{k}_2)\cdot\vec{r}$

We know that the condition for maximum is

$\varphi = 2m\pi$, where $m = 0, 1, 2, \ldots\ldots$

$\therefore$ $(\vec{k}_1 - \vec{k}_2)\cdot\vec{r} = 2m\pi$

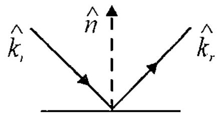
Correct option is (b)

6. A monochromatic plane wave at oblique incidence undergoes reflection at a dielectric interface. If $\hat{k}_i$, $\hat{k}_r$ and $\hat{n}$ are the unit vectors in the directions of incident wave, reflected wave and the normal to the surface respectively, which one of the following expression is correct?

(a) $(\hat{k}_i - \hat{k}_r)\times\hat{n} \neq 0$ (b) $(\hat{k}_i - \hat{k}_r)\cdot\hat{n} = 0$ (c) $(\hat{k}_i\times\hat{n})\cdot\hat{k}_r = 0$ (d) $(\hat{k}_i\times\hat{n})\cdot\hat{k}_r \neq 0$

Soln. We know that $\vec{k}_i, \vec{k}_r, \hat{n}$ all are in the same plane.

The vector $(\hat{k}_i\times\hat{n})$ have direction perpendicular to the plane, that contain by the $\hat{k}_i, \hat{n}, \hat{k}_r$



$\therefore$ $(\hat{k}_i\times\hat{n})\cdot\hat{k}_r = 0$ because $(\hat{k}_i\times\hat{n})\perp\hat{k}_r$

Correct option is (c)

7. The electric field of a uniform plane wave propagating in a dielectric, non-conducting medium is given by,

$$\vec{E} = \hat{x}10\cos(6\pi\times10^7\,t - 0.4\pi z)V/m$$

The phase velocity of the wave is ________ ×10⁸ m/s.

Soln. We have, $\vec{E} = \hat{x}\,10\cos\left(6\pi\times10^{7}t - 0.4\pi z\right)\frac{V}{m}$

Compaired with electric field $\vec{E} = \hat{x}\left(\omega t - kz\right)\frac{V}{m}$

We get, $\omega = 6\pi\times10^{7},\quad k = 0.4\pi$

Phase velocity, $V_{\varphi} = \frac{\omega}{k} = \frac{6\pi}{0.4\pi}\times10^{7} = 1.5\times10^{8}\ m/\sec$

18. An electromagnetic wave with $\vec{E}(z,t) = E_0\cos(\omega t - kz)\hat{i}$ is travelling in free space and crosses a disc of radius 2m placed perpendicular to the z-axis. If $E_0 = 90V\ m^{-1}$, the average power, in watt, crossing the disc along the z-direction is

(a) 30 (b) 60 (c) 120 (d) 270

Soln. We have, $I = 150\,\frac{\omega}{m^{2}}$

We know, $I = \langle u\rangle c = \frac{1}{2}\varepsilon_0 E_0^2 c$

Therefore, the average power $= I\times\pi R^2 = \frac{1}{2}\times8.85\times10^{-12}\times(90)^2\times\pi\times(2)^2\times3\times10^{8} = 135 \simeq 120$

Correct option is (c)

19. The intensity of a laser in free space is $150\ mW/m^2$. The corresponding amplitude of the electric field of the laser is ______ V/m. $\left(\varepsilon_0 = 8.854\times10^{-12}\ C^2/N.m^2\right)$

Soln. We know that, $I = \langle u\rangle c = \frac{1}{2}\varepsilon_0 E_0^2 c$

$$E_0 = \sqrt{\frac{2I}{c\varepsilon_0}} = \sqrt{\frac{2\times150\times10^{-3}}{3\times10^{8}\times8.85\times10^{-12}}} = 10.62\,\frac{V}{m}$$

20. A plane electromagnetic wave travelling in a vacuum is characterised by the electric and magnetic fields

$$\vec{E} = \hat{i}\left(30\pi\ \mathrm{Vm^{-1}}\right)\exp i\,(\omega t + kz)$$

$$\vec{H} = \hat{j}\left(H_0\ \mathrm{Am^{-1}}\right)\exp i\,(\omega t + kz)$$

If $\omega, k > 0$, the value of H_0 must be **[TIFR 2013]**

(a) 2π (b) 0.67 (c) 0.25 (d) 0.94

Soln. $|\vec{H}| = H_0 = \left|\frac{\vec{k}\times\vec{E}}{\omega\mu_0}\right| = \frac{kE_0}{\omega\mu_0} = \varepsilon_0 c E_0 = 8.85\times10^{-12}\times3\times10^{8}\times30\pi = 0.25$

Correct option is (c)

21. The instantaneous electric and magnetic fields created at a distance 'r' by a point source at the origin are given by

$$\vec{E} = \frac{A\cos\omega t}{2\pi\varepsilon_0 r}\hat{\theta} \qquad \hat{H} = \frac{B\cos\omega t}{\mu_0 r}\hat{\varphi}$$

[TIFR 2014]

where ω, A, B are constants, and the unit vectors $(\hat{r}, \hat{\theta}, \hat{\varphi})$ form an orthonormal set. The time-averaged power radiated by the source is

(a) $\frac{\omega\varepsilon_0}{\mu_0}AB$ (b) $\frac{c^3}{2\pi}AB$ (c) c^2AB (d) $\frac{2\pi\omega}{c}AB$

ln. We have,

$$\vec{E} = \frac{A\cos\omega t}{2\pi\varepsilon_0 r}\hat{\theta} \qquad \vec{H} = \frac{B\cos\omega t}{\mu_0 r}\hat{\varphi}$$

$$\therefore \qquad \langle s\rangle = \langle E\times H^*\rangle = \frac{AB}{2\pi\varepsilon_0 r^2\mu_0}\frac{1}{T}\int_0^T\cos^2\omega t dt = \frac{AB}{4\pi\varepsilon_0\mu_0 r^2}$$

Therefore, time-average power radiated by the source is

$$P = \int_0^{\pi}\int_0^{2\pi}\frac{AB}{4\pi\varepsilon_0\mu_0 r^2}r^2\sin\theta d\theta d\varphi = \frac{AB}{4\pi\varepsilon_0\mu_0}4\pi = c^2AB$$

Correct option is (c)

2. A light source has a small filament at the centre of a sphereical glass bulb of radius 5 cm and negligible thickness. If this source emits 100 Watts of power in the form of spherical electromagnetic waves, the r.m.s. electric field E at the surface of the bulb (in units of Volt/m) will be approximately **[TIFR 2015]**

(a) 1547 (b) 1094 (c) 109.4 (d) 15.47

oln. We know power,

$$p = I\times 4\pi R^2 = \frac{1}{2}\varepsilon_0 E_0^2 c\times 4\pi R^2$$

$$\Rightarrow \qquad E_0 = \sqrt{\frac{2p}{\varepsilon_0 c\times 4\pi R^2}} = \sqrt{\frac{2\times 100}{8.85\times 10^{-12}\times 3\times 10^8\times 4\times 3.14\times(0.5)^2}} = 1094\ volt/m$$

Correct option is (b)

3. Measurement of the magnitudes of the electric field (E) and the magnetic field (B) in a plane polarized electromagnetic wave in vaccum leads to the following results **[TIFR 2015]**

$$\frac{\partial E}{\partial y} = -\frac{\partial B}{\partial t} \qquad \frac{\partial B}{\partial y} = -\frac{1}{c^2}\frac{\partial E}{\partial t}$$

at all points where the measurement is made. In this case the electric vector $\vec{E}$, the magnetic vector $\vec{B}$ and the wave vector $\vec{k}$ (with magnetic k) can be written in terms of the unit vectors $(\hat{x}, \hat{y}, \hat{z})$ along the Cartesian axes as

(a) $\vec{E} = E\hat{x},\ \ \vec{B} = B\hat{y},\ \ \vec{k} = k\hat{z}$ (b) $\vec{E} = E\hat{x},\ \ \vec{B} = B\hat{z},\ \ \vec{k} = -k\hat{y}$

(c) $\vec{E} = E\hat{x},\ \ \vec{B} = -B\hat{z},\ \ \vec{k} = k\hat{y}$ (d) $\vec{E} = -E\hat{y},\ \ \vec{B} = -B\hat{z},\ \ \vec{k} = -k\hat{x}$

oln. From the given two equations.

We can write, $\left.\begin{array}{l}\vec{k}\times\vec{E} = \omega\vec{B}\\ \vec{k}\times\vec{B} = -\varepsilon_0\omega\vec{E}\end{array}\right\}$

The option (c) only satisfy the above two equation simultaneously.

Correct option is (c)

24. A time-dependent magnetic field $\vec{B}(t)$ is produced in a circular region of space, infinitely long and of radius R. The magnetic field is given as $\vec{B} = B_0 t\hat{z}$ for $0 \le r < R$ and is zero for $r > R$. Where B_0 is a positive constant. The electric field for $r > R$ is **[JEST 2012]**

(a) $\frac{B_0 R^2}{r}\hat{r}$ (b) $\frac{B_0 R^2}{2r}\hat{\theta}$ (c) $-\frac{B_0 R^2}{r}\hat{r}$ (d) $-\frac{B_0 R^2}{2r}\hat{\theta}$

Soln. We have,

$$\vec{B} = B_0 t\hat{z} \qquad for\ \ 0 < r < R$$
$$= 0 \qquad n > R$$

Therefore, the electric field for $r > R$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = 0 = \vec{\nabla}\times\left(-\frac{B_0 R^2}{2r}\hat{\theta}\right)$$

$$\therefore \qquad \vec{E} = -\frac{B_0 R^2}{2r}\hat{\theta}$$

Correct option is (d)

25. An electromagnetic wave of frequency ω travels in the x-direction through vacuum. It is polarized in the y-direction and the amplitude of the electric field is E_0. With $k = \omega/c$ where c is the speed of light in vacuum, the electric and the magnetic fields are then conventionally given by **[JEST 2013]**

(a) $\vec{E} = E_0\cos(ky - \omega t)\hat{x}$ and $\vec{B} = \frac{E_0}{c}\cos(ky - \omega t)\hat{z}$

(b) $\vec{E} = E_0\cos(kx - \omega t)\hat{y}$ and $\vec{B} = \frac{E_0}{c}\cos(kx - \omega t)\hat{z}$

(c) $\vec{E} = E_0\cos(kx - \omega t)\hat{z}$ and $\vec{B} = \frac{E_0}{c}\cos(kx - \omega t)\hat{y}$

(d) $\vec{E} = E_0\cos(kx - \omega t)\hat{x}$ and $\vec{B} = \frac{E_0}{c}\cos(ky - \omega t)\hat{y}$

Soln. An electromagnetic wave frequency ω travels in the x-direction through the vacuum and $\hat{y}$ polarized electric field with amplitude E_0 is given by

$$\vec{E} = E_0\cos(kx - \omega t)\hat{y}$$

$$\therefore \qquad \vec{B} = \frac{\vec{k}\times\vec{E}}{\omega} = \frac{k}{\omega}\left(\hat{k}\times\vec{E}\right) = \frac{1}{c}\left[E_0\cos(kx - \omega t)\right]\left(\hat{x}\times\hat{y}\right) = \frac{1}{c}E_0\cos(kx - \omega t)\hat{z}$$

Correct option is (b)

PRACTICE SET

. Current flowing in a circuit containing a capacitor. To satisfy the continuity equation, $\vec{\nabla}.\vec{J}=-\frac{\partial\rho}{\partial t}$, for charge and current, the current density $\vec{J}$ must be replaced by

(a) $\frac{1}{4\pi}\frac{\partial\vec{D}}{\partial t}$ (b) $\vec{J}-\frac{1}{4\pi}\frac{\partial\vec{D}}{\partial t}$ (c) $\vec{D}+\frac{1}{4\pi}\frac{\partial\vec{J}}{\partial t}$ (d) $\vec{J}+\frac{1}{4\pi}\frac{\partial\vec{D}}{\partial t}$

2. For a medium of electric and magnetic field the relation between E and B with propagation vector k

(a) $\bar{E}\times\bar{k}=\omega\bar{B}$ (b) $\bar{E}\times\bar{R}=\bar{\omega}\times\bar{B}$ (c) $\bar{E}=\frac{k}{\omega}\times\bar{B}$ (d) $\vec{B}=\frac{\vec{k}\times\vec{E}}{\omega}$

3. In a free space, which is the correct set of Maxwell's equations in absence of any charges and current densities :

(a) $\vec{\nabla}.\vec{E}=\rho/\varepsilon_0, \vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$
$\vec{\nabla}.\vec{B}=0, \vec{\nabla}\times\vec{B}=\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$

(b) $\vec{\nabla}.\vec{E}=0, \vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$
$\vec{\nabla}.\vec{B}=0, \vec{\nabla}\times\vec{B}=\mu_0\vec{J}$

(c) $\vec{\nabla}.\vec{E}=\rho/\varepsilon_0, \vec{\nabla}\times\vec{E}=0$ (d)
$\vec{\nabla}.\vec{B}=0, \vec{\nabla}\times\vec{B}=\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$

(d) $\vec{\nabla}.\vec{E}=0, \vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$
$\vec{\nabla}.\vec{B}=0. \vec{\nabla}\times\vec{B}=\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$

4. For plane electromagnetic waves varying in time as e^{-iwt}, the time averaged Poynting vector is:

(a) $\frac{1}{2}R_e(\vec{E}\times\vec{B})$ (b) $R_e(\vec{E}\times\vec{H})$ (c) $\frac{1}{2}R_e(\vec{E}\times\vec{H})$ (d) $\frac{1}{2}(R_e\vec{E})\times(R_e\vec{H})$

5. Electromagnetic wave in vacuum is given by $\vec{E}(\vec{r},t)=\hat{i}E_0e^{-i(\omega t-k.z)}$. Magnetic field $\vec{B}$ is:

(a) $\hat{j}\frac{E_0}{c}\hat{k}\exp[-i(\omega t-k.x)]$ (b) $\hat{j}\frac{E_0}{c}\exp[-i(\omega t-k_z z)]$

(c) $\hat{k}\frac{E_0}{c}\exp[-i(\omega t-k.x)]$ (d) $\hat{j}\frac{E_0}{c}\exp[-i(\omega t-k.x)]$

6. The electric field of an electromagnetic wave is given by $\vec{E}=E_0(3\hat{i}+4\hat{j})\cos k(4x-3y-ct)$. The directions of the magnetic field $\vec{B}$ and the poynting vector $\vec{p}$ are

(a) $-\hat{k},\frac{4\hat{i}+3\hat{j}}{5}$ (b) $\hat{k},\frac{4\hat{i}-3\hat{j}}{5}$ (c) $\frac{4\hat{i}-3\hat{j}}{5},\hat{k}$ (d) $\frac{4\hat{i}-3\hat{j}}{5},\frac{4\hat{i}+3\hat{j}}{5}$

7. For a wave propagating in an air filled metallic rectangular waveguide, the frequency and the wave vector are related as (c is the speed of light and ω_c is the cut off frequency)

(a) $c^2k^2=\omega^2+\omega_c^2$ (b) $c^2k^2=\omega^2$ (c) $c^2k^2=\omega^2-\omega_c^2$ (d) $c^2k^2=\sqrt{\omega^4-\omega_c^4}$

8. In a region of space it is found that $\vec{\nabla}.\vec{E}=0$ but $\vec{\nabla}\times\vec{E}\neq 0$. Which of the following statements is not true?

(a) The charge density must be zero
(b) A displacement current must exist
(c) The divergence of magnetic field must be zero
(d) A time-varying magnetic field must exist

9. Which of the following statement is ***CORRECT.***
(a) Electromagnetic wave is longitudinal wave
(b) Electromagnetic wave can not propagate through the vacuum.
(c) Electromagnetic wave is transverse wave.
(d) Velocity of electromagnetic wave in vacuum is less than the velocity is light in free space.

10. The wavelength of red helium-neon laser in air is 6328 Å. What happens to its frequency in glass that has a refractive index of 1.50 ?
(a) Increases by a factor of 3
(b) Decreases by a factor of 1.5
(c) Remains the same
(d) Decreases by a factor of 0.5

ANSWER KEY					
Questions	1	2	3	4	5
Option	(d)	(d)	(d)	(c)	(b)
Questions	6	7	8	9	10
Option	(b)	(a)	(b)	(c)	(c)

Chapter 11

Circuit Theory

Basic circuit

The emf of a battery is equal to the voltage across its terminals when the current is zero. That is, the emf is equivalent to the **open-circuit voltage** of the battery. The **equivalent resistance** of a set of resistors connected in series is

$$R_{eq} = R_1 + R_2 + R_3 + ... \qquad ...(1)$$

The **equivalent resistance** of a set of resistors connected in **parallel** is

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + ... \qquad ...(2)$$

If it is possible to combine resistors into series or parallel equivalents, the preceding two equations make it easy to determine how the resistors influence the rest of the circuit.

Circuits involving more than one loop are conveninetly analyzed with the use **Kirchhoff's rules.**

(1) The sum of currents entering any junction in an electric circuit must equal the sum of the currents leaving that junction:

$$\Sigma I_{in} = \Sigma I_{out} \qquad ...(3)$$

(2) The sum of the potential differences across all elements around any circuit loop must be zero.

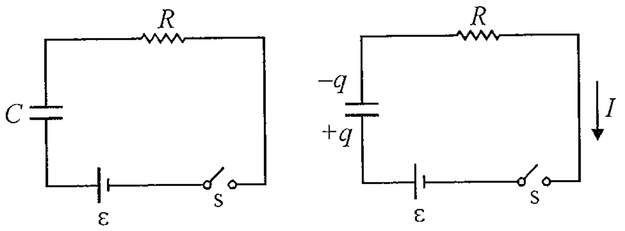
$$\sum_{\substack{\text{closed}\\ \text{loop}}} \Delta V = 0 \qquad ...(4)$$

The first rule is a statement of conservation of charge; the second is equivalent to a statment of conservation of energy.

When a resistor is traversed in the direction of the current, the change in poptential ΔV across the resistor is $-IR$. When a resistor is traversed in the direction opposite the current, $\Delta V = +IR$. When a source of emf is traversed in the direction of the emf (negative terminal to positive terminal), the change in potential is $+\varepsilon$. When a source of emf is traversed opposite the emf (positive to negative), the change in potential is $-\varepsilon$. The use of these rules allows you to analyze electric circuits.

RC Circuit

Figure represents a simple RC circuit with DC voltage. If ε be the voltage applied, and C and R are the capacitance and Resistance then the circuit equation is



$t < 0$ $\qquad$ $t > 0$

$$\varepsilon - \frac{q}{C} - IR = 0$$

Substituting, I = dq/dt we get in above equation

$$\frac{dq}{dt} = \frac{\varepsilon}{R} - \frac{q}{RC}$$

To find an expression for q, we first combine the terms on the right-hand side:

$$\frac{dq}{dt} = \frac{C\varepsilon}{RC} - \frac{q}{RC} = -\frac{q - C\varepsilon}{RC}$$

Now we multiply by dt and divide by $q - C\varepsilon$ to obtain

$$\frac{dq}{q - C\varepsilon} = -\frac{1}{RC}dt$$

Integrating this expression, using the fact that q = 0 at t = 0, we obtain

$$\int_0^q \frac{dq}{q - C\varepsilon} = -\frac{1}{RC}\int_0^t dt$$

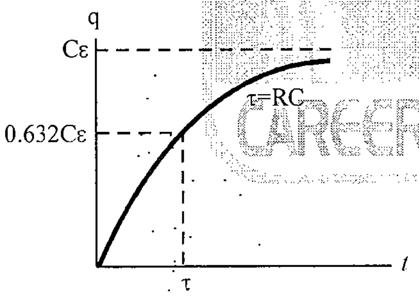
$$\ell n\left(\frac{q - C\varepsilon}{-C\varepsilon}\right) = -\frac{1}{RC}t$$

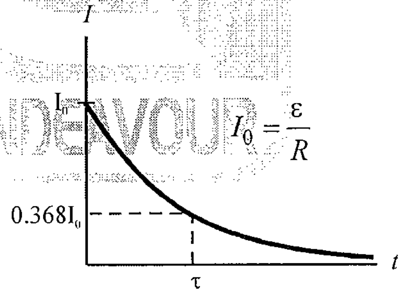
Or, $$q(t) = C\varepsilon\left(1 - e^{-t/RC}\right) = Q\left(1 - e^{-t/RC}\right) \quad \ldots (5)$$

By differentiating the expression we get the current

$$I(t) = \frac{\varepsilon}{R}e^{-1/RC} \quad \ldots (6)$$

Now we can represent q and I graphically





While discharging, no voltage in the circuit. So, the circuit equation becomes

$$-\frac{q}{C} - IR = 0$$

When we substitute I = dq/dt into this expression, it becomes

$$-R\frac{dq}{dt} = \frac{q}{C}$$

$$\frac{dq}{q} = -\frac{1}{RC}dt$$

Integrating this expression, using the fact that q = Q at t = 0, gives

$$\int_Q^q \frac{dq}{q} = -\frac{1}{RC}\int_0^t dt$$

$$\ell n\left(\frac{q}{Q}\right)=-\frac{1}{RC}$$

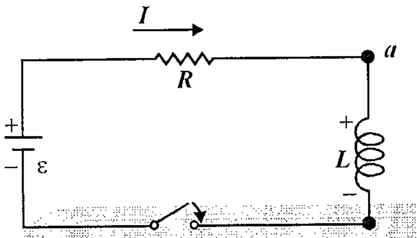
$$q(t)=Qe^{-t/RC} \qquad ...(7)$$

Again differntiating we get the expression of I.

$$I(t)=\frac{dq}{dt}=\frac{d}{dt}\left(Qe^{-t/RC}\right)=-\frac{Q}{RC}e^{-t/RC} \qquad ...(8)$$

LR Circuit

Consider a inductor of inductance L and a resistor with resistance R connected to a battery



The back emf created by inductors is

$$\varepsilon_L=-L\frac{dI}{dt}$$

The circuit equation becomes

$$\varepsilon-IR-L\frac{dI}{dt}=0$$

$$\Rightarrow \quad L\frac{dI}{dt}=(\varepsilon-IR)$$

$$\Rightarrow \quad \frac{dI}{(\varepsilon-IR)}=\frac{dt}{L}$$

Integrating this equation, we get

$$\ell n\left(\frac{\varepsilon-IR}{\varepsilon}\right)=-\frac{R}{L}t$$

$$\Rightarrow \quad \frac{\varepsilon-IR}{\varepsilon}=e^{-Rt/L} \Rightarrow I=\frac{\varepsilon}{R}\left(1-e^{-Rt/L}\right)$$

$$\Rightarrow \quad I=I_0\left(1-e^{-Rt/L}\right) \qquad ...(9) \qquad \text{where, } I_0=\frac{\varepsilon}{R}$$

We can also write this expression as

$$I=\frac{\varepsilon}{R}\left(1-e^{-t/\tau}\right) \qquad ...(10)$$

Where the constant τ is the time constant of the RL circuit:

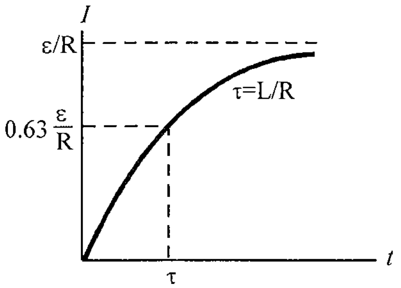
$$\tau=L/R$$

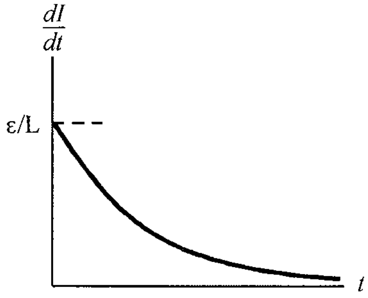
Physically, τ is the time it takes the current in the circuit to reach $\left(1-e^{-1}\right)=0.63$ of its final value ε/R. The time constant is a useful parameter for comparing the time response of various circuits.

dI/dt also decreases exponentially

$$\frac{dI}{dt}=\frac{\varepsilon}{L}e^{-t/\tau} \qquad \text{...(12)}$$

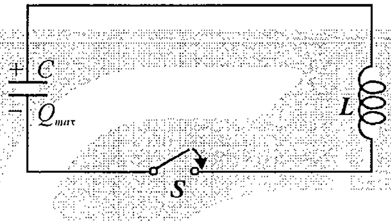
Graphically





LC circuit

Consider a LC circuit as shown below. A capacitor and an Inductor is conneted.



The circuit equation is

$$\frac{Q}{C}+L\frac{dI}{dt}=0$$

We know

$$dI/dt=d^2Q/dt^2$$

So, the circuit equation becomes

$$\frac{Q}{C}+L\frac{d^2Q}{dt^2}=0$$

$$\frac{d^2Q}{dt^2}=-\frac{1}{LC}Q$$

This is the same equation of simple harmonic oscillator.

$$\frac{d^2Q}{dt^2}=-\frac{k}{m}x=-\omega^2x$$

So, solving for Q we get

$$Q=Q_{max}\cos(\omega t+\phi) \qquad \text{...(13)}$$

where Q_{max} is the maximum charge of the capacitor and the angular frequency ω is

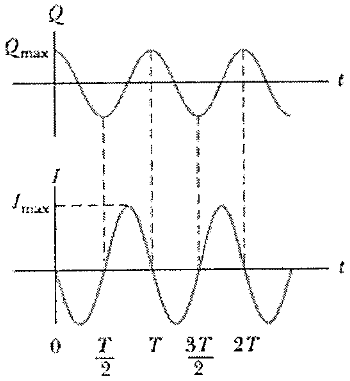
$$\omega=\frac{1}{\sqrt{LC}} \qquad \text{...(14)}$$

Note that the angular frequency of the oscillations depends slowly on the inductance and capacitane of the circuit. This is the natural frequency of oscillation of the LC circuit.

Differentiating Q, we get I

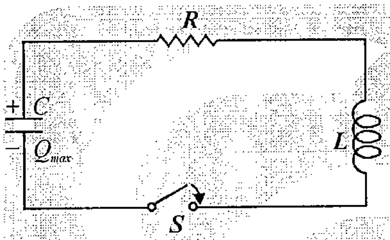
$$= \frac{dQ}{dt} = -\omega Q_{max}\sin(\omega t + \phi) \qquad \dots(15)$$

Charge and current varies like



RLC circuit

Shows a RLC circuit



The circuit equation is

$$L\frac{dI}{dt} + \frac{Q}{C} + RI = 0$$

Or,
$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0$$

This is the same equation of damped harmonic oscillator.

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

So solving for small R we get

$$Q = Q_{max}e^{-Rt/2L}\cos\omega_d t \qquad \dots(16)$$

Where,
$$\omega_d = \left[\frac{1}{LC} - \left(\frac{R}{2L}\right)^2\right]^{1/2} \qquad \dots(17)$$

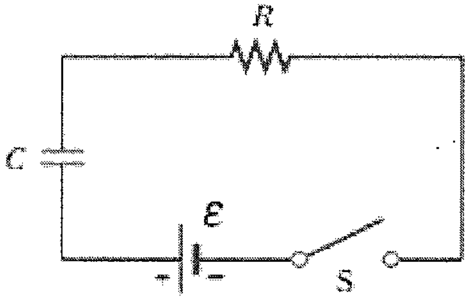
ADDITINOAL SOLVED PROBLEMS

1. An unchanged capacitor and a resistor are connected in series to battery, as shown in figure 8.8. If $\varepsilon = 12.0\text{V}$, $C = 5.00\ \mu\text{F}$ and $R = 8.00\times10^5\Omega$, find the time constant of the circuit, the maximum charge on the capacitor, the maximum current in the circuit, and the charge and net current as functions of time.

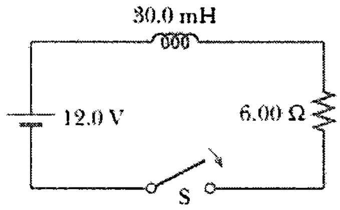
Soln. The time constant of the circuit is $\tau = RC = (8.00\times10^3\ \Omega)(5.00\times10^{-6}\ \text{F}) = 4.00\ \text{s}$. The maximum charge on the capacitor is $Q = C\varepsilon = (5.00\ \mu\text{F})(12.0\ \text{V}) = 60.0\ \mu\text{C}$. The maximum current in the circuit is $I_0 = \varepsilon / R = (12.0\ \text{V})/(8.00\times10^5\Omega) = 15.0\ \mu\text{A}$. Using these values and equation .... and ...., we find that

$$q(t) = (60.0\ \mu\text{C})\left(1 - e^{-t/4.00\ \text{s}}\right)$$

$$I(t) = (15.0\ \mu\text{A})e^{-t/4.00\ \text{s}}$$

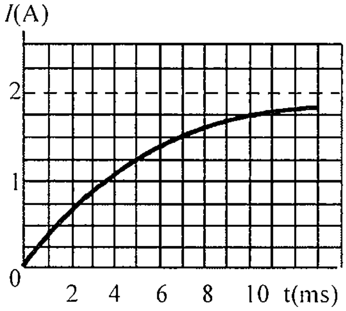


2. The time constant of Fig. 8.9 is closed at $t = 0$
(a) Find the time constant (b) Calculate the current at $t = 200$ ms

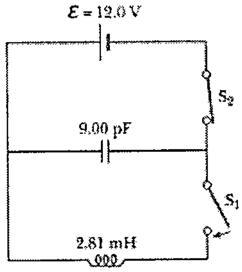


Soln. (a) The time constant is $\tau = \dfrac{L}{R} = \dfrac{30.0\times10^{-3}\text{H}}{6.00\ \Omega} = 5.00\ \text{ms}$

(b) The current is, $I = \dfrac{\varepsilon}{R}\left(1 - e^{-t/\tau}\right) = \dfrac{12.0\ \text{V}}{6.00\ \Omega}\left(1 - e^{-0.400}\right) = 0.659\text{Å}$



(a) Find the frequency of oscillation of the circuit.
(b) What are the maximum values of charge on the capacitor and current in the circuit?
(c) Determine the charge and current as functions of time.



oln. (a) $$f = \frac{\omega}{2\pi} = \frac{1}{2\pi\sqrt{L.C}} = \frac{1}{2\pi\left(2.81\times10^{-3}\ \text{H}\right)\left(9.00\times10^{-12}\ \text{F}\right)^{1/2}} = 1.00\times10^{6}\ \text{Hz}$$

(b) $$Q_{\max} = C\varepsilon = \left(9.00\times10^{-12}\ \text{F}\right)\left(12.0\ \text{V}\right) = 1.08\times10^{-10}\ \text{C}$$

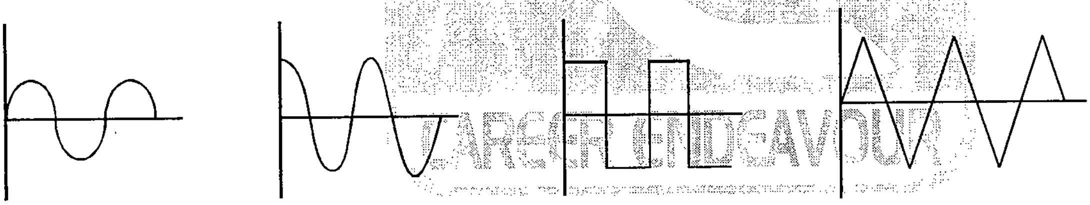
$$I_{\max} = \omega Q_{\max} = 2\pi f\, Q_{\max} = \left(2\pi\times10^{6}\ \text{s}^{-1}\right)\left(1.08\times10^{-10}\ \text{C}\right) = 6.49\times10^{-4}\ \text{A}$$

(c) $$Q = Q_{\max}\cos\omega t = \left(1.08\times10^{-10}\ \text{C}\right)\cos\left[\left(2\pi\times10^{6}\ \text{rad/s}\right)t\right]$$

$$I = -I_{\max}\sin\omega t = \left(-6.79\times10^{-4}\ \text{A}\right)\sin\left[\left(2\pi\times10^{6}\ \text{rad/s}\right)t\right]$$

AC circuit(Alternating current):

If the direction of the current in a resistor or in any other element changes alternatively, the current is called an alternative current (AC).



- Harmonic current is a current on which current is either sine or cosine,

House Hold supply:

$$V = V_0 \sin\omega t$$

$\omega = 2\pi f$, $f = 50$ Hz

$V_0 = 311$ (Peak voltage)

$$V_{avg} = \frac{1}{T}\int_0^T V_0 \sin\omega t\,dt$$

$= 0$ for a full cycle.

$V_{avg} = \frac{2V_0}{\pi}$ for half cycle

R.M.S value:

$$V_{rms} = \sqrt{\frac{1}{T}\int_0^T \left(V_0\sin\omega t\right)^2 dt} = V_0\sqrt{\frac{1}{2T}\times T}$$

$$= \frac{V_0}{\sqrt{2}} = \frac{311}{\sqrt{2}} \approx 220 \text{ V}$$

$$I_{rms} = {}^{I_0}\!/\!_{\sqrt{2}}$$

Pure resistive circuit:

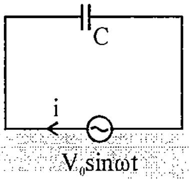
$$iR = V_0 \sin \omega t\,;\ i = i_0 \sin \omega t$$

$\Rightarrow$ Current and voltage are in same phase.

$$H = \int i^2 R dt = R \int i_0^2 \sin^2 \omega t dt = i_0^2 RT \frac{1}{T}\int_0^T \sin^2 \omega\, t\, dt = \frac{i_0^2 RT}{2} = \left(\frac{i_0}{\sqrt{2}}\right)^2 RT$$

$$\boxed{H = i_{rms}^2 RT}$$

2. **Pure capacitive circuit:**



$$Q = CV$$

$$Q = CV_0 \sin \omega t$$

$$\frac{dQ}{dt} = \omega C V_0 \cos \omega t$$

$$i = \frac{V_0}{{}^1\!/\!_{\omega C}} \sin\left(\omega t + {}^{\pi}\!/\!_2\right)$$

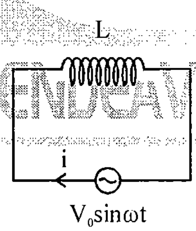
$$\boxed{i = \frac{V_0}{|X_C|} \sin\left(\omega t + {}^{\pi}\!/\!_2\right)}$$

Where $|X_C| = \dfrac{1}{\omega C}$ reactive (capacitive) $\quad X_C = \dfrac{1}{j\omega C}$

Here current leads voltage by ${}^{\pi}\!/\!_2$.

- It is a high pass filter.
- It blocks dc and pass a.c

3. **Pure inductive circuit:**



$$L\frac{di}{dt} = V_0 \sin \omega t$$

$$di = \frac{V_0}{L} \sin \omega\, t\, dt$$

$$i = \frac{V_0}{\omega L} \sin\left(\omega t - {}^{\pi}\!/\!_2\right)$$

$$\boxed{i = \frac{V_0}{|X_L|} \sin\left(\omega t - {}^{\pi}\!/\!_2\right)}$$

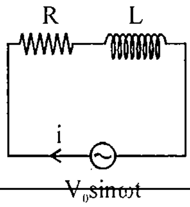
Where $|X_L| = \omega L \rightarrow$ reactance (inductive) $\quad X_L = j\omega L$

$\Rightarrow$ current in lags behind the voltage by ${}^{\pi}\!/\!_2$

- Low pass filter.
- Easy path for d.c. and hard for a.c.

4. **Series:**

L–R circuit:



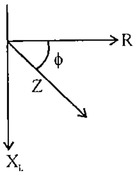
$$V = V_0 \sin \omega t$$

$$\boxed{i = \frac{V_0}{z}\sin(\omega t + \phi)}$$

where $z \rightarrow$ impedance

$$z = \sqrt{R^2 + X_L^2} = \sqrt{R^2 + \omega^2 L^2}$$

$$\phi = \tan^{-1}\left(\frac{\omega L}{R}\right)$$



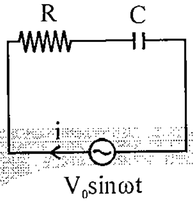
Series C–R circuit:

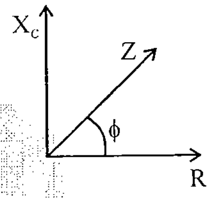
$$V = V_0 \sin\omega t$$

$$i = \frac{V_0}{z}\sin(\omega t + \phi)$$

$$Z = \sqrt{R^2 + X_C^2}$$

$$\tan\phi = \frac{X_C}{R}$$

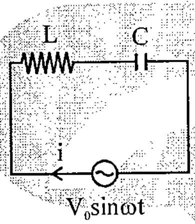




Series LC circuit:

$$V = V_0 \sin\omega t$$

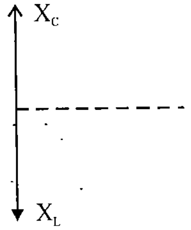
$$i = \frac{V_0}{z}\sin(\omega t + \phi)$$

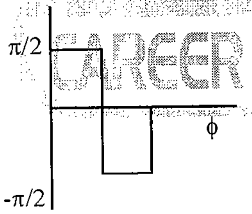


$$Z = |X_C + X_L| = \left|\left(\frac{1}{\omega C} - \omega L\right)\right|$$

$\phi = +\pi/2$ if $X_C > X_L$ (Capacitance circuit)

$\phi = -\pi/2$ if $X_L > X_C$ (Inductive circuit)





$$\left.\begin{array}{l} \text{If } X_L = X_C \\ Z = 0 \Rightarrow i = \infty \end{array}\right\}\left[\text{Resonance condition (Pure resistive circuit)}\right]$$

If $X_L = X_C$ & $V = 0$; $i = \frac{V}{Z} = \frac{0}{0}$ indeterminant

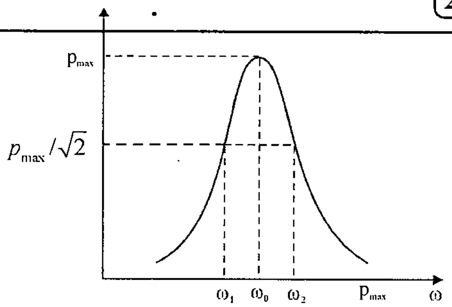
- Here current is due to the oscillation of energy between capacitor and inductor.

7. **Series LCR circuit:**

$$Z = \sqrt{(X_L - X_C)^2 + R^2}\ ; \tan\phi = \frac{X_L - X_C}{R} \qquad X_C = \frac{1}{\omega C}; X_L = \omega L$$

And, $V = \sqrt{V_R^2 (V_C - V_L)^2}$

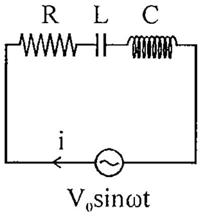
At resonance frequency

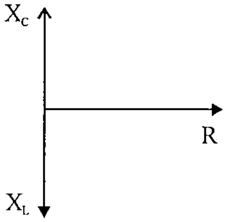


$$|X_L| = |X_C|; \quad \omega_0 L = \frac{1}{\omega_0 C} \;\because\; \omega_0 = \frac{1}{\sqrt{LC}}$$

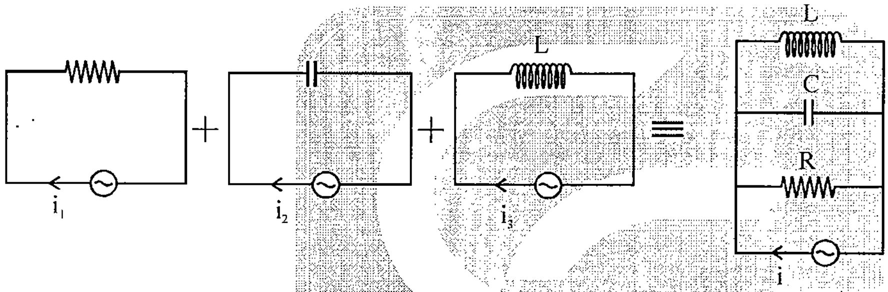
$$\Delta\omega = \frac{\omega_0}{Q}$$

$$Q = \text{Quality factor} = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR} \Rightarrow |QV_R| = |V_L| = |V_C|$$





Parallel LCR circuit:



$$i_1 = \frac{V_0}{R}\sin\omega t$$

$$i_2 = \frac{-V_0}{X_L}\cos\omega t, \qquad i_3 = \frac{V_0}{X_C}\cos\omega t$$

$$i = i_1 + i_2 + i_3$$

Power consumption:

$$P = \frac{1}{T}\int_0^T Vi\,dt = \frac{1}{T}\int_0^T V_0 \sin\omega t\, i_0 \sin(\omega t + \phi)\,dt$$

$$\boxed{P = V_{rms} I_{rms} \cos\phi}$$

Where $\cos\phi \rightarrow$ phase factor equal to $\frac{R}{|Z|}$

- For pure resistive circuit $\phi = 0$
- For pure inductive, $\phi = -\pi/2$; $P = 0$
- For pure capacitive, $\phi = \pi/2$; $P = 0$
- In tubelight, we prefer choke coil over a resistance to control the current because power consumption by induction is minimum.

. A 150 Ω resistor, a 10 μF capacitor and a 0.1 H inductor are connected in series to an A.C. source operating at an angular frequency ω.

(a) Find the value of ω for which the combination acts as a pure resistive load.

(b) The a.c. source is operated at a peak voltage of $300\sqrt{2}$ and a frequency equal to hlaf of the resonance frequency of the circuit. Find the peak value of the current in the ckt and the phase difference between the current and voltage. Also find the peak voltage across the inductor.

oln. At resonant state $X_L = X_C$ (Pure resistive circuit)

Given that $R = 150\ \Omega,\ C = 10\mu F = 10\times10^{-6}\,F$

$$L = 0.1\,H;\ X_L = X_C;\ \omega L = \frac{1}{\omega C};\ \omega^2 = \frac{1}{LC} = \frac{1}{0.1\times10^{-5}} = 10^6;\ \omega = 1000\ \text{rad/sec}$$

(b) $V_0 = 300\sqrt{2}$

Now, $\omega' = \dfrac{\omega}{z} = 500\ \text{rad/sec}$

$$Z = \sqrt{(X_L - X_C)^2 + R^2}$$

$$X_C = \omega' L = 500\times0.1 = 50$$

$$X_\ell = \frac{1}{\omega C} = \frac{1}{500\times10^{-5}} = 200$$

$$\Rightarrow \quad z = \sqrt{(200-50)^2 + (150)^2} = 150\sqrt{2}$$

Peak value of current.

$$i_0 = \frac{V_0}{Z} = \frac{300\sqrt{2}}{150\sqrt{2}} = 2A$$

Phase difference, $\tan\phi = \dfrac{X_C - X_L}{R} = \dfrac{200-50}{150} = 1$

$\Rightarrow \quad \phi = 45°$

$\Rightarrow$ current leads by 45°.

Peak voltage across the inductor.

$X_L = \omega L = 50;\ V_0 = i_0 X_L = 2\times50 = 100$ volt

5. A resistor of 1 kΩ and an inductor of 5 μH are connected in series with a battery of emf. 4V through a switch. The switch is closed at t = 0. In the following, you may use $e^3 = 20$

(a) Find the current flowing in the circuit at t = 15 μsec.

(b) Find the heat dissipated through the resistor during the first 15 μs. **[IIT-JAM 2006]**

Soln. Given that $R = 1\ k\Omega = 10^3\Omega$

$L = 5\ mH = 5\times10^{-3} H$

emf = 4 V

$t = 15\mu\,\text{sec} = 15\times10^{-6}$

(a) Growth of current i through the circuit is given by

$$i = i_0\left(1 - e^{-t/\tau}\right)$$

where $i_0 = \dfrac{\varepsilon mf}{R} = \dfrac{V}{R}$

and $\lambda = \frac{L}{R}$ = time constant of the circuit

Here, $i_0 = \frac{4}{10^3} = 4\times10^{-3}\,A$; $\lambda = \frac{L}{R} = \frac{5\times10^{-3}}{10^3} = 5\times10^{-6}$

$$\Rightarrow \quad i = 4\times10^{-3}\left(1-e^{-15\times10^{-6}/5\times10^{-6}}\right) = 4\times10^{-3}\left(1-\frac{1}{20}\right) = \frac{4\times19}{20}\times10^{-3} = 3.8\times10^{-3} \text{ amp.}$$

(b) Heat dissipated to the resistor in time t

$$H = i^2Rt = \left(3.8\times10^{-3}\right)^2\times1\times10^3\times15\times10^{-6} = 216.6\times10^{-9} = 2.166\times10^{-7}\ J$$

6. Consider a single loop circuit with a resistor $(R = 160\ \Omega)$, a capacitor $(C = 10\ \mu F)$, an inductor $(L = 200\ mH)$ and an alternating source with $E_{max} = 36V$. If the circuit in driven at frequency f = 50 Hz.

(a) Find the impedance z for the circuit.

(b) Find the amplitude of the current I and phase constant ϕ **[JNU - 2006]**

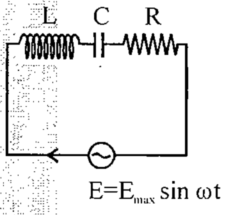
Soln. Given $R = 160\ \Omega, \quad C = 10\ \mu F = 10^{-5} F$

$L = 200\ mH, = 0.2H$

$f = 50\ Hz,\ E_{max} = 36V$

$$X_L = \omega L = 2\pi f L = 2\times3.14\times50\times0.2 = 62.8\ \Omega$$

$$X_C = \frac{1}{\omega C} = \frac{1}{2\pi fC} = \frac{1}{2\times3.14\times50\times10^{-5}} = 318.47\Omega$$



(a) Impedance $z = \sqrt{\left(X_C - X_L\right)^2 + R^2}$

$$= \sqrt{(318.47 - 62.80)^2 + 160^2}$$

$$= \sqrt{65367.1489 + 25600} = \sqrt{90967.15}\ \Omega$$

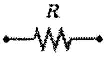
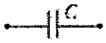
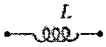
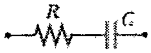
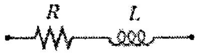
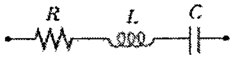
(b) Current in circuit (I_{rms}).

$$I_0 = \frac{E_{max}}{Z} = 36\ ;\ I_{rms} = \frac{I_0}{\sqrt{2}}$$

Phase, $\tan\phi = \frac{X_C - X_L}{R} = \frac{318.47 - 62.8}{160}$

SUMMARY TABLE

TABLE **Impedance Values and Phase Angles for Various Circuit-Element Combinations**[a]

Circuit Elements	Impedance Z	Phase Angle ϕ
	R	0°
	X_C	$-90°$
	X_L	$+90°$
	$\sqrt{R^2+X_C^2}$	Negative, between $-90°$ and 0°
	$\sqrt{R^2+X_L^2}$	Positive, between 0° and 90°
	$\sqrt{R^2+(X_L-X_C)^2}$	Negative if $X_C > X_L$ Positive if $X_C < X_L$

PRACTICE SET

1. In an ac circuit the current and the emf are $i = 5\sin(100\pi t - \pi/2)A$ and $v = 200\sin(100\pi t)V$. Then the power dissipation in the circuit is
(a) 1000W (b) 40W (c) 500W (d) zero

2. In a pure inductive circuit impressed with an ac emf ε current
(a) lags behind ε by 90° (b) is in phase with ε
(c) leads over ε by 90° (d) is 180° out of phase with ε

3. A capacitor acts as an open circuit for
(a) ac (b) both ac and dc (c) dc (d) none of these

4. The average value of the ac votlage $v = V_0 \sin\omega t$ over a complete period is
(a) $\frac{2V_0}{\pi}$ (b) $\frac{V_0}{2}$ (c) $\frac{V_0}{\pi}$ (d) zero

5. A pure LC parallel circuit is impressed with an ac source of frequency ω. If $\omega > \frac{1}{\sqrt{LC}}$, then the circuit reactance is
(a) inductive (b) resistive (c) capacitive (d) zero

ANSWER KEY					
Questions	1	2	3	4	5
Option	(d)	(a)	(c)	(d)	(c)

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Kinetic Theory of Gases & Thermodynamics

for

M.Sc. Entrance

IIT-JAM | JNU | TIFR | JEST | BHU | ISM | JNCASR | DU

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Chapter 1

Kinetic Theory of Gases

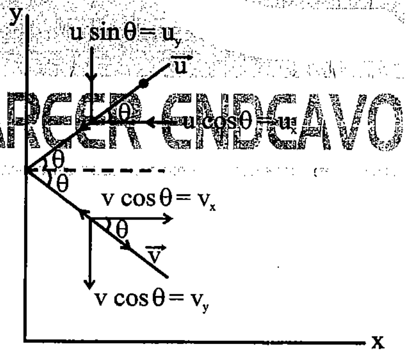
Postulates of Kinetic Theory

Suppose a container with volume V contains a very large number N of identical molecules, each with mass m. Then, the kinetic theory postulates the following

(1) The molecules behave as point particles that are small compared to the size of the container and to the average distance between molecules.
(2) The molecules are in constant motion. Each molecule collides occasionally with a wall of the container. These collisions are perfectly elastic.
(3) There is no force of attraction among the gas molecules and between the gas molecules and walls of the container.
(4) The container walls are rigid and infinitely massive and do not move.
(5) The molecules obey Newton's law of motion.
(6) The time of collision is much less than the averaged time between two successive collision.

Meaning of elastic collision

If a molecule with velocity $\vec{u}$ strikes a wall align along y-axis, then the collision is *elastic* if



- Velocity component parallel to wall (*y*-component) does not change, i.e., $v_y = u_y$.
- Velocity component perpendicular to wall (*x*-component) reverses direction, i.e., $v_x = -u_x$.
- Speed v does not change, i.e., $|\vec{v}| = |\vec{u}|$.

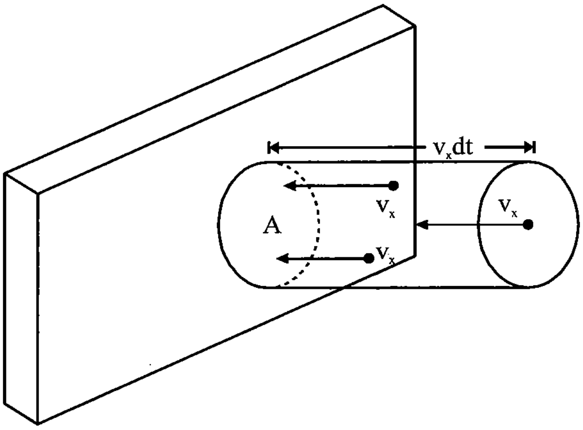
Note : We will follow these notations throughout the book.

N = Number of molecules in the gas
n = Number of moles in the gas
N_A = Avagadro number

where $\boxed{N = nN_A}$

Pressure Exerted by a Gas

During collisions, the molecules exert forces on the walls of the container, this is the origin of the pressure that the gas exerts on the wall of the container.



Steps to calculate the pressure exerted by gas molecules on the walls of the container

1. Number of collisions that occur per unit time for a certain area A of the wall.
2. Total momentum associated with these collisions
3. Force needed to cause this momentum change.

Assuming that all molecules in the gas have the same magnitude of x-velocity, $|v_x|$.

For each collision the x-component of velocity changes from $-|v_x|$ to $|v_x|$

So, the change in the momentum is $\Delta p_x = m|v_x| - (-m|v_x|) = 2m|v_x|$ ... (1)

If a molecule is going to collide with a given wall area A during a small time interval dt, then at the beginning of time dt if must be within a distance $|v_x|dt$ from the wall and it must be headed toward the wall.

⇒ Total number of molecules that collide with area A during time interval dt ≡ Number of molecules with base area A and length $|v_x|dt$ and leading towards the wall.

where volume of cylinder = $A|v_x|dt$

Assuming uniform distribution of molecules, number of molecules in this cylinder = $\left(\frac{N}{V}\right)\left(A|v_x|dt\right)$

On the average, half of these molecules are moving towards the wall (that causes pressure) and half are moving away from it.

⇒ Number of collision with A during dt = $\frac{1}{2}\left(\frac{N}{V}\right)\left(A|v_x|dt\right)$ ... (2)

Due to all molecules in the gas, the total momentum change during dt is

$$dp_x = (\text{No. of collisions}) \times 2m|v_x|$$

$$= \frac{1}{2}\left(\frac{N}{V}\right)\left(A|v_x|dt\right) \times 2m|v_x|$$

$$= \frac{NAm}{V} v_x^2\, dt \qquad \left[\because\ |v_x|^2 = v_x^2\right]$$

$$\Rightarrow \quad \frac{dp_x}{dt} = \frac{NAm\,v_x^2}{V} \qquad \ldots (3)$$

According to Newton's second law, this rate of change of momentum equals the force exerted by the wall area A on the gas molecules. From Newton's III law, this is equal and opposite to the force exerted on the wall by the molecules.

So pressure P is the magnitude of the force exerted on the wall per unit area is

$$P = \frac{F}{A} = A\frac{dp_x}{dt} = \left(\frac{N}{V}\right) m v_x^2 \qquad \ldots (4)$$

⇒ Pressure exerted by the gas depends on the number of molecules per unit volume (N/V), the mass m per molecule, and the speed of the molecules.

Note: $|v_x|$ is really not the same for all the molecules. But we could have sorted the molecules into groups having the same $|v_x|$ with each group, where net effect of the molecules is (v_x^2)avg.

Also, speed v of 'a' molecules is $v^2 = v_x^2 + v_y^2 + v_z^2$

For all molecules.

$$\left(v^2\right)_{avg} = \left(v_x^2\right)_{avg} + \left(v_y^2\right)_{avg} + \left(v_z^2\right)_{avg} \qquad \ldots (5)$$

Since there is no real difference in our model between $x-$, $y-$ and $z-$direction,

$$\langle v_x^2 \rangle_{avg} = \left(v_y^2\right)_{avg} = \left(v_z^2\right)_{avg}$$

$$\therefore \quad \langle v^2 \rangle_{avg} = 3\langle v_x^2 \rangle_{avg}$$

$$\Rightarrow \quad \left(v_x^2\right)_{avg} = \frac{1}{3}\left(v^2\right)_{avg} \qquad \ldots (6)$$

Using (4), we have

$$P = \frac{1}{3}\frac{N}{V} m\left(v^2\right)_{avg}$$

$$\Rightarrow \quad PV = \frac{1}{3}Nm\left(v^2\right)_{avg} \quad \left[\because \left(v^2\right)_{avg} = \left(\overline{v}^2\right) = \langle v^2 \rangle\right] \qquad \ldots (7)$$

Now the average translational Kinetic energy (KE) of a single molecule $= \frac{1}{2}m\left(v^2\right)_{avg}$.

So the total random kinetic energy K_{tr} of translational motion of all the molecules is

$$K_{tr} = \frac{N}{2}m\left(\mathrm{v}^2\right)_{avg} \qquad \ldots (8)$$

(There may be other energies also associated with rotation and vibration of molecules but we are considering only translational energy)

$$\text{Equations (7) and (8)} \Rightarrow \boxed{PV = \frac{2}{3}K_{tr.}} \qquad \ldots (9)$$

Comparing will ideal gas equation

$$PV = nRT$$

which is based on experimental studies of gas behavior, we have

$$\boxed{K_{tr} = \frac{3}{2}nRT}\ ;\ T = \text{ absolute temperature} \qquad \ldots (10)$$

This is average translational kinetic energy of 'n' moles of ideal gas which clearly shows that translational kinetic energy depends only on the temperature of the gas.

Now, average translational kinetic energy of 'a' single molecule

$$= \frac{K_{tr.}}{N} = \frac{3\,nRT}{2\,N} \qquad \ldots (11)$$

$$\text{where } n = nN_A \qquad \ldots (12)$$

$$\text{So, } \frac{K_{tr}}{N} = \frac{3}{2}\left(\frac{R}{N_A}\right)T \qquad \ldots (13)$$

The ratio $\frac{R}{N_A}$ occurs requently in molecular theory. It is called Boltzmann's constant k_B.

$$\Rightarrow \quad k_B = \frac{R}{N_A} = \frac{8.314 \text{ J/mol}\cdot\text{K}}{6.022\times10^{23} \text{ molecules/mol}} = 1.381\times10^{-23} \text{ J/molecules}\cdot\text{K}$$

So, average kinetic translational energy of a molecule $= \frac{3}{2}k_BT$

Molecular Speeds

The average translational kinetic energy is equals to $\frac{3}{2}k_BT$. Also, from the basic definition of kinetic energy, the average translational kinetic energy must also be equals to $\frac{1}{2}m\left(\mathrm{v}^2\right)_{avg.}$ So,

$$\frac{1}{2}m\left(\mathrm{v}^2\right)_{avg.} = \frac{3}{2}k_BT$$

where square-root of $\left(v^2\right)_{avg}$ = Root-mean square speed (or rms speed) v_{rms}

$$\therefore \quad v_{rms} = \sqrt{\left(v^2\right)_{avg}} = \sqrt{\frac{3k_BT}{m}} = \sqrt{\frac{3RT}{M}} \qquad \left[\because k_B = \frac{R}{N_A}\right]$$

where, M is the molar mass of the gas and is equal to mN_A.

Problem: Why there is hardly any hydrogen gas (H_2) in the earth's atmosphere, despite its being the most common element in the universe ?

Soln. Molar mass of hydrogen gas, $M = 2\times10^{-3}$ kg

$$\therefore \quad v_{\text{r.m.s.}} = \sqrt{\frac{3RT}{M}} = \sqrt{\frac{3\times8.314\times300}{2\times10^{-3}}} \qquad [\because T = 300 \text{ K (room temperature)}]$$

$$= 1.934 \text{ km/sec}$$

A sizeable fraction of H_2 molecule in the atmosphere would have speeds greater than the earth's escape of 11.2 km/sec.

On the other hand, the heavier, slowly moving gases, can't escape so easily which is why they predominate in our atmosphere.

Problem: At room temperature, the rms speed of the molecules of a certain diatomic gas is found to be 1930 m/s. The gas is

(a) H_2 (b) F_2 (c) O_2 (d) Cl_2

[IIT-JEE : 1984]

Soln. $v_{\text{rms}} = \sqrt{\dfrac{3RT}{M}}$

Room temperature $T \approx 300\,K$

$$\therefore \quad 1930 = \sqrt{\frac{3 \times 8.31 \times 10^3 \times 300}{M}}$$

$\therefore \quad M = 2.0$ g/mol or the gas is H_2.

Correct option is (a)

Problem: What is the average translational kinetic energy of a molecule in 1 mole of a gas at 27°C ?

(a) 6.21×10^{-21} J (b) 6.21×10^{-22}J (c) 6.21×10^{-23}J (d) 6.21×10^{-20}J

Soln. Given : $T = 27\,^\circ\text{C} = 300$ K

Avgerage K_{tr} per molecule $= \dfrac{3}{2} k_B T = \dfrac{3}{2} \times 1.381 \times 10^{-23} \times 300 = 6.21 \times 10^{-21}$ J

Correct option is (a)

Problem: Five gas molecules chosen at random are found to have speeds of 500, 600, 700, 800 and 900 m/s? What is the rms speed and average speed?

(a) v_{rms}=700 m/s and v_{av} = 714 m/s (b) v_{rms}=71.4 m/s and v_{av} = 70.4 m/s

(c) v_{rms}=714 m/s and v_{av} = 700 m/s (d) v_{rms}=70.4 m/s and v_{av} = 71.4 m/s

Soln. Given : $v_1 = 500$ m/s, $v_2 = 600$ m/s, $v_3 = 700$ m/s, $v_4 = 800$ m/s, $v_5 = 900$ m/s

$$v_{\text{avg.}} = \frac{\sum_{i=1}^{5} v_i}{5} = \frac{500 + 600 + 700 + 800 + 900}{5} = 700 \text{ m/s}$$

$$\text{and } v_{\text{r.m.s.}} = \left(\frac{\sum_{i=1}^{5} v_i^2}{5} \right)^{1/2} = \left(\frac{500^2 + 600^2 + 700^2 + 800^2 + 900^2}{5} \right)^{1/2} = 714.14 \text{ m/s}$$

Correct option is (c)

Problem: Assuming that the hydrogen molecules have a root-mean-square speed of 1270 m/s at 300 K, calculate the rms at 600 K.

(a) 179.6 m/s (b) 1796 m/s (c) 2796 m/s (d) 279.6 m/s

Soln. Given : $(v_{\text{r.m.s.}})_1 = 1270$ m/s, $T_1 = 300$ K and $(v_{\text{r.m.s.}})_2 = ?$, $T_2 = 600$ K

$$\text{Since, } (v_{\text{r.m.s.}}) = \sqrt{\frac{3 k_B T}{m}} \Rightarrow \frac{(v_{\text{r.m.s.}})_1}{(v_{\text{r.m.s.}})_2} = \sqrt{\frac{T_1}{T_2}}$$

$$\Rightarrow (v_{\text{r.m.s.}})_2 = \sqrt{\frac{T_2}{T_1}}\,(v_{\text{r.m.s.}})_1 = \sqrt{2} \times 1270 = 1796 \text{ m/s}.$$

Correct option is (b)

Problem: One gram mole of oxygen at 27°C and one atmospheric pressure is enclosed in a vessel.

(a) Assuming the molecules to be moving with v_{rms}, find the number of collisions per second which the molecules make with one sqaure metre area of the vessel wall.

(b) The vessel is next thermally insulated and moved with a constant speed v_0. It is then suddenly stopped. The process results in a rise of the temperature of the gas by 1°C. Calculate the speed v_0. **[IIT-JEE : 1983]**

Soln. **(a)** $v_{\text{rms}} = \sqrt{\dfrac{3RT}{M}} = \sqrt{\dfrac{3 \times 8.31 \times 300}{32 \times 10^{-3}}} = 483.4 \text{ m/s}$

Given, $P_0 = 1.01 \times 10^5$ N/m² = Force per unit area. Let n molecules of oxygen strike the wall per second per m² and recoil with same speed.
Change in momentum is $(2nmv_{\text{rms}})$.
The change in momentum per unit time is the force.

Hence, $P_0 = 2nmv_{\text{rms}}$

$$\therefore \quad n = \frac{P_0}{2mv_{\text{rms}}} = \frac{1.01 \times 10^5}{2\left[\dfrac{32}{6.02 \times 10^{26}}\right](483.4)} = 1.96 \times 10^{27}\text{/s}$$

(b) $\dfrac{1}{2}(m_{\text{gas}})v_0^2 = nC_V\Delta T$

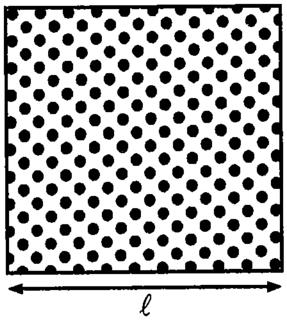
$$\therefore \quad v_0 = \sqrt{\frac{2nC_V\Delta T}{m_{\text{gas}}}} = \sqrt{\frac{(2)(n)\left(\dfrac{5}{2} \times 8.31\right)(1)}{(n)(32 \times 10^{-3})}} = 36 \text{ m/s}$$

Problem: A cubical box of side 1 m contains helium gas (atomic weight 4) at a pressure of 100 N/m². During an observation time of 1 s, an atom travelling with the root mean square speed parallel to one of the edges of the cube, was found to make 500 hits with a particular wall, without any collision with other atoms. Take, $R = \dfrac{25}{3}$ J/mol-K and $k_B = 1.38 \times 10^{-23}$ J/K. **[IIT-JEE : 2002]**

(a) Evaluate the temperature of the gas.
(b) Evaluate the average kinetic energy per atom.
(c) Evaluate the total mass of helium gas in the box.

Soln. Volume of the box = 1 m³
Pressure of the gas = 100 N/m²
Let T be the temperature of the gas. Then,

(a) Time between two consecutive collisions with one wall $= \dfrac{1}{500}$ s. This time should be equal to $\dfrac{2l}{v_{\text{rms}}}$, where l is the side of the cube.



$$\Rightarrow \quad \frac{2l}{v_{\text{rms}}} = \frac{1}{500}$$

$$\Rightarrow \quad v_{\text{rms}} = 1000 \text{ m/s} \qquad (\text{as } l = 1 \text{ m})$$

$$\Rightarrow \quad \sqrt{\frac{3RT}{M}} = 1000$$

$$\therefore \quad T = \frac{(1000)^2 M}{3R} = \frac{(10)^6 (4\times10^{-3})}{3(25/3)} = 160 \text{ K}$$

(b) Average kinetic energy per atom $= \frac{3}{2}k_B T = \frac{3}{2}(1.38\times10^{-23})(160) \text{ J} = 3.312 \times 10^{-21} \text{ J}$

(c) From $PV = nRT = \frac{m}{M}RT$

We get mass of helium gas in the box, $m = \frac{PVM}{RT}$

Substituting the values, we get

$$m = \frac{(100)(1)(4)}{(25/3)(160)} = 0.3 \text{ g}$$

Molecular Speeds Distribution

The molecules in a gas don't have the same speed. We will study a function $f(v)$ that will help us to know how a particular number of molecules possess a certain amount of speed. This function is called distribution function.

If we observe total N molecules, the number dN having speeds in the range between v and $v + dv$ is given by

$$dN = N f(v) dv \qquad \dots (17)$$

Probability that a random chosen molecule will have a speed in the interval v to $v + dv$ is

$$f(v) dv \qquad \dots (18)$$

Hence $f(v)$ is the probability per unit speed interval.

It is not equal to the probability that a molecule has speed exactly equals to v.

Since probability is pure number, $f(v)$ has units of reciprocal speed (s/m).

The Maxwell-Boltzmann Distribution

The function $f(v)$ describing the actual distribution of molecular speeds is called the *Maxwell-Boltzmann distribution.*

Deriving from the statistical mechanics. it is equal to

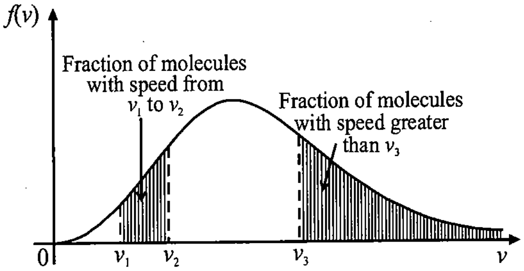
$$\boxed{f(v) = 4\pi\left(\frac{m}{2\pi k_B T}\right)^{3/2} v^2 e^{-\frac{mv^2}{2k_B T}}}, \quad 0 \le v < \infty \qquad \dots (19)$$

where m is the mass of a molecule

v is the velocity, different for different group of molecules.

T is the absolute temperature.

The graph representing Maxwell-Boltzmann distribution is



Each molecule must have some value of v, so the integral of $f(v)$ over all v must be unity for any T i.e.

$$\int_0^\infty f(\mathrm{v})\,d\mathrm{v} = 1.$$

At each temperature, the height of the curve for any value v is proportional to the number of molecules with speeds near v.

The various speeds possess by molecules:

(1) The average speed (v_{av})

It is the average speed possesses by the molecules of the gas.

It is given by

$$v_{av} = \int_0^\infty v f(v)\,dv = 4\pi\left(\frac{m}{2\pi k_B T}\right)^{3/2}\int_0^\infty v^3 e^{-\frac{mv^2}{2k_BT}}\,dv$$

Using $\int_0^\infty x^n e^{-ax^2}dx = \dfrac{\Gamma\left(\frac{n+1}{2}\right)}{2a^{\frac{n+1}{2}}}$, we have ... (20)

$$v_{av} = 4\pi\left(\frac{m}{2\pi k_B T}\right)^{3/2}\frac{\Gamma 2}{2\left(\frac{m}{2k_BT}\right)^2} = \frac{4\pi}{\pi^{3/2}}\left(\frac{m}{2k_BT}\right)^{-\frac{1}{2}}\frac{1}{2} = \sqrt{\frac{8k_BT}{\pi m}} \quad ...(21)$$

$$\boxed{v_{av} = \sqrt{\frac{8k_BT}{\pi m}}}$$

where $\Gamma n = (n-1)!$ if n is an integer.

(2) Most probable speed (v_{mp})

It is the speed possesses by the maximum number of molecules. For this speed, the value of $f(v)$ is maximum. So using (19) and concept of maxima and minima, we have $\frac{d}{dv}f(v) = 0$

$$\Rightarrow 4\pi\left(\frac{m}{2\pi k_B T}\right)^{3/2}\left[2ve^{-\frac{mv^2}{2k_BT}} - \frac{2mv^3}{2k_BT}e^{-\frac{mv^2}{2k_BT}}\right] = 0$$

$$\Rightarrow 4\pi\left(\frac{m}{2\pi k_B T}\right)^{3/2} 2ve^{-\frac{mv^2}{2k_BT}}\left[1 - \frac{mv^2}{2k_BT}\right] = 0$$. This gives

Either $v = 0$

or, $e^{-\frac{mv^2}{2k_BT}} = 0 \Rightarrow v = \infty$

or, $1 - \frac{mv^2}{2k_BT} = 0 \Rightarrow v = \sqrt{\frac{2k_BT}{m}}$

The speeds of the molecule can neither be zero nor infinite (as per as the assumption of the kinetic theory as the molecules are in constant motion with finite speed).

So $$\boxed{v_{mp} = \sqrt{\frac{2k_BT}{m}}} \qquad \text{... (22)}$$

(3) The rms speed (v_{rms})

It is the speed observed during the experiments and is given by

$$v_{rms} = \sqrt{\overline{v^2}} \qquad \text{... (23)}$$

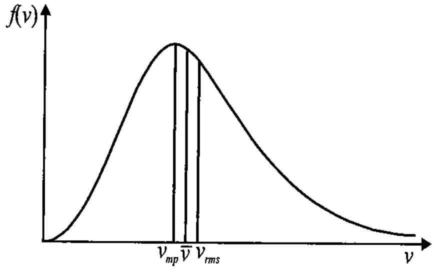
Now $$\overline{v^2} = \int_0^\infty v^2 f(v)\,dv = 4\pi\left(\frac{m}{2\pi k_BT}\right)^{3/2} \int_0^\infty v^4 e^{-\frac{mv^2}{2k_BT}}\,dv$$

Using $$\int_0^\infty x^n e^{-ax^2}\,dx = \frac{\Gamma\left(\frac{n+1}{2}\right)}{2a^{\frac{n+1}{2}}}$$

$$\therefore \quad \overline{v^2} = 4\pi\left(\frac{m}{2\pi k_BT}\right)^{3/2} \frac{\Gamma\left(5/2\right)}{2\left(\frac{m}{2k_BT}\right)^{5/2}} = \frac{3k_BT}{m}$$

So $$\boxed{v_{rms} = \sqrt{\overline{v^2}} = \sqrt{\frac{3k_BT}{m}}} \qquad \text{... (24)}$$

The graph representing v_{rms}, v_{avg}, v_{mp} is



Problem: A quantity of gas containing N molecules has a speed distribution function $f(v)$. How many molecules have speeds between v_1 and $v_2 > v_1$?

(a) $\int_0^{v_2} f(v)\,dv - \int_0^{v_1} f(v)\,dv$ (b) $N\left[\int_0^{v_2} f(v)\,dv - \int_0^{v_1} f(v)\,dv\right]$

(c) $\int_0^{v_1} f(v)\,dv - \int_0^{v_2} f(v)\,dv$ (d) $n\left[\int_0^{v_1} f(v)\,dv - \int_0^{v_2} f(v)\,dv\right]$

Soln. Required number $= \int_{v_1}^{v_2} Nf(v)\,dv = N\left[\int_0^{v_2} f(v)\,dv - \int_0^{v_1} f(v)\,dv\right]$

Correct option is (b)

Problem: Which of the inequality is valid for molecular distribution of gas?

(a) $v_{rms} > \overline{v} > v_{m.p.}$ (b) $v_{rms} > v_{m.p.} > \overline{v}$ (c) $v_{m.p.} > \overline{v}_{rms} > \overline{v}$ (d) $v_{m.p.} > \overline{v} > v_{rms}$

Soln. We know that, $v_{\text{r.m.s.}} = \sqrt{\dfrac{3RT}{M}}$; $v_{\text{m.p.}} = \sqrt{\dfrac{2RT}{M}}$; $v_{\text{avg.}} = \overline{v} = \sqrt{\dfrac{8RT}{\pi M}}$

Clearly, $v_{\text{r.m.s.}} : v_{\text{m.p.}} : \overline{v} = \sqrt{3} : \sqrt{2} : \sqrt{\dfrac{8}{\pi}}$

$$= 1.732 : 1.414 : 1.596$$

$\Rightarrow v_{\text{r.m.s.}} : \overline{v} : v_{\text{m.p.}} = 1.732 : 1.596 : 1.414$

So, $v_{\text{r.m.s.}} > \overline{v} > v_{\text{m.p.}}$

Correct option is (a)

Another forms of Maxwellian distribution

It is given by

$$f(v)\,dv = 4\pi\left(\frac{m}{2\pi k_B T}\right)^{3/2} v^2 e^{-\frac{mv^2}{2k_B T}}\,dv \qquad \text{... (25)}$$

(1) Energy form

Since $\varepsilon = \dfrac{1}{2}mv^2 \Rightarrow v^2 = \dfrac{2\varepsilon}{m}$

Using this in equation (25),

$$f(\varepsilon)\,d\varepsilon = 4\pi\left(\frac{m}{2\pi k_B T}\right)^{3/2}\left(\frac{2\varepsilon}{m}\right)e^{-\varepsilon/k_B T}\left(\frac{d\varepsilon}{\sqrt{2\varepsilon m}}\right)$$

$$= 4\sqrt{2}\,\pi\left(\frac{1}{2\pi k_B T}\right)^{3/2}\sqrt{\varepsilon}\,e^{-\varepsilon/k_B T}\,d\varepsilon \qquad \text{... (26)}$$

(2) Momentum form

Now, $p = mv$ or $v = \dfrac{p}{m}$

Using this in (25),

$$f(p)\,dp = 4\pi\left(\frac{m}{2\pi k_B T}\right)^{3/2}\frac{p^2}{m^3}e^{-\frac{1}{2k_B T}\frac{p^2}{m}}\,dp \qquad \text{... (27)}$$

Maxwellian speed distribution in different dimensions

(i) In three dimension : $f(v)=4\pi\left(\frac{m}{2\pi k_BT}\right)^{3/2} v^2 e^{-\frac{mv^2}{2k_BT}}$

(ii) In two dimension : $f(v)=2\pi\left(\frac{m}{2\pi k_BT}\right)^{2/2} v e^{-\frac{mv^2}{2k_BT}}$

(iii) In one dimension : $f(v)=\left(\frac{m}{2\pi k_BT}\right)^{1/2} e^{-\frac{mv^2}{2k_BT}}$

Problem : Three closed vessels A, B and C at the same temperature T and contain gases which obey the Maxwellian distribution of velocities. Vessels A contains only O_2, B only N_2 and C a mixture of equal quantities of O_2 and N_2. If the average speed of the O_2 molecules in vessel A is v_1, that of the N_2 molecules in vessel B is v_2, the average speed of the O_2 molecules in vessel C is **[IIT-JEE : 1992]**

(a) $(v_1+v_2)/2$ (b) v_1 (c) $(v_1 v_2)^{1/2}$ (d) $\sqrt{3k_BT/M}$

Soln. The average speed of molecules of an ideal gas is given by

$\langle v\rangle=\sqrt{\frac{8RT}{\pi M}}$ *i.e.*, $\langle v\rangle\propto\sqrt{T}$ for same gas.

Since temperature of A and C are same, average speed of O_2 molecules will be equal in A and C i.e., v_1.

So, the correct option is (b)

Problem: The temperature of an ideal gas is increased from 120 K to 480 K. If at 120 K at the root mean sqaure velocity of the gas molecules is v, at 480 K it becomes **[IIT-JEE : 1996]**

(a) 4 v (b) 2 v (c) v/2 (d) v/4

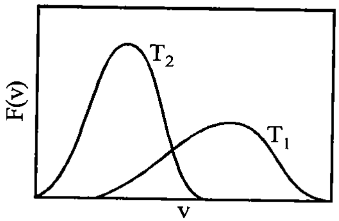
Soln. $v_{rms}=\sqrt{\frac{3RT}{M}}$ i.e., $v_{rms}\propto\sqrt{T}$

When temperature is increased from 120 K to 480 K (i.e., four times), the root mean square speed will becomes $\sqrt{4}$ or 2 times i.e., 2v.

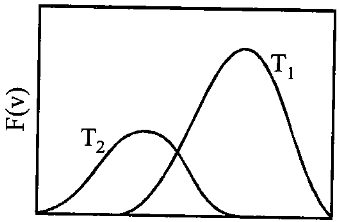
So, the correct option is (b)

Problem: For temperature $T_1 > T_2$, the qualitative temperature dependence of the probability distribution $F(v)$ of the speed v of a molecule in three dimensions is correctly represented by the following figure:

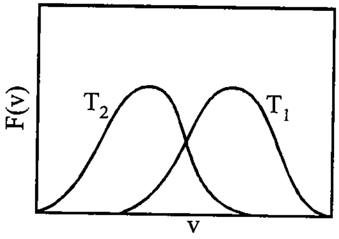
(a)



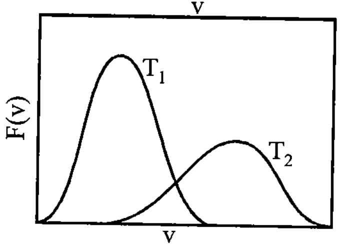
(b)



(c)



(d)

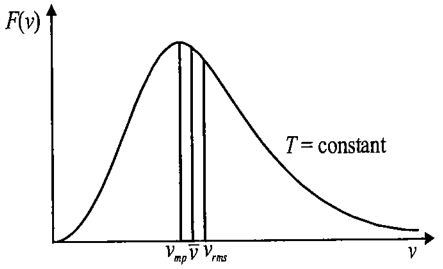


Soln. For N-particles system in thermal equilibrium, the number of molecules having speed between v to $v + dv$ is given by

$$F(\mathrm{v})d\mathrm{v} = 4\pi N\left(\frac{m}{2\pi k_B T}\right)^{3/2} e^{-\frac{m\mathrm{v}^2}{2kT}}\mathrm{v}^2 d\mathrm{v}$$

$$\Rightarrow F(\mathrm{v}) = 4\pi N\left(\frac{m}{2\pi k_B T}\right)^{3/2} e^{-\frac{m\mathrm{v}^2}{2kT}}\mathrm{v}^2 \text{ and } \int_0^\infty F(\mathrm{v})d\mathrm{v} = N$$

$$\text{and } \bar{\mathrm{v}} = \sqrt{\frac{8k_BT}{\pi m}}, \mathrm{v}_p = \sqrt{\frac{2k_BT}{m}}, \mathrm{v}_{rms} = \sqrt{\frac{3k_BT}{m}}$$



If we increase temperature, then $\bar{\mathrm{v}}$, $\bar{\mathrm{v}}_p$ and v_{rms} all will increase and particles have more values of velocities. More particles will distribute in large velocity region but total number of particles in the system will be constant (N). In $F(\mathrm{v})$ Vs. v plot area under the curve remain constant. So on increasing the temperature, peak will decrease and shifted towards higher velocity side.

Correct answer is (a)

Problem: Let $\bar{\mathrm{v}}$, $\bar{\mathrm{v}}_p$ and v_{rms} respectively denote the mean speed, root mean square speed and most probable speed of the molecules in an ideal monoatomic gas at absolute temperature T. The mass of a molecule is m. Then,

(a) no molecule can have a speed greater than $\sqrt{2}\,\mathrm{v}_{\mathrm{rms}}$

(b) no molecule can have a speed less than $\mathrm{v}_p/\sqrt{2}$

(c) $\mathrm{v}_p < \bar{\mathrm{v}} < \mathrm{v}_{\mathrm{rms}}$

(d) the average kinetic energy of a molecule is $\frac{3}{4}m\mathrm{v}_p^2$

[IIT-JEE : 1998]

Soln. $\mathrm{v}_{\mathrm{rms}} = \sqrt{\frac{3RT}{M}}, \bar{\mathrm{v}} = \sqrt{\frac{8}{\pi}\cdot\frac{RT}{M}} = \sqrt{\frac{2.5RT}{M}}$ and $\mathrm{v}_p = \sqrt{\frac{2RT}{M}}$

From these expressions we can see that, $\mathrm{v}_p < \bar{\mathrm{v}} < \mathrm{v}_{\mathrm{rms}}$

Secondly, $\mathrm{v}_{\mathrm{rms}} = \sqrt{\frac{3}{2}}\,\mathrm{v}_p$

and average kinetic energy of a gas molecule

$$= \frac{1}{2}m\mathrm{v}_{\mathrm{rms}}^2 = \frac{1}{2}m\left(\sqrt{\frac{3}{2}}\,\mathrm{v}_p\right)^2 = \frac{3}{4}m\mathrm{v}_p^2.$$

Hence, correct options are (c, d)

Problem: Consider a Maxwellian distribution of the velocity of the molecules of an ideal gas. Let v_{mp} and v_{rms} denote the most probable velocity and the root mean square velocity, respectively. The magnitude of the ratio $\mathrm{v}_{mp}/\mathrm{v}_{rms}$ is:

(a) 1 (b) $2/3$ (c) $\sqrt{2/3}$ (d) $3/2$

Soln. $f(\text{v}) \propto \text{v}^2 \exp\left(-\dfrac{m\text{v}^2}{2k_BT}\right) \Rightarrow \dfrac{df}{d\text{v}} = 0 \Rightarrow \text{v}_{mp} = \sqrt{\dfrac{2k_BT}{m}}$

and $\quad \text{v}_{rms} = \left[\int_0^\infty \text{v}^2 f(\text{v})\,d\text{v}\right]^{1/2} = \sqrt{\dfrac{3k_BT}{m}}$

$\therefore \quad \dfrac{\text{v}_{mp}}{\text{v}_{rms}} = \sqrt{\dfrac{2}{3}}$

Correct answer is (c).

Problem: A gas of molecules each having mass m is in the thermal equilibrium at a temperature T. Let $\text{v}_x, \text{v}_y, \text{v}_z$ be the Cartesian components of velocity $\bar{\text{v}}$ of a molecule. The mean value of $(\text{v}_x - \alpha\text{v}_y + \beta\text{v}_z)^2$ is

(a) $(1+\alpha^2+\beta^2)\dfrac{k_BT}{m}$ (b) $(1-\alpha^2+\beta)\dfrac{k_BT}{m}$

(c) $(\beta^2-\alpha^2)\dfrac{k_BT}{m}$ (d) $(\alpha^2+\beta^2)\dfrac{k_BT}{m}$

[IIT-JAM : 2010]

Soln. $\langle \text{v}_x^2\rangle = \langle \text{v}_y^2\rangle = \langle \text{v}_z^2\rangle = \dfrac{k_BT}{m}$

$\langle \text{v}_x\text{v}_y\rangle = \langle \text{v}_y\text{v}_z\rangle = \langle \text{v}_z\text{v}_x\rangle = 0$

So, $\langle(\text{v}_x - \alpha\text{v}_y + \beta\text{v}_z)^2\rangle = (1+\alpha^2+\beta^2)\dfrac{k_BT}{m}$

Hence, correct option is (a)

Problem: Let v_x, v_y and v_z denote the components of the velocity along x-, y- and z-directions, respectively, of an ideal gas particle. At the absolute temperature T, the average value of the product $\text{v}_x^2\text{v}_y^2\text{v}_z^2$ is proportional to

(a) T (b) $T^{3/2}$ (c) T^3 (d) T^6

[JNU : 2010]

Soln. $\langle \text{v}_x^2\,\text{v}_y^2\,\text{v}_z^2\rangle$

$$= N\left(\frac{m}{2\pi k_BT}\right)^{3/2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \text{v}_x^2\,\text{v}_y^2\,\text{v}_z^2\, e^{-m(\text{v}_x^2+\text{v}_y^2+\text{v}_z^2)/2k_BT}\, d\text{v}_x\, d\text{v}_y\, d\text{v}_z$$

[$\because$ v_x is velocity, so limit goes from $-\infty$ to $+\infty$]

$$= N\left(\frac{m}{2\pi k_BT}\right)^{3/2} \int_{-\infty}^{\infty} \text{v}_x^2\, e^{-m\text{v}_x^2/2k_BT} d\text{v}_x \int_{-\infty}^{\infty} \text{v}_y^2\, e^{-m\text{v}_y^2/2k_BT} d\text{v}_y \int_{-\infty}^{\infty} \text{v}_z^2\, e^{-m\text{v}_z^2/2k_BT} d\text{v}_z$$

Let $\dfrac{m\text{v}_x^2}{2k_BT} = x \Rightarrow \text{v}_x = \sqrt{\dfrac{2k_BT}{m}}\,x^{1/2} \Rightarrow d\text{v}_x = \sqrt{\dfrac{2k_BT}{m}}\cdot\dfrac{1}{2}\cdot x^{-1/2}dx$

So, $$\int_{-\infty}^{\infty} v_x^2 e^{-mv_x^2/2k_BT} dv_x = \frac{2k_BT}{m}\int_{-\infty}^{\infty} xe^{-x}\sqrt{\frac{2k_BT}{m}}\cdot\frac{1}{2}x^{-1/2}dx = \frac{2k_BT}{m}\sqrt{\frac{2k_BT}{m}}\int_0^{\infty} x^{1/2}e^{-x}dx$$

$$= \left(\frac{2k_BT}{m}\right)^{3/2}\overline{|3/2} = \left(\frac{2k_BT}{m}\right)^{3/2}\cdot\frac{1}{2}\sqrt{\pi}$$

Similarly, $$\int_{-\infty}^{\infty} v_y^2 e^{-mv_y^2/2k_BT} dv_y = \int_{-\infty}^{\infty} v_z^2 e^{-mv_z^2/2k_BT} dv_z = \frac{1}{2}\left(\frac{2k_BT}{m}\right)^{3/2}\sqrt{\pi}$$

$$\langle v_x^2 v_y^2 v_z^2\rangle = N\left(\frac{m}{2\pi k_BT}\right)^{3/2}\frac{1}{8}\cdot\left(\frac{2k_BT}{m}\right)^{9/2}\pi^{3/2} \propto T^3$$

Hence, correct option is (c)

Problem: A classical gas of molecules, each of mass m, is in thermal equilibrium at the absolute temperature T. The velocity components of the molecules along the Cartesian axes are v_x, v_y and v_z. The mean value of $(v_x+v_y)^2$ is

(a) $\frac{k_BT}{m}$ (b) $\frac{3k_BT}{2m}$ (c) $\frac{1}{2}\frac{k_BT}{m}$ (d) $\frac{2k_BT}{m}$

Soln. $\langle(v_x+v_y)^2\rangle = \langle v_x^2\rangle + \langle v_y^2\rangle + 2\langle v_xv_y\rangle$

Putting $\langle v_x^2\rangle = \langle v_y^2\rangle = \langle v_z^2\rangle = \frac{k_BT}{m}$ $\left[\because \frac{1}{2}mv^2 = \frac{1}{2}m(v_x^2+v_y^2+v_z^2) = \frac{3}{2}k_BT\right]$

and $\langle v_xv_y\rangle = 0$

So, $\langle(v_x+v_y)^2\rangle = \frac{k_BT}{m} + \frac{k_BT}{m} + 0 = \frac{2k_BT}{m}$

Hence correct option is (d).

Degree of Freedom

The total number of independent variables required to describe completely the state of motion of a body are called its *degree of freedom* (f).

It is given by $f = 3N - C$ for 3-dimension.

where N = Number of particles

C = Number of constraints (Genrally number of bonds).

(1) For monoatomic gas (e.g., Ar, Xe, Re)

$N = 1; C = 0$

So, $f = 3$ (All translational)

(2) For diatomic gas (e.g., O_2, N_2)

$N = 2; C = 1$

So, $f = 5$ (3 translational; 2 rotational)

(3) For triatomic gas

(i) linear triatomic gas (e.g., O_3)

$N = 3;\ C = 3$

So, $f = 6$ (3 translational; 3 rotational)

(ii) linear triatomic gas (e.g., CO_2)

$N = 3; \quad C = 2$

So, $f = 7$ (3 translational; 4 rotational)

Problem: The average translational kinetic energy of O_2 (molar mass 32) molecules at a particular temperature is 0.048 eV. The translational kinetic energy of N_2 (molar mass 28) molecules in eV at the same temperature is **[IIT-JEE : 1997]**

(a) 0.0015 (b) 0.003 (c) 0.048 (d) 0.768

oln. Average translational kinetic energy of an ideal gas molecule is $3/2\, k_BT$ which depends on temperature only. Therefore, if temperature is same, translational kinetic energy of O_2 and N_2 both will be equal.

Hence, correct option is (c)

Law of Equipartition of Energy

For 1-D, the Maxwell's speed distribution function is

$$f(v) = \left(\frac{m}{2\pi k_BT}\right)^{1/2} e^{-\frac{mv^2}{2k_BT}}$$

So avg kinetic energy is

$$\overline{\in} = \frac{\int\limits_{-\infty}^{\infty} \in f(v)\,dv}{\int\limits_{-\infty}^{\infty} f(v)\,dv} = \frac{\int\limits_{-\infty}^{\infty} \frac{1}{2}mv^2\left(\frac{m}{2\pi k_BT}\right)^{1/2} e^{-\frac{mv^2}{2k_BT}}dv}{\int\limits_{-\infty}^{\infty}\left(\frac{m}{2\pi k_BT}\right)^{1/2} e^{-\frac{mv^2}{2k_BT}}dv}$$

$$= \frac{m}{2}\frac{\int\limits_{-\infty}^{\infty} v^2 e^{-\frac{mv^2}{2k_BT}}dv}{\int\limits_{-\infty}^{\infty} e^{-\frac{mv^2}{2k_BT}}dv} = \frac{m}{2}\frac{\int\limits_{0}^{\infty} v^2 e^{-\frac{mv^2}{2k_BT}}dv}{\int\limits_{0}^{\infty} e^{-\frac{mv}{2k_BT}}dv}$$

Using $\int\limits_0^{\infty} x^n e^{-ax^2}dx = \dfrac{\Gamma\left(\frac{n+1}{2}\right)}{2a^{\frac{n+1}{2}}}$

$$\Rightarrow \quad \overline{\in} = \frac{m}{2}\,\frac{\Gamma\left(\frac{3}{2}\right)\Big/ 2\left(\frac{m}{2k_BT}\right)^{3/2}}{\Gamma\left(\frac{1}{2}\right)\Big/ 2\left(\frac{m}{2k_BT}\right)^{1/2}} = \frac{m}{2}\,\frac{\frac{1}{2}\sqrt{\pi}}{\sqrt{\pi}}\,\frac{2k_BT}{m}$$

$$\Rightarrow \quad \overline{\in} = \frac{1}{2}k_BT$$

Thus, in thermal equilibrium, any degree of freedom (such as component of the position or velocity of a particle) which appears only "quadratically" in the energy has an average energy per molecule of $\frac{1}{2}k_BT$.

This is the *law of equipartition of energy*

- Average energy per molecule

$$U = \frac{f}{2} k_B T \qquad \text{... (28)}$$

- Average energy per mole

$$U = N_A \frac{f}{2} k_B T = \frac{f}{2} RT \qquad [\because \; R = k_B N_A]$$

Problem: A system of N non-interacting classical point particles constrained to move on the two-dimensional surface of a sphere. The internal energy of the system is

(a) $\frac{3}{2} N k_B T$ (b) $\frac{1}{2} N k_B T$ (c) $N k_B T$ (d) $\frac{5}{2} N k_B T$

Soln. Since, particle is moving on two-dimensional surface its degree of freedom = 2

Hence, for N non-interacting particle total degree of freedom = $2N$

The energy per degree of freedom = 1/2 $k_B T$

Hence, total energy $U = 2N \times \frac{1}{2} k_B T$

$\Rightarrow U = N k_B T$

Hence, the correct option is (c).

Problem: A system of N classical non-interacting particles, each of mass m, is at a temperature T and is confined by the external potential $V(r) = \frac{1}{2} A r^2$ (where A is a constant) in three dimensions. The internal energy of the system is

(a) $3 N k_B T$ (b) $\frac{3}{2} N k_B T$

(c) $N (2mA)^{3/2} k_B T$ (d) $N \sqrt{\frac{A}{m}} \ln\left(\frac{k_B T}{m}\right)$

Soln. The contribution of thermal energy with each quadratic term in Hamiltonian is $\frac{1}{2} k_B T$.

The Hamiltonian of a classical particle $H = \frac{p^2}{2m} + \frac{1}{2} A r^2$

In 3-D, hamiltonian has 6 quadratic term. So, the internal energy of a particle is $6 \times \frac{1}{2} k_B T = 3 k_B T$

The internal energy of a system of N particles = $3 N k_B T$.

Correct answer is (a)

Problem: A vessel contains a mixture of one mole of oxygen and two moles of nitrogen at 300 K. The ratio of the average rotational kinetic energy per O_2 molecule to per N_2 molecule is

(a) 1 : 1 (b) 1 : 2 (c) 2 : 1

(d) depends on the moment of inertia of the two molecules

[IIT-JEE : 1998]

Soln. Average kinetic energy per molecule per degree of freedom = 1/2 k_BT. Since, both the gases are diatomic and at same temperature (300 K), both will have the same number of rotational degree of freedom i.e., two. Therefore, both the gases will have the same average rotational kinetic energy per molecule = 2 × 1/2 k_BT or k_BT. Thus ratio will be 1 : 1.

Correct option is (a).

Problem: A gas mixture consists of 2 moles of oxygen and 4 moles of argon at temperature T. Neglecting all vibrational modes, the total internal energy of the system is

(a) 4 RT (b) 15 RT (c) 9 RT (d) 11 RT

[IIT-JEE : 1999]

Soln. Internal energy of n moles of an ideal gas at temperature T is given by

$$U = n\left(\frac{f}{2}RT\right)$$

where, f = degree of freedom.
= 5 for O_2 and 3 for Ar

Hence, $$U = U_{O_2} + U_{Ar} = 2\left(\frac{5}{2}RT\right) + 4\left(\frac{3}{2}RT\right) = 11\,RT.$$

Correct option is (d)

Problem: At a given temperature, T, the average energy per particle of a non-interacting gas of two-dimensional classical harmonic oscillator is ____________ k_BT.

Soln. According to two-dimensional harmonic oscillator, we consider the vibration of molecules in two dimensions. Vibrational mode in one dimension provides k_BT energy. In two dimension, total energy is vibrational molecules.

Hence, 2 is correct answer = 2 × (k_BT) energy.

Specific Heat Capacity

- The "molar" specific heat at constant volume is

$$C_V = \left(\frac{dU}{dT}\right)_V = \frac{f}{2}R \qquad \text{... (29)}$$

- Also, the molar specific heat at constant pressure is

$$C_P = C_V + R = \left(\frac{f}{2}+1\right)R \qquad \text{... (30)}$$

- Also, the ratio of C_P and C_V is

$$\gamma = \frac{C_P}{C_V} = 1 + \frac{2}{f} \qquad \text{... (31)}$$

(1) For monoatomic gas, $(f = 3)$

$$U = \frac{3}{2}RT;\ \ C_V = \frac{3}{2}R;\ \ C_P = \frac{5}{2}R;\ \ \gamma = \frac{5}{3} = 1.67$$

(2) For diatomic gas $(f = 5)$

$$U = \frac{5}{2}RT;\ \ C_V = \frac{5}{2}R;\ \ C_P = \frac{7}{2}R;\ \ \gamma = 1 + \frac{2}{5} = \frac{7}{5} = 1.40$$

(3) For triatomic non-linear gas $(f=6)$

$$U=\frac{6}{2}RT;\;\; C_V=\frac{6}{2}R;\;\; C_P=\frac{8}{2}R;\;\; \gamma=1+\frac{2}{6}=\frac{4}{3}=1.33$$

(4) For triatomic linear gas $(f=7)$

$$U=\frac{7}{2}RT;\;\; C_V=\frac{7}{2}R;\;\; C_P=\frac{9}{2}R;\;\; \gamma=1+\frac{2}{7}=\frac{9}{7}=1.28$$

Problem: Consider oxygen gas at room temperature and pressure and assume that only translational and rotational motions of the molecules contribute to the specific heat. If C_P and C_V denote respectively, specific heats at constant pressure and volume, then the ratio C_P/C_V is

(a) $\frac{3}{5}$ (b) $\frac{5}{3}$ (c) $\frac{5}{7}$ (d) $\frac{7}{5}$ **[IISc : 2006]**

Soln. $\gamma=\frac{C_P}{C_V}=1+\frac{2}{f}$

when f is degree of freedom

There are two degree of freedom associated with rotational motion and three degree of freedom associated with translational motion.

So, total number of degree of freedom associated with motion of diatomic molecule, $f=5$

So, $\gamma=1+\frac{2}{5}=1.4$

Hence, correct option is (d)

Problem: Keeping density and sound speed constant, doubling the pressure implies that the ratio of specific heats C_P/C_V. **[IISc : 2009]**

(a) doubles (b) halves (c) remains constant (d) quadruples

Soln. Velocity, $\mathrm{v}=\sqrt{\frac{\gamma P}{\rho}} \Rightarrow \gamma=\frac{\rho \mathrm{v}^2}{P}$

Keeping ρ and v same and doubling pressure will halves γ.

Hence, correct option is (b)

Mean Free Path

In the assumption of kinetic theory of gases, we assumed that the molecules of gas are moving with very high velocity, which means, that molecules of gas contained in vessel should disappear in no time which is contrary to actual observation, because of absence of force to restrain their motion. Thus there was a serious objection against the assumption that the molecules of a gas move with large rectilinear velocities. This difficulty was solved by the Clausius who assumed that the molecules of a gas have a small, but finite size. These molecule collide with one another after short intervals resulting in a change in both the magnitude and direction of the velocities of the molecules.

Since there is no force acting on the molecules except during collision, they move in straight lines with constant speeds between two successive collision. Thus the paths of diffierent molecules is a series of short zig-zag paths of different length. These paths of different length are called the *free paths* of the molecule and their mean is called *mean free path.*

- The *mean free path* is defined as the average distance travelled by a molecule between two successive collision.
- It is given by

$$\boxed{\lambda=\frac{1}{\sqrt{2}\,\pi d^2 n}} \qquad \ldots (32)$$

where d is the diameter of the molecule and n is the particle density i.e. number of molecules per unit volume.

- $$\lambda=\frac{N}{\sqrt{2}\,\pi d^2 V}\qquad\left[\because\quad n=\frac{N}{V}\right]\qquad\ldots(33)$$
- $$\lambda=\frac{k_BT}{\sqrt{2}\,\pi d^2 P}\qquad\ldots(34)$$

  where we have used $PV=Nk_BT$
- The *collision time* is defined as the time between two successive collision. It is equals to

  $$\tau=\frac{\lambda}{\bar{v}}=\frac{k_BT}{\sqrt{2}\,\pi d^2 P\bar{v}}=\frac{1}{\sqrt{2}\,\pi d^2 n\bar{v}}\qquad\ldots(35)$$

  where $\bar{v}$ is the average speed of the molecules
- The *collision frequency* is defined as the number of collisions in one second. It is equals to

  $$=\frac{1}{\tau}=\sqrt{2}\,\pi d^2 n\bar{v}$$
- Note: The $\bar{v}$ is usually taken to be v_{rms}.

Problem: At what pressure will the mean free path be 50 cm for spherical molecules of radius 3×10^{-10}m? Assume an ideal gas at 20°C.

Soln. From the expression for the mean free path we obtain.

$$\frac{N}{V}=\frac{1}{4\pi\sqrt{2}\,b^2\lambda}\qquad\text{where }\lambda=\text{mean free path.}$$

Combining this with the ideal gas law in the form $PV=Nk_BT$ gives

$$P=\frac{k_BT}{4\pi\sqrt{2}\,b^2\lambda}=\frac{(1.38\times10^{-23}J/K)(293K)}{4\pi\sqrt{2}(3.0\times10^{-10}m)^2(0.50m)}=5.1mPa$$

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Problem: Estimate the mean free path of a molecule of air t 27 °C and 1 atm. Model the molecules as sphere with radius $r=2.0\times10^{-10}$ m.

(a) 5.8×10^{-9}m (b) 5.8×10^{-8}m (c) 5.8×10^{-10}m (d) 5.8×10^{-7}m

Soln. Given : $T=27\,^\circ\text{C}=300\text{ K},\ P=1\text{ atm}=1.013\times10^5\text{ N/m}^2,\ r=2.0\times10^{-10}\text{ m}$

The mean free path is $\lambda=\dfrac{k_BT}{\sqrt{2}\pi d^2P}$, where d is the diameter of the molecules $=2r$

$$\therefore\ \lambda=\frac{k_BT}{4\sqrt{2}\pi r^2P}=5.8\times10^{-8}\text{ m}$$

Correct option is (b)

PROBLEMS TO BE PLAYED

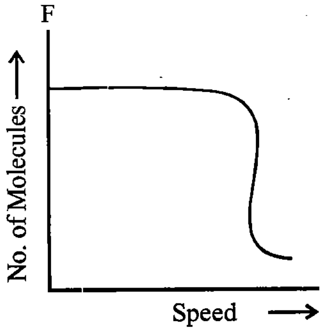
For polyatomic gases such as hydrogen the corresponding degree of freedom would be–
(a) Two, three (b) Five, six (c) Three, nine (d) Three, four

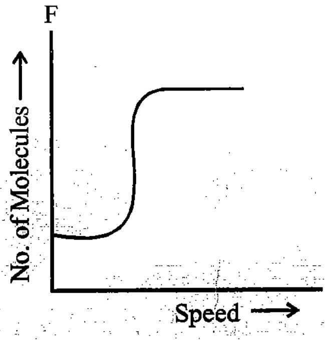
The ratio of two specific heats of a diatomic gas is–
(a) 1.66 (b) 1.33 (c) 1.40 (d) 1.52

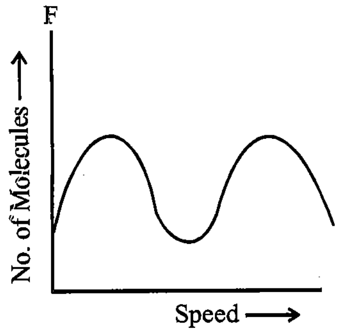
If the degree of freedom of a gas is n, then the ratio of C_P and C_V is–
(a) $1+\frac{2}{n}$ (b) $1+\frac{1}{n}$ (c) $1+\frac{1}{2n}$ (d) $\frac{2n}{2n+1}$

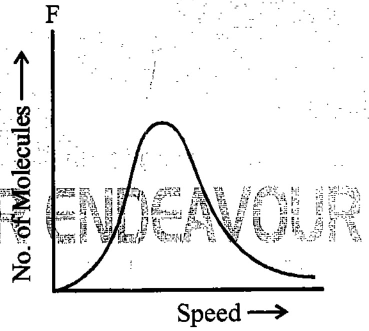
In a gas the expression for the average speed of molecule is given by–
(a) $\sqrt{\frac{3k_B T}{m}}$ (b) $\sqrt{\frac{2k_B T}{m}}$ (c) $\sqrt{\frac{8k_B T}{m}}$ (d) $\sqrt{\frac{3k_B T}{2m}}$

Which of the following curves represents Maxwell's distribution law of speed of molecules?

(a)


(b)


(c)


(d)


The gas exerts pressure on the walls of the container because the gas molecules–
(a) Have finite size (b) Obey Boyle's law
(c) Have momentum (d) Collide with one another

7. The most probable speed v_{mp} and the root mean square speed 'c' are related by–
(a) $v_{mp}=\sqrt{\frac{1}{2}}c$ (b) $v_{mp}=\sqrt{\frac{2}{3}}c$ (c) $v_{mp}=\sqrt{\frac{3}{8}}c$ (d) $v_{mp}=\sqrt{\frac{1}{3}}c$

The speed of sound V_s in a gas, r.m.s. speed of the molecule of a gas V_{rms} and the ratio of specific heats γ of the gas is related by
(a) $V_s = V_{(rms)}$ (b) $V_s = V_{rms}\times\sqrt{3/\gamma}$ (c) $V_s = V_{rms}\sqrt{\gamma}$ (d) $V_s = V_{rms}\sqrt{\gamma/3}$

9. The mean free path is
(a) Inversely proportional to the pressure (P) (b) directly proportional to the pressure
(c) proportional to P^2 (d) proportional to P^4

10. The correct expression for the most probable velocity of the molecules of a gas in a steady state is given by
(a) $\left(\frac{2}{3}\right)^{1/2}$ r.m.s. speed (b) equal to r.m.s. speed
(c) $\frac{2}{3}$ r.m.s speed (d) $\left(\frac{3}{2}\right)^{1/2}$ r.m.s. speed

1. According to the kinetic theory of gases, pressure exerted by a perfect gas is equal to
(a) $\frac{1}{2}mv^2$, where v is average velocity (b) mgh
(c) $\frac{2}{3}E$, where E is the mean kinetic energy per unit volume of gas
(d) all of these

2. Assuming the Maxwellian distribution of speeds for molecules of a liquid also, at temperature below the normal boiling point of a liquid
1. molecules with higher speeds escape from the surface of the liquid.
2. molecules with lower speeds escape from the surface
3. average kinetic enegy of the molecules is the liquid increases as evaporation proceeds
4. average kinetic energy decreases with evaporation
(a) 2 and 3 (b) 1 and 4 (c) 2 and 4 (d) 1 and 3

3. Maxwell's law of distribution of speed show that the number of molecules with average speed is
(a) Very small (b) large (c) zero (d) exactly equal to 1.

4. The root-mean square speed of a particle of mass m in the kinetic theory is given by
(a) $\sqrt{\frac{k_BT}{m}}$ (b) $\sqrt{\frac{2k_BT}{m}}$ (c) $\sqrt{\frac{3k_BT}{m}}$ (d) $\sqrt{\frac{8k_BT}{m}}$

5. Specific heats of a gas at constant volume (C_V) and at constant pressure (C_P) are related as–
(a) $\frac{C_P}{C_V} = 1 - \frac{R}{J}$ (b) $C_P - C_V = \frac{R}{J}$ (c) $C_P - C_V = \frac{J}{R}$ (d) $C_P - C_V = \frac{R}{J}$

6. If molar $C_P = 4.97$ cal/mole, then
(a) $C_v = 2.97$ (b) $C_v = 1.97$ (c) $C_v = 5.97$ (d) $C_v = 0.97$

ANSWER KEY

1. (a) **2. (c)** **3. (a)** **4. (c)** **5. (d)** **6. (c)** **7. (b)**
8. (d) **9. (a)** **10. (a)** **11. (c)** **12. (b)** **13. (b)** **14. (c)**
15. (b) **16. (a)**

Chapter 2

The Gas Laws & the First Law of Thermodynamics

The Gas Laws

(1) Boyle's Law:

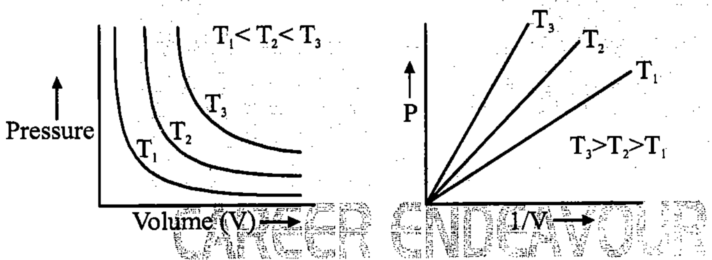
- "At constant temperature, the pressure of a fixed amount (i.e. number of moles n) of gas varies inversely with its volume.

  Mathematically, $P \propto \frac{1}{V}$ (at constant T and n) ... (1)

  $\Rightarrow \quad \boxed{PV = \text{Constant}}$ ... (2)

  $\Rightarrow \quad P_1V_1 = P_2V_2$ ... (3)

  where constant depends upon the amount of gas, temperatue of the gas and the units in which P and V are expressed.
- The variation of pressure with volume and its inverse is



Each graph is called *isotherm* as temperature is constant in each one.

- Equation (2) $\Rightarrow \frac{Pm}{\rho} = \text{Constant}; \quad \rho \rightarrow$ density of gas

  $\Rightarrow \quad P = \rho \times \text{Constant}$

At constant temperature, pressure is directly proportional to the density of a fixed mass of the gas.

Problem: A balloon is filled with hydrogen at room temperature. It will burst if pressure exceeds 0.2 bar. If at 1 per pressure the gas occupies 2.27 L volume, upto what volume can the baloon be expanded?

Soln. Given : $P_1 = 1\text{ bar},\ V_1 = 2.27\text{ L}$ and $P_2 = 2\text{ bar},\ V_2 = ?$

Using Boyle's law, $P_1V_1 = P_2V_2 \Rightarrow V_2 = \frac{P_1}{P_2}V_1 = \frac{1}{0.2} \times 2.27\text{ L} = 11.35\text{ L}$

Correct option is (a)

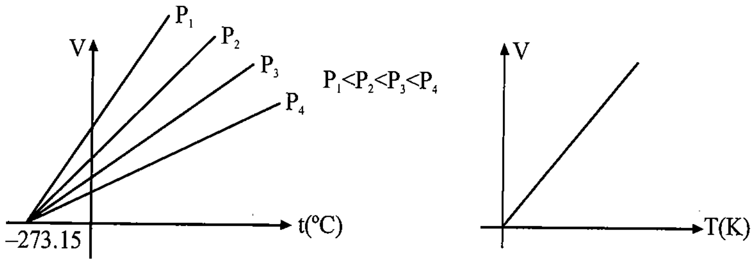
(2) Charle's Law:

- "For a fixed mass of a gas at constant pressure, volume of a gas is directly related to the absolute temperature."

Mathematically, $V \propto T$ ... (4)

Or $\boxed{\frac{V}{T} = \text{Constant}}$ ... (5)

- The variation of volume with temperature in Kelvin and Celsius is



(1) Each Graph is called *isobar* as pressure is constant in all graphs.
(2) All the lines meet the temperature aries at –273.15°C.

Problem: On a ship sailing in Pacific Ocean where temperature is 23.4 °C, a balloon is filled with 2 L air. What will be the volume of the balloon when the ship reaches Indian Ocean where temperature is 26.1 °C ?

(a) 201.8 L (b) 20.18 L (c) 0.2018 L (d) 2.018 L

Soln. Given : $T_1 = 23.4\,°\text{C} = (23.4 + 273)\,\text{K} = 296.4\,\text{K}$

$$V_1 = 2\,\text{L}$$

$$T_2 = 26.1\,°\text{C} = (26.1 + 273)\,\text{K} = 299.1\,\text{K}$$

$$V_2 = ?$$

Since the pressure during the sailing remains constant due to the atmosphere. i.e., atmospheric pressure, the pressure remains constant. So using Charle's law, we have

$$\frac{V_1}{T_1} = \frac{V_2}{T_2}$$

$$\Rightarrow V_2 = \frac{T_2}{T_1} V_1 = \frac{299.1}{296.4} \times 2 = 2.018\,\text{L}$$

Correct option is (d)

(3) Avagadro Law

- "Equal volume of all gases under the same conditions of temperature and pressure contain the equal number of molecules".

Mathematically, $V \propto N \Rightarrow V \propto n \quad [\because N = nN_A]$ ... (6)

Or, $\boxed{\frac{V}{n} = \text{constant}}$ ... (7)

where number of moles $= \frac{w}{W}$

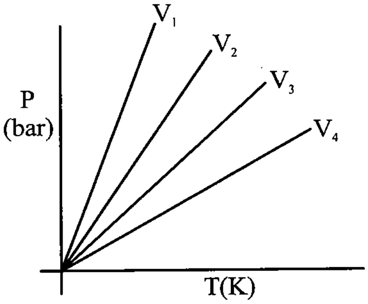
where, w = Given mass
W = Molar mass

(4) Gas-Lussac's Law:

- "At constant volume, pressure of a fixed amount of gas varies directly with the temperature."
  Mathematically, $P \propto T$ ... (8)

Or $\boxed{\dfrac{P}{T} = \text{Constant}}$ ... (9)

- The variation of pressure with temperature is



- Each graph is called *isochore* as volume is constant in all graphs.

The Ideal-Gas Equation:

The three gas-laws are

$N \propto P$ (Boyle's law)

$V \propto T$ (Charle's law)

$V \propto n$ (Avagadro's law)

Combining, we have

$$V \propto \frac{nT}{P}$$

$\Rightarrow \quad V = \dfrac{RnT}{P};$ where R = Universal Gas constant

$\Rightarrow \quad \boxed{PV = nRT}$ ...(10)

This is called an *ideal gas equation*.

Problem: A balloon will carry a total load of 175 kg when the temperature and pressure are normal. What load will the balloon carry on rising to a height at which the barometric pressure is 50 cm of mercury and the temperature is –10 °C, assuming the envelope maintains a constant volume ?

(a) 1195.0 kg (b) 119.5 kg (c) 11.95 kg (d) 1.195 kg

Soln. Using $\dfrac{\rho V}{T} = \text{constant}$, we have

$$\Rightarrow \frac{\rho_1 V_1}{T_1} = \frac{\rho_2 V_2}{T_2}$$

Here, $\rho_1 = \dfrac{M_1}{V}$ and $\rho_2 = \dfrac{M_2}{V}$

$$\therefore \quad \frac{M_1}{M_2} = \frac{\rho_1}{\rho_2} = \frac{P_1 T_2}{P_2 T_1} \Rightarrow M_2 = \frac{P_2}{P_1}\frac{T_1}{T_2} M_1 = \frac{50}{76} \times \frac{273}{263} \times 175 = 119.5\ kg$$

Problem: A monoatomic gas is described by the equation of state $P(V-bn)=nRT$, where b and R are constants and other quantities have their usual meanings. The maximum density (in moles per unit volume to which this gas can be compressed is

(a) $\frac{1}{\text{bn}}$ (b) b (c) $\frac{1}{\text{b}}$ (d) infinity

Soln. Given : $P(V-bn)=nRT$

$$\Rightarrow PV-Pbn=nRT \Rightarrow n=\frac{PV}{Pb+nRT}$$

So density, $\rho=\frac{n}{V}=\frac{P}{Pb+nT}$

For maximum of ρ,

$$\frac{d\rho}{dP}=\frac{(Pb+nT)1-P(b)}{(Pb+nT)^2}=0 \Rightarrow nT=0$$

$$\therefore\ \rho=\frac{P}{Pb}=\frac{1}{b}$$

Correct option is (c)

Problem: An ideal gas expands in a process in which its pressure depends on the volume as $p=p_0e^{-\alpha V}$, where p_0 and α are two positive constants. In n is the number of moles in the gas, the maximum temperature that it will attain in this process is (R is the universal gas constant)

(a) $p_0/e\alpha nR$ (b) $p_0/\alpha n^3R$ (c) $ep_0/\alpha nR$ (d) $p_0^2/\alpha nR$

[JNU : 2009]

Soln. $p=p_0e^{-\alpha V}$

So, $T=\frac{pV}{nR}=\frac{p_0e^{-\alpha V}\cdot V}{nR}$

To maximise T, take its derivative and make it equals to zero

$$\frac{dT}{dV}=\frac{p_0}{nR}\left[e^{-\alpha V}-\alpha Ve^{-\alpha V}\right]=0$$

$$\Rightarrow \quad V=\frac{1}{\alpha}$$

Now, $\frac{d^2T}{dV^2}=\frac{p_0}{nR}\left[-\alpha e^{-\alpha V}-\alpha e^{-\alpha V}+\alpha^2Ve^{-\alpha V}\right]<0$ for $V=\frac{1}{\alpha}$

So, $T_{\max}=\left.\frac{p_0e^{-\alpha V}\cdot V}{nR}\right|_{V=\frac{1}{\alpha}}=\frac{p_0}{e\alpha nR}$

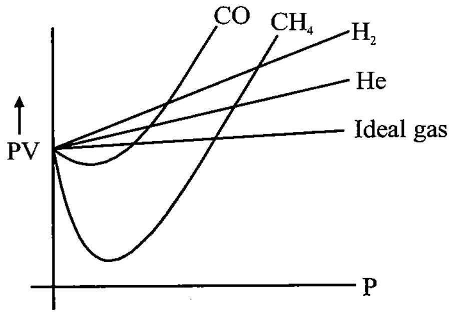
Hence, correct option is (a)

Deviation from ideal gas behavior:

A gas which obeys the gas laws and the gas equation $PV = nRT$ strictly at all temperature and pressure is said to be an *ideal gas*. The molecules of ideal gases are assumed to be volume less points with no attractive forces between one another. But no real gas strictly obeys the gas equation at all temperatures and pressures. Deviation from ideal behavior are observed particularly at high pressures or low temperatures.

The deviation from ideal behavior is expressed by introducing a factor Z known as *compressibility factor* in the ideal gas equation. Z may be expressed as $Z = \dfrac{PV}{nRT}$

- In case of ideal gas, $PV = nRT$. $\therefore\ Z = 1$
- In case of real gas, $PV \neq nRT$. $\therefore\ Z \neq 1$



- When $Z < 1$, it is a negative deviation. It shows that the gas is more compressible than expected from ideal behavior.
- When $Z > 1$, it is a positive deviation. It shows that the gas is less compressible than expected from ideal behavior.

Causes of Deviation from Ideal Behavior

1. The volume occupied by gas molecules is not negligible as compared to the volume occupied by the gas.
2. The forces of attraction between gas molecules are negligible.

Problem: One mole of SO$_2$ occupies a volume of 350 mL at 300 K and 50 atm pressure. Calculate the compressibility factor of the gas.

Soln. $Z = \dfrac{PV}{nRT} = \dfrac{50 \times 1.01 \times 10^5 \times 0.350 \times 10^{-3}}{1 \times 8.314 \times 300} = 0.71.$

Thus, SO$_2$ is more compressibility than expected from ideal behavior.

Different forms of van der Waals' equation

1. **At very low pressure :** V becomes so large that both b and a/V^2 become negligible and the van der Waals' equation reduces to

$$\left(P + \frac{a}{V^2}\right)(V - b) = RT \Rightarrow PV = RT$$

Gases approach ideal behavior at very low pressure.

2. **At low pressure :** V is large and 'b' is negligible in comparison with V.

Therefore, $\left(P + \dfrac{a}{V^2}\right)V = RT \Rightarrow PV + \dfrac{a}{V} = RT$

$$\Rightarrow \quad PV = RT - \frac{a}{V} \text{ or } \frac{PV}{RT} < 1 - \frac{a}{RTV} \text{ or } \frac{PV}{RT} < 1$$

This accounts for the dip in isotherm at low pressure.

3. At fairly high pressure : $\frac{a}{V^2}$ may be neglected in comparison with P.

Therefore, $P(V-b)=RT \Rightarrow \frac{PV}{RT}=1+\frac{Pb}{RT}>1$

This accounts for the rising parts of the isotherm at high pressure.

Hydrogen and Helium are two lightest gases known. Their molecules have very small masses. The attractive forces between such molecules will be extensively small. So a/V^2 is negligible even at ordinary pressure.

Thus, $\frac{PV}{RT}>1$, even at low pressure.

rob. One litre of a gas at 300 atm and 473 K is compressed to a pressure of 600 atm and 273 K. The compressibility factors found are 1.072 and 1.375 respectively at initial and final states. Calculate the final volume.

oln. $P_1V_1 = Z_1 nRT_1$ and $P_2V_2 = Z_2 nRT_2$

$$\therefore \frac{P_1V_1}{Z_1T_1}=\frac{P_2V_2}{Z_2T_2} \Rightarrow V_2=\frac{P_1V_1Z_2T_2}{P_2Z_1T_1}$$

$$\Rightarrow V_2=\frac{300}{600}\times\frac{1.375}{1.072}\times\frac{273}{473}\times 1\text{ L}=370.9\text{ mL}.$$

vander Waal Gas (The Real Gas)

(1) Correction in pressure :

Due to the presence of molecular attraction the molecule striking the wall of container is pulled back by the other molecules, and hence the observed pressure on the face of wall is less than the actual pressure within the body of the gas. The decrease in pressure p is proportional to

- Number of attracting molecules / Volume.
- Number of molecules striking per unit area of the wall of the container per second.

Decrease in pressure, $p \propto \frac{1}{\rho^2} \propto \frac{1}{V^2} \Rightarrow p=\frac{a}{V^2}$

Real pressure $= p + p = P + \frac{a}{V^2}$

(2) Correction in Volume:

Since the particles have finite size the volume left for the particles for their free movement is less than actual volume V. If b is the volume occupied by the size of the particle, then

free volume $= V - b$

So, from ideal gas equation PV = RT, we have

$$\left(P+\frac{a}{V^2}\right)(V-b)=RT \qquad \text{... (11)}$$

For n mole, $\left(P+\frac{an^2}{V^2}\right)(V-nb)=nRT$ ... (12)

Equations (11) and (12) are called *vander Waal gas equation* for one mole and n moles respectively.

Problem: A real gas behaves like an ideal gas if its
(a) pressure and temperature are both high
(b) pressure and temperature are both low
(c) pressure is high and temperature is low
(d) pressure is low and temperature is high **[IIT-JEE : 2010]**

Soln. A real gas behaves like an ideal gas at low pressure and high temperature.
Hence, correct option is (d).

Different Types of Temperature in Thermodynamics

(1) Boyle's Temperature (T_b)

It is a temperature at which vander Waal gas behave like an ideal gas.

If at $T = T_b$, the Vander Waal's equation i.e. $PV = RT$, then

$$\frac{RT_bV}{V-b} - \frac{a}{V} = RT_b \Rightarrow RT_b\left(\frac{V}{V-b} - 1\right) = \frac{a}{V}$$

$$\Rightarrow RT_b = \frac{a}{b}\left(\frac{V-b}{V}\right) \Rightarrow T_b = \frac{a}{Rb}\left(\frac{1-b}{V}\right) = \frac{a}{Rb}\left(1-\frac{b}{V}\right)$$

Since b << V, so $\boxed{T_b = \frac{a}{Rb}}$

(2) Critical temperature (*Tc*)

It is the temperature above which a gas can't be liquified by the application of pressure alone.
According to vander Waal gas equation

$$\left(P + \frac{a}{V^2}\right)(V-b) = RT \Rightarrow P = \frac{RT}{V-b} - \frac{a}{V^2}$$

Now $\left(\frac{dP}{dV}\right)_{T=T_C} = -\frac{RT_C}{(V_C-b)^2} + \frac{2a}{V_C^3} = 0 \Rightarrow \frac{2a}{V_C^3} = \frac{RT_C}{(V_C-b)^2}$ ... (14)

Again differentiating

$$\left(\frac{d^2P}{dV^2}\right)_{T=T_C} = \frac{2RT_C}{(V_C-b)^3} - \frac{6a}{V_C^4} = 0$$

$$\Rightarrow \frac{2RT_C}{(V_C-b)^3} = \frac{6a}{V_C^4}$$

Using (14), $\frac{V_C}{3} = \frac{V_C - b}{2} \Rightarrow V_C = 3b$ ... (15)

Using (15) in (14), $T_C = \frac{2a \cdot 4b^2}{27b^3R} \Rightarrow T_C = \frac{8a}{27Rb}$ ... (16)

Using (16) in vander Waal's equation

$$P_C = \frac{a}{27b^2}$$

where the *critical pressure* of a substance is the pressure required to liquefy a gas at its critical temperature

Also, $\frac{P_CV_C}{T_C} = \frac{a}{27b^2} \cdot \frac{3b}{8a} \cdot 27Rb = \frac{3R}{8}$ ... (17)

Problem: The equation of state for a gas is given by $\left[p+\left(\frac{\alpha N}{V}\right)^2\right](V-\beta N)=Nk_BT$, where p, V, T, N and k_B respectively represent pressure, volume, temperature, number of atoms and the Boltzmann constant, respectively, while α and β are constants specific to the gas.

If the critical point C corresponds to a point of inflexion of the p-V curve, then the critical volume V_c and the critical pressure p_c, for this gas are given by

(a) $V_c=3\beta N,\ p_c=\frac{\alpha^2}{3\beta^2}$

(b) $V_c=3\beta N,\ p_c=\frac{\alpha}{27\beta^2}$

(c) $V_c=3\beta N,\ p_c=\frac{8\alpha^2}{27\beta}$

(d) $V_c=3\beta N,\ p_c=\frac{\alpha^2}{27\beta^2}$

Soln. Given: $\left[p+\left(\frac{\alpha N}{V}\right)^2\right](V-\beta N)=Nk_BT$

Comparing with vander Waals' equation

$$\left(p+\frac{an^2}{V^2}\right)(V-nb)=nRT$$

or in terms of number of molecules (N),

$$\left(p+\frac{aN^2}{\gamma^2}\right)(V-Nb)=Nk_BT,\text{ we have } a=\alpha^2,\ b=\beta$$

So $V_c=3Nb=3N\beta$

$$p_c=\frac{a}{27b^2}=\frac{\alpha^2}{27\beta^2}$$

Correct option is (c)

Terms in Thermodynamics

1. **Thermodynamics :** It is the branch of physics that deals with the concepts of heat and temperature and the inter conversion of heat and other forms of energy.
- It is a macroscopic science. It deals with bulk systems and does not go into the molecular constitution of matter.

2. **System:** According to Oxford dictionary, a *system* is defined as complex whole set of connected things or parts.

3. **Thermodynamic systems:** A definite quantities of substance bounded by some closed surface is known as a *system*.
   A system always interacts with the environment surrounding the system. Thus, surrounding has to play a considerable/significant role in affecting the working of the system.
4. **State of a system:** The quantities which are used to determine the characteristic of a system are said to be the *state* of a system.

The equation connecting the pressure, volume and the absolute temperature of a gas is called the equation of state of that gas. Mathematically, the equation of state is represented as

$$F(P,V,T)=0$$

5. **Equilibrium :** If the macroscopic variables (P, V, T) that characterise the system do not change in time, the system is said to be in an *equilibirium*.
   (i) If T is not changing with time → Thermal Equilibrium
   (ii) If P is not changing with time → Mechanical Equilibrium
   (iii) If composition is not changing with time → Chemical Equilibrium

6. **Quasi-static process:** A *quasi-static* process is defined as the process in which the deviation from thermodyanmic equilibrium is infinitesimal and all the state through which the system passes may be considered as the succession of equilbrium states e.g. if a turbine is moved at infinitesimally slow rate, the system is at all time in quiescent equilibrium state and then process can be considered *quasi-static*.

7. **Reversible process :** A *reversible process* is one which can be reversed in such a way that all changes taking place in the direct process are exactly repeated in the opposite order and sense and no changes are left in any of the bodies taking part in the process or in the surrounding.

   The condition for reversibility :
   A process can be reversible only when it satisfy two conditions

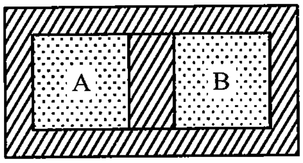
(i) Process must be quasistatic (If the process is performed in such a way that at any instant during the process the system is very nearly in the thermodynamic equilibrium).
A quasistatic process is an idealised process in which all changes take place infinitely and slowly.

(ii) Process must be non-dissipiative (dissipiative force such as friction, resistance, hysteresis etc must be completely absent).

8. **Irreversible Process:** Any process inevitably is an irreversible process because we always have some departure from initial system in carrying out the process again. Thus, those processes which produce a permanent change in the thermodynamic state of the system and cannot be retraced in the opposite order are known as *irreversible process*.

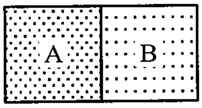
Types of Walls

(1) Adiabatic Wall : It is an insulating wall (can be movable) that does not allow flow of energy (heat) from one system to another.



All possible pairs (P_A, V_A) and (P_B, V_B) are in equilibrium.

(2) Diathermal Wall : It is a conducting wall that allows energy (heat) from system to another.



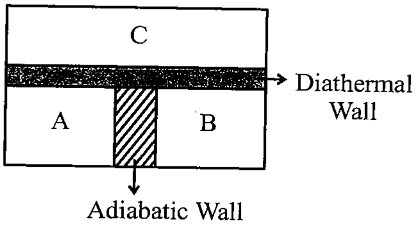
After the exchange of heat, a new equilibrium state is achieved i.e.

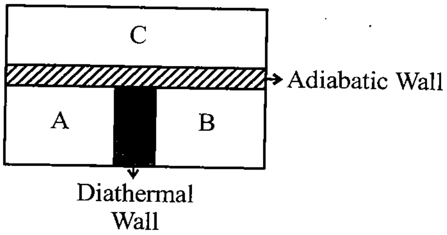
$$(P_A, V_A), (P_B, V_B) \Rightarrow (P_A', V_A') \text{ and } (P_B', V_B')$$

Zeroth Law of Thermodynamics

"It states that two systems in thermal equilibrium with a third system separately are in thermal equilibrium with each other."

Suppose there are three systems A, B and C. If initially, there is an adiabatic wall between A and B, and a diathermal wall between both A and C, and B and C as shown in the figure. Then after some time, the systems A and B both will be in thermal equilibrium with system C individually. Now, we have interchanged the adiabatic and diathermal wall as shown in the figure below. Then, we observed that A and B are already in thermal equilibrium.





- This law gives the definition of temperature.
- Mathematically, if $T_A = T_C$ and $T_B = T_C$

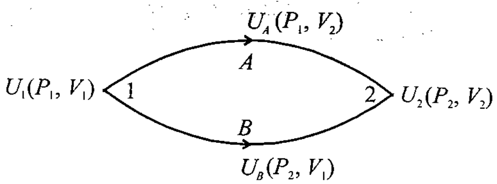
$\Rightarrow$ $T_A = T_B$

i.e. system A and system B are also in thermal equilibrium

- Temperature is the marker of the hotness and coldness of a body.
- Heat flows from a body at high temperature to a body at low temperature. The flow stops when the temperature equalise i.e. bodies come in thermal equilibrium.

Internal Energy (U)

- It is the sum of molecular KE and PE in the frame of reference relative to which the centre of mass of the system is at rest.
- It is a state variable i.e., its value depends only on the state of the system and not on the path taken to arrive that state. For example, if you change the volume first and then pressure through the path A or change the pressure first and then volume through the path B, the change in internal energy between initial and final states remains same.



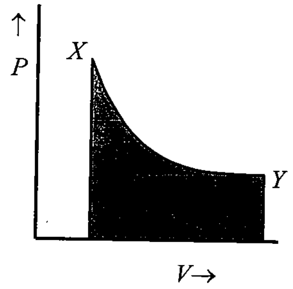
i.e. $\Delta U_{1 \xrightarrow[A]{} 2} = \Delta U_{1 \xrightarrow[B]{} 2}$

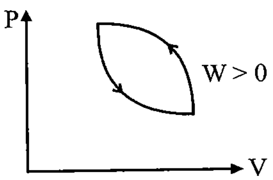
Work (W)

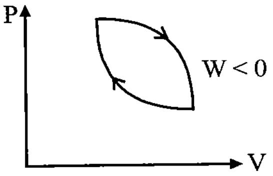
- It is defined as

$$\boxed{W = \int_{V_1}^{V_2} P\,dV}$$

- It is a path variable i.e., depends on the path taken and given by the area in the PV-curve.

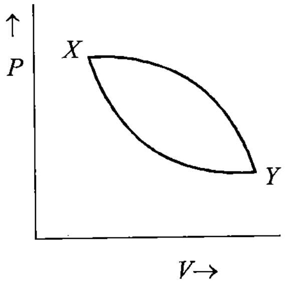






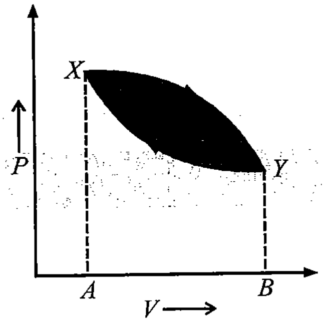
(i) Work done in anti-clockwise direction is positive and done by the system.
(ii) Work done in clockwise direction is negative and done on the system.

Problem: A piston containing an ideal gas is originally in the state X (see figure). The gas is taken through a thermal cycle $X \to Y \to X$ as shown:



The work done by the gas is positive if the direction of the thermal cycle is:
(a) clockwise
(b) counter - clockwise
(c) neither clockwise nor counter-clockwise
(d) clockwise from $X \to Y$ and counter-clockwise from $Y \to X$.

Soln. Since work done = Area between P-V curve and volume axis



For clockwise cycle, work done is

W = area of shaded portion (in clockwise sense)

= area of $XYBAX$ – area of $YXABY$ = positive

Hence correct option is (a).

Heat (Q)

- It is the defined as the flow of energy by virtue of the temperature difference.
- It is also a path function.
- If $\Delta Q > 0$, heat is given to the system

  If $\Delta Q < 0$, heat is taken from the system.

Sign Convention

$$\text{Work}(W)=\begin{cases}+\text{ve, by the system}\\-\text{ve, on the system}\end{cases}$$

$$\text{Heat}(Q)=\begin{cases}+\text{ve, given to the system}\\-\text{ve, taken away from the system}\end{cases}$$

First Law of Thermodynamics

- It is simply the principle of conservation of energy applied to the system.
- If $\Delta Q=$ Heat supplied to the system by the surroundings
  $\Delta W=$ Work done by the system on the surroundings
  $\Delta U=$ Change in internal energy of the system.

then $\Delta Q=\Delta U+\Delta W$ ... (18)

or $dQ=dU+dW$ ... (19)

where $\Delta W=P\Delta V$ ... (20)

$\Rightarrow$ $\Delta Q=\Delta U+P\Delta V$ ... (21)

Problem: A thermos bottle containing coffee is vigorously shaken and thereby the temperature of coffee rises. considering the coffee as the system, which of the following is CORRECT:

(a) $\Delta Q=0; \Delta W<0; \Delta U>0$
(b) $\Delta Q=0; \Delta W>0; \Delta U>0$
(c) $\Delta Q=0; \Delta W<0; \Delta U<0$
(d) $\Delta Q=0; \Delta W>0; \Delta U<0$

oln. Since no heat is given, $\Delta Q=0$.

Also, the bottel is shaken and hence work is done on the bottle.

$\Rightarrow$ $\Delta W<0$

So from first law of thermodynamic,

$\Delta U=\Delta Q-\Delta W$

$\Rightarrow$ $\Delta U=-\Delta W$ $[\because\ \Delta Q=0]$

$\Rightarrow$ $\Delta U>0$ $[\because\ \Delta W<0]$

Correct option is (a)

Problem: Let ΔW be the work done in a quasistatic reversible thermodynamics process. Which of the following statements about ΔW is correct?

(a) ΔW is a perfect differential if the process is isothermal
(b) ΔW is a perfect differential if the process is adiabatic
(c) ΔW is always a perfect differential
(d) ΔW cannot be a perfect differential.

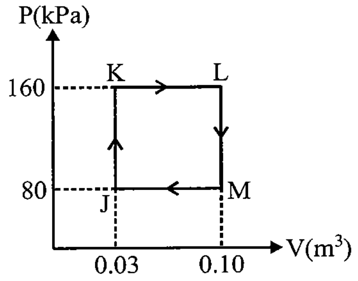
ln. According to the first law of thermodynamics $dQ=dU+dW$
where dQ is the amount of heat absorbed by a system, dU is the change in the internal energy of the system which depends only on the temperature, and dW is the amount of work done. If $dQ\neq 0$, then dW is not a perfect differential as it depends upon the path chosen. However for an adiabatic process $dQ=0$,

$\therefore$ $dU=-dW$

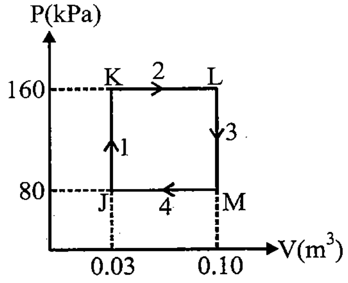
Since dU is a perfect differential, therefore dW is also a perfect differential.

Correct option is (b)

Problem: The PV diagram below represents an ideal monoatomic gas cycle for 1 mole of a gas. In terms of the gas constant R, calculate the temperatures at the points J, K, L and M.
Also calculate the heat rejected and heat absorbed during the cycle and efficiency of the cycle.



Soln. Using, $PV = nRT$ or $T = \dfrac{PV}{nR}$



At point K: $T = \dfrac{160\times10^3\times0.03}{R}$

At point L: $\text{T} = \dfrac{160\times10^3\times0.1}{R} = \dfrac{16000}{R}$

At point M : $\text{T} = \dfrac{80\times10^3\times0.1}{R} = \dfrac{8000}{R}$

At point J : $\text{T} = \dfrac{80\times10^3\times0.03}{R} = \dfrac{2400}{R}$

Total work done during the cycle = Area of loop JKLM
$= (160-80)\times10^3\times(0.1-0.03) = 5600\text{J}$

In process J → K,

$$dQ_1 = dU_1 + dW_1 = \frac{3}{2}R(T_K - T_J) + 0$$

$$= \frac{3}{2}R\frac{2400}{R} = 3600\text{ J}$$

In process K → L,

$$dQ_2 = dU_2 + dW_2$$

$$= \frac{3}{2}R(T_L - T_K) + 160\times10^3(0.1-0.03) = \frac{3}{2}R.\frac{11200}{R} + 11200 = 28000\text{J}$$

In process L → M,

$$dQ_3 = dU_3 + dW_3 = \frac{3}{2}R(T_M - T_L) + 0$$

$$= \frac{3}{2}.R.\left(\frac{-8000}{R}\right) = -12000J$$

In process, M → J,

$$dQ_4 = \frac{3}{2}R(T_J - T_M) + 80\times10^3(-0.07)$$

$$= -8400 - 5600 = -14000\text{J}$$

Heat absorbed, $dQ^{+} = dQ_1 + dQ_2 = 31600J$

Heat rejected, $dQ^{-} = dQ_3 + dQ_4 = -26000J$

$$\text{Efficiency} = \frac{\text{Total work done}}{\text{Total heat absorbed}} = \frac{5600}{31600} = \frac{14}{79}$$

Problem: When a system is taken from the state A to the state B along the path ACB, 80 Joules of heat flows into the system and the system does 30 Joules of work.

(i) How much heat flows into the system along the path ADB if work done is 10 J?

(ii)The system is returned from state B to state A along the curved path. The work done on the system is 20 joule. Does the system absorb or liberate heat and how much?

(iii) If $U_A = 0$, $U_D = 40$ J. Find the heat absorbed in the process AD & DB

Soln. Along the path ACB,

$$\Delta U = \Delta Q - \Delta W \quad \Rightarrow \quad U_B - U_A = 80 - 30 = 50$$

(i) Along the path ACD,

$$\Delta Q = \Delta U + \Delta W = (U_B - U_A) + \Delta W \qquad [\because \text{ internal energy is a state function}]$$

$= 50 + 10 = 60$ J

(ii) Along the curved path from B to A,

$$\Delta Q = \Delta U + \Delta W = (U_A - U_B) + \Delta W = (-50) + (-20) = -70 \text{ J}$$

So, liberated heat is 70 J

(iii) Given: $U_A = 0, U_0 = 40J$

Since, $U_B - U_A = 50 \Rightarrow U_B = 50J$

Now, for the path DB, $\Delta W = P\Delta V = 0$

So, $\Delta Q = \Delta U = U_B - U_D = 50 - 40 = 10\ J$

Now, for the path AD,

$$\Delta Q = \Delta U + \Delta W = (U_D - U_A) + \Delta W = (40 - 0) + 10 = 50\ J \qquad [\because \Delta W = 10\ J\ (\text{Given})]$$

Problem: A vessel contains an ideal monoatomic gas, which expands at constant pressure when heat Q is supplied to it. Then work done in the expansion is

(a) Q (b) $\frac{3}{5}Q$ (c) $\frac{2}{5}Q$ (d) $\frac{2}{3}Q$

Soln. The internal energy is

$$\Delta U = C_V dT = \frac{f}{2} R\,dT = \frac{3}{2} R\,dT = \frac{3}{2} P\,dV \qquad [\because PV = RT]$$

where $f = 3$ for monoatomic gas.

Now from first law of thermodynamic,

$$Q = \Delta U + PdV$$

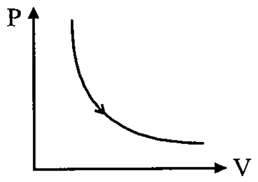
$$\Rightarrow Q = \frac{3}{2} PdV + PdV \Rightarrow PdV = \frac{2}{5}Q \Rightarrow W = \frac{2}{5}Q$$

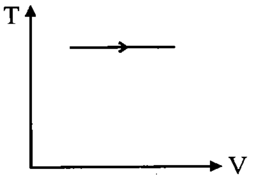
Correct option is (c)

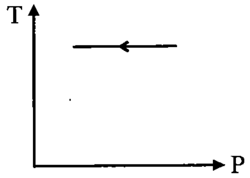
Application of First Law of Thermodynamics

(1) Isothermal Process $(\Delta T = 0)$:

- It is the process in which the temperature of the system is kept fixed throughout.
- E.g. (i) Melting of ice at melting point
  (ii) Vaporization of water at 100°C.
- Ideal gas equation $PV =$ constant ... (22)
- The graphs are







- First Law of Thermodynamics says

$$\Rightarrow dQ = dU + dW = C_v\, dT + dW \qquad \text{... (23)}$$

where $dU = C_v\, dT$ ... (24)

Since $dT = 0,\ dQ = dW$

$\Rightarrow$ All heat supplied to the system gets converted into the work.

- Work done

$$W = \int_{V_1}^{V_2} P\,dV = \int_{V_1}^{V_2} \frac{nRT}{V} dV$$

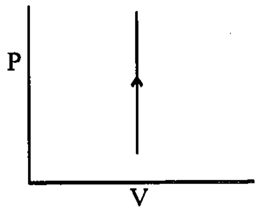
$$\Rightarrow W = nRT\, \ell n \frac{V_2}{V_1} \qquad \text{... (25)}$$

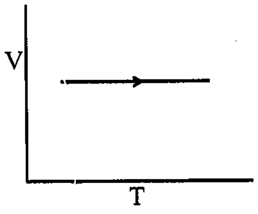
or $$\boxed{W = nRT\, \ell n\left(\frac{P_1}{P_2}\right)} \qquad \text{... (26)}$$

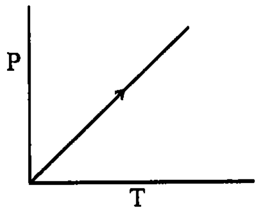
- Iso thermal process are generally slow.

(2) Isochoric process $(\Delta V = 0)$:

- It is the process in which volume remains constant.
- Ideal gas equation $\Rightarrow \frac{P}{T} =$ constant.
- The graphs are







- First law of thermodynamics gives $dQ = dU + PdV$
  Since $dV = 0$,

$$\Rightarrow dQ = dU = C_v dT \qquad \left[\because\ C_v = \left(\frac{dQ}{dT}\right)_v\right] \qquad \text{... (27)}$$

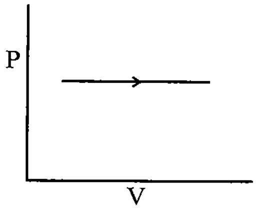
$\Rightarrow$ All heat given to the system is converted to the increase in internal energy.

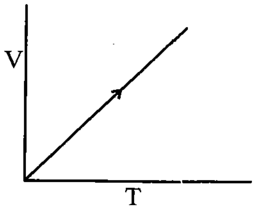
- Work done

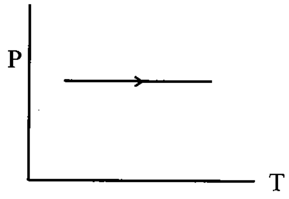
$$\boxed{W=\int_{V_1}^{V_2} PdV=0} \qquad \ldots (28)$$

(3) Isobaric Process $(\Delta P=0)$:

- It is the process for which pressure of the gas remains constant.
- Ideal gas equation $\Rightarrow$ $V \propto T$ or $\frac{V}{T}=$ constant ... (29)
- The graphs are





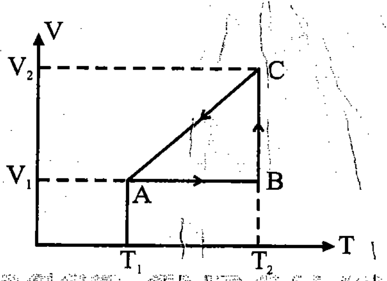


- First law of thermodynamics says $dQ=dU+PdV$ ... (30)

  where $dQ=C_P dT$ ... (31)

- Work done, $W=\int PdV = P\int_{V_1}^{V_2} dV = P(V_2-V_1)$

Problem: Figure shows a process ABCA performed on an ideal gas. The net heat given to the system during the process is



(a) $R\left[T_2 \ln\left(\frac{V_1}{V_2}\right)+(T_1-T_2)\right]$ (b) $R\left[T_2 \ln\left(\frac{V_1}{V_2}\right)+(T_2-T_1)\right]$

(c) $R\left[T_2 \ln\left(\frac{V_2}{V_1}\right)+(T_1-T_2)\right]$ (d) $R\left[T_2 \ln\left(\frac{V_2}{V_1}\right)+(T_2-T_1)\right]$

ln. Since change in internal energy in a cyclic process is zero, the net heat given to the system is the total work.

i.e. $\Delta Q=\Delta W$

$$\Rightarrow \quad \Delta Q = \underset{(\text{isochoric process})}{W_{A\to B}} + \underset{(\text{isothermal process})}{W_{B\to C}} + \underset{(\text{isobaric process})}{W_{C\to A}}$$

$$=0+RT_2\ell n\left(\frac{V_2}{V_1}\right)+P(V_2-V_1)$$

$$=RT_2\ell n\left(\frac{V_2}{V_1}\right)+R(T_2-T_1)=R\left[T_2\ell n\left(\frac{V_2}{V_1}\right)+(T_2-T_1)\right]$$

Correct option is (d).

(4) Adiabatic Process $(\Delta Q = 0)$:

- When a gas expands or contracts without exchanging heat with the surrounding, the process is called *adiabatic process.*
- For this process, the system is insulated from the surrounding.
- Mathematically,

$$\Delta Q = 0 \qquad \text{... (32)}$$

Equation govering is

$$PV^{\gamma} = \text{Constant} \quad \text{(for ideal gas)} \qquad \text{... (33)}$$

Also, $\dfrac{nRT}{V}\cdot V^{\gamma} = \text{Constant} \quad (\because \; PV = nRT)$

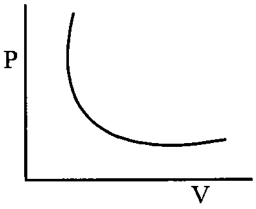
$$\Rightarrow \quad TV^{\gamma-1} = \text{Constant} \qquad \text{... (34)}$$

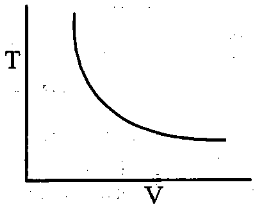
Also, from equation (33), $P\left(\dfrac{nRT}{P}\right)^{\gamma} = \text{Constant}$

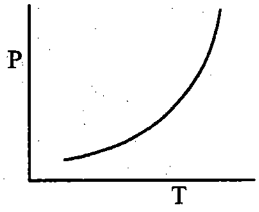
$$\Rightarrow \quad P^{1-\gamma}\, T^{\gamma} = \text{Constant} \qquad \text{... (35)}$$

$$\text{where } \gamma = \frac{C_P}{C_V} = 1 + \frac{2}{f}$$

- The graphs are







- First Law of Thermodynamics : $dQ = dU + dW$

Since $dQ = 0, \; dW = -dU$

$\Rightarrow$ The workdone by the gas is at the expense of its internal energy.

- Work done

$$W = \int_{V_1}^{V_2} \frac{K}{V^{\gamma}}\, dV = \frac{K\left[V_2^{1-\gamma} - V_1^{1-\gamma}\right]}{1-\gamma} = \frac{KV_2^{1-\gamma} - KV_1^{1-\gamma}}{1-\gamma}$$

where $K = P_1 V_1^{\gamma} = P_2 V_2^{\gamma}$ =constant of proportionality

$$\therefore \; W = \frac{P_2 V_2^{\gamma} \cdot V_2^{1-\gamma} - P_1 V_1^{\gamma} V_1^{1-\gamma}}{1-\gamma}$$

$$\Rightarrow \; W = \frac{P_2 V_2 - P_1 V_1}{1-\gamma} = \frac{P_1 V_1 - P_2 V_2}{\gamma - 1}$$

Or $\boxed{W = \dfrac{nR}{\gamma-1}(T_1 - T_2)}$

- Adiabatic process are generally very fast e.g. the bursting of a tyre.

Problem: A motor car tyre has a pressure of 2 atomsphere at room temperature 27°C. If the tyre suddenly bursts, find the resulting temperature.

Soln. $P_1 = 2$ atomsphere $\quad P_2 = 1$ atomsphere

$T_1 = 27°C = 27 + 273 = 300$ K $\quad T_2 = ?$ $\quad \gamma = 1.4$ for air

Using, $\dfrac{P_1^{\gamma-1}}{T_1^{\gamma}} = \dfrac{P_2^{\gamma-1}}{T_2^{\gamma}}$

$$\Rightarrow \left(\frac{P_2}{P_1}\right)^{\gamma-1} = \left(\frac{T_2}{T_1}\right)^{\gamma}$$

$$\Rightarrow \left(\frac{1}{2}\right)^{1.4-1} = \left(\frac{T_2}{300}\right)^{1.4} \qquad \Rightarrow \left(\frac{1}{2}\right)^{0.4} = \left(\frac{T_2}{300}\right)^{1.4}$$

Taking log both side,

$-0.4 \log 0.5 = 1.4 \log T_2 - 1.4 \log 300$

$\Rightarrow \quad -0.1204 = 1.4 \log T_2 - 3.4680$

$\Rightarrow \quad 1.4 \log T_2 = 3.4680 - 0.1204 = 3.3476$

$\Rightarrow \quad \log T_2 = 3.3476/1.4 = 2.3911$

$\Rightarrow \quad T_2 = 246.1 \text{ K} = -26.9°C$

Problem: A quantity of air at 27°C and atmospheric pressure is suddenly compressed to half its original volume. Find the (i) pressure and (ii) temperature.

Soln. Given $P_1 = 1$ atm, $P_2 = ?$, $\gamma = 1.4$

$V_1 = V$, $V_2 = V/2$, $T_1 = 27°C = 300K$, $T_2 = ?$

For an adiabatic process, $P_1V_1^{\gamma} = P_2V_2^{\gamma}$

$$\Rightarrow \quad P_2 = P_1\left(\frac{V_1}{V_2}\right)^{\gamma} = 1\left(\frac{V}{\frac{V}{2}}\right) = 1(2)^{1.4} = 2.636\,atm$$

Also, $\quad T_1V_1^{\gamma-1} = T_2V_2^{\gamma-1}$

$$\Rightarrow \quad T_2 = T_1\left(\frac{V_1}{V_2}\right)^{\gamma-1} = 300\left(\frac{V}{\frac{V}{2}}\right)^{1.4-1} = 300\,(2)^{0.4} = 395.9\text{K} = 122.9°\text{C}$$

Problem: Air is compressed adiabatically of half its volume. Calculate the change in its temperature.

Soln. Let the initial temperature be T_1K and final temperature T_2K

Initial volume $V_1 = V$ and final volume $V_2 = \dfrac{V}{2}$

For adiabatic process, $\quad T_1V_1^{\gamma-1} = T_2V_2^{\gamma-1}$

$$\Rightarrow \quad T_2 = T_1\left(\frac{V_1}{V_2}\right)^{\gamma-1} \quad or \quad T_2 = T_1(2)^{1.4-1}$$

$$\Rightarrow \quad T_2 = (2)^{0.4}\,T_1 = 1.319T_1$$

So change in temperture is

$$T_2 - T_1 = 1.319T_1 - T_1 = 0.319T_1 K$$

Problem: 1 gm molecule of a manoatomic $\left(\gamma = \frac{5}{3}\right)$ perfect gas at 27°C adiabatically compressed in reversible process from an initial pressure 1 atmosphere to a final pressure of 50 atmosphere. Calculate the resulting difference in temperature.

Soln. Given: $P = 1$ atm, $P_2 = 50$ atm, $T_1 = 27°\text{C} = 300\text{K}$, $T_2 = ?\ \gamma = \frac{5}{3}$

Using $\frac{P_1^{\gamma-1}}{T_1^{\gamma}} = \frac{P_2^{\gamma-1}}{T_2^{\gamma}}$ or $\left(\frac{P_2}{P_1}\right)^{\gamma-1} = \left(\frac{T_2}{T_1}\right)^{\gamma}$

$$\Rightarrow \quad (50)^{\frac{5}{3}-1} = \left(\frac{T_2}{300}\right)^{\frac{5}{3}} \Rightarrow 50^{\frac{2}{3}} = \left(\frac{T_2}{300}\right)^{\frac{5}{3}}$$

$$\Rightarrow \quad \frac{2}{3}\log 50 = \frac{5}{3}\log T_2 - \frac{5}{3}\log 300$$

$$\Rightarrow \quad T_2 = 1434\,K = 1161°C$$

Problem: If the equation of state for a gas with internal energy U is $PV = \frac{1}{3}U$ then the equation for an adiabatic process is:

(a) $PV^{1/3} = \text{constant}$ (b) $PV^{2/3} = \text{constant}$ (c) $PV^{4/3} = \text{constant}$ (d) $PV^{3/5} = \text{constant}$

Soln. For adiabatic process,

$$dQ = 0$$

$$\Rightarrow \quad dU = -dW = -PdV$$

But $PV = \frac{1}{3}U \Rightarrow U = 3PV$

$$\Rightarrow \quad d(3PV) = -PdV \Rightarrow 3\,PdV + 3\,VdP = -PdV$$

$$\Rightarrow \quad 3\,VdP = -4PdV \quad \text{or} \quad \frac{dP}{P} = -\frac{4}{3}\frac{dV}{V}$$

Integrating, $\log P = -\frac{4}{3}\log V + \text{constant} \Rightarrow \log PV^{4/3} = \text{constant}$

$$\Rightarrow \quad \text{or} \quad PV^{4/3} = \text{constant}$$

Hence, correct option is (c).

Problem: An ideal gas expands adiabatically but quasistatically from state $i(P_i V_i T_i)$ to a state $f(P_f V_f T_f)$. What is the change in its internal energy?
What would this change be if the expansion were isothermally?

Soln. If the process is performed in such a way that at any instant during the process the system is verynearly in the thermodynamic equilibrium, the process is called *quasistatic,* this means we can specify pressure, volume, temperature uniquely at any instant during such a process. Thus a quasistatic process is an idealised process in which all changes take place infinitely slowly.

From first law $dQ = dU + dW$
But for adiabatic process $dQ = 0$
So $dU = -dW$

Total change in internal energy = $\int_i^f dU = -\int_i^f PdV = -\int_i^f \frac{K}{V^\gamma}dV$

So $U_f - U_i = -K\left(\frac{V^{-\gamma+1}}{-\gamma+1}\right)_i^f$ $\left(\because PV^\gamma = K \Rightarrow P = \frac{K}{V^\gamma}\right)$

$= \frac{K}{1-\gamma}\left(V^{-\gamma+1}\right)_i^f$

But $P_iV_i^\gamma = P_fV_f^\gamma = K$

So $U_f - U_i = \frac{1}{1-\gamma}\left(P_fV_f^\gamma V_f^{-\gamma+1} - P_iV_i^\gamma V_i^{-\gamma+1}\right)$

$= \frac{1}{1-\gamma}\left(P_fV_f - P_iV_i\right)\ (\because PV = nRT) = \frac{nR}{1-\gamma}\left(T_f - T_i\right)$

(ii) for isothermal expansion $T_i = T_f$

$$U_f - U_i = \frac{f}{2}nR\left(T_f - T_i\right) = 0$$

So for adiabatic isothermal process, the change in internal energy is zero.

Problem: A monoatomic ideal gas, initially at temperature T_1, is enclosed in a cylinder fitted with a frictionless piston. The gas is allowed to expand adiabatically to a temperature T_2 by releasing the piston suddenly. If L_1 and L_2 are the lengths of the gas column before and after expansion respectively, then T_1/T_2 is given by

(a) $(L_1/L_2)^{2/3}$ (b) (L_1/L_2) (c) L_2/L_1 (d) $(L_2/L_1)^{2/3}$ **[IIT-JEE : 2000]**

Soln. During adiabatic expansion, we know

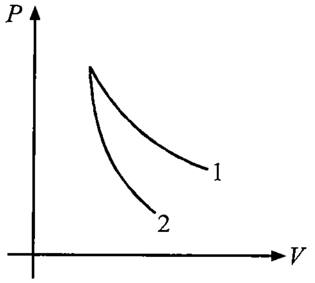
$TV^{\gamma-1} = \text{constant}$ or $T_1V_1^{\gamma-1} = T_2V_2^{\gamma-1}$

For a monoatomic gas, $\gamma = \frac{5}{3}$

$\therefore \frac{T_1}{T_2} = \left(\frac{V_2}{V_1}\right)^{\gamma-1} = \left(\frac{AL_2}{AL_1}\right)^{(5/3)-1}$ (A = Area of cross-section of piston)

$= \left(\frac{L_2}{L_1}\right)^{2/3}$

Problem: P-V plots for two gases during adiabatic processes are shown in the figure. Plots 1 and 2 should correspond respectively to



(a) He and O_2 (b) O_2 and He (c) He and Ar (d) O_2 and N_2

[IIT-JEE : 2001]

oln. In adiabatic process, PV^γ = constant

Therefore, slope of P-V graph, $\frac{dP}{dV} = -\gamma\frac{P}{V}$

$\Rightarrow$ slope $\propto \gamma$ (with negative sign)

From the given graph, $(\text{slope})_2 > (\text{slope})_1$

$\therefore \gamma_2 > \gamma_1$

Therefore, 1 should correspond to O_2 $(\gamma = 1.4)$ and 2 should correspond to He $(\gamma = 1.67)$.

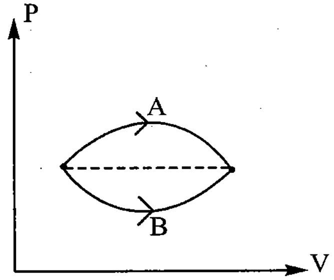
Hence, correct option is (b)

(5) Cyclic Process

- In a *cyclic process*, the system returns to its intial state.
- Since internal energy is a state function, $\Delta U = 0$
- First Law of Thermodynamics $\Rightarrow \Delta Q = \Delta U + \Delta W \Rightarrow \Delta Q = \Delta W$

Problem: Figure shows the process A and B on a system. Let ΔQ_1 and ΔQ_2 be the heat given to the system in the process A and B respectively then

(a) $\Delta Q_1 > \Delta Q_2$ (b) $\Delta Q_1 = \Delta Q_2$ (c) ΔW (d) $\Delta Q_1 \leq \Delta Q_2$



oln. Since ΔU is same for both the process and ΔW is the amount of work which is represented by the area under the curve and since area under A is greater than area under B.

$$\Delta W_1 > \Delta W_2 \Rightarrow \Delta W_1 + \Delta U > \Delta W_2 + \Delta U$$

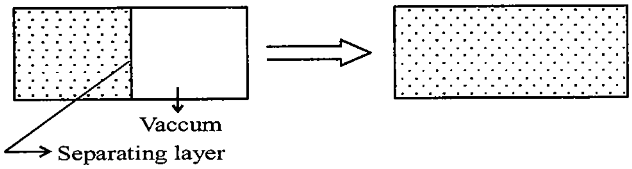
So $\Delta Q_1 > \Delta Q_2$

If both the arrow signs are changed then $\Delta Q_1 < \Delta Q_2$

(6) Free expansion

- In a system say if a gas expands in such a way that no heat enters or leave the system (adiabatic process) and also no work is done by or on the system, the gas is said to be expanded freely.
- For free expansion, $dQ = 0$ and $dW = PdV = 0$
  (There is no pressure acting on the gas during free expansion)
- First Law of Thermodynamics $\Rightarrow dU = 0$

e.g. If initially the gas is contained in half of the box and the other half has vacuum in it. When the separating layer is suddenly removed, the gas expands freely in the whole box.



Problem: The internal energy of n moles of a gas is given by $E = \frac{3}{2}nRT - \frac{a}{v}$ where V is the volume of the gas at temperature T and a is a positive constant. One mole of the gas in state (T_1, V_1) is allowed to expand adiabatically into vacuum to a final state (T_2, V_2). The temperature T_2 is:

(a) $T_1 + \frac{a}{R}\left(\frac{1}{V_2} + \frac{1}{V_1}\right)$ (b) $T_1 - \frac{2}{3}\frac{a}{R}\left(\frac{1}{V_2} - \frac{1}{V_1}\right)$

(c) $T_1 + \frac{2}{3}\frac{a}{R}\left(\frac{1}{V_2} - \frac{1}{V_1}\right)$ (d) $T_1 - \frac{1}{3}\frac{a}{R}\left(\frac{1}{V_2} - \frac{1}{V_1}\right)$

ln. Since the process is free expansion process. So the change in internal energy for free expansion process is zero.

$$\Rightarrow \quad U_2 - U_1 = 0$$

$$\Rightarrow \quad \frac{3}{2}nRT_1 - \frac{a}{V_1} = \frac{3}{2}nRT_2 - \frac{a}{V_2}$$

$$\Rightarrow \quad \frac{3}{2}nRT_2 = \frac{3}{2}nRT_1 - \frac{a}{V_1} + \frac{a}{V_2} = \frac{3}{2}nRT_1 + a\left(\frac{1}{V_2} - \frac{1}{V_1}\right)$$

For $n = 1$,

$$\Rightarrow \quad \boxed{T_2 = T_1 + \frac{2a}{3R}\left(\frac{1}{V_2} - \frac{1}{V_1}\right)}$$

Hence correct option is (c).

Problem: Two gases separated by a impermeable but movable partition are allowed to freely exchange energy. At equilibrium, the two sides will have the same

(a) pressure and temperature (b) volume and temperature

(c) pressure and volume (d) volume and energy

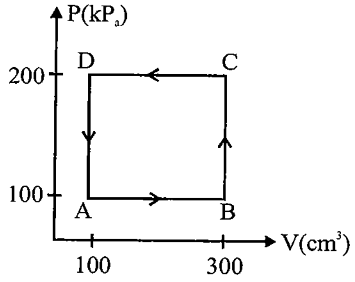
ln. Once the system is released freely, due to exchange of energy both sides will attain thermal equilibrium by attaining the same temperature. Also, due to unbalanced pressure, both the sides will attain mechanical equilibrium by balancing the pressure on both sides of the movable wall.

Hence correct option is (a).

Problem: A thermodynamic system is taken through the cycle $ABCD$ as shown in figure

(i) Calculate the work done by the gas (system) during the path AB, BC, CD, DA

(ii) Find the total heat rejected by the gas during the process $ABCDA$.



n. (i) $W_1 = PdV = 100\ kPa \times 200\ cm^3$ (along AB)

$$= 100 \times 1000 \times \frac{200}{10^6} \frac{Pa}{m^3} = 20\ J$$

$W_2 = 0$ because the change in volume is zero. (along BC)

$W_3 = -P \times dV = -200\ kPa \times 200\ cm^3$ (along CD)

$$= -200 \times 1000 \times \frac{200}{10^6} \frac{Pa}{m^3}$$

$$= -40\ J$$

The negative sign is due to decrease in volume

$W_4 = 0$ because the change in volume is zero. (along DA)

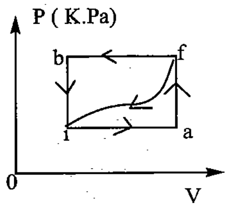
$\therefore\ W = W_1 + W_2 + W_3 + W_4 = 20 - 40 = -20J$

(ii) $dQ = dU + dW$

For cyclic process $dU = 0$

$dQ = dW = -20J \Rightarrow$ Heat rejected by the gas

Problem: When a system is taken from a state i to f along the path iaf as shown in figure. It is found that heat Q absorbed by the system is 50 cal and the work done by the system is 20 cal. Along the path ibf $Q = 36$ cal.



(i) what is the work done along the path ibf?

(ii) If the work done on the system is equal to 13 cal for the curved returned path fi. What is the value Q for this path?

(iii) If internal energy of the system in initial state is equal to 10 cal. What is the internal energy of the system in final state ?

(iv) If internal energy of the system for state b = 22 cal. What are the values Q for the process bf and ib?

Soln. Given $U_i = 10$ cal, $Q = 50$ cal and $W = 20$ cal for the path iaf,

(i) $Q = U_f - U_i + W$

$\Rightarrow\ 50 = U_f - 10 + 20$

$\Rightarrow\ \boxed{U_f = 40\ cal}$

So work done along path ibf,

As $dQ = 36$

$dQ = dU + dW$

$\Rightarrow\ dW = dQ - dU = 36 - (40 - 10) = 6$ cal

(ii) $W = 13$ cal for $(f \rightarrow i)$

$W = -13$ cal

$dQ = dU + dW$

$= (U_i - U_f) - 13 = -30 - 13 = -43$ Cal

(iii) $\Delta U = 30 \Rightarrow U_f - U_i = 30$

$\Rightarrow U_f - 10 = 30 \Rightarrow U_f = 40$

(iv) $U_b = 22$ cal, $U_f = 40$ cal

So for path $bf \Rightarrow U_f - U_b = 18$ cal

As $\Delta W = 6$ $\{\because\ i \to b, \Delta W = 0\}$

$dQ = 18 + 6 = 24\ cal$

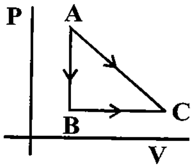
For bath $i \to b$

$U_b = 22\ cal \quad U_i = 10\ cal$

$\Delta U = 12\ cal, \quad \Delta W = 0, \quad \Delta Q = \Delta U + \Delta W$

$\Rightarrow \quad \Delta Q = 12\ cal$

Problem: A given quantity of gas is taken from the state $A \to C$ reversibly, by two paths, $A \to C$ directly and $A \to B \to C$ as shown in the figure below.



During the $A \to C$ the work done by the gas is 100 J and the heat absorbed is 150 J. If during the process $A \to B \to C$ the work done by the gas is 30 J, the heat absorbed is:

(a) 20J (b) 80J (c) 220 J (d) 280 J

Soln. Using $dQ = dU + dW_1$

For path AC: $150\ \text{J} = dU + 100\ \text{J}$

$\Rightarrow dU = 150\ \text{J} - 100\text{J} = 50\ \text{J}$

For path ABC: $dQ = dU + dW_2 = 50\text{J} + 30\text{J} = 80\text{J}$ ($\because$ internal energy is a state function)

Correct option is (b)

Problem: Figure shows a process $ABCA$ performed on an ideal gas. Find the net heat given to the system during the process.

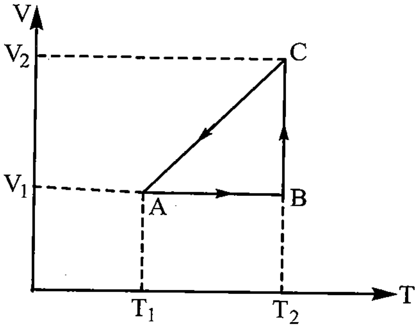
Soln. Since the process is cyclic so the change in internal energy $dU = 0$

Since $dU = 0 \Rightarrow dQ = dW = PdV$

For the process AB change in volume = Zero

$\Rightarrow W_1 = 0$

For process BC, temperature is constant



$$W_2 = \int_{V_1}^{V_2} dW_2 = \int_{V_1}^{V_2} PdV = \int_{V_1}^{V_2} \frac{TR}{V} dV$$

$$W_2 = RT\left[\ell n\, V\right]_{V_1}^{V_2} = RT_2 \ell n \frac{V_2}{V_1}$$

For process $CA, \frac{V}{T} = \text{constant}$

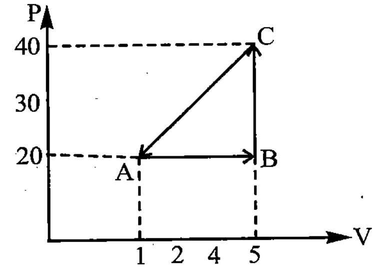
Also, $PV = RT \Rightarrow \frac{V}{T} = \frac{R}{P} \Rightarrow V = \frac{RT}{P} \Rightarrow dV = \frac{R}{P}dT$

$$W_3 = \int_{T_2}^{T_1} dW = \int_{T_2}^{T_1} PdV = \int_{T_2}^{T_1} P\frac{R}{P}dT = [RT]_{T_2}^{T_1} = R(T_1 - T_2)$$

$$W_3 = R(T_1 - T_2)$$

Now, $Q = W = W_1 + W_2 + W_3 = R\left[T_2 \ell n \frac{V_2}{V_1} + (T_1 - T_2)\right]$

Problem: A gas is carried through a cycle $ABCA$ as shown in the P–V diagram. Determine weather the heat is given to the system or by the system, work is done by the system or on the system, internal energy of the system increases or decreases in the following process.



(a) For the process AB if $\Delta U = +\text{Ve}$

(b) for the process BC if $Q = +\text{Ve}$

(c) For process CA

(d) Net work done in the complete cycle.

Soln. (i) $\Delta U = +Ve$ (given) $\Rightarrow$ so internal energy of the system increase

$\Delta W = +Ve$ (work done by the system because volume increase)

$\Delta Q = \Delta U + \Delta W = +Ve$ (Heat given to the system)

(ii) For BC

$Q = +Ve$ (given) $\Rightarrow$ Heat given to the system

$\Delta W = 0$ ($W = PdV$ and V is constant)

$\Delta U = +Ve$ $\Delta Q = \Delta U + \Delta W$. Internal energy of the system increase)

(iii) For process CA

For process ABC ΔU = positive $\left(\because (\Delta U)_{AC} = (\Delta U)_{AB} + (\Delta U)_{BC} = (+Ve) + (+Ve)\right)$

For CA Process $(\Delta U)_{CA} = -(\Delta U)_{AC}$ = negative

$(\Delta U)_{CA} = -ve$ (Internal energy of the system decrease)

$\Delta W = -ve$ ($\because$ volume decrease so work done on the system)

$\Delta Q = -ve$ ($\Delta Q = \Delta U + \Delta W = -ve$, heat is given by the system)

(iv) Net workdone in complete cycle = area under the curve

$$= \frac{1}{2} \times AB \times BC = \frac{1}{2} \times 4 \times 20 = 40 \text{ Joule.}$$

Slope of Adiabatic and Isothermal Process

(a) Isothermal Process :

$$PV = \text{constant} \Rightarrow PdV + VdP = 0 \Rightarrow \left(\frac{dP}{dV}\right)_{\text{iso}} = -\frac{P}{V}$$

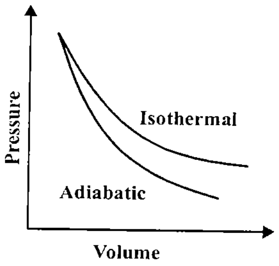
(b) Adiabatic Process :

$$PV^{\gamma} = \text{Constant} \Rightarrow P\gamma V^{\gamma-1}dV + dP\cdot V^{\gamma} = 0 \Rightarrow \frac{dP}{dV} = -\frac{PV^{\gamma-1}\gamma}{V^{\gamma}}$$

$$\Rightarrow \left(\frac{dP}{dV}\right)_{\text{adia}} = -\gamma\frac{P}{V}$$

$$\therefore \quad \boxed{\frac{(dP/dV)_{\text{adia.}}}{(dP/dV)_{\text{iso.}}} = \gamma > 1}$$

(Slope of adiabatic is γ-times than that of isothermal).



Bulk Modulus or Incompressibility

The *bulk modulus* or *incompressibility* of a substance is a measure of how incompressible/resistant to compressibility that substance is. It is defined as the ratio of the infinitesimal pressure increase to the resulting relative decrease of the volume.

Mathematically, $K = -V\left(\frac{\partial P}{\partial V}\right)_T$

Isothermal Compressibility

In thermodynamics and fluid mechanics, *compressibility*, (also known as coefficient of compressibility or isothermal compressibility) is a measure of the relative volume change of a fluid or solid as a response to a pressure (or mean stress) change.

Mathematically, $\beta = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T$

Coefficient of Volume Expansion

It is a measure of how easily a substance expands on increasing its temperature. It is defined as the ratio of the infinitesimally increase in the volume of the substance to the infinitesimally increase in the temperature per unit original volume.

Mathematically, $\alpha = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P$

Problem: For a gas under isothermal conditions, its pressure P varies with volume V as $P \propto V^{-5/3}$. The bulk modulus B is proportional to

(a) $V^{-1/2}$ (b) $V^{-2/3}$ (c) $V^{-3/5}$ (d) $V^{-5/3}$

ln. According to ideal gas equation, we get $PV = nRT$. For isothermal condition, we get $dPV + PdV = 0$.

$$\Rightarrow P = -\frac{dP}{dV/V} = B$$

So, $P = B$ ...(i)

But given that $P \propto V^{-5/3}$ ...(ii)

Comparing equations (i) and (ii), we get $B \propto V^{-5/3}$.

Hence correct option is (d).

Problem: The isothermal compressibility, κ of an ideal gas at temperature T_0 and volume V_0, is given by

(a) $-\frac{1}{V_0}\frac{\partial V}{\partial P}\bigg|_{T_0}$ (b) $\frac{1}{V_0}\frac{\partial V}{\partial P}\bigg|_{T_0}$ (c) $-V_0\frac{\partial P}{\partial V}\bigg|_{T_0}$ (d) $V_0\frac{\partial P}{\partial V}\bigg|_{T_0}$

Soln. Isothermal incompressibility is defined as $-\frac{\partial P}{\partial V/V_0}\bigg|_{T_0}$

Hence correct option is (a).

Problem: Express the isothermal compressibility $-\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T$ of one mole of vander Waal's gas as a function of temperature and volume.

Soln. Isothermal compressibility is defined as

$$\beta = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T \qquad \text{(for one mole)}$$

The vander Waal gas equation, is

$$\left(P+\frac{a}{V^2}\right)(V-b) = RT$$

$$\Rightarrow \quad P = \frac{RT}{V-b} - \frac{a}{V^2}$$

$$\therefore \quad \left(\frac{\partial P}{\partial V}\right)_T = -\frac{RT}{(V-b)^2} + \frac{2a}{V^3}$$

$$\therefore \quad \beta = -\frac{1}{V}\left(\frac{\partial P}{\partial V}\right)_T = \frac{-1}{V\left[\frac{RT}{(V-b)^2} + \frac{2a}{V^3}\right]} \Rightarrow \boxed{\beta = \frac{1}{\frac{RTV}{(V-b)^2} - \frac{2a}{V^2}}}$$

PROBLEMS TO BE PLAYED

1. The change in the internal energy of the gas is directly proportional to–
(a) The change in volume (b) Change in pressure
(c) Change in temperature (d) None of these

2. The first law of thermodynamics is conservation of–
(a) Momentum (b) Energy (c) Both (a) and (b) (d) None of these

3. A thermos bottle containing coffee is vigorously shaken and thereby the temperature of coffee rises. Considering the coffee as the system, which of the following is ***CORRECT***.
(a) $Q = 0$; $W = -$ve; $dU = +$ ve (b) $Q = 0$; $W = +$ ve; $dU = +$ ve
(c) $Q = 0$; $W = -$ve; $dU = -$ve (d) $Q = 0$; $W = +$ ve; $dU = -$ ve

4. A resistor immersed in running water carries an electric current. Consider the resistor as the system–
(a) $Q = 0$; $W = -$ve; $dU = -$ve (b) $Q = 0$; $W = -$ve; $dU = +$ ve
(c) $Q = 0$; $W = +$ ve; $dU = -$ve (d) $Q = 0$; $W = +$ ve; $dU = +$ ve

5. The quantity remaining constant in the isothermal expansion of an ideal gas is–
(a) Heat (b) Internal energy
(c) Pressure (d) Temperature and pressure

6. The work done W during an isothermal process in which the gas expands from an initial volume V_1 to a final voluem V_2 is given by–
(R = Gas constant, T = Temperature)
(a) $R(V_2 - V_1)\log_e\left(\frac{T_1}{V_2}\right)$ (b) $(T_2 - T_1)\log_e\left(\frac{V_2}{V_1}\right)$
(c) $RT\log_e\left(\frac{V_2}{V_1}\right)$ (d) $RT\log_e\left(\frac{V_1}{V_2}\right)$

. If a system A is in thermal equilibrium separately with B and C, then B and C are also in thermal equilibrium with each other. This is the statement of–
(a) Zeroth law of thermodynamics (b) First law of thermodynamics
(c) Second law of thermodynamics (d) Third law of thermodynamics

. In a cyclic process
(a) Work done is zero
(b) Work done by the system is equal to the quantity of heat given to the system
(c) Work done does not depend on the quantity of heat given to the system
(d) The internal energy of the system increases

. One mole of a perfect gas expands adiabatically. As a result of this, its pressure, temperature and volume change from P_1, T_1, V_1 to P_2, T_2 and V_2 respectively. If molar specific heat at constant volume is C_V, then the work done by the gas is–
(a) $2.303\ P_1V_1\ \log\frac{V_2}{V_1}$ (b) $RT_1\log\frac{V_2}{V_1}$ (c) $\frac{P_1^2V_1^2 - P_2^2V_2^2}{R(T_2 - T_1)}$ (d) $C_V(T_2 - T_1)$

0. For a thermodynamic system work done in a process depends upon–
(a) The path (b) State of the system
(c) External pressure (d) Nature of the system

. Work done in an isothermal change in a gas is
(a) $W = T\ \log_e\frac{V_2}{V_1}$ (b) $W = R\ \log_e\left(\frac{V_2}{V_1}\right)$
(c) $W = RT\ \log_{10}\left(\frac{V_1}{V_2}\right)$ (d) $W = RT\cdot\log_{10}\left(\frac{V_2}{V_1}\right)$

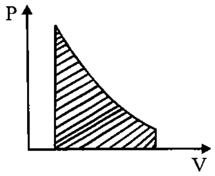
12. In an isobaric process work done is

(a) $W = RT \log_e\left(\frac{V_2}{V_1}\right)$ (b) $W = \frac{R}{\gamma - 1}(T_1 - T_2)$

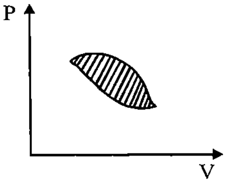
(c) $W = P(V_2 - V_1)$ (d) $W = 0$

13. The work done by a gas in adiabatic expansion is represented by the diagram–

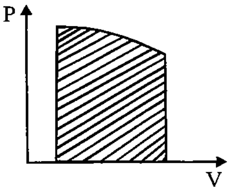
(a)



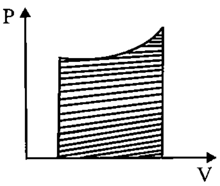
(b)



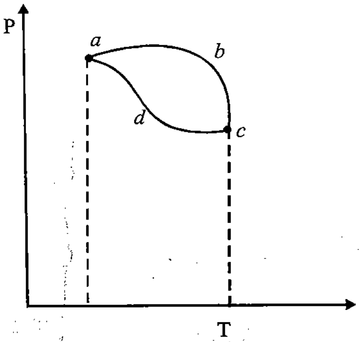
(c)



(d)



14. In the *PV* diagram below which is the correct statement?



(a) Any point on the line *abc* represents a higher temperature than any point on the line *adc*

(b) The area *abcda* represents the work done by the system for the process

(c) Heat is lost to the surroundings during the process is represented by path *adc*

(d) The path *abc* and *adc* represents isothermals

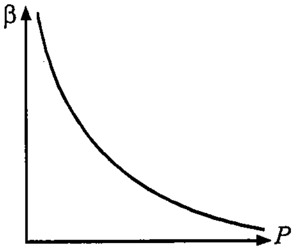
15. In an isothermal change the internal energy of molecules–

(a) May increase or decrease (b) Does not change

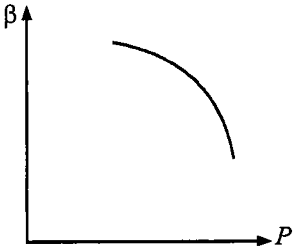
(c) Increases (d) Decreases

16. Which of the following graph correctly represent the variation of $\beta = -\frac{dV/dP}{V}$ with *P* for an ideal gas at constant temperature ?

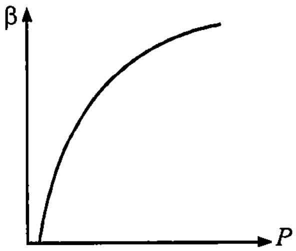
(a)



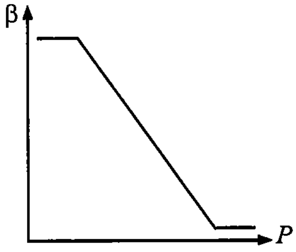
(b)



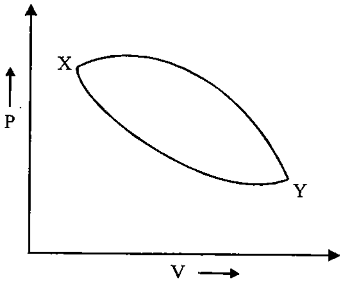
(c)



(d)



17. A piston containing an ideal gas is originally in the state X (see fig.). The gas is taken through a thermal cycle $X \to Y \to X$ as shown. They work done by the gas is positive if the direction of the thermal cycle is



(a) Clockwise (b) Counter clockwise
(c) Neither clockwise nor counter clockwise
(d) Clockwise from $X \to Y$ and counter clockwise from $Y \to X$

18. An ideal gas in a cylinder is compressed adiabatically to one-third of its initial volume. During this process 20J work is done on the gas by compressing agent. Which of the following statements is true for this case?
(a) Change in the internal energy in this process is zero
(b) The internal energy increases by 20 J
(c) The internal energy decreases by 20 J
(d) Temperature of the gas decreases

19. The relation connecting pressure and volume in an adiabatic expansion is given by PV^n = constant. If the specific heat at constant pressure is l and the specific heat at constant volume is m, then n is given by–

(a) $\frac{m}{l}$ (b) $\frac{l}{m}$ (c) $l-m$ (d) $m-l$

20. The internal energy of a system is the
(a) potential energy of the particle of the system
(b) kinetic energy of the particle of the system
(c) sum of kinetic and mutual potential energies of the particles of the system
(d) difference between the kinetic and mutual potential energies of the system

21. A vessel contains an ideal monoatomic gas, which expands at constant pressure when heat Q is supplied to it. Then work done in the expansion is

(a) Q (b) $\frac{3}{5}Q$ (c) $\frac{2}{5}Q$ (d) $\frac{2}{3}Q$

22. The slopes of isothermal and adiabatic curves are related as
(a) Isothermal curve slope = Adiabatic curve slope
(b) Isothermal curve slope = γ × Adiabatic curve slope
(c) Adiabatic curve slope = γ × Isothermal curve slope

(d) Adiabatic curve slope = $\frac{\gamma}{2}$ × Isothermal curve slope

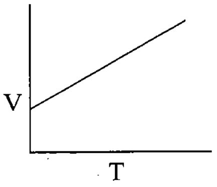
23. Two bodies are said to be in thermal equilibrium when they are equal in
(a) temperature (b) heat content
(c) heat capacity (d) specific heat

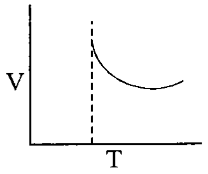
24. If the equation of state for a gas with internal energy U is $PV = \frac{1}{3}U$, then the equation for an adiabatic process is:

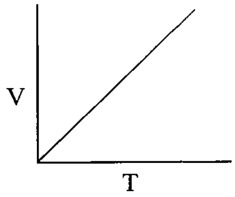
(a) $PV^{1/3}$ = constant (b) $PV^{2/3}$ = constant (c) $PV^{4/3}$ = constant (d) $PV^{3/5}$ = constant

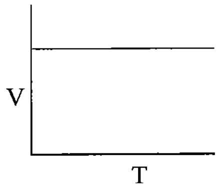
25. When a perfect gas is supposed to expand freely against vacuum in a insulated vessel, the gas has undergone–
(a) An increase in pressure (b) A change in temperature
(c) A change in entropy (d) A change in phase

26. A gas obey the equation of state $P(V-b) = RT$ where b is a constant. For this gas a plot of V versus T at constant P (isobar) will be–

(a)


(b)


(c)


(d)


27. The general gas equation $PV = RT$ was improved by Vander Waal who introduced a correction factor $\frac{a}{V^2}$ which is added to P. The term, $\frac{a}{V^2}$, relates to

(a) Volume of the molecules
(b) Effective area of the molecules
(c) Average velocity of the molecules
(d) Force of attraction between the molecules

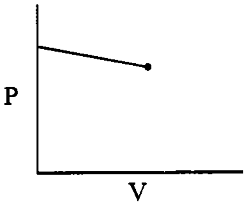
28. The pressure of a gas filled in a vessel is P, if the masses of all the molecules of the gas are halved and their speeds doubled, the new pressure will be–

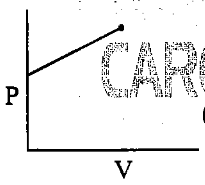
(a) $4P$ (b) $2P$ (c) P (d) $\frac{P}{2}$

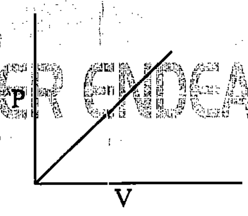
29. For a gas obeying Vander Waal's equation the critical constants can be related correctly by

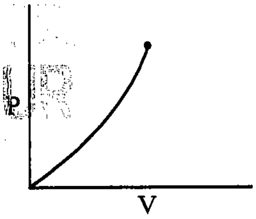
(a) $\frac{RT_c}{P_cV_c} = \frac{8}{3}$ (b) $\frac{P_cV_c}{RT_c} = \frac{3}{8}$ (c) $\frac{RT_c}{P_cV_c} = \frac{3}{8}$ (d) none of these

30. Which of the following graphs satisfy Boyle's Law?

(a)


(b)


(c)


(d)


31. On suffering adiabatic expansion the internal energy of a gas–

(a) Increases
(b) Decreases
(c) Remains unchanged
(d) May increase or decrease

ANSWER KEY

1. (c)	**2. (b)**	**3. (a)**	**4. (b)**	**5. (b)**	**6. (c)**	**7. (a)**
8. (b)	**9. (d)**	**10. (a)**	**11. (d)**	**12. (c)**	**13. (a)**	**14. (b)**
15. (b)	**16. (a)**	**17. (d)**	**18. (c)**	**19. (b)**	**20. (c)**	**21. (c)**
22. (c)	**23. (a)**	**24. (c)**	**25. (a)**	**26. (a)**	**27. (d)**	**28. (b)**
29. (a)	**30. (c)**	**31. (b)**				

Chapter 3

The Carnot Engine

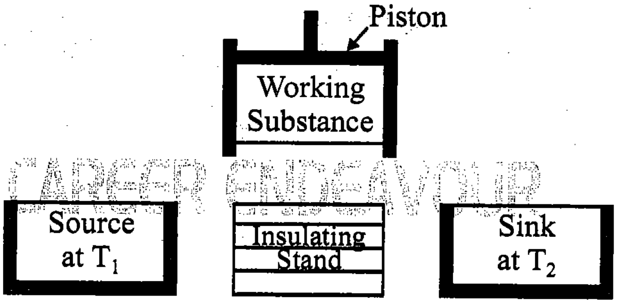
The Carnot Engine

Suppose we have a hot reservoir at temperature T_1 and a cold reservoir at temperature T_2. What is the maximum efficiency possible for a heat engine operating between the two reservoir and what cycle of processes should be adopted to achieve the maximum efficiency ? Sadi Carnot, a French engineer first considered these questions in 1824.

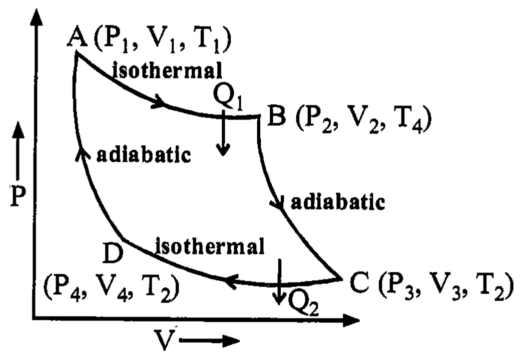
A *Carnot engine* governs by *Carnot cycle* which is defined as any cyclic process bounded by two reversible isothermals and two reversible adiabatics.

It consists of the following :

1. A cylinder fitted with piston
2. A source at higher temperature (T_1)
3. A source at lower temperature (T_2)
4. An insulating stand


Piston
Working Substance
Source at T_1
Insulating Stand
Sink at T_2

The Carnot cycle is shown as :


A (P_1, V_1, T_1)
isothermal
Q_1
B (P_2, V_2, T_4)
adiabatic
P
adiabatic
D
isothermal
(P_4, V_4, T_2)
Q_2
C (P_3, V_3, T_2)
V

The various steps in the Carnot engine are as follows :

(a) Step $A \to B$: Isothermal expansion of the gas.

Since the internal energy of an ideal gas is a function of its temperature only, the internal energy is constant in the isothermal process $A \to B$ and the heat flow Q_1 into the gas in this process is equal to the work done by the gas W_1. So we have

$$Q_1 = W_1 = nRT_1 \ln\left(\frac{V_2}{V_1}\right) \qquad \text{... (1)}$$

where V_1 and V_2 are the volumes in states A and B, respectively.

(b) Step $B \to C$: Adiabatic expansion of the gas.

To expand adiabatically, the gas needs to perform work W_2 which it does at the expense of its internal energy and hence fall in its temperature. So, we have

$$W_2 = nR\frac{(T_1 - T_2)}{\gamma - 1} \qquad \text{... (2)}$$

(c) Step $C \to D$: Isothermal compression of the gas.

The gas is compressed isothermally from the volume V_3 to V_4. This requires work W_3 need to be done on the gas. This is the amount of heat Q_2 released by the gas to the reservoir. This gives

$$Q_2 = W_3 = nRT_2 \ln\left(\frac{V_4}{V_3}\right) \qquad \text{... (3)}$$

(d) Step $D \to A$: Adiabatic compression of the gas.

The gas in the system is further compressed adiabatically. The necessary energy required to perform this task increases the internal energy of the gas resulting in the rise of temperature of the system. The work done involves in the process is given by

$$W_4 = nR\frac{(T_2 - T_1)}{\gamma - 1} \qquad \text{... (4)}$$

So the total workdone by the gas in one complete cycle is

$$W = W_1 + W_2 + W_3 + W_4$$

$$= nRT_1 \ln\left(\frac{V_2}{V_1}\right) + nR\frac{(T_1 - T_2)}{\gamma - 1} - nRT_2 \ln\left(\frac{V_4}{V_3}\right) - nR\frac{(T_2 - T_1)}{\gamma - 1}$$

$$= nRT_1 \ln\left(\frac{V_2}{V_1}\right) - nRT_2 \ln\left(\frac{V_4}{V_3}\right) \qquad \text{... (5)}$$

Now from equations (1) and (3),

$$\frac{Q_1}{Q_2} = \frac{nRT_1 \ln(V_2/V_1)}{nRT_2 \ln(V_4/V_3)} = \frac{T_1}{T_2}\,\frac{\ln(V_2/V_1)}{\ln(V_4/V_3)} \qquad \text{... (6)}$$

Since states B and C lie on the same adiabatic and hence

$$T_1 V_2^{\gamma-1} = T_2 V_3^{\gamma-1} \qquad \text{... (7)}$$

Similarly, since states A and D lie on the same adiabatic,

$$T_1 V_1^{\gamma-1} = T_2 V_4^{\gamma-1} \qquad \text{... (8)}$$

Now dividing equation (7) by (8), we have $\frac{V_2}{V_1} = \frac{V_3}{V_4}$ ... (9)

Now using equation (9) in (6), we get

$$\frac{Q_1}{Q_2} = \frac{T_1}{T_2}\frac{\ln(V_3/V_4)}{\ln(V_4/V_3)} \quad \text{... (10)}$$

$$\Rightarrow \frac{Q_1}{Q_2} = -\frac{T_1}{T_2} \text{ or } \frac{Q_2}{Q_1} = -\frac{T_2}{T_1} \quad \text{... (11)}$$

So the magnitude is $\boxed{\frac{Q_2}{Q_1} = \frac{T_2}{T_1}}$ ... (12)

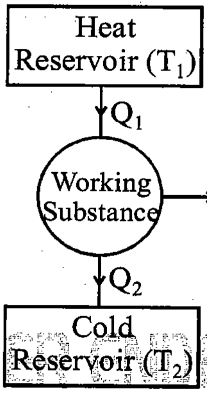
Thus for an "ideal gas", the ratio Q_2/Q_1 depends only on the temperature of the source (T_1) and of the sink (T_2) and independent of the nature of the working substance.

The Heat Engine

It is a device by which a system is made to undergo a cyclic process that results in conversion of heat to work. Since the process is cyclic, the net flow of heat Q into the gas or the system, in any complete cycle, is equal to the work W done by the engine per cycle.

i.e. $Q = W$ ... (13)

The Schematic representation of a heat engine is



Here Q_1 is the amount of heat given to the working substance from the hot reservoir at temperature T_1. A part of this Q_1 amount of heat is used to perform work W by the system and the rest of the heat Q_2 is rejected to the sink at temperature T_2. So by the conservation of energy

$$Q_1 = W + Q_2 \quad \text{... (14)}$$

$$\Rightarrow W = Q_1 - Q_2 \quad \text{... (15)}$$

The efficiency (η) is defined as $\eta = \frac{\text{what you get}}{\text{what you pay}}$.

For the heat engine, we have given Q_1 amount of heat to the system and as a result, it performs W amount of work. So the thermal efficiency is $\eta = \frac{W}{Q_1}$ ... (16)

$$\Rightarrow \eta = \frac{Q_1 - Q_2}{Q_1} = 1 - \frac{Q_2}{Q_1} \quad \text{... (17)} \quad \text{[Using (15)]}$$

$$\Rightarrow \boxed{\eta = 1 - \frac{T_2}{T_1}} \qquad \ldots (18) \qquad [\text{Using } (12)]$$

The definition in equation (17) is valid for every type of heat engine whereas equation (18) is valid only when the working substance is an ideal gas.

Clearly, the efficiency of a heat engine depends only on the temperatures of the source (T_1) and of the sink (T_2) and independent of the nature of the working substance.

For $Q_2 = 0$ i.e., if no heat gets wasted, the efficiency of the heat engine is 1 or 100 %. It means that all the heat given to the system is utilized in the work. This is an ideal case and is not practically possible.

Moreover, the value of 'η' lies between 0 and 1.

Problem: Find the efficiency of the Carnot engine working between the steam point and the ice point.

Soln. Given $T_1 = 100°\text{C} = 100 + 273 = 373\text{ K}$ (steam point)

$T_2 = 0°\text{C} = 0 + 273 = 273\text{ K}$ (ice point)

So, $$\eta = 1 - \frac{T_2}{T_1} = 1 - \frac{273}{373} = \frac{373 - 273}{373} = \frac{100}{373}$$

$$\%\text{ of efficiency } = \frac{100}{373} \times 100 = 26.81\%$$

Problem: A Carnot engine whose temperature of the source is 400 K takes 200 calories of heat at this temperature and reject 150 calories of heat to the sink. What is temperature of the sink. Also calculate the efficiency of the engine?

Soln. $Q_1 = 200\ Cal$, $Q_2 = 150$ Cal, $T_1 = 400$ K, $T_2 = ?$

Using $$\frac{Q_1}{T_1} = \frac{Q_2}{T_2} \Rightarrow T_2 = \frac{Q_2}{Q_1} \times T_1 = \frac{150}{200} \times 400 = 300\ K$$

$$\eta = 1 - \frac{T_2}{T_1} = 1 - \frac{300}{400} = 0.25$$

% of efficiency $= 0.25 \times 100 = 25\%$.

Problem: A Carnot engine is operated between two reservoirs at temperature of 450 K and 350 K. If the engine receives 1000 calories of heat from the source in each cycle. Calculate the amount of heat rejected to the sink in each cycle. Calculate the efficiency of the engine & the work done by the engine in each cycle (1 calorie = 4.2J)

Soln. $T_1 = 450\text{ K}$, $T_2 = 350\text{K}$, $Q_1 = 1000$ cal, $Q_2 = ?$

Using, $$\frac{Q_2}{Q_1} = \frac{T_2}{T_1} \Rightarrow Q_2 = \frac{T_2}{T_1} \times Q_1 = \frac{350}{450} \times 1000 = 777.77\ cal.$$

So, $$\eta = 1 - \frac{T_2}{T_1} = 1 - \frac{350}{450} = \frac{100}{450} = 0.2222$$

So, % of efficiency =22.22%

Also, work done in each cycle $= Q_1 - Q_2$

$= (1000 - 777.77)$ cal $= 222.23$ cal

$= 222.23 \times 4.2\text{ J} = 933.336\text{ J}$

Problem: A Carnot engine whose low temperature reservoir is at 7°C has an efficiency of 50%. It is desired to increase the efficiency to 70%. By how many degrees should the temperature of the high temperature reservoir be increased?

Soln. In the Case-I:

$\eta = 50\% = 0.5$, $T_2 = 273 + 7 = 280$ K, $T_1 = ?$

$$So, \quad \eta = 1 - \frac{T_2}{T_1} \Rightarrow 0.5 = 1 - \frac{280}{T_1}$$

$$\Rightarrow \quad \frac{280}{T_1} = 0.5 \quad \Rightarrow \quad T_1 = \frac{280}{0.5} = \frac{2800}{5} = 560\,K$$

In the Case-II:

$$\eta = 70\% = 0.7$$

$$T_2 = 280K, \; T_1 = ?$$

$$So, \eta = 1 - \frac{T_2}{T_1}$$

$$\Rightarrow 0.7 = 1 - \frac{280}{T_1} \Rightarrow \frac{280}{T_1} = 0.3 \Rightarrow T_1 = \frac{280}{0.3} = \frac{280}{3} = 933.33\,K$$

So increase in temperature = 933.33 – 560 = 373.33 K

Problem: In a Carnot cycle isothermal expansion of an ideal gas take place at 400 K and the isothermal compression at 300 K during the expansion 500 cal of heat energy are transfered to the gas. Determine

(1) The work performed by the gas during isothermal expansion.

(2) The heat rejected from the gas during an isothermal compression

(3) The work done in isothermal compression of gas?

Soln. (i) The work performed by the gas during the isothermal expansion is equivalent to heat Q_1 absorbed by the gas

Here $Q_1 = 500$ cal $= W/J = Q$

$W = 500 \times 4.18 = 2090$ Joule.

(ii) Let Q_2 be the amount of heat rejected. Then,

$$\frac{Q_1}{Q_2} = \frac{T_1}{T_2} \Rightarrow Q_2 = \frac{T_2}{T_1} \times Q_1 = 500 \times \frac{300}{400} = 375\,cal$$

(iii) The work done on the gas during isothermal compression is equivalent to the heat rejected

$$W = Q_2 = 375\,cal = 375 \times 4.18 = 1567.5\,Joule$$

Problem: A reversible engine works between two temperature whose difference is 100°C. If 746 J of heat from the source is taken and 546 J heat is given to the sink by working substance. Calculate the temperature of the source and sink.

Soln. Given $T_1 - T_2 = 100°C$, $Q_1 = 746$ J, $Q_2 = 546$ J

$$\text{Using } \frac{Q_1}{Q_2} = \frac{T_1}{T_2} \Rightarrow \frac{Q_1}{Q_2} = \frac{746}{546} = \frac{373}{273}$$

$$\Rightarrow \quad \frac{Q_1 - Q_2}{Q_2} = \frac{T_1 - T_2}{T_2} \Rightarrow \frac{200}{546} = \frac{100}{T_2} \Rightarrow T_2 = \frac{546 \times 100}{200} = 273°C$$

Now, $T_1 - T_2 = 100 \Rightarrow T_1 = T_2 + 100 = 273 + 100 = 373°C$.

Problem: An ideal Carnot heat engine takes 1000 cal of heat at 627°C from a source and rejects a part at 27°C to the sink. Find the work done by the engine per cycle. The heat rejected to the sink per cycle and the efficiency of the engine.

Soln. $Q_1 = 1000$ cal, $T_1 = 627°\text{C} = 900$ K

$Q_2 = ?$ $T_2 = 27°\text{C} = 300$ K

$$\text{Now,}\quad \frac{Q_1}{Q_2} = \frac{T_1}{T_2} \Rightarrow \frac{1000}{x} = \frac{900}{300} \Rightarrow x = \frac{1000 \times 300}{900} = 333.3\,cal$$

So the heat rejected per cycle to the sink = 333.3 cal

Now $W = Q_1 - Q_2 = 1000 - 333.3 = 666.7$ cal $= (666.7 \times 4.18)$ Joules $= 2.786 \times 10^3$ Joules

$$\text{Now,}\quad \eta = 1 - \frac{T_2}{T_1} = 1 - \frac{300}{900} = \frac{600}{900} = \frac{2}{3}$$

Problem: The efficiency of a Carnot cycle is 1/6. On reducing the temperature of the sink by 65°C the efficiency increases to 1/3. Find the initial and final temperature between which the cycle is working.

Soln. Using $\eta = 1 - \frac{T_2}{T_1} \Rightarrow \frac{1}{6} = \frac{T_1 - T_2}{T_1} \Rightarrow 6T_1 - 6T_2 = T_1$

or $5T_1 - 6T_2 = 0$ ... (i)

Now, according to question $1 - \frac{(T_2 - 65)}{T_1} = \frac{1}{3} \Rightarrow \frac{T_1 - T_2 + 65}{T_1} = \frac{1}{3} \Rightarrow 3T_1 - 3T_2 + 195 = T_1$

Or $2T_1 - 3T_2 = -195$ ... (ii)

Solving equations (i) and (ii),

$5T_1 - 6T_2 = 0 \Rightarrow 5T_1 - 6T_2 = 0 \Rightarrow 2T_1 - 3T_2 = -195$

Solving $T_1 = 390$ K. Also, $5T_1 = 6T_2 \Rightarrow 5 \times 390 = 6T_2$

$$\Rightarrow T_2 = \frac{5 \times 390}{6} = 325K$$

Problem: For a Carnot engine using an ideal gas the adiabatic expansion ratio (ρ) is 5 and value of $\gamma = 1.4$ calculate the efficiency of the engine

Soln. $$\eta = 1 - \left(\frac{1}{\rho}\right)^{\gamma - 1} = 1 - \left(\frac{1}{5}\right)^{1.4-1} = 1 - \left(\frac{1}{5}\right)^{0.4} = 0.48$$

Problem: Two Carnot engines are operating in series. The I engine absorbs a quantity of heat Q_1 at a temperature T_1 and after doing work W_1 rejects the remaining heat Q_2 at lower temperature T_2. The II engine absorbs the heat Q_2 at temperature T_2 (rejected by I) and after doing work W_2 rejects the remaining heat Q_3 at a still lower temperature T_3. Compute the efficiency of the combination. **[IISc. 2008]**

Soln. $W = (Q_1 - Q_2) + (Q_2 - Q_3) = Q_1 - Q_3$

$$\eta\ (\text{efficiency}) = \frac{W}{Q_1} = \frac{Q_1 - Q_3}{Q_1} = 1 - \frac{Q_3}{Q_1}$$

$$\text{But}\quad \frac{Q_1}{Q_2} = \frac{T_1}{T_2} \text{ and } \frac{Q_1}{Q_3} = \frac{T_1}{T_3}$$

$$\text{So,}\quad \eta = 1 - \frac{T_3}{T_1}$$

Adiabatic Expansion Ratio (ρ)

It is defined as the ratio of the initial and final volumes in an adiabatic expansion process.

Since, BC is an adiabatic process, $\rho = \frac{V_3}{V_2}$

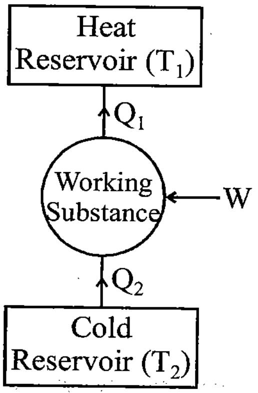
Then, from (7), $\frac{T_2}{T_1} = \left(\frac{V_2}{V_3}\right)^{\gamma-1} = \left(\frac{1}{\rho}\right)^{\gamma-1}$

$$\therefore \qquad \eta = 1 - \frac{T_2}{T_1} = 1 - \left(\frac{1}{\rho}\right)^{\gamma-1}$$

Refrigerator and Heat Pump

A *refrigerator* is the reverse of a heat engine. Here the working substance extracts heat Q_2 from the cold reservoir at temperature T_2, some external work W is done on it and heat Q_1 is released to the hot reservoir at temperature T_1.

The Schematic diagram of a refrigerator is



The efficiency of a refrigerator is measured by the coefficient of performance (K). It is defined as the ratio of the heat extracted from the cold reservoir Q_2 to the work done W, by an external agent.

So we have

$$K = \frac{Q_2}{W} = \frac{Q_2}{Q_1 - Q_2} \qquad \text{... (19)}$$

$$\Rightarrow \qquad \boxed{K = \frac{T_2}{T_1 - T_2}} \qquad \text{... (20)} \qquad \text{[From (12)]}$$

The coefficient of performance of a refrigerator, unlike the thermal efficiency of a heat engine, can be much larger than 1 or 100 % but can't be infinite. If its value gets infinity, it means that there is no need of any work to be performed to extract heat from the cold reservoir and to release it into the hot reservoir. A good refrigerator should have high value of K.

A heat pump is the same as refrigerator. What term we use depends on the purpose of the device. If the purpose is to cool a portion of space, the device is said to be a refrigerator. If the idea is to pump heat into a portion of space, the device is called a *heat pump*.

Problem: A heat pump working on the Carnot cycle maintains the inside temperature of a house at 22°C by suppling 450 kJ s^{-1}. If the outside temperature is 0°C, the heat taken, in kJ s^{-1}, from the outside air is approximately.

(a) 487 (b) 470 (c) 467 (d) 417

Soln. The efficiency of Carnot heat engine may be defined as

$$\eta = \frac{Q_1 - Q_2}{Q_1} = \frac{T_1 - T_2}{T_1}$$

where, Q_1 is heat taken from the source by working substance and Q_2 is the heat given by working substance to sink respectively. Also, T_1 and T_2 are the temperatures of source and sink respectively. Hence, in question

$Q_1 = 450$ kJ/s, $Q_2 = ?$

$T_1 = 22 + 273 = 295$ K,

$T_2 = 273\ K$

Hence, $\frac{450 - Q_2}{450} = \frac{295 - 273}{295}$ or $Q_2 = \frac{273 \times 450}{295} = 416.44$ kJ/s $\cong 417$ kJ/s

Hence correct option is (d).

Negative Temperature

From efficiency of Carnot engine, $\eta = 1 - \frac{T_2}{T_1}$

If suppose the efficiency gets greater than 1, i.e., $\eta > 1$

$$\Rightarrow 1 - \frac{T_2}{T_1} > 1$$

$$\Rightarrow T_2 < 0 \qquad \ldots(21)$$

This is not possible as T_2 should always be greater than zero (by definition).

Problem: A Carnot refrigerator taken heat from water at 0 °C and discard it to a room at 27 °C. What is the value of the coefficient of its performance ?

(a) 11.1 (b) 111.0 (c) 1.11 (d) 0.111

Soln. Given : $T_1 = 27\,°\text{C} = 300$ K and $T_2 = 0\,°\text{C} = 273$ K

The coefficient of performance is

$$K = \frac{Q_2}{W} = \frac{Q_2}{Q_1 - Q_2} = \frac{T_2}{T_1 - T_2}$$

$$= \frac{273}{300 - 273} = \frac{273}{27} = 111.0$$

Correct option is (b)

Problem: A Carnot engine working as a refrigerator between 260 K and 300 K receives 500 calories of heat from the reservoir at the lower temperature calculate the amount of heat rejected to the reservoir at the higher temperature. Calculate also the amount of workdone in each cycle to operate the refrigerator.

Soln. $Q_1 = ?$, $Q_2 = 500$ cal , $T_1 = 260$K, $T_2 = 300$K

For Carnot engine:

$$\frac{Q_1}{T_1}=\frac{Q_2}{T_2}\Rightarrow Q_1=\frac{Q_2}{T_2}\times T_1=\frac{5000\times 260}{300}=576.92cal$$

So, $W=Q_1-Q_2=76.92$ cal $=76.92\times 4{\cdot}2=323.06$ Joule.

Problem: A Carnot's refrigerator takes heat from water at 0°C and discard it to a room at 27°C. 1 kg of water at 0°C is to be changed into ice at 0°C. How many calories of heat are discarded to the room? What is the work done by the refrigerator in this process? What is the co-efficient of performance of the machine? (L_f=80 cal/g)

Soln. $Q_1=?$, $Q_2=mL_f=1000\times 80=80{,}000$ cal

$T_1=300K$, $T_2=273K$

For Carnot engine,

$$\frac{Q_1}{Q_2}=\frac{T_1}{T_2}\Rightarrow Q_1=\frac{T_1}{T_2}\times Q_2=\frac{80000\times 300}{273}=87{,}900\,cal.$$

Work done by the refrigerator is

$$W=(Q_1-Q_2)\ =\ 4.2(87900-80000)J$$

$$=4.2\times 7900=3.183\times 10^4\ J$$

$$\text{Co-efficient of performance}=\frac{Q_2}{Q_1-Q_2}=\frac{80{,}000}{87900-80{,}000}=\frac{80{,}000}{7900}=10.13$$

Relation between 'η' and 'K'

By definition

$$\eta=1-\frac{T_2}{T_1}=\frac{T_1-T_2}{T_1} \qquad \text{... (22)}$$

$$\text{and } K=\frac{T_2}{T_1-T_2} \qquad \text{... (23)}$$

$$\text{Now, } K+1=\frac{T_2}{T_1-T_2}+1=\frac{T_1}{T_1-T_2}=\frac{1}{\eta} \qquad \text{[From (22)]}$$

$$\Rightarrow \eta=\frac{1}{K+1} \qquad \text{... (24)}$$

$$\Rightarrow \boxed{K=\frac{1}{\eta}-1} \qquad \text{... (25)}$$

Carnot Theorem

1. No engine working between two given temperatures can be more "efficient" than a reversible (Carnot engine) engine, working between the same limits of temperature.
2. All reversible engine (Carnot engine) working between the same limits of temperature have the same efficiency, whatever be the working substance.

Problem: Two identical bodies of constant heat capacities at temperature T_1 and T_2 $(T_1>T_2)$ are used as reservoirs for a heat engine. Assuming that the bodies don't undergo any phase change and remain at constant pressure P. Show that the amount of work obtained is given by

$$W=C_P(T_1+T_2-2T_f)$$

where, T_f is the final temperature of the bodies. Also show that W is maximum when $T_f=\sqrt{T_1T_2}$

Soln. The working substance takes heat from the body at temperature T_1 and after doing work gives the balance heat to the body at temperature T_2. This process continues until T_1 drops to T_f and T_2 rises to T_f

The total heat taken from the hot body

$$Q_1 = C_P\left(T_1 - T_f\right) \qquad \text{... (i)}$$

and the heat given to the cold body

$$Q_2 = C_P\left(T_f - T_2\right) \qquad \text{... (ii)}$$

The balance $(Q_1 - Q_2)$ converts into work, i.e.,

$$W = Q_1 - Q_2 = C_P\left(T_1 - T_f\right) - C_P\left(T_f - T_2\right)$$

$$W = C_P\left(T_1 - T_f - T_f + T_2\right) = C_P\left(T_1 + T_2 - 2T_f\right)$$

The heat Q_1 is taken in at a mean temperature $= \dfrac{T_1 + T_f}{2}$

and the heat Q_2 is given out at a mean temperature $= \dfrac{T_2 + T_f}{2}$

Carnot engine have maximum efficiency so the work done must be maximum

$$\frac{Q_1}{Q_2} = \frac{\dfrac{T_1 + T_f}{2}}{\dfrac{T_2 + T_f}{2}} = \frac{T_1 + T_f}{T_2 + T_f}$$

From equations (i) and (ii),

$$\frac{Q_1}{Q_2} = \frac{T_1 - T_f}{T_f - T_2}$$

So $$\frac{T_1 + T_f}{T_2 + T_f} = \frac{T_1 - T_f}{T_f - T_2}$$

$$\Rightarrow \quad T_f = \sqrt{T_1 T_2}$$

PROBLEMS TO BE PLAYED

1. Which of the following is correct?

(a) $\frac{T_1}{H_2}+\frac{T_2}{H_1}=0$ (b) $\frac{H_1}{T_1}=\frac{H_2}{T_2}$ (c) $H_1T_1=H_2T_2$ (d) $H_1T_1+H_2T_2=0$

2. The device which converts heat into mechanical work is–

(a) Motor (b) Generator (c) Heat engine (d) Energy converter

3. A reversible heat engine can have 100% efficiency if the temperature of sink is–

(a) Less than that of source (b) Equal to that of source
(c) 0°C (d) 0 K

4. In a heat engine the maximum heat that can be converted into mechanical work–

(a) Depends upon friction (b) Depends upon working temperatures
(c) Is 100% (d) Depends upon the working

. The change in entropy is–

(a) Positive in a reversible change (b) Negative in an irreversible change
(c) Positive in an irreversible change (d) Negative in a reversible change

. If T_1 and T_2 are the temperatures of two reservoirs (where $T_1 > T_2$) and a Carnot engine being operated between them will have the efficiency

(a) $\frac{(T_1-T_2)}{T_2}$ (b) $\frac{(T_1-T_2)}{T_1}$ (c) $\frac{T_1}{(T_1-T_2)}$ (d) $\frac{T_2}{(T_1-T_2)}$

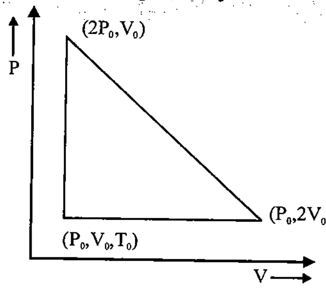
. The efficiency of a heat engine working between heat reservoirs at temperature 327 °C and 27°C respectively is

(a) 25° (b) 50° (c) 75° (d) 100°

. In a reversible cycle, the value of the integral $\oint \frac{dQ}{T}$ is–

(a) $\oint \frac{dQ}{T}>0$ (b) $\oint \frac{dQ}{T}<0$ (c) $\oint \frac{dQ}{T}=0$ (d) $\oint \frac{dQ}{T}=\text{constant}$

. The work of a reversible heat engine performing this cycle is–



(a) $\frac{1}{2}$ (b) $\frac{1}{6}$ (c) $\frac{1}{8}$ (d) $\frac{1}{10}$

. If an ideal (frictionless) heat enegine absorbs heat at a temperature 100°C and exhausts heat at a temperature 0 °C, its efficiency is 100%. This statements is

(a) True (b) False
(c) Uncertain (d) Needs more information

11. A heat engine works on a Carnot cycle with a heat sink at a temperature of 27°C. The efficiency of the engine is 10%. The approximate temperature of the heat source is
(a) 30 K (b) 270 K (c) 60°C (d) 30 °C

12. An ideal engine absorbs heat at a temperature of 127 °C and exhausts heat at a temperature of 77°C. Its efficiency is approximately
(a) 13% (b) 39% (c) 61% (d) 74%

13. The efficiency of a Carnot engine using an idal gas as the working substance is
(a) $\frac{T_1-T_2}{T_1}$ (b) $\frac{T_1}{T_1-T_2}$ (c) $T_1(T_1-T_2)$ (d) $T_2(T_1-T_2)$

14. The efficiency of a Carnot's cycle is given by (in terms of compression ratio represented by ρ)
(a) $\eta=1-\left(\frac{1}{\rho}\right)^{\gamma-1}$ (b) $\eta=1-(\rho)^{\gamma-1}$ (c) $\eta=1-\left(1-\frac{1}{\rho}\right)^{\frac{1}{\gamma-1}}$ (d) $\eta=1-(\rho)^{\gamma}$

15. The efficiency of a Carnot engine between the source at absolute temperature T_1 and the sink at an absolute tem,perature T_2 is given by
(a) $\frac{1}{T_1}\times T_2\times 100$ (b) $\left(\frac{T_1}{T_2}-1\right)\times 100$ (c) $\left(1-\frac{T_2}{T_1}\right)\times 100$ (d) $\frac{T_2}{T_1}\times 100$

16. In a Carnot engine for a certain difference in temperature between the source T_1 and T_2, the efficiency η
1. can be increased to a larger extent by increasing the source temperature
2. can be increased to a larger extent by decreasing the sink temperature
3. is independent of the working substance in the cycle
4. is same as that of an irreversible engine based on the Carnot cycle with the same T_1 and T_2
(a) only 4 (b) 1 and 3 (c) 2 and 3 (d) only 3

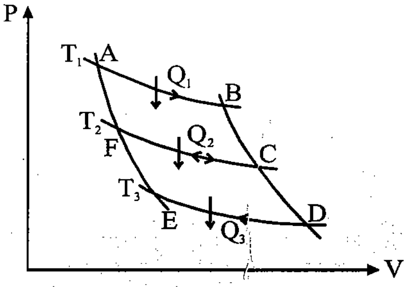
ANSWER KEY

1. (b) **2. (c)** **3. (d)** **4. (b)** **5. (c)** **6. (b)** **7. (b)**
8. (c) **9. (a)** **10. (b)** **11. (a)** **12. (a)** **13. (a)** **14. (b)**
15. (c) **16. (c)**

Chapter 4

The Entropy & The Second Law of Thermodynamics

Entropy (S)



[Diagram showing two Carnot ABCF and FCDE]

Let ABCF and FCDE represent two Carnot cycles. Let Q_1 amount of heat entered in the system in the process A to B at temperature T_1 and let Q_2 amount of heat rejected from the system in the process C to D at temperatue T_2.

Then from Carnot's theorem,

$$\frac{Q_1}{T_1} = \frac{Q_2}{T_2} \qquad \ldots (1)$$

Similarly in the cycle if Q_2 be the heat entered in the system at temperature T_2 and Q_3 be the heat liberated out of the system at temperature T_3, then by the Carnot's theorem

$$\frac{Q_2}{T_2} = \frac{Q_3}{T_3} \qquad \ldots (2)$$

where the curves AFE and BCD are the adiabatics.

So if Q be the amount of heat absorbed or rejected at a temperature T in going from one adiabatic to the other then,

$$\frac{Q}{T} = \text{Constant} \qquad \ldots (3)$$

This constant quantity is called *change in entropy* between the state represented by the two adiabatics. If two adiabatics are very closed then, change in entropy is given by

$$dS = \frac{dQ}{T} \qquad \ldots (4)$$

It is a state function. The change in the entropy for a finite change is

$$\int_1^2 dS = \int_1^2 \frac{dQ}{T} \qquad \ldots (5)$$

$$\Rightarrow \quad \boxed{S_2 - S_1 = \Delta S = \int_1^2 \frac{dQ}{T}} \qquad \ldots (6)$$

Problem: The entropy function of system is given by $S(E) = a\,E\,(E_0 - E)$ where a and E_0 are positive constants. The temperature of the system is

(a) negative for some energies
(b) increases monotonically with energy
(c) decreases monotonically with energy
(d) zero

Soln. $\frac{1}{T} = \frac{dS}{dE} \quad \Rightarrow \frac{1}{T} = a\,E_0 - 2aE \quad \Rightarrow T = \frac{1}{(a\,E_0 - 2aE)}$.

This means that T is continuous and will be positive for $E > 2E_0$ and negative for $E < 2E_0$.

Hence correct option is (a).

Problem: Experimental measurements of heat capacity per mole of aluminium at low temperatures. Show that the data can be fitted to the formula, $C_v = aT + bT^3$, where $a = 0.00135\,JK^{-2}\,\text{mol}^{-1}$, $b = 2.48 \times 10^{-5}\,J\,K^{-4}\,\text{mol}^{-1}$ and T is the temperature in Kelvin. The entropy of a mole of aluminium at such temperature is given by

(a) $aT + \frac{bT^3}{3} + C$, C is a constant
(b) $\frac{aT}{2} + \frac{b}{4}T^3 + C$, C is a constant
(c) $aT + \frac{6}{3}T^3$
(d) $\frac{aT}{2} + \frac{b}{4}T^3$

Soln. Given : $C_v = aT + bT^3$

The entropy of a mole is

$$\Delta S = \int \frac{dQ}{T} = \int \frac{C_v\,dT}{T} = \int \left(a + bT^2\right) dT$$

$= aT + \frac{bT^3}{3} + C$, where C is a constant.

Correct option is (a)

Problem: In a certain process, the temperature (T) and entropy (S) are related by $T \propto S^n$, where n is a number. What is the heat capacity in terms of the entropy of the process?

(a) $\frac{S^2}{k_B n^2}$
(b) $\frac{S}{n}$
(c) $n\,k_B e^{S/k_B}$
(d) $n\sqrt{S\,k_B}$

Soln. Given : $T = K\,S^n$, K is a constant

Since $dS = \frac{dQ}{T} = \frac{CdT}{T}$

$$\Rightarrow \quad C = T\frac{dS}{dT} = \frac{T}{dT/dS}$$

$$\Rightarrow \; C = \frac{KS^n}{nKS^{n-1}} = \frac{S}{n}$$

Correct option is (b)

Problem: Which of the following integrals depends only on the initial and final states of a thermodynamic system (i.e., independent of the path of transformation)?

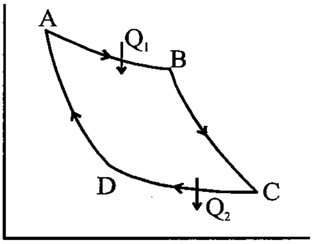
(a) $\int PdV$ (b) $\int dQ$ (c) $\int \frac{dQ}{T}$ (d) $\int T^2 dS$ **[JNU : 2009]**

Ans. (c) Since, entropy is a state function and $\Delta S = \int \frac{dQ}{T}$

Change in Entropy in Some Processes

(1) Reversible Process: Consider a reversible process (Carnot cycle) ABCD. Then, change in entropy in process A to B

$$\Delta S_1 = \frac{Q_1}{T_1}$$



In process B to C,

$$\Delta S_2 = 0 \text{ (BC is an adiabatic process)}$$

In process C to D,

$$\Delta S_3 = -\frac{Q_2}{T}$$

In process D to A,

$$\Delta S_4 = 0 \text{ (DA is an adiabatic process)}$$

So total change in the entropy is

$$\Delta S = \Delta S_1 + \Delta S_2 + \Delta S_3 + \Delta S_4 = \frac{Q_1}{T_1} - \frac{Q_2}{T_2} \qquad \dots(7)$$

By Carnot theorem,

$$\frac{Q_1}{Q_2} = \frac{T_1}{T_2} \quad \text{or} \quad \frac{Q_1}{T_1} = \frac{Q_2}{T_2}$$

Therefore, equation (7), ⇒ $\Delta S = 0$

"So there is no change in the entropy in a reversible process."

Now from equation (5),

$$\boxed{\oint dS = \oint \frac{dQ}{T} = 0} \qquad \dots(8)$$

Problem: A Carnot cycle operates as a heat engine between two bodies of equal heat capacity until their temperatures become equal. If the initial temperatures of the bodies are T_1 and T_2, respectively, and $T_1 > T_2$ then their common final temperature is

(a) T_1^2 / T_2 (b) T_2^2 / T_1 (c) $\sqrt{T_1 T_2}$ (d) $\frac{1}{2}(T_1 + T_2)$

Soln. Carnot cycle is reversible cycle so the entropy change must be zero.

$$\Delta S = 0$$

Let the equilibrium temperature be T then

$$\Delta S = C\int_{T_1}^{T} \frac{dT}{T} + C\int_{T_2}^{T} \frac{dT}{T} = 0; \text{ where } C \text{ is the heat capacity}$$

$$\Rightarrow \ln\left(\frac{T}{T_1}\right)+\ln\left(\frac{T}{T_2}\right)=0$$

$$\Rightarrow \ln\left(\frac{T^2}{T_1T_2}\right)=0 \Rightarrow T^2=T_1T_2 \Rightarrow T=\sqrt{T_1T_2}$$

Correct answer is (c)

Problem: Consider two identical blocks of constant specific heat capacity, one of which is at temperature T_1 and the other at T_2. A reversible engine transfers heat from the hotter block to the colder until they reach the same temperature which is

(a) $\sqrt{T_1T_2}$ (b) $\frac{T_1+T_2}{2}$ (c) $\frac{T_1T_2}{T_1+T_2}$ (d) T_2

Soln. Let the final temperature be T. Then,

$$\Delta S=(\Delta S)_1+(\Delta S)_2$$

$$=C\int_{T_1}^{T}\frac{dT}{T}+C\int_{T_2}^{T}\frac{dT}{T}=C\ln\left(\frac{T^2}{T_1T_2}\right)$$, where C is the specific heat capacity

For reversible process $\Delta S=0$

$$\Rightarrow \ln\left(\frac{T^2}{T_1T_2}\right)=0 \Rightarrow T=\sqrt{T_1T_2}$$

Correct option is (a)

(2) Irreversible Process:

The efficiency in any heat engine (may be reversible or irreversible) is

$$\eta=1-\frac{Q_2}{Q_1} \qquad ...(9)$$

and in Carnot heat engine (reversible engine) is

$$\eta'=1-\frac{T_2}{T_1} \qquad ...(10)$$

The Carnot theorem says that the efficiency of a reversible Carnot engine is maximum. So

$$1-\frac{T_2}{T_1}\ge 1-\frac{Q_2}{Q_1} \Rightarrow \frac{Q_2}{Q_1}\ge\frac{T_2}{T_1} \Rightarrow \frac{Q_2}{T_2}-\frac{Q_1}{T_1}\ge 0$$

$$\Rightarrow \boxed{\Delta S\ge 0} \qquad ...(11)$$

The equality holds here for reversible engine.

Equation (11) says that the entropy always increases in an irreverssible process.

(3) Reversible Adiabatic Process:

In any adiabatic process, $dQ=0$ and hence $dS=0$.

It means that there is no change in the entropy in a reversible adiabatic process. Such an adiabatic process is called "isentropic".

(4) Reversible Isothermal Process:

The change in entropy is

$$\Delta S = \int \frac{dQ}{T}$$

Since T remains constant in an isothermal process, so we have

$$\Delta S = \frac{1}{T}\int_{Q_2}^{Q_1} dQ = \frac{1}{T}(Q_1 - Q_2)$$

where Q_1 and Q_2 are the amount of heat absorbed and rejected respectively. Henceforth, $Q_1 - Q_2$ is the amount of work done (ΔW).

So, $\Delta S = \dfrac{\Delta W}{T}$

where $(\Delta W)_{\text{isothermal}} = nRT\,\ell n\left(\dfrac{V_2}{V_1}\right)$

$$\therefore \quad \boxed{\Delta S = nR\,\ell n\left(\frac{V_2}{V_1}\right)} \qquad \text{... (12)}$$

where V_1 and V_2 are the initial and final volumes of the gas in the system.

Problem: An ideal gas undergoes an isothermal expansion (at a constant temperature T) from an initial volume V_1 to a final volume V_2. The change in the entropy per mole is

(a) $R\left(\dfrac{V_1}{V_2}\right)$ (b) $R\ln|V_1 - V_2|$ (c) $R\ln\left(\dfrac{V_1}{V_2}\right)$ (d) $R\ln\left(\dfrac{V_2}{V_1}\right)$

Soln. For isothermal expansion, $Q = W = nRT\ln\left(\dfrac{V_2}{V_1}\right)$

$$\therefore\ (\Delta S)_{\text{isothermal}} = \int \frac{dQ}{T} = \frac{1}{T}\int dQ = nR\ln\left(\frac{V_2}{V_1}\right)$$

$$\Rightarrow\ (\Delta S)_{\text{isothermal}} = R\ln\left(\frac{V_2}{V_1}\right) \qquad [\because \text{ For 1 mole}]$$

Correct option is (d)

Problem: A sample of ideal gas with initial pressure P and volume V is taken through an isothermal expansion process during which the change in entropy is found to be ΔS. The universal gas constant is R. Then the work done by the gas is given by

(a) $(PV\Delta S)/(nR)$ (b) $nR\Delta S$ (c) PV (d) $(P\Delta S)/(nRV)$

oln. By ideal gas equation, $PV = nRT$ ...(i)

and $\Delta S = \dfrac{\Delta Q}{T}$ ...(ii)

Multiplying equations (i) and (ii),

$$PV\Delta S = nR\Delta Q$$

$$\Rightarrow \quad \Delta Q = \frac{PV\Delta S}{nR}$$

$\because$ For isothermal expansion, $\Delta U = 0$

$$\Rightarrow \quad W = \Delta Q = \frac{PV\Delta S}{nR}$$

Hence correct option is (a).

(5) Reversible Isobaric Process:

The change in the entropy is

$$\Delta S = \int \frac{dQ}{T} \quad \text{where} \quad dQ = C_P dT$$

$$\therefore \quad \Delta S = \int_{T_1}^{T_2} C_P \frac{dT}{T}$$

If C_P remains constant in the range T_1 and T_2, then

$$\boxed{\Delta S = C_P \ \ell n\left(\frac{T_2}{T_1}\right)} \qquad \text{... (13)}$$

(6) Reversible Isochoric Process:

The entropy change is

$$\Delta S = \int \frac{dQ}{T}$$

where $dQ = C_V \, dT$

$$\therefore \quad \Delta S = \int_{T_1}^{T_2} C_V \frac{dT}{T}$$

If there is no change in the value of C_V for the range T_1 and T_2, then

$$\boxed{\Delta S = C_V \ \ell n\left(\frac{T_2}{T_1}\right)} \qquad \text{... (14)}$$

(7) Irreversible Process (an another approach)

The another way of concluding the entropy always increases in a irrevsible process is as follows:

Let our system having heat capacity C and at temperature T_1 is in contact with a reservoir at temperature $T_2\left(T_2 > T_1\right)$. So some amount of heat flows from reservoir to the system. If there is a finite temperature difference between initial and final state of the system and if the process is isochoric then, the change in the system's entropy is

$$(\Delta S)_{sys} = \int \frac{dQ}{T} = \int_{T_1}^{T_2} C\frac{dT}{T} = C\ \ell n\left(\frac{T_2}{T_1}\right) \qquad \text{... (15)}$$

Since $T_2 > T_1$, $(\Delta S)_{sys.} > 0$ ... (16)

Also, heat flows in to the system is $Q = C\left(T_2 - T_1\right)$

Since the heat capacity of the reservoir is so large, there is no change in the temperature if heat flows in or out of it and since heat flows out of the reservoir, we have

$$(\Delta S)_{\text{Res.}} = \int \frac{dQ}{T_2} = \frac{1}{T_2}\int dQ = -\frac{Q}{T_2} = -\frac{C_V(T_2-T_1)}{T_2} = -C_V\left(1-\frac{T_1}{T_2}\right) \qquad \text{... (17)}$$

Since $T_1 < T_2 \Rightarrow \frac{T_1}{T_2} < 1$, So $(\Delta S)_{res} < 0$.

So the entropy of the reservoir decreases.

The total change in entropy of the composite system, body plus reservoir, is

$$\Delta S = (\Delta S)_{sys.} + (\Delta S)_{res.} = C_V\left[\ell n\left(\frac{T_2}{T_1}\right) - \frac{T_2-T_1}{T_2}\right]$$

$$= -C_V\left[\ell n\left(\frac{T_1}{T_2}\right) - \frac{T_1}{T_2} + 1\right] \qquad \text{... (18)}$$

Since $\frac{T_1}{T_2} < 1$, $\ell n\left(\frac{T_1}{T_2}\right) - \frac{T_1}{T_2} + 1 < 0$,

$\Rightarrow \quad \Delta S > 0$

which means the entropy of the composite system always increases in the irreversible process.

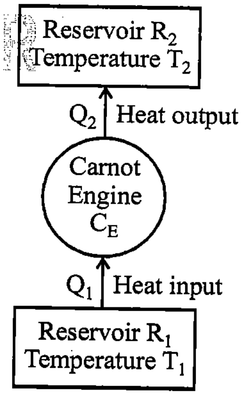
Problem: A Carnot cycle operates on a working substance between two reservoirs at temperature T_1 and T_2, where $T_1 > T_2$. During each cycle an amount of heat Q_1 is extracted from the reservoir at T_1 and an amount Q_2 is delivered to the reservoir at T_2. Which of the following statements is incorrect ?

(a) Work done in one cycle is $Q_1 - Q_2$

(b) $\frac{Q_1}{T_1} = \frac{Q_2}{T_2}$

(c) Entropy of the reservoir decreases

(d) Entropy of the universe (consisting of the working substance and the two reservoirs) increases



Soln. The given Carnot engine is shown schematically in the figure aside.

In ideal case (no energy dissipation)

Work done = Heat input – Heat output

$= Q_1 - Q_2 \qquad \text{...(i)}$

($\because$ given $T_1 > T_2 \Rightarrow Q_1 > Q_2$)

Now efficiency of a Carnot cycle $\eta = \frac{Q_1 - Q_2}{Q_1} = \frac{T_1 - T_2}{T_1}$

$$\Rightarrow 1 - \frac{Q_2}{Q_1} = 1 - \frac{T_2}{T_1} \Rightarrow \boxed{\frac{Q_2}{T_2} = \frac{Q_1}{T_1}} \qquad \text{...(ii)}$$

Now, $\Delta S = \frac{dQ}{T}$ (definition of entropy)

Since heat is taken away from R_1,

$\Rightarrow \Delta S_{R_1} = -\dfrac{Q_1}{T_1} \Rightarrow$ change in entropy for R_1 is negative

$\Rightarrow$ Entropy of the hotter reservoir decreases ...(iii)

Finally $\Delta S_{universe} = \Delta S_{R_1} + \Delta S_{R_2} + \Delta S_{C_E} = -\dfrac{Q_1}{T_1} + \left(\dfrac{Q_1}{T_1} - \dfrac{Q_2}{T_2}\right) + \dfrac{Q_2}{T_2}$

$\Rightarrow \boxed{\Delta S_{universe} = 0} \Rightarrow (d)$ is incorrect.

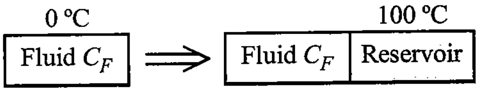
Hence, from (i), (ii) and (iii) we find (a), (b) and (c) are correct. Therefore one has to choose option (d) which is incorrect.

Hence, the correct option is (d).

Problem: A certain amount of fluid with heat capacity C_F Joules/°C is initially at a temperature 0°C. It is then brought into contact with a heat bath at a temperature of 100°C, and the system is allowed to come into equilibrium. In this process, the entropy (in Joules/°C) of the universe changes by

(a) $100\ C_F$ (b) 0 (c) $0.055\ C_F$ (d) $0.044\ C_F$

Soln.



$$(\Delta S)_{\text{universe}} = (\Delta S)_{\text{fluid}} + (\Delta S)_{\text{reservoir}}$$

$$= \int \frac{dQ}{T} + \int \frac{dQ}{T}$$

$$= C_F \int_{T_1}^{T_2} \frac{dT}{T} + \frac{1}{T_2} \int dQ = C_F \int_{T_1}^{T_2} \frac{dT}{T} - \frac{C_F(T_2 - T_1)}{T_2}$$

$$= C_F \ln\left(\frac{T_2}{T_1}\right) - C_F\left(\frac{T_2 - T_1}{T_2}\right)$$

where $T_1 = 0\,°\text{C} = 273$ K, $T_2 = 100\,°\text{C} = 373$ K

Therefore, $\Delta S = C_F\left[\ln\left(\dfrac{373}{273}\right) - \dfrac{100}{373}\right] = 0.044\ C_F$

Correct option is (d)

Change in Entropy in Conversion Process System

(1) When a liquid is heated from T_1 to $T_2\ (T_2 > T_1)$.

Let a liquid of mass 'm' and specific heat C is heat from T_1 to T_2. If dQ is the amount of heat taken to rise its temperature by dT, then

$$dQ = mC\,dT \qquad ...(19)$$

The corresponding change in entropy is

$$dS = \frac{dQ}{T} = \frac{mC\,dT}{T} \qquad ...(20)$$

The total entropy change is

$$(\Delta S)_{liq} = \int_{T_1}^{T_2} mC\frac{dT}{T} = mC\,\ell n\left(\frac{T_2}{T_1}\right) \qquad \text{... (21)}$$

Since, $T_2 > T_1$, $(\Delta S)_{liq} > 0$. It means that the entropy increases if a liquid is heated. Similarly, if $T_2 < T, (\Delta S)_{liq.} < 0$. So the entropy decreases if a liquid is cooled down.

(2) Change in entropy when change of state takes place.

If a substance of mass 'm' and latent heat L is heated to convert into another state, the amount of heat required is

$$Q = mL$$

So, the change in entropy is

$$\Delta S = \int\frac{dQ}{T} = \frac{1}{T}\int dQ$$

where temperature remains constant during change of state.

$$\Rightarrow \qquad \Delta S = \frac{mL}{T} \qquad \text{... (22)}$$

(3) If ice (0°C) melts into water then changes to steam (100°C).

The total change in the entropy is

$$\Delta S_1 = (\Delta S)_{\substack{ice \to water \\ (0°C) \quad (0°C)}} = \frac{mL_f}{T_1} \quad [\because\ T_1 = 273K \text{ and } L_f \text{ is latent heat of fusion}]$$

$$(\Delta S)_2 = (\Delta S)_{\substack{water \to water \\ (0°C) \quad (100°C)}} = mC\,\ell n\left(\frac{T_2}{T_1}\right) \quad [\because\ T_2 = 373K \text{ and C is specific heat}]$$

$$(\Delta S)_3 = (\Delta S)_{\substack{water \to steam \\ (100°C) \quad (100°C)}} = \frac{mL_v}{T_2} \quad [\because L_v \text{ is latent heat of vapourisation}]$$

$$\therefore \quad (\Delta S)_{Total} = \frac{mL}{T_1} + mC\,\ell n\left(\frac{T_2}{T_1}\right) + \frac{mL}{T_2} \qquad \text{... (23)}$$

Problem: One kilogram of ice melts at 0°C into water at the same temperature. The change in entropy (cal/K) is $\left(L_f = 3.35\times10^5 J\,kg^{-1}\right)$

(a) ∞ (b) 0 (c) 0.293 (d) 293

Soln. Given: $m = 1$ kg, $L_f = 3.35\times10^5$ J/kg^{-1}

$$\Delta S = \frac{Q}{T} = \frac{mL}{T} = \frac{1\times3.35\times10^5}{273}\text{ J}$$

$$= \frac{3.35\times10^5}{273\times4.2}cal \quad [\because\ 1\text{ cal} = 4.2\text{ J}]$$

$$= 292.168\text{ cal/K} \simeq 293\text{ cal/K}$$

Correct option is (d)

(4) If a body of mass m_1, specific heat C_1 and temperature T_1 is thermally connected with another body of mass m_2, specific heat C_2 and temperature T_2.

Let $T_1 > T_2$ and T be the equilibrium temperature. Then, according to the principle of calorimetry,

$$m_1 C_1 (T_1 - T) = m_2 C_2 (T - T_2)$$

$$\Rightarrow \quad T = \frac{m_1 C_1 T_1 + m_2 C_2 T_2}{m_1 C_1 + m_2 C_2}$$

The change in the entropy is

$$\Delta S = \Delta S_1 + \Delta S_2 = m_1 C_1 \int_{T_1}^{T} \frac{dT}{T} + m_2 C_2 \int_{T_2}^{T} \frac{dT}{T}$$

$$= m_1 C_1 \, \ell n\left(\frac{T}{T_1}\right) + m_2 C_2 \, \ell n\left(\frac{T}{T_2}\right) = \ell n\left[\left(\frac{T}{T_1}\right)^{m_1 C_1} \left(\frac{T}{T_2}\right)^{m_2 C_2}\right] > 0$$

Change in Entropy in Terms of Different Parameters

(1) Change in entropy of a 'perfect gas' in terms of its pressure, volume and specific heat

Let us consider one mole of perfect gas at pressure P, volume V and temperature T. When an infinitesimal amount of heat dQ is added to it, the increase in entropy is

$$dS = \frac{dQ}{T}$$

From I law, $dQ = dU + dW$

But $dU = C_V dT$ and $dW = PdV = RT \frac{dV}{V}$

So, $$dQ = C_V dT + RT \frac{dV}{V}$$

Now, $$dS = \frac{dQ}{T} = C_V \frac{dT}{T} + R \frac{dV}{V}$$

The total entropy is

$$S = \int \frac{dQ}{T} = C_V \int \frac{dT}{T} + R \int \frac{dV}{V} \qquad \text{... (i)}$$

From ideal gas equation, $PV = RT$, we have

$$dT = \frac{PdV + VdP}{R}$$

and $R = C_P - C_V$

$$\therefore \; S = C_V \int \frac{PdV + VdP}{RT} + R \int \frac{dV}{V}$$

$$= C_V \int \frac{PdV + VdP}{PV} + (C_P - C_V) \int \frac{dV}{V}$$

$$= C_V \int\frac{dV}{V} + C_V\int\frac{dP}{P} + C_P\int\frac{dV}{V} - C_V\int\frac{dV}{V}$$

$$= C_V \int\frac{dP}{P} + C_P\int\frac{dV}{V}$$

$$\boxed{S = C_V \ \log P + C_P \ \log V + \text{constant}}$$

If a gas passes from an initial state P_i, V_i, T_i to P_f, V_f, T_f then the change in entropy of the gas is

$$\boxed{\Delta S = C_V \log\frac{P_f}{P_i} + C_P \log\frac{V_f}{V_i}}$$

(2) Change in entropy of a perfect gas in terms of its temperature, pressure and specific heat at constant pressure

$$dQ = dU + dW = C_V dT + PdV = C_V dT + RT\frac{dV}{V} = TdS$$

$$dS = C_V\frac{dT}{T} + R\frac{dV}{V}$$

$$\Delta S = \int_i^f dS = \int_i^f \frac{C_V dT}{T} + R\int_i^f \frac{dV}{V} = C_V \ln\frac{T_f}{T_i} + R\ln\frac{V_f}{V_i}$$

Also, for perfect gas, we have

$$\frac{PV}{T} = \text{constant} \Rightarrow \frac{P_iV_i}{T_i} = \frac{P_fV_f}{T_f} \Rightarrow \frac{V_f}{V_i} = \frac{P_iT_f}{P_fT_i}$$

$$\Delta S = C_V \ln\frac{T_f}{T_i} + R\ln\frac{V_f}{V_i} = C_V \ln\frac{T_f}{T_i} + R\ln\frac{P_i}{P_f} + R\ln\frac{T_f}{T_i}$$

$$= C_V \ln\frac{T_f}{T_i} - R\ln\frac{P_f}{P_i} + R\ln\frac{T_f}{T_i}$$

$$= (C_P - R)\ln\frac{T_f}{T_i} - R\ln\frac{P_f}{P_i} + R\ln\frac{T_f}{T_i} \quad (\because C_P - C_V = R)$$

$$= C_P \ln\frac{T_f}{T_i} - R\ln\frac{P_f}{P_i}$$

(3) Change in entropy of a vander Waals' gas as its state changes from V_i, T_i to V_f, T_f:

Equation of state for one mole of vander Waal's gas is

$$\left(P + \frac{a}{V^2}\right)(V - b) = RT \qquad \text{where } a, b \text{ are vander Waals' constant}$$

$$dS = \frac{dQ}{T} \qquad \text{(Second law of thermodynamics)}$$

$$dQ = dU + dW \qquad \text{(First law of thermodynamics)}$$

where, $dU = C_V dT$ and $dW = \left(P + \frac{a}{V^2}\right)dV = \frac{RT}{V-b}dV \quad \left\{\because P_{eff} = P + \frac{a}{V^2}\right\}$

$$dS = \frac{dQ}{T} = \frac{dU + dW}{T} = \frac{C_V dT + \frac{RT}{(V-b)} dV}{T}$$

$$dS = C_V \frac{dT}{T} + R \frac{dV}{V-b}$$

The change in entropy when the gas passes from the state V_i, T_i to V_f, T_f is:

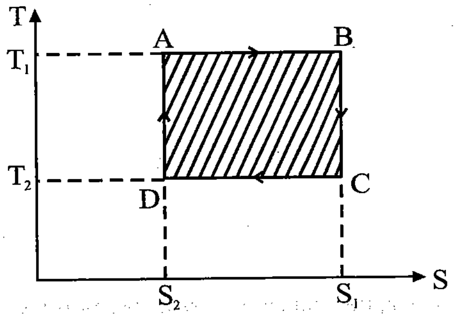
$$\Delta S = S_f - S_i = \int_i^f dS = C_V \int_i^f \frac{dT}{T} + R \int_i^f \frac{dV}{V-b}$$

$$= C_V \ell n \frac{T_f}{T_i} + R \ell n \frac{V_f - b}{V_i - b}$$

$$\boxed{\Delta S = C_V \ell n \frac{T_f}{T_i} + R \ell n \frac{V_f - b}{V_i - b}}$$

Temperature - Entropy Diagram

The Carnot cycle is T-S diagram is represented as:



Since adiabatic process occurs at constant entropy, BC and DA are adiabatics. On the other hand, AB and CD are isothermals.

The area under the curve AB representing any reversible process in a T-S diagram is

$$\int_A^B T\, dS = \int_A^B dQ = Q_{A \to B}$$

So the area under such a curve represents the heat flow in the same way that the area under a curve in a P-V diagram represents work. The area enclosed by the graph of a reversible cyclic process corresponds to the net heat flow into a system in the process.

Efficiency

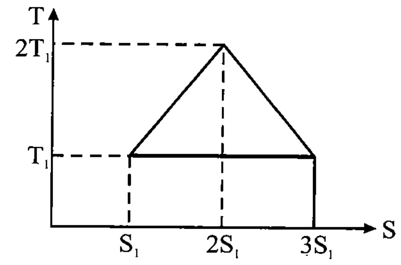
Heat entered into the system $= T_1(S_1 - S_2)$

Heat goes out of the system $= T_2(S_1 - S_2)$

The heat converted into work $= (T_1 - T_2)(S_1 - S_2)$

So the efficiency is $\eta = \frac{(T_1 - T_2)(S_1 - S_2)}{T_1(S_1 - S_2)} \Rightarrow \boxed{\eta = 1 - \frac{T_2}{T_1}}$

Problem: A reversible engine cycle is shown in the following T-S diagram. The efficiency of the engine is-



(a) $\frac{1}{3}$ (b) $\frac{1}{2}$ (c) $\frac{1}{5}$ (d) $\frac{1}{4}$

Soln. The net flow of heat is the area of T-S diagram

i.e., $Q_1 - Q_2 = W = \frac{1}{2} \times (3S_1 - S_1) \times (2T_1 - T_1) = S_1 T_1$

The heat goes out at low temperature T_1. So heat goes out is

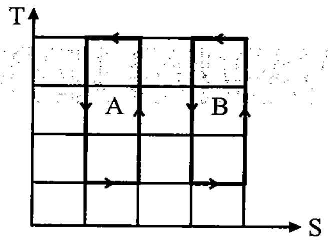
$Q_2 = (3S_1 - S_1) \times T_1 = 2 S_1 T_1$

Therefore, $Q_1 = Q_2 + S_1 T_1 = 3 S_1 T_1$

So, the efficiency is $\eta = 1 - \frac{Q_2}{Q_1} = 1 - \frac{2 S_1 T_1}{3 S_1 T_1} = \frac{1}{3}$

Correct option is (a)

Problem: The entropy temperature diagram of two Carnot engines, A and B are shown in the figure. The efficiencies of the engine are η_A and η_B, respectively. Which one of the following equalities is correct?



(a) $\eta_A = \frac{\eta_B}{2}$ (b) $\eta_A = \eta_B$ (c) $\eta_A = 3\eta_B$ (d) $\eta_A = 2\eta_B$

Soln. The efficiency for a Carnot engine is $\eta = 1 - \frac{T_2}{T_1}$. It means the efficiency depends only on the temperature of the source (T$_1$) and the temperature of the sink(T$_2$). Now for engine A and B, T$_1$ and T$_2$ are same.

So, $\eta_A = \eta_B$

Correct option is (b)

The Second Law of Thermodynamics

"It states that every process taking place in an isolated system, the entropy of the system either increases (irreversible process) or remains constant (reversible process)."

i.e. $(dS \geq 0)$

- Condition for equilibrium: The necessary condition for the equation of an isolated system is that its entropy shall be a maximum.
- **Clausius Statement:** No process is possible whose sole result is a flow out of one system at a given temperature and a heat flow of the same magnitude into a second system at a higher temperature. It means we must need some work to perform on the system to make the heat to flow from colder region to hotter region.

- **Kelvin-Planck Statement:** No process is possible whose sole result is heat flow Q out of a reservoir at a single temperature out the performance of work W equal in magnitude Q. It means that some heat must extract out of the system to the reservoir. It means that the efficiency of heat engine can never be greater than unity.

The Third Law of Thermodynamics

It states that the entropy of all the perfect crystalline solids is zero at absolute to zero temperature. It is also referred as *Nernst law.* It provides the basis for the calculation of absolute entropies of the substances. Mathematically,

$$\lim_{T \to 0} S = 0$$

PROBLEMS TO BE PLAYED

1. The combined form of first and second law of thermodynamics is given by–
(P = Pressure, V= Volume, T = Temperature, U = Internal energy, S = Entropy, Q = Quantity of heat)
(a) $TdS = dU + PdV$ (b) $dQ = TdS$
(c) $dU = TdS + dQ$ (d) $TdS = dU - PdV$

2. A given amount of heat cannot be completely converted into work. However it is possible to convert a given amount of work completely into heat. This statement results from the
(a) Zeroth law of thermodynamics (b) First law of thermodynamics
(c) Second law of thermodynamics (d) Third law of thermodynamics

3. One kilogram of ice melts at 0°C into water at the same temperature. The change in entropy is (in cal/K)–
(a) ∞ (b) 0
(c) 0.293 (d) 293

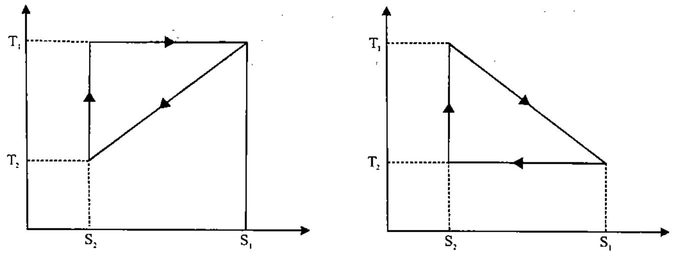
4. "A physical or chemical change takes place in such a way that the entropy either decreases or remains unchanged." This statement is
(a) True universally (b) True only for open system
(c) True only for closed system (d) Not true

5. Change in entropy depends–
(a) Only on the transfer of heat (b) Only on change of temperature
(c) On transfer of mass (d) On the thermodynamic state

6. Two ends of a rod are kept at 127° C and 227°C. When 2000 cal of heat flows in this rod, trhen the change in entropy is
(a) 1.0 cal/K (b) 20 cal/K (c) 6.9 cal/K (d) 0.7 cal/K

7. On a (T – f) diagram, *i.e.*, temperature (T) and entropy (f), the isothermals are
(a) Parallel to f axis
(b) Parallel to T-axis
(c) Some parallel to T and some parallel to f axis
(d) None of the these

8. The T – S diagram of two cycle for the operation of an engine are shown in figures below:



The numerical values of the parameters T_1, T_2, S_1 and S_2 in the two figures are the same which cycle has greater efficiency?
(a) First (b) Second
(c) Both have equal efficiency (d) None

. For any process, the second law of thermodynamics requires that the change of entropy of the universe be
(a) positive only (b) positive or zero (c) zero only (d) negative or zero

0. On a (T-S) diagram, adiabatics are lines
(a) having any indication
(b) parallel to the T-axis
(c) parallel to the S-axis
(d) some parallel to the S-axis and some parallel to the T-axis.

11. Which of the following is a property of entropy?
(a) Entropy increase during an irreversible operation
(b) Net change in entropy in a reversible cycle is zero
(c) Change in entropy during an adiabatic operation is zero
(d) All of the above

12. At 0 K fluids are assumed to have
(a) Minimum entropy (b) Maximum entropy
(c) Zero entropy (d) Fixed value of entropy

13. The difference in entropy between a state of volume V_i and a state of volume V_f (temperature and no. of molecules remaining constant) is equal to
(a) $nR\log\frac{V_f}{V_i}$ (b) $nR\log\frac{V_i}{V_f}$ (c) $nR^2\log\frac{V_f}{V_i}$ (d) $nR^2\log\frac{V_i}{V_f}$

14. An engine absorbs heat at a temperature of 1000 K and rejects heat at 600 K. If the engine operates at maximum possible efficiency, the amount of work performed by the engine for 200 J heat input is
(a) 1600 J (b) 1200 J (c) 800 J (d) 400 J

15. A mass m of water at temperature T_1 K is isobarically and adiabatically mixed with an equal mass of water at T_2 K. The entropy change of the universe is–
(a) $2mC_P\frac{T_1}{T_2}$ (b) $2mC_P In\frac{T_1+T_2}{2\sqrt{T_1T_2}}$ (c) $2mC_P In\frac{(T_1+T_2)}{2}$ (d) None of these

16. In a reversible process the change in entropy of the system–
(a) Increase (b) Decreases (c) Zero (d) Remains constant

17. In an irreversible adiabatic change in entropy–
(a) Increases (b) Decreases (c) Remains constant (d) Zero

18. Entropy remains constant in–
(a) Isothermal process (b) Adiabatic process (c) Cyclic process (d) Isobaric process

19. The are of the Carnot cycle on a T–S diagram represents–
(a) Heat absorbed from the source (b) Work done in a cycle
(b) Heat rejected to the sink (d) Efficiency of the engine

20. The value of entropy at absolute zero of temperature would be
(a) zero for all the materials
(b) finite for all the materials
(c) zero for some materials and non-zero for others
(d) unpredictable for any materials

ANSWER KEY

1. (a)	**2. (c)**	**3. (d)**	**4. (d)**	**5. (d)**	**6. (b)**	**7. (a)**
8. (b)	**9. (a)**	**10. (c)**	**11. (d)**	**12. (c)**	**13. (a)**	**14. (c)**
15. (b)	**16. (c)**	**17. (a)**	**18. (c)**	**19. (b)**	**20. (a)**	

Chapter 5

Thermodynamic Potentials & Maxwell Relations

Thermodynamic Potentials

Thermodynamic potentials are scalar quantities used to represent the themodynamic state of a system. They measure the energy of a system in terms of different variables because often we can only measure certain properties of the systems. For example, we might know the pressure and temperature of system but not the volume or the entropy. Thermodynamic potential allow us to measurse more state variables of the system. There are four thermodynamics potentials. They are :

1. Internal energy (U)
2. Enthalpy (H)
3. Helmholtz function (F)
4. Gibbs function (G)

1. Internal energy (U) : It is the sum total of internal molecular kinetic and potential energy of the system. It is a state function. By first law of thermodynamics the internal energy is given by

$$dU = dQ - dW \quad \text{... (1)}$$

where $dW = PdV$ and $dQ = TdS$

So, we have $dU = TdS - PdV$ ... (2)

Now keeping volume (V) constant in Eq. (2), we get

$$T = \left(\frac{\partial U}{\partial S}\right)_V \quad \text{... (3)}$$

and now keeping entropy (S) constant in Eq. (2), we have

$$P = -\left(\frac{\partial U}{\partial V}\right)_S \quad \text{... (4)}$$

Now, from Eq. (3), $\left(\frac{\partial T}{\partial V}\right)_S = \left(\frac{\partial^2 U}{\partial V\,\partial S}\right)_S$ ... (5)

and from Eq. (4), $-\left(\frac{\partial P}{\partial S}\right)_V = \left(\frac{\partial^2 U}{\partial S\,\partial V}\right)_V$ ... (6)

Exploiting $\frac{\partial^2 U}{\partial V\,\partial S} = \frac{\partial^2 U}{\partial S\,\partial V}$, we get $\boxed{\left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial P}{\partial S}\right)_V}$ ... (7)

2. **Enthalpy (*H*) :** Changes of phase are always associated with changes in volume, so that work is always done on or by a system in a phase change. If the change takes place at constant temperature, the pressure is constant also and the work done by the system is therefore,

$$W = P(V_2 - V_1) \qquad \text{... (8)}$$

where V_1 and V_2 are the initial and final volumes. Then from the first law, the change in internal energy is

$$U_2 - U_1 = L - P(V_2 - V_1) \qquad \text{... (9)}$$

where *L* is the heat supplied by or extracted out the system and known as latent heat or heat of transformation. This equation gives,

$$L = (U_2 + PV_2) - (U_1 + PV_1) \qquad \text{... (10)}$$

The sum $(U + PV)$ is defined as the enthalpy (*H*)

i.e., $H = U + PV$ ... (11)

Therefore, $L = H_2 - H_1$

The heat of transformation in any change of phase is equal to the difference between the enthalpies of the system in the two phase.

Now, using Eq. (11), we have

$$dH = dU + PdV + VdP$$

$$= TdS - PdV + PdV + VdP \qquad \text{[Since using (2)]}$$

Equation (12) gives at constant pressure, $dH = TdS$. It means that the enthalpy change represents the total heat of the system at constant pressure.

$\Rightarrow dH = TdS + VdP$ ... (12)

Now keeping entropy (*S*) and pressure (*P*) constant one by one, we have

$$V = \left(\frac{\partial H}{\partial P}\right)_S \qquad \text{... (13)}$$

and $T = \left(\frac{\partial H}{\partial S}\right)_P$ ... (14)

Using $\left(\frac{\partial H}{\partial S\,\partial P}\right) = \left(\frac{\partial H}{\partial P\,\partial S}\right)$, we get $\boxed{\left(\frac{\partial V}{\partial S}\right)_P = \left(\frac{\partial T}{\partial P}\right)_S}$ ... (15)

3. **Helmholtz function (*F*) :** If *Q* amount of heat flows from a single reservoir at a temperature *T* into a system, the entropy change of the reservoir is

$$(\Delta S)_{\text{res.}} = -\frac{Q}{T} \qquad \text{... (16)}$$

By the principle of entropy $(\Delta S)_{\text{system}} + (\Delta S)_{\text{res.}} \geq 0$

$$\Rightarrow S_2 - S_1 - \frac{Q}{T} \geq 0 \quad \left[\because\ (\Delta S)_{system} = S_2 - S_1\right] \qquad \text{[Since using (16)]}$$

$$\Rightarrow T(S_2 - S_1) \geq Q \qquad \text{... (17)}$$

From the first law, when a system performs any process, reversible or irreversible, between two equilibrium states, the work in the process is

$$W = -(U_2 - U_1) + Q \qquad \text{... (18)}$$

where $(U_2 - U_1)$ is the change in the internal energy of the system. So, we have

$$W \le -(U_2 - U_1) + T(S_2 - S_1)$$

$$\Rightarrow W \le (U_1 - TS_1) - (U_2 - TS_2)$$

Defining $U - TS \equiv F$, the Helmholtz function, we have

$$W \le F_1 - F_2 \qquad \text{... (19)}$$

That is the decrease in the Helmholtz function of a system sets an upper limit to the work in any process between two equilibrium states at the same temperature, during which there is a heat flow into the system from a single reservoir at this temperature.

The equality sign in Eq. (19) holds when the process is reversible and hence the work is maximum. If the process is irreversible, the work is less than this maximum.

Moreover at constant volume $W = PdV = 0$. So, Eq. (19) gives $F_1 - F_2 \ge 0$ or $F_1 \ge F_2$. It means that a process at constant volume and temperature is allowed only if $F_1 \ge F_2$, i.e. Helmholtz function decreases.

Using the definition, $F = U - TS$, we have

$$dF = dU - TdS - SdT$$

$$= TdS - PdV - TdS - SdT \qquad \text{[Since using (2)]}$$

$$\Rightarrow dF = -PdV - SdT \qquad \text{... (20)}$$

Keeping volume (V) and temperature (T) constant one by one in Eq. (20), we have

$$S = -\left(\frac{\partial F}{\partial T}\right)_V \qquad \text{... (21)}$$

and $P = -\left(\frac{\partial F}{\partial V}\right)_T$ ... (22)

Differentiating Eq. (21) w.r.t. volume at constant temperature and differentiating Eq. (22) with respect to temperature at constant volume, we have

$$\boxed{\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V} \qquad \text{... (23)}$$

where we have used $\dfrac{\partial^2 F}{\partial V\,\partial T} = \dfrac{\partial^2 F}{\partial T\,\partial V}$.

4. **Gibbs function (G) :** Consider a process at a constant external pressure P. The PdV work is

$$W = P(V_2 - V_1)$$

If these is non-PdV work (W_{non}) also in the system then from Eq. (19), we have

$$P(V_2 - V_1) + W_{\text{non}} \le F_1 - F_2$$

$$\Rightarrow W_{\text{non}} \le (F_1 + PV_1) - (F_2 + PV_2)$$

Writing $F + PV \equiv G$, the Gibbs function, we have

$$W_{\text{non}} \le G_1 - G_2 \qquad \text{... (24)}$$

That is the decrease in the Gibbs function of a system sets an upper limit to the "non-PdV" work in any process between two equilibrium states at the same temperature and pressure. The equality in equation (24) holds when the process is reversible.

Since $G = F + PV$

$$\Rightarrow G = U - TS + PV \qquad \text{... (25)}$$

where $F = U - TS$

Moreover Eq. (25) gives $G = H - TS$ [Using (11)]

Using Eq. (25), we have

$$dG = dU - TdS - SdT + PdV + VdP$$

$$= TdS - PdV - TdS - SdT + PdV + VdP \qquad \text{[Since using (2)]}$$

$$\Rightarrow dG = -SdT + VdP \qquad \text{... (26)}$$

Keeping temperature and pressure (P) constant one by one in Eq. (26), we get

$$S = -\left(\frac{\partial G}{\partial T}\right)_P \qquad \text{... (27)}$$

$$\text{and } V = \left(\frac{\partial G}{\partial P}\right)_T \qquad \text{... (28)}$$

From equations (27) and (28) and exploiting $\left(\frac{\partial^2 G}{\partial P \partial T}\right) = \left(\frac{\partial^2 G}{\partial T \partial P}\right)$, we have

$$\boxed{\left(\frac{\partial V}{\partial T}\right)_P = -\left(\frac{\partial S}{\partial P}\right)_T} \qquad \text{... (29)}$$

Problem: If U, F, H and G represent internal energy, Helmholtz free energy, enthalpy, and Gibbs free energy respectively, then which one of the following is a correct thermodynamic relation? **[JAM-2016]**

(a) $dU = PdV - TdS$ (b) $dH = VdP + TdS$ (c) $dF = -PdV + SdT$ (d) $dG = VdP + SdT$

Soln. The enthalpy is given by

$$H = U + PV$$

$$dH = dU + PdV + VdP$$

$$\Rightarrow dH = TdS - PdV + PdV + VdP \quad [\because \; dU = TdS - PdV]$$

$$\Rightarrow dH = TdS + VdP$$

Correct option is (b)

Problem: Let U, T, S and P denote, respectively, the internal energy, temperature, entropy and pressure of a thermodynamic system. Then a change ΔF in the free energy $F = U - TS$ measures **[JAM-2009]**

(a) the heat exchanged at constant pressure
(b) the work done by the system at constant entropy
(c) the work done on the system at constant temperature
(d) the heat exchanged at constant temperature

Soln. Given $F = U - TS$

$$\Rightarrow \Delta F = \Delta U - T\Delta S - S\Delta T = T\Delta S - P\Delta V - T\Delta S - S\Delta T = -P\Delta V - S\Delta T$$

At constant temperature,

$$\Delta F = -P\Delta V$$

So ΔF represents the work done on the system at constant temperature.

Correct option (c)

Problem: A thermodynamic system is maintained at constant temperature and pressure. In thermodynamic equilibrium, its

(a) Gibbs free energy is minimum (b) Enthalpy is maximum

(c) Helmholtz free energy is minimum (d) Internal energy is zero

Soln. The Gibb's free energy is

$$G = U - TS + PV$$

$$\Rightarrow \quad dG = dU - TdS - SdT + PdV + VdP$$

$$= TdS - PdV - TdS - SdT + PdV + VdP$$

$$= -SdT + VdP \qquad [\because \; dU = TdS - PdV]$$

At constant temperature and pressure,

$$dG = 0$$

So Gibbs free energy is minimum.

Correct option is (a)

Problem: The Gibb's function G in thermodynamics is defines as $G = H - TS$ where H is the enthalpy, T is the temperature and S is the entropy. In an isothermal, isobaric, reversible process, value of G

(a) remains constant but not zero (b) varies linearly

(c) varies non linearly (d) is zero

Soln. $G = H - TS = U + PV - TS \qquad [\because \; H = U + PV]$

$$\therefore \quad dG = dU + pdV + VdP - TdS - SdT$$

$$= TdS - PdV + PdV + VdP - TdS - SdT = VdP - SdT$$

In isobaric $(dP = 0)$ and isothermal $(dT = 0)$ process, $dG = 0$

$\Rightarrow$ G remains constant but not zero.

Correct option is (a)

Problem: With U and F as thermodynamic potentials, the Gibb's Helmholtz equation is

(a) $U = F - T\left(\frac{\partial F}{\partial T}\right)_V$ (b) $U = F + T\left(\frac{\partial F}{\partial T}\right)_V$ (c) $S = F - T\left(\frac{\partial F}{\partial T}\right)_V$ (d) $S = F + T\left(\frac{\partial F}{\partial T}\right)_V$

Soln. The definition of F is

$$F = U - TS$$

or $U = F + TS$ ... (1)

Also, $dF = dU - TdS - SdT$

$$= TdS - PdV - TdS - SdT = -PdV - SdT$$

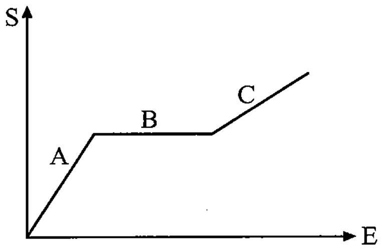
$$\therefore \quad S = -\left(\frac{\partial F}{\partial T}\right)_V \qquad \text{... (2)}$$

Using equation (2) in (1),

$$U = F - T\left(\frac{\partial F}{\partial T}\right)_V$$

Correct option is (a).

Problem: The entropy S of a thermodynamic system as a function of energy E is given by the following graph



The temperatures of the phases A, B and C, denoted by T_A, T_B and T_C, respectively, satisfy the following inequalities :

(a) $T_C > T_B > T_A$ (b) $T_A > T_C > T_B$ (c) $T_B > T_C > T_A$ (d) $T_B > T_A > T_C$

Soln. We know that,

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_{N,V}$$

From figure,

$$\left(\frac{\partial S}{\partial E}\right)_A > \left(\frac{\partial S}{\partial E}\right)_C > \left(\frac{\partial S}{\partial E}\right)_B$$

$$\Rightarrow T_B > T_C > T_A$$

Correct option is (c)

Problem: The internal energy E of a system is given by $E = \frac{bS^3}{VN}$, where b is a constant and other symbols have their usual meaning. The temperature of this system is equal to

(a) $\frac{bS^2}{VN}$ (b) $\frac{3bS^2}{VN}$ (c) $\frac{bS^3}{V^3N}$ (d) $\left(\frac{S}{N}\right)^2$

Soln. We know that,

$$T = \left(\frac{\partial E}{\partial S}\right)_{N,V}$$

Where $E = \frac{bS^3}{VN}$ $\therefore$ $T = \frac{3bS^2}{VN}$

Correct option is (b)

Problem : Let U, T, S and P denote, respectively, the internal energy, temperature, entropy and pressure of a thermodynamic system. Then a change ΔF in the free energy $F = U - TS$ measures

(a) the heat exchanged at constant pressure
(b) the work done by the system at constant entropy
(c) the work done on system at constant temperature
(d) the heat exchanged at constant temperature

[JNU : 2009]

Soln. $F = U - TS$

$$dF = dU - TdS - SdT$$

$$= TdS - PdV - TdS - SdT$$

$$= -PdV - SdT$$

At constant temperature, $dF = -PdV$

So, ΔF measures the work done by system at constant temperature.

Hence, correct option is (c)

Maxwell Equations

The equations (7), (15), (23) and (29) are called Maxwell equations. They are :

$$\left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial P}{\partial S}\right)_V ; \left(\frac{\partial T}{\partial P}\right)_S = \left(\frac{\partial V}{\partial S}\right)_P ; \left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V ; \left(\frac{\partial S}{\partial P}\right)_T = -\left(\frac{\partial V}{\partial T}\right)_P$$

Note :

1. In each of the Maxwell relations, the cross product of the differentials has the dimensions of energy.

2. The independent variable in the denominator of one side of an equation is the constant on the other side.

Problem: Establish the following thermodynamic relation $\left(\frac{\partial Q}{\partial P}\right)_T = -T\left(\frac{\partial V}{\partial T}\right)_P = -TV\alpha$ where α in the coefficient of volume expansion at constant pressure. Hence, show that increase in pressure heats a body that expands with rise of temperature and colds that contracts on heating.

Soln. The Maxwell thermodynamic relation is:

$$\left(\frac{\partial S}{\partial P}\right)_T = -\left(\frac{\partial V}{\partial T}\right)_P$$

Multiplying by T both side

$$T\left(\frac{\partial S}{\partial P}\right)_T = -T\left(\frac{\partial V}{\partial T}\right)_P$$

$$\Rightarrow \quad \left(\frac{\partial Q}{\partial P}\right)_T = -T\left(\frac{\partial V}{\partial T}\right)_P \qquad (\because dQ = TdS)$$

$$\Rightarrow \quad \left(\frac{\partial Q}{\partial P}\right)_T = -TV\frac{I}{V}\left(\frac{\partial V}{\partial T}\right)_P$$

$$\Rightarrow \quad \left(\frac{\partial Q}{\partial P}\right)_T = -TV\alpha \qquad \text{where} \quad \alpha = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P \text{ – Coefficient of Volume expansion}$$

(i) If substance expands with rise in temperature then α will be positive

Hence, $\left(\frac{\partial Q}{\partial P}\right)_T < 0$

If we increase the pressure, ∂Q at constant temperature will be negative. So, to maintain temperature constant heat must be drawn from the system. It means increase in pressure will heat the system.

(ii) If substance contracts with rise in temperature then α will be negative. Hence,

$$\left(\frac{\partial Q}{\partial P}\right)_T = +Ve$$

If we increase the pressure $(\partial Q)_T$ will be positive, to maintain temperature constant heat must be given to the system. It means increase in pressure will cools the system.

Problem: Show that $\left(\frac{\partial \alpha}{\partial P}\right)_T + \left(\frac{\partial \beta}{\partial T}\right)_P = 0$

where α and β are the coefficient of thermal expansion and coefficient of compressibility respectively

Soln. Since V is a state function so we can write

$$\frac{\partial^2 V}{\partial T \partial P} = \frac{\partial^2 V}{\partial P \partial T} \quad \text{or} \quad \frac{\partial}{\partial T}(-V\beta)_P = \frac{\partial}{\partial P}(V\alpha)_T \qquad \left[\because -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T = \beta \text{ and } \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P = \alpha\right]$$

Carrying out the differentiation we get

$$-\beta\left(\frac{\partial V}{\partial T}\right)_P - V\left(\frac{\partial \beta}{\partial T}\right)_P = \alpha\left(\frac{\partial V}{\partial P}\right)_T + V\left(\frac{\partial \alpha}{\partial P}\right)_T$$

or
$$-\frac{\beta}{V}\left(\frac{\partial V}{\partial T}\right)_P - \left(\frac{\partial \beta}{\partial T}\right)_P = \frac{\alpha}{V}\left(\frac{\partial V}{\partial P}\right)_T + \left(\frac{\partial \alpha}{\partial P}\right)_T$$

$$\left(\frac{\partial \alpha}{\partial P}\right)_T + \left(\frac{\alpha \beta}{\partial T}\right)_P = -\frac{\beta}{V}\left(\frac{\partial V}{\partial T}\right)_P - \frac{\alpha}{V}\left(\frac{\partial V}{\partial P}\right)_T$$

$$= -\frac{\beta}{V}(\alpha V) - \frac{\alpha}{V}(-\beta V) = 0$$

Energy Equation

First law of thermodynamics ⇒ $dQ = dU + PdV$
Second law of thermodynamics ⇒ $dQ = T\,d\,S$
Combining these two laws ⇒ $TdS = dU + PdV$

This equation is called combined first and second law of thermodynamics.

$$\Rightarrow \quad dS = \frac{1}{T}dU + \frac{P}{T}dV$$

Differentiate w.r.t. volume at constant temperature, we get

$$\left(\frac{\partial S}{\partial V}\right)_T = \frac{1}{T}\left(\frac{\partial U}{\partial V}\right)_T + \frac{P}{T} \quad \Rightarrow \quad T\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial U}{\partial V}\right)_T + P$$

Using Maxwell's thermodynamic relation $\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V$ the above equation become

$$T\left(\frac{\partial P}{\partial T}\right)_V = \left(\frac{\partial U}{\partial V}\right)_T + P$$

So,
$$\boxed{\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_V - P} \qquad \dots (30)$$

This relation is called *Energy equation*.

(i) For an ideal gas:

The equation of state for an ideal gases is

$$PV = RT \Rightarrow P = \frac{RT}{V} \Rightarrow \left(\frac{\partial P}{\partial T}\right)_V = \frac{R}{V}$$

Put this value in equation (30), we have

$$\left(\frac{\partial U}{\partial V}\right)_T = T.\frac{R}{V} - P$$

$$\left(\frac{\partial U}{\partial V}\right)_T = P - P = 0 \quad \left(\because \frac{RT}{V} = P\right)$$

Hence internal energy of an ideal gas is independent of volume

(ii) For a vander Waal gas (or Perfect gas) :

The equation of state is

$$\left(P + \frac{a}{V^2}\right)(V - b) = RT$$

$$P = \frac{RT}{V-b} - \frac{a}{V^2}$$

$$\left(\frac{\partial P}{\partial T}\right)_V = \frac{R}{V-b}$$

Putting this value in equation (30), we have

$$\left(\frac{\partial U}{\partial V}\right)_T = T.\frac{R}{V-b} - P$$

$$\left(\frac{\partial U}{\partial V}\right)_T = \frac{a}{V^2}$$

Hence internal energy of an real gas depends on volume.

Problem: A gas obeys the equation of state $PV = nRT + nB(T)P$, where $B(T)$ is a constant which depends on temperature. Find W, ΔU, Q and ΔH.

Soln. Here $W = -\int_{V_1}^{V_2} PdV = -\int_{V_1}^{V_2} \frac{RT}{V-B}dV = -RT\ln\left(\frac{V_2 - B}{V_1 - B}\right)$, W $= RT\ln\left(\frac{P_2}{P_1}\right)$

Now, $dU = TdS - PdV$,

The energy equation is $\left(\frac{\partial U}{\partial V}\right)_T = -P + T\left(\frac{\partial P}{\partial T}\right)_V$

We know, $\Delta U = \int_{V_1}^{V_2}\left(\frac{\partial U}{\partial V}\right)_T dV = \int_{V_1}^{V_2}\left[-P + T\left(\frac{\partial P}{\partial T}\right)_V\right]dV$

Here, $P = \frac{RT}{V-B}$

or $\left(\frac{\partial P}{\partial T}\right)_v = \frac{R}{V-B} + \frac{RT}{(V-B)^2}\frac{dB}{dT}$

$\therefore \quad T\left(\frac{\partial P}{\partial T}\right)_v = \frac{RT}{V-B} + \frac{RT^2}{(V-B)^2}\frac{dB}{dT}$

Hence $\Delta U = RT^2\frac{dB}{dT}\int_{V_1}^{V_2}\frac{dV}{(V-B)^2} = -RT^2\frac{dB}{dT}\left(\frac{1}{V_2-B}-\frac{1}{V_1-B}\right) = -T\frac{dB}{dT}\left(\frac{RT}{V_2-B}-\frac{RT}{V_1-B}\right)$

Or, $\boxed{\Delta U = T\frac{dB}{dT}(P_1-P_2)}$

To calculate Q, we use first law so

$Q = \Delta U - W \quad \Rightarrow Q = T\frac{dB}{dT}(P_1-P_2) + RT\ln\left(\frac{P_1}{P_2}\right)$

For ΔH, $\Delta H = \Delta U + \Delta(PV)$

$\Rightarrow \Delta(PV) = \Delta(RT+BP) = (RT+BP_2)-(RT+BP_1) = B(P_2-P_1)$

Hence $\Delta H = T\frac{dB}{dT}(P_1-P_2) + B(P_2-P_1)$

or $\boxed{\Delta H = \left(B - T\frac{dB}{dT}\right)(P_2-P_1)}$

Problem: A certain gas obeys the equation of state $P(V-b) = RT$

(i) Derive an expression for heat & work in a reversible isothermal expansion of the gas

(ii) Derive an expression for heat & work in a reversible adiabatic expansion of the gas

(iii) Calculate ΔU, Q and W when 1 mol of the gas expands reversible at 300K from 0.220 to 20.02 litre.

(a) isothermally (b) adiabatically. Assume that b = 0.020 litre mol^{-1}, $C_V = \frac{3}{2}R$ and $\gamma = \frac{5}{3}$

Soln. (i) $dW = PdV$ or $W = \int_{V_1}^{V_2} PdV$

Given equation of state is $P = \frac{RT}{(V-b)}$

Thus, $W = \int_{V_1}^{V_2}\frac{RT}{V-b}dV = RT\ell n\left(\frac{V_2-b}{V_1-b}\right)$

Now, $\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_V - P = T\left(\frac{\partial\left(\frac{RT}{V-b}\right)}{\partial T}\right)_V - P$

$$= \frac{RT}{V-b} - P = P - P = 0$$

$\Rightarrow \quad \Delta U = 0$

Thus, $Q = \Delta U + W = W = RT\,\ell n\left(\frac{V_2 - b}{V_1 - b}\right)$

(ii) For adiabatic reversible expansion, Q = 0

Hence $dU = dW = \frac{RT}{V-b}dV$

$$\Rightarrow \int_{T_1}^{T_2}\frac{C_V dT}{T} = \int_{V_1}^{V_2}\frac{R}{V-b}dV \Rightarrow C_V \ell n\left(\frac{T_2}{T_1}\right) = R\ell n\left(\frac{V_2-b}{V_1-b}\right)$$

$$\Rightarrow \frac{T_2}{T_1} = \left(\frac{V_2-b}{V_1-b}\right)^{\frac{R}{C_V}} = \left(\frac{V_2-b}{V_1-b}\right)^{(\gamma-1)}$$

$$\therefore W = \Delta U = C_V\left(T_2 - T_1\right)$$

$$= C_V T_1\left[\left(\frac{V_1-b}{V_2-b}\right)^{\gamma-1} - 1\right]$$

(iii) $T_1 = T_2 = 300K, V_1 = 0.220$ litre, $V_2 = 20.020$ litre, $b = 0.020$ litre mol^{-1}

(a) For isothermal reversible expansion

$$W = Q = RT_1\,\ell n\left(\frac{V_2-b}{V_1-b}\right) = (8.314)(300)\,\ell n\left(\frac{20.0}{0.20}\right) = 11.548\ KJ\,mol^{-1}$$

$$\Delta U = 0$$

(b) For an adiabatic reversible expansion

$Q = 0$ thus

$$\Delta U = W = C_V T_1\left[\left(\frac{V_1-b}{V_2-b}\right)^{\gamma-1} - 1\right]$$

Given, $C_V = \frac{3}{2}R,\ \gamma = \frac{5}{3}$ and $T_1 = 300\ K$

$$\Delta U = W = \left(\frac{3}{2}R\right)(300)\left[\left(\frac{200}{20000}\right)^{2/3} - 1\right] = -3.554\ \text{kJ}$$

General Relation between C_V and C_P

The specific heat at constant pressure and constant volume are given as

$$C_P = \left(\frac{\partial Q}{\partial T}\right)_P = T\left(\frac{\partial S}{\partial T}\right)_P \qquad [\because dQ = TdS]$$

and $C_V = \left(\frac{\partial Q}{\partial T}\right)_V = T\left(\frac{\partial S}{\partial T}\right)_V$

Now, if the entropy S is regarded as a function of T and V and since dS is perfect differential

$$S = S(T, V)$$

$$dS = \left(\frac{\partial S}{\partial T}\right)_V dT + \left(\frac{\partial S}{\partial V}\right)_T dV$$

$$\Rightarrow \left(\frac{\partial S}{\partial T}\right)_P = \left(\frac{\partial S}{\partial T}\right)_V + \left(\frac{\partial S}{\partial V}\right)_T \left(\frac{\partial V}{\partial T}\right)_P$$

$$\Rightarrow T\left(\frac{\partial S}{\partial T}\right)_P = T\left(\frac{\partial S}{\partial T}\right)_V + T\left(\frac{\partial S}{\partial V}\right)_T \left(\frac{\partial V}{\partial T}\right)_P$$

From Maxwell's relation $\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V$

$$\therefore \quad T\left(\frac{\partial S}{\partial T}\right)_P = T\left(\frac{\partial S}{\partial T}\right)_V + T\left(\frac{\partial P}{\partial T}\right)_V \left(\frac{\partial V}{\partial T}\right)_P$$

Now, $$C_P - C_V = T\left(\frac{\partial S}{\partial T}\right)_P - T\left(\frac{\partial S}{\partial T}\right)_V = T\left(\frac{\partial P}{\partial T}\right)_V \left(\frac{\partial V}{\partial T}\right)_P \qquad \text{...(31)}$$

(a) For a perfect gas
The equation of state is $PV = RT$.

$$\therefore \quad \left(\frac{\partial P}{\partial T}\right)_V = \frac{R}{V} \text{ and} \left(\frac{\partial V}{\partial T}\right)_P = \frac{R}{P}$$

$$\therefore \quad C_P - C_V = T\left(\frac{\partial P}{\partial T}\right)_V \left(\frac{\partial V}{\partial T}\right)_P = T\left(\frac{R}{V}\right)\left(\frac{R}{P}\right) = \frac{TR^2}{PV} = \frac{R^2T}{RT} = R$$

(b) For a vander Waal's gas

The equation of state is $\left(P + \frac{a}{V^2}\right)(V - b) = RT$

or $$\left(P + \frac{a}{V^2}\right) = \frac{RT}{V-b} \qquad \text{...(32)}$$

where a and b are constant
Differentiating equation (32) w.r.t. T at constant volume, we get

$$\left(\frac{\partial P}{\partial T}\right)_V = \frac{R}{V-b}$$

Differentiating equation(32) w.r.t. T at constant pressure ,we get

$$0 - \frac{2a}{V^3}\left(\frac{\partial V}{\partial T}\right)_P = -\frac{RT}{(V-b)^2}\left(\frac{\partial V}{\partial T}\right)_P + \frac{R}{V-b}$$

$$\Rightarrow \left(\frac{\partial V}{\partial T}\right)_P \left[\frac{RT}{(V-b)^2} - \frac{2a}{V^3}\right] = \frac{R}{V-b} \Rightarrow \left(\frac{\partial V}{\partial T}\right)_P = \frac{\frac{R}{V-b}}{\left[\frac{RT}{(V-b)^2} - \frac{2a}{V^3}\right]}$$

Substituting the value of $\left(\frac{\partial P}{\partial T}\right)_V$ and $\left(\frac{\partial V}{\partial T}\right)_P$ in equation (31) we have

$$C_P - C_V = T\frac{R}{V-b}\cdot\frac{\frac{R}{V-b}}{\left[\frac{RT}{(V-b)^2}-\frac{2a}{V^3}\right]} = \frac{R^2T}{(V-b)^2}\frac{1}{\left[\frac{RT}{(V-b)^2}-\frac{2a}{V^3}\right]}$$

So $$C_P - C_V = \frac{R}{1-\frac{2a}{V^3}\frac{(V-b)^2}{RT}}$$

Neglecting b in comparison to V

$$C_P - C_V = \frac{R}{1-\frac{2a}{V^3}\frac{V^2}{RT}} = \frac{R}{1-\frac{2a}{RTV}} = R\left(1-\frac{2a}{RTV}\right)^{-1}$$

Expanding binomially and neglecting higher powers we get

$$C_P - C_V = R\left(1+\frac{2a}{RTV}\right)$$

TdS Eequations

(i) $TdS = C_V dT + T\left(\frac{\partial P}{\partial T}\right)_V dV$ (ii) $TdS = C_P dT - T\left(\frac{\partial V}{\partial T}\right)_P dP$

Prove :

(i) Let the entropy S of a thermodynamic system be a function of temperature T and volume V i.e. $f(T, V)$. Since dS is a perfect differential, so we can write

$$dS = \left(\frac{\partial S}{\partial T}\right)_V dT + \left(\frac{\partial S}{\partial V}\right)_T dV$$

Multiplying by T, we have

$$TdS = T\left(\frac{\partial S}{\partial T}\right)_V dT + T\left(\frac{\partial S}{\partial V}\right)_T dV \qquad \text{... (33)}$$

Since $TdS = dQ$ and for any substance the specific heat at constant volume is given by

$$C_V = \left(\frac{\partial Q}{\partial T}\right)_V = T\left(\frac{\partial S}{\partial T}\right)_V$$

From Maxwell's equation, we get

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V$$

Making these substitution in equation (33), we get

$$\boxed{TdS = C_V dT + T\left(\frac{\partial P}{\partial T}\right)_V dV}$$ which is the required relation

(ii) Let the entropy S of a thermodynamic system be function of temperature T and pressure P i.e., $S=f(T,P)$ Since dS is a perfect differential

$$dS = \left(\frac{\partial S}{\partial T}\right)_P dT + \left(\frac{\partial S}{\partial P}\right)_T dP$$

Multiplying by T, we get

$$TdS = T\left(\frac{\partial S}{\partial T}\right)_P dT + T\left(\frac{\partial S}{\partial P}\right)_T dP \qquad \text{... (34)}$$

For any substance the specific heat at constant pressure is given by

$$C_P = \left(\frac{\partial Q}{\partial T}\right)_P = T\left(\frac{\partial S}{\partial T}\right)_P$$

From Maxwell's equation, $\left(\frac{\partial S}{\partial P}\right)_T = -\left(\frac{\partial V}{\partial T}\right)_P$

Making these substitution in equation (34), we get

$$\boxed{TdS = C_P dT - T\left(\frac{\partial V}{\partial T}\right)_P dP}$$

PROBLEMS TO BE PLAYED

1. Which of the following is not Maxwell's thermodynamic relation?

(a) $\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V$ (b) $\left(\frac{\partial S}{\partial P}\right)_T = -\left(\frac{\partial V}{\partial T}\right)_P$ (c) $\left(\frac{\partial T}{\partial P}\right)_S = \left(\frac{\partial V}{\partial S}\right)_P$ (d) $\left(\frac{\partial P}{\partial V}\right)_T = \left(\frac{\partial S}{\partial T}\right)_V$

2. Which of the following Maxwell's relation leads of Clausius–Clapeyron equation?

(a) $\left(\frac{\partial T}{\partial V}\right)_S = \left(\frac{\partial P}{\partial T}\right)_V$ (b) $\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V$ (c) $\left(\frac{\partial T}{\partial P}\right)_S = \left(\frac{\partial V}{\partial S}\right)_P$ (d) $\left(\frac{\partial V}{\partial T}\right)_P = \left(\frac{\partial S}{\partial P}\right)_T$

3. The thermodynamical relation expressing the change at temperature with volume at constant entropy is–

(a) $\left(\frac{\partial T}{\partial V}\right)_S = -T\left(\frac{\partial P}{\partial Q}\right)_V$ (b) $\left(\frac{\partial T}{\partial V}\right)_S = T\left(\frac{\partial P}{\partial Q}\right)_V$

(c) $\left(\frac{\partial T}{\partial V}\right)_S = -V\left(\frac{\partial P}{\partial Q}\right)_T$ (d) $\left(\frac{\partial T}{\partial V}\right)_S = T\left(\frac{\partial P}{\partial Q}\right)_T$

4. The thermodynamical relation expressing T*d*S equation–

(a) $TdS = C_V dT - T\left(\frac{\partial P}{\partial T}\right)_V dV$ (b) $TdS = C_V dT + T\left(\frac{\partial P}{\partial T}\right)_V dV$

(c) $TdS = C_V dT - T\left(\frac{\partial S}{\partial V}\right)_T$ (d) $TdS = C_V dT + T\left(\frac{\partial S}{\partial V}\right)_T$

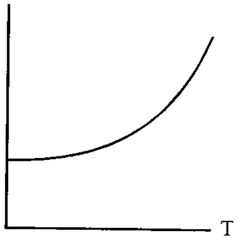
5. The Gibb's function G in thermodynamics is defined as

$$G = H - TS$$

(where H = Enthalpy, T = Temperature, S = Entropy)
In an isothermal, isobaric, reversible process, G
(a) Remains constant but not zero (b) Varies linearly
(c) Varies non linearly (d) Is zero

The Gibb's potential is defined as–
(a) $G = U - PV + TS$ (b) $G = U + PV + TS$
(c) $G = U - PV - TS$ (d) $G = U + PV - TS$

For a stable thermodynamics system, the temperature dependence of the free energy $F(T)$ is represented by

(a)


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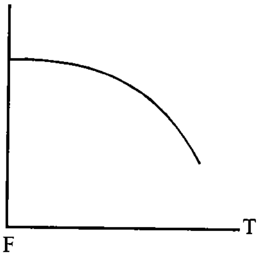

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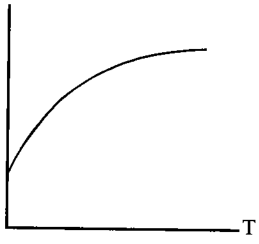

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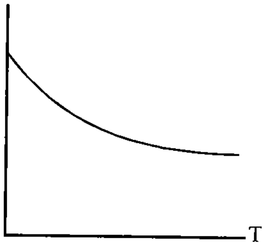
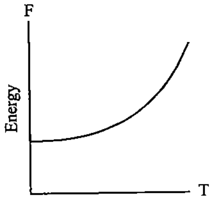
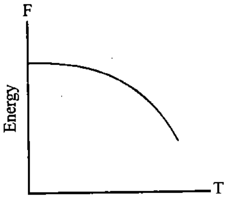


8. For a stable thermodynamic system, the temperature dependence of the internal energy U(T) is represented as

(a)


(b)


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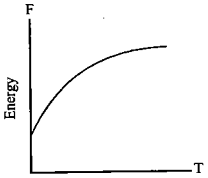
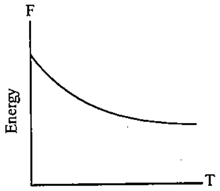


(d)


9. With U and F as thermodynamic potentials, the Gibb's Helmholtz equation is given as

(a) $U = F - T\left(\frac{\partial F}{\partial \mathrm{T}}\right)_V$ (b) $U = F + T\left(\frac{\partial F}{\partial \mathrm{T}}\right)_V$ (c) $S = F - T\left(\frac{\partial F}{\partial \mathrm{T}}\right)_V$ (d) $S = F + T\left(\frac{\partial F}{\partial \mathrm{T}}\right)_V$

10. A fluid at high pressure is throuttled through a narrow porous opening in a region of lower pressure without any transfer of heat. In such a process

(a)The entropy does not change
(b) The Gibb's free energy remains constant
(c) The entropy is decreased
(d) The enthalpy of the fluid is constant

11. Which of the following is not an exact differential?

(a) dQ (Q = heat absorbed)
(b) dU (U = internal energy)
(c) dS (S = entropy)
(d) dF (F = free energy)

ANSWER KEY

1. (d) **2. (b)** **3. (a)** **4. (b)** **5. (a)** **6. (d)** **7. (a)**

8. (b) **9. (a)** **10. (d)** **11. (a)**

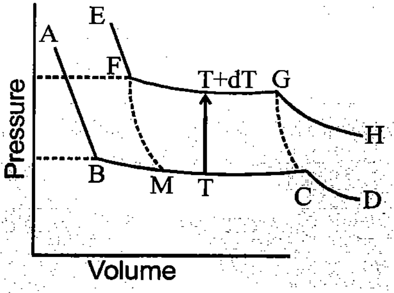
Chapter 6

The Phase Transition

Clausius Clapeyron Equation

The phase diagram shows that only two phase of water co-exist at a time, either liquid and gas or solid and liquid or solid and gas. When there is change in the states from solid to liquid state, the characteristic temperature is called the *melting point* of solid and when it is from liquid to vapour the temperature is called *boiling point* of the liquid.

Clausius developed the relation which shows the variation in the boiling and melting with the pressure by applying II law of thermodynamics that relation is known as *Clausius Clapeyron equation.*



ABCD and EFGH are two isothermals which are maintained at infinitesimally close temperature T and T+dT. In figure AB and EF represents the liquid state of the substance. At point B and F, substance exists in pure liquid state. FG and BC are that states where liquid and vapour phase of the substance co-exist. Substance is purely in vapour state in GH and CD states.

In order to form a Carnot cycle, we draw two adiabatic from F and G to M and C respectively. Now, FGCMF forms a Carnot cycle in which FG represents the isothermal expansion, GC shows the adiabatic expansion, NM is the process along which isothermal boiling point of liquid rises with increase in pressure and vice versa. The amount of heat Q_1 absorbed along FG is equal to the latent heat of vapourization (L + dL) at temperature (T + dT) as substance changing from liquid state at F to the vapourization state FG. From C to B, the amount of heat rejected is Q_2. Here, Q_2 is equal to the latent heat of vapourization L at temperature T. By increasing the pressure a little, the original point F is restored. So, the cycle FGCBF is completely reversible. Then applying the principle of Carnot's reversible cycle, we have
From the Carnot cycle.

$$\frac{Q_1}{T_1} = \frac{Q_2}{T_1}$$

$$\frac{Q_1 - Q_2}{Q_2} = \frac{T_1 - T_2}{T_2}$$

Since, $Q_1 = L + dL \;\&\; Q_2 = L$

Also, $T_1 = T + dT \;\&\; T_2 = T$

Then $\dfrac{L+dL-L}{L}=\dfrac{T+dT-T}{T}$

$$dL=\frac{L}{T}dT \qquad \text{... (1)}$$

The amount of heat converted into work $=Q_1-Q_2=L+dL-L=dL$ ... (2)

The work done by the substance is also represented by the area under curve, i.e. area of rhombus.

$$dL=dP\left(V_2-V_1\right) \qquad \text{... (3)}$$

Using equation (1), (2) and (3)

We have $\dfrac{L}{T}dT=dP\left(V_2-V_1\right) \Rightarrow \boxed{\dfrac{dP}{dT}=\dfrac{L}{T\left(V_2-V_1\right)}}$

This equation is called *Clausius–Clapeyron Equation*. This is true for both the change of state, i.e., from liquid to vapour and solid to liquid.

1. **Effect of melting point:**

When solid melts, there occurs two cases.

(a) When $V_2 > V_1$ for substance like wax, sulphur, $\dfrac{dP}{dT}>0$, therefore melting point of solid rise with rise in pressure and vice versa.

(b) When $V_2 < V_1$ for substance like ice, gallium and bismuth, V_2-V_1 is negative quantity. Hence $\dfrac{dP}{dT}$ is also negative, which means melting point of such substance decreases with increase in pressure.

2. **Effect of pressure on boiling points of liquid.**

When liquid boil i.e. changes its state from liquid to gaseous, there is an increase in volume, which means $\left(V_2-V_1\right)$ is positive and hence $\dfrac{dP}{dT}>0$.

Problem: The pressure versus temperature diagram of a given system at certain low temperature range is found to be parallel to the temperature axis in the liquid to solid transition region. The change in the specific volume remains constant in this region. The conclusion one can get from the above is

(a) the entropy of solid is zero in this temperature region

(b) the entropy increases when the system goes from liquid to solid phase in this temperature region

(c) the entropy decreases when the system transforms from liquid to solid phase in this region

(d) the change in entropy is zero in the liquid to solid transition region

Soln. *P-T* curve is parallel to the temperature axis

$\Rightarrow$ slope of the curve = 0

$$\Rightarrow \frac{dP}{dT}=0 \qquad \text{...(i)}$$

Now, from Clausius-Clapeyron equation

$$\frac{dP}{dT}=\frac{L}{T\Delta V}=\frac{\Delta S}{\Delta V} \quad \left[\because \Delta S=\frac{L}{T}\right] \qquad \text{...(ii)}$$

We know $\dfrac{dP}{dT}=\dfrac{\Delta S}{\Delta V}$

Therefore, from (i), $\Delta S=0$.

Hence, the correct option is (d).

Problem: A solid melts into a liquid via first order phase transition. The relationship between the pressure P and the temperature T of the phase transition is $P = -2T + P_0$, where P_0 is a constant. The entropy change associated with the phase transition is 1.0 J mole^{-1} K^{-1}. The Clausius Clapeyron equation for the latent heat is $L = T\left(\frac{dP}{dT}\right)\Delta V$. Here, $\Delta V = V_{\text{liquid}} - V_{\text{solid}}$ is the change in molar volume at the phase transition. The correct statement relating the values of the volumes is

(a) $V_{\text{liquid}} = V_{\text{solid}}$ (b) $V_{\text{liquid}} = V_{\text{solid}} - 1$ (c) $V_{\text{liquid}} = V_{\text{solid}} - 1/2$ (d) $V_{\text{liquid}} = V_{\text{solid}} + 2$

[IIT-JAM : 2006]

Soln. $P = -2T + P_0$

$$\frac{dP}{dT} = -2$$

Entropy changes $\Delta S = \frac{L}{T} = 1$ (Given)

From Clausius Clapeyron equation, $\frac{dP}{dT} = \frac{\Delta S}{\Delta V}$, we have

$$\Delta V = V_{\text{liquid}} - V_{\text{solid}} = -\frac{1}{2}$$

So, $V_{\text{liquid}} = V_{\text{solid}} - \frac{1}{2}$

Correct option is (c)

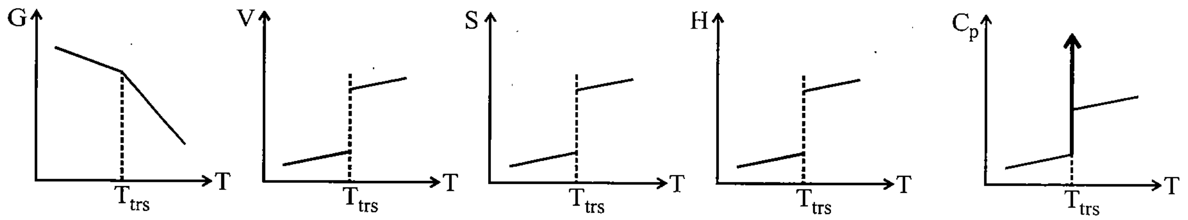
Phase Transition

The term phase transition is most commonly used to describe transitions between solid, liquid and gaseous states of matter, and, in rare cases, plasma. A phase of a thermodynamic system and the states of matter have uniform physical properties. During a phase transition of a given medium certain properties of the medium change, often discontinuously, as a result of the change of some external condition, such as temperature, pressure, or others. For example, a liquid may become gas upon heating to the boiling point, resulting in an abrupt change in volume. The measurement of the external conditions at which the transformation occurs is termed the *phase transition.*

Types of phase transition

(1) First order phase transition

- The change in phase, in which heat is either absorbed or released and occured at constant temperature and pressure are known as *first order phase transition.*
- The Gibb's function G remains constant in both phases at transition point.
- The first derivative of G with respect to temperature, i.e., entropy and with respect to pressure, i.e., volume (or density) is discontinuous at transition point.
- The examples are transformation of liquid into vapour and solid into liquid.
- The graphs are :



where $\left(\frac{\partial G}{\partial T}\right)_P = -S$, $\left(\frac{\partial G}{\partial P}\right)_T = V$

- $(\Delta G)_{\text{trans.}} = 0$

  $(\Delta S)_{\text{trans.}} = \frac{\Delta H_{\text{trans.}}}{T_{\text{trans.}}} \neq 0$

- $(\Delta V)_{\text{trans.}} \neq 0$

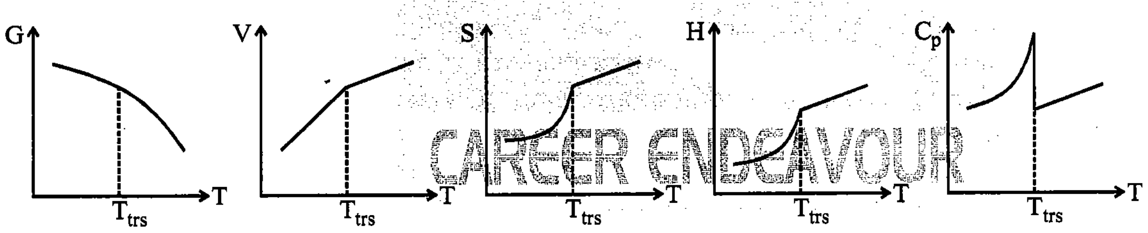
(2) The Second Order Phase Transition

- The first derivative of Gibb's function with respect to temperature and pressure are continuous at transition point, i.e., entropy and volume respectively in second order phase transition.
- So $S = -\left(\frac{\partial G}{\partial T}\right)_P$ and $V = \left(\frac{\partial G}{\partial P}\right)_T$ are continuous at the transition point.
- The second derivatives with respect to temperature and pressures are discontinuous.

  i.e., $\left(\frac{\partial^2 G}{\partial T\,\partial P}\right) = \left(\frac{\partial V}{\partial T}\right)_P = \alpha V$ and $\left(\frac{\partial^2 G}{\partial P\,\partial T}\right) = -\left(\frac{\partial S}{\partial T}\right)_T = \frac{-C_V}{T}$ are discontinuous

Note : C_P is also discontinuous at transition point.

- It means that the coefficient of isothermal compressibility (α) and specific heat at constant volume, (C_V) or at constant pressure (C_P) at transition point are discontinuous and have different values in different phases.
- For example, liquid-vapour transition at the critical point, the transition of a super-conductor from the superconducting to the normal state in zero magnetic field, ferromagnetic to paramagnetic transitions in simple model etc.
- The Graphs are:



Problem: For a liquid to vapour phase transition at T_{tr}, which of the plots between specific Gibbs free energy 'g' and temperature 'T' is correct?

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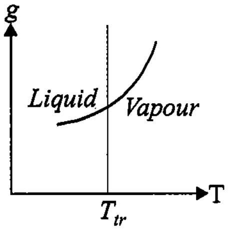

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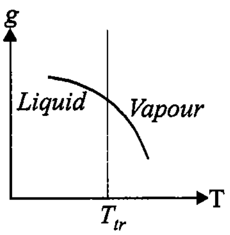

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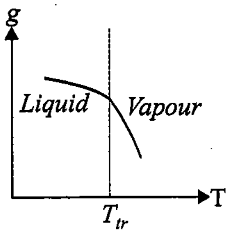

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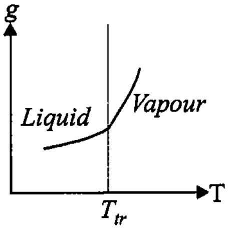


[IIT-JAM : 2012]

Soln. **(c)** Actually, variation of g with T is determined by the entropy. Because the entropy of gaseous phase of substance is greater than that of liquid phase. Therefore, *g* for gas changes more steeply than that for liquid.

Problem: A thermodynamic system is maintained at constant temperature and pressure. In thermodynamic equilibrium, its **[IIT-JAM : 2008]**

(a) Gibbs free energy is minimum
(b) Enthalpy is maximum
(c) Helmholtz free energy is minimum
(d) Internal energy is zero

Soln. The Gibb's free energy is $G = U - TS + PV$

$\Rightarrow \quad dG = dU - TdS - S\,dT + PdV + VdP = -SdT + VdP$

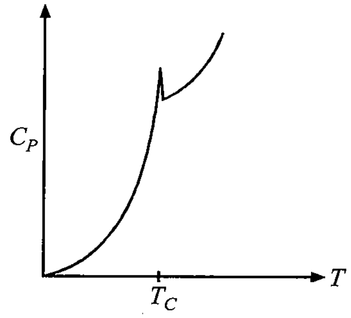
At constant T and P, $dG = 0$

So, G is minimum.

Correct option is (a)

Problem: The plot of specific heat versus temperature across the superconducting transition temperature (T_C) is most appropriately represented by

(a)



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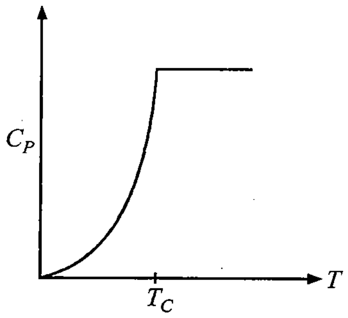

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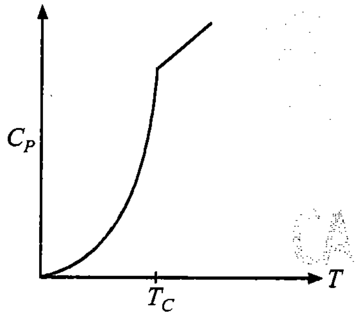

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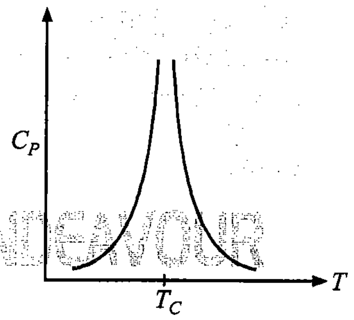


Soln. Phase transition for superconducting state is a second order phase transition. For second order phase transition the C_p has a finite discontinuity at the transition temperature.

Hence correct option is (a).

Problem: In a first order phase transition, at the transition temperature, specific heat of the system

(a) diverges and its entropy remains the same
(b) diverges and its entropy has finite discontinuity
(c) remains unchanged and its entropy has finite discontinuity
(d) has finite discontinuity and its entropy diverges

Soln. The key points are:

(i) specific heat diverges for both first and second order phase transitions.
(ii) The entropy has a finite discontinuity only for first order phase transition is continuous for second order phase transition.

Keeping the above key points in mind, we can see that correct answer is (b).

Hence, the correct option is (b).

Problem: Across a first order phase transition, the free energy is
(a) proportional to the temperature
(b) a discontinuous function of the temperature
(c) a continuous function of the temperature but its first derivative is discontinuous
(d) such that the first derivative with respect to temperature is continuous

Soln.

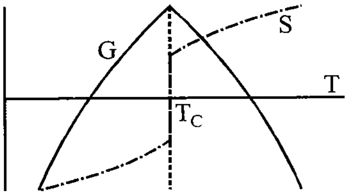


Figure : Schematic diagram for the Gibbs free energy and the entropy in a first order phase transition.

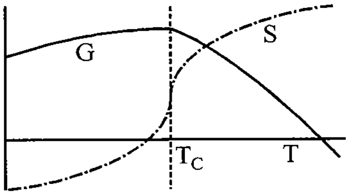


Figure : Schematic diagram for the Gibbs free energy and the entropy in a continuous phase transition.

From the above we can see that for first order transition the Gibbs free energy is itself continuous function of temperature but at the transition temperature there is a discontinuity in its first derivative withrespect to temperature i.e., entropy.

Hence correct option is (c).

Problem: Identity which one is a first order phase transition?
(a) A liquid to gas transition at its critical temperature
(b) A liquid to gas transition close to its triple point
(c) A paramagnetic to ferromagnetic transition in the absence of a magnetic field
(d) A metal to superconductor transition in the absence of a magnetic field

Soln. Clearly option (c) and (d) are not first order phase transition as it involves symmetry variation. So now the confusion can arise only with option (a) and (b). In order to avoid that confusion, the key point is the definition of critical point. Beyond the critical point, no phase transition is possible, therefore it cannot be the right answer.

Hence, the correct option is (b).

Problem: Which one of the following is a first order phase transition?
(a) Vaporization of a liquid at its boiling point
(b) Ferromagnetic to paramagnetic
(c) Normal liqiuid He to superfluid He
(d) Superconducting to normal state

Soln. Vaporization of a liquid at its boiling point, in which only phase is changed, is the first order phase transition but in other three options, the structure of material is changed, so these are second order phase transition.

Hence correct option is (a).

Problem: Consider the transition of liquid water to steam as water boils at a temperature of 100°C under a pressure of 1 atmosphere. Which one of the following quantities does not change discontinuously at the transition?

(a) The Gibbs free energy (b) The internal energy
(c) The entropy (d) The specific volume

Soln. Liquid water to steam is first order phase transition. So, G is continuous and I order derivative of G is discontinuous.

$\therefore$ $\left(\frac{dG}{dP}\right)_T = V$ is discontinuous.

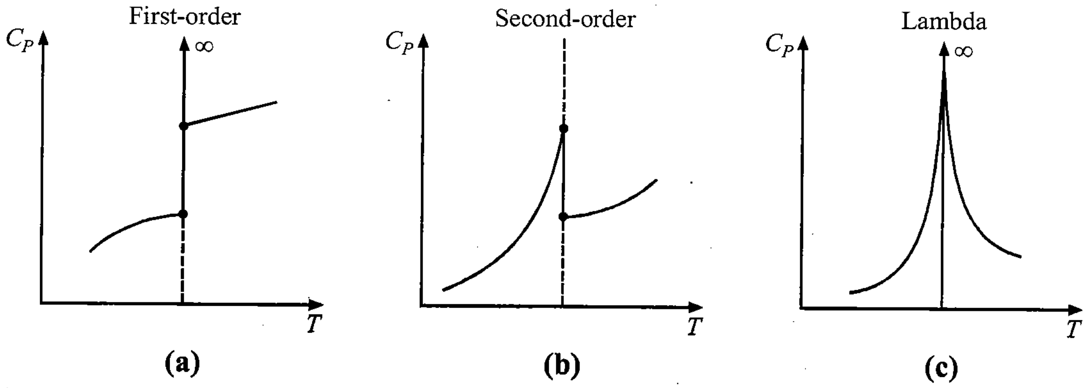
$$\therefore \qquad -\left(\frac{dG}{dT}\right)_P = S \text{ is discontinuous.}$$

$$U = G + TS - PV \text{ is also discontinuous.}$$

Correct answer : (a)

Problem: A second order phase transition is one in which
(a) the plot of entropy as a function of temperature shows a discontinuity
(b) the plot of volume heat as a function of temperature shows a discontinuity
(c) the plot of volume as a function of pressure shows a discontinuity
(d) the plot of comprehensility as a function of temperature is continuous

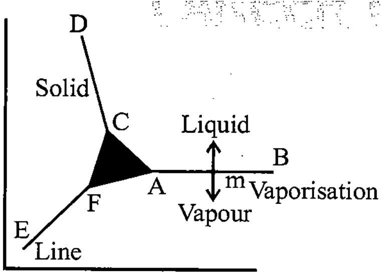
Soln.



Distinguish characteristics among the three types of phase transitions: (a) first order; (b) second order; (c) lambda.

Hence correct option is (b).

Triple Point



In the figure AB represent the vaporisation state.
If at point m, pressure is increased keeping temperature constant then boiling point will increase and all vapour condense into liquid line. CD represents the ice line or fusion line separating solid and liquid phase of water. Line EF indicates when the pressure on ice is raised evaporation from its surface slows down. The equilibrium between solid and vapour state of a substance can be represented by a curve CD called *sublimation line.* All the state of water co-exist only at the triple point state.

Problem: The vapour pressure p (in mm of Hg) of a solid, at temperature T, is expressed by $\ell n\ p = 23 - 3863/T$ and that of its liquid phase by $\ell n\ p = 19 - 3063/T$. The triple point (in Kelvin) of the material is:
(a) 185 (b) 190 (c) 195 (d) 200

Soln. At triple point the vapour pressure of solid and liquid phases are equal.

Hence, $23-\dfrac{3863}{T}=19-\dfrac{3063}{T}$

$\Rightarrow 4-\dfrac{3863}{T}=-\dfrac{3063}{T}$

$\Rightarrow 4T-3863=-3063 \Rightarrow 4T=3863-3063$

or $4T=800$

$T=200\text{ K}$

Hence correct option is (d).

PROBLEMS TO BE PLAYED

1. Using second latent heat equation $C_2 - C_1 = \frac{dL}{dT} - \frac{L}{T}$ show that the specific heat of steam is negative–

(a) If L = 800 – 0.705 T　　(b) If L = 800 T – 0.705

(c) If L = 800 T^2 – 0.705 T　　(d) If L = 800 T 0.705 T^2

2. The Clausius-Clapeyron equation indicates that an increase in pressure increases the melting point, in case of–

(a) All substances

(b) Substances which expand on solidification

(c) Substances which contract on solidification

(d) Substances which neither expand nor contraction on solidification

3. When a liquid is heated but its state remains unchanged, its molecule gain–

(a) P. E.　　(b) K.E.　　(c) Both (a) & (b)　　(d) None of these

4. In the process of phase transition–

(a) Gibb's potential function remains constant–

(b) Only entropy remains constant

(c) Only volume remains constant

(d) Only temperature remains constant

5. A second order phase transition is characterised by–

(a) A latent heat

(b) A discontinuous change in specific heat

(c) A change in volume

(d) Irreversible behaviour during warming and cooling

6. At the critical point

(1) The difference in densities of the liquid and gaseous phase is small

(2) The pressure does not depend on the volume

(3) Density fluctuations are large

(4) The latent heat for liquid vapour transition becomes very large

(a) 1 Only　　(b) 2 and 3　　(c) 3 and 4　　(d) 4 Only

7. A second order phase transition is one in which–

(a) The plot of entropy as a function of temperature shows a discontinuity

(b) The plot of specific heat as a function of temperature shows a discontinuity

(c) The plot of volume as a function of pressure shows a discontinuity

(d) The plot of comprehensibility as a function of temperature is continuous

8. At the critical point

1. difference in densities of the liquid and gaseous phase is small

2. pressure does not depend on the volume

3. density fluctuations are large

4. latent heat for liquid-vapour transition becomes very large.

(a) 1 only　　(b) 2 and 3　　(c) 3 and 4　　(d) 4 only

ANSWER KEY

1. (a)　**2. (c)**　**3. (a)**　**4. (a)**　**5. (b)**　**6. (a)**　**7. (a)**

8. (b)

Chapter 7

Statistical Mechanics

Fundamental Concepts and Theorems

- **The macroscopic and microscopic states**

We consider a physical system composed of N identical particles confined to a space of volume V. Next we consider the total energy of the system is E.

The specification of the actual values of the parameters N, V and E then define a *macrostate* of the given system.

At the molecular level, however a large number of possibilities still exist because at that level there will in general be a large number of different ways in which the macrostate (N, V, E) of the given system can be realized. In other words there will be a large number of different ways in which the total energy E of the system can be distributed among N particles constituting it. Each of these (different) ways specifies a *microstate.*

- **Equal a priori probabilities theorem**

In any case to a given macrostate of the system there does in general correspond a large number of microstates and it seems natural to assume when there are no constraints, then at any time t the system is equally likely to be in any one of these microstates. This assumption forms the backbone of our formalism and is generally referred to as the postulate of "*equal a priori probabilities*" for all microstates consistent with the given macrostate.

- **The ensemble theory**

We noted that, for a given macrostate (N, V, E), a statistical system, at any time t, is equally likely to be in any one of an extremely large number of distinct microstates. As time passes, the system continually swtiches from one microstate to another, with the result that, over a reasonable span of time, all one observes is a behavior "averaged" over the variety of microstates through which the system passes. It may, therefore, make sense if we consider, at a single instant of time, a rather large number of system-all being some sort of "mental copies" of the given system-which are characterized by the same macrostate as the original system but are, naturally enough in all sorts of possible microstates. Then, under ordinary circumstances, we may expect that the average behavior of any system in this collection, which we call an ensemble, would be identical with the time-averaged behavior of the given system. It is on the basis of this expectation that we proceed to develop the so-called ensemble theory.

For classical systems, the most appropriate workshop for developing the desired formalism is the phase space.

- **Phase Space**

The microstate of a given classical system, at any time t, may be defined by specifying the instantaneous positions and momenta of all the particles constituting the system. Thus, if N is the number of particles in the system, the definition of a microstate requires the specification of $3N$ position coordinates $q_1, q_2, \dots, q_{3N}$ and $3N$ momentum coordinates $p_1, p_2, \dots, p_{3N}$. Geometrically, the set of coordinates (q_i, p_i) where $i = 1, 2, \dots, 3N$, may be regarded as a point in a space of $6N$ dimensions. We refer to this space as the *phase space*, and the phase point (q_i, p_i) as a representative point, of the given system.

Of course, the coordinates q_i and p_i are functions of the time t; the precise manner in which they vary with time t is determined by the canonical equations of motion, where $H(q_i, p_i)$ is the Hamiltonian of the system.

Now, as time passes, the set of coordinates (q_i, p_i), which also defines the microstate of the system, undergoes a continual change. Correspondingly, our representative point in the phase space carves out a trajectory.

Problem: The dimension of phase space of ten rigid diatomic molecules is:

(a) 5 (b) 10 (c) 50 (d) 100

Soln. A rigid diatomic molecule has degree of freedom = 5 so, each molecule will require total 5 coordinate variables and 5 momentum variables for their complete description (3-translational, 2-rotational). Hence, each molecule requires 10 phase space dimensions.
Therefore, Total dimension of 10 such molecules in the phase space = $10 \times 10 = 100$.
Hence correct option is (d).

(5 degree of freedom for a rigid diatomic molecule can also be understood by the fact that there are 2 particles each with 3 degree of function but there is one constraint that being rigid, the distance between them is fixed, therefore, total degree of freedom $= 3 + 3 - 1 = 5$).

- **Liouville's Theorem**

The theorem may be stated in two parts

(1) According to this theorem, the "local" density of the representative points, as viewed by an observer moving with a representative point, stays constant in time. i.e.,

$$\frac{dP}{dt} = \frac{\partial P}{\partial t} + [P, H] = 0$$

P is the probability density. Thus this theorem is also known as the *principle of conservation of the probability density* in phase space.

(2) Any arbitrary element of volume or extension-in-phase in the phase space, bounded by a moving surface and containing a number of phase points does not change with time despite the displacements and distortions. This part represent the principle of conservation of execution in phase space.

- **Distinguishable and indistinguishable particles:**

Two particles are said to be *distinguishable* when the average distance between them is large compared to their de Broglie wavelength and they don't have overlapping wavepackets.
Distinguishable means they can be described classically (No need of quantum mechanics). *Indistinguishable* particles have overlapped wavepackets. Bosons and fermions are indistinguishable particles. Bosons have integral spin. Fermions have half-integral spin and obeys Pauli's exclusion principle.

- **Wave function of two identical particles**

According to quantum mechanical theory of identical particles:

For two particles in states n and s represented by wave functions $\phi_n(r_1)$ & $\phi_s(r_2)$ respectively,

We have $\psi_{ns}(r_1, r_2) \equiv$ wave function representing the combined state of the two particles.

Case I. Distinguishable particles (Maxwell-Boltzmann statistics)

$$\psi_{ns}(r_1, r_2) \equiv \phi_n(r_1)\,\phi_s(r_2)$$

Case II. Indistinguishable particles

Case II (A) Fermions (Fermi-dirac statistics)

$$\psi_{ns}(r_1, r_2) \equiv \frac{1}{\sqrt{2}}[\phi_n(r_1)\,\phi_s(r_2) - \phi_n(r_2)\,\phi_s(r_1)]$$

Case II (B) Bosons (Bose-Einstein statistics)

$$\psi_{ns}(r_1, r_2) \equiv \frac{1}{\sqrt{2}}[\phi_n(r_1)\,\phi_s(r_2) + \phi_n(r_2)\,\phi_s(r_1)]$$

- **Density of States**

If we consider the phase space made up of the unit cells where each cell corresponds to a microstate of the given system. Each unit cell has the volume of h^N where h is the Planck's constant and N is the degree of freedom of the particle. For a single particle this volume also can be written as h^D where D is the dimension of the space.

In the case of a single particle in 3-dimentional space, $N = 3$; accordingly the number of microstates available would asymptotically be equal to the volume of the allowed region of the phase space divided by h^3. Let $\Sigma(P)$ denote the number of microstates available to a free particle confined to volume V of the physical space, its momentum p being less than or equal to a specified value P. Then

$$\sum(P) \approx \frac{1}{h^3} \int_{p \le P} \dots \int (d^3q\, d^3p) = \frac{V}{h^3} \frac{4\pi}{3} P^3$$

whence we obtain for the number of microstates with momentum lying between p and $p + dp$

$$g(p)\,dp = \frac{d\sum(p)}{dp}\,dp \approx \frac{V}{h^3} 4\pi p^2 dp$$

Expressed in terms of the particles energy, these expression assume the form

$$\sum(E) \approx \frac{V}{h^3} \frac{4\pi}{3} (2mE)^{3/2} \qquad \left[\because\ p^2 = 2mE\right]$$

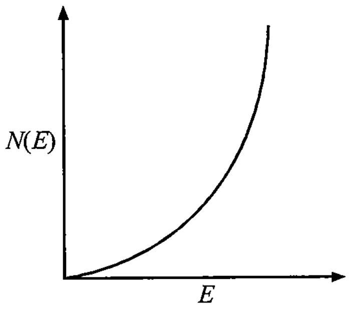
and

$$a(\varepsilon)\,d\varepsilon = \frac{d\sum(\varepsilon)}{d\varepsilon}\,d\varepsilon \approx \frac{V}{h^3} 2\pi (2m)^{3/2} \varepsilon^{1/2} d\varepsilon$$

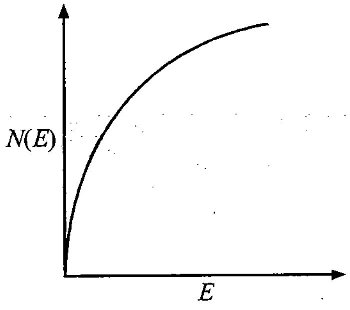
where, $a(\varepsilon)$ is the density of state.

Problem: For a free electron gas in two-dimensions, the variation of the density of states, $N(E)$ as a function of energy E, is best represented by

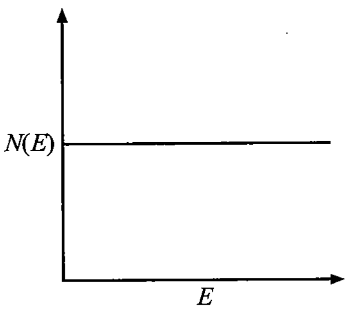
(a)



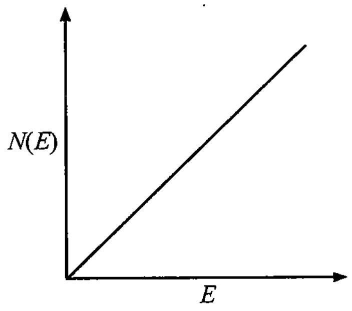
(b)



(c)



(d)



Soln. For free electron gas in two dimensions, the density of states $N(E)$ has following property $N(E)\,dE \propto d^2p$.

Now, $d^2p = 2\pi p\, dp$

Since, $p^2 = 2mE \;\Rightarrow\; 2p\,dp = 2m\,dE$

$\Rightarrow\; 2p\pi\,dp = 2\pi m\,dE \equiv d^2p$

$\therefore\; N(E)dE \propto 2\pi m\,dE$

$\Rightarrow\; N(E)$ is independent of E.

Hence correct option is (c).

Problem: The energy dependence of the density of states for a two dimensional non-relativistic electron gas is given by, $g(E) = CE^n$, where C is constant. The value of n is_________.

Soln. For a two dimensional non-relativistic electron gas the density of states is given by

$$g(E)dE = 2\frac{V\,d^2p}{(2\pi\hbar)^2} = 2\frac{V\,2\pi p\,dp}{(2\pi\hbar)^2} = 2\frac{V\,2\pi m\,dE}{(2\pi\hbar)^2}.$$

Thus, $g(E)$ is independent of E

$\Rightarrow\; g(E) = CE^0$.

Hence the value of n is zero.

Problem: In a classical micro-canonical ensemble for a system of N non-interacting particles, the fundamental volume in phase space which is regarded as "equivalent to one micro-state" is

(a) h^{3N} (b) h^{2N} (c) h^N (d) h

where h is the Planck's constant.

Soln. For one particle, $V = dxdp_x \cdot dydp_y \cdot dzdp_z = h^3$

For N particle, $V = (h^3)^N = h^{3N}$

Hence correct option is (a).

Problem: For an energy state E of a photon gas, the density of states is proportional to

(a) $\sqrt{E}$ (b) E (c) $E^{3/2}$ (d) E^2

Soln. Density of states is defined as

$$\mathcal{D}(E) = \frac{4\pi g p^2}{(2\pi\hbar)^3}\left(\frac{dp}{dE}\right)$$

where p = momentum of the particle.

For photons we have $E = pc \;\Rightarrow\; p^2 = \frac{E^2}{c^2}$ and $\frac{dp}{dE} = \frac{1}{c}$.

$$\therefore\; \mathcal{D}(E) = \frac{4\pi g E^2}{(2\pi\hbar c)^3} \;\Rightarrow\; \mathcal{D}(E) \propto E^2$$

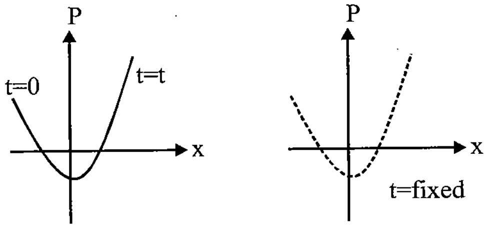
Hence correct option is (d).

Ensembles in Statistical Physics

The description of any system in statistical physics is done by considering ensembles. Once we know our ensemble (collection of systems etc.), we can quickly apply the laws of statistical mechanics to it and understand the physics thereof.

Part of phase line constructed due to the motion of phase point with time. Each phase point on the line corresponds to different time co-ordinates.

Gibbs replace this time dependent picture by a static picture (time fix) in which the entire phase line exists at one time. Then each phase point represents a seperate system that are macroscopically identical but microscopically different to the system of interest. These systems are non-interacting independent system. All members are identical in features like N,V,E. (macrostate) but differing position and momenta co-ordinates of individual molecules (microscopically)



A collection of large no. of independent systems which are macroscopically identical but microscopically different is called *ensemble.*

The concept of ensemble helps in understanding the actual system more completely. It represents at one time the microstate of actual system which would occur in the course of time. We study the statistical behaviour of the ensemble by the knowledge of the behaviour of the ensemble. We can predict the probable behaviour of the system under consideration.

The various types of ensembles are :

1. **Microcanonical ensemble**

   It is the ensemble in which all element have same macrostate represented by same no. of particle N, same volume V and same energy E i.e. macrostate is specified by (N,V, E) for all elements (systems). All elements of microcanonical ensemble are separated by impenetrable, rigid and non-conducting walls such that values of (N, V, E) for a particular system are not affected by the presence of other system i.e. all systems are isolated to each other.

2. **Canonical ensemble**

   It is the ensemble in which all elements will have same macrostate represent by same N,V, T for all elements. The elements of a canonical ensemble are separated from each other by impeneterable, rigid and conducting walls such that the values of N, V, T for a particular system are not affected by others. In canonical ensemble each element is in thermal contact (equilibrium) with heat reservoir or with all other elements of the ensemble at temperature T. In this ensemble elements can exchange energy but not particle.

3. **Grand canonical ensemble**

   It is the ensemble in which all elements will have same macrostate represent by same μ, V, T for all elements. The elements of a grand canonical ensemble are separated from each other by penetrable, rigid and conducting walls such that the values of μ, V, T for a particular system aren't affected by others. In this ensemble the elements are in contact with particle energy reservoir. The elements can exchange energy as well as particle.

Microcanonical Ensemble

For microcanonical ensemble, energy remains constant (systems are isolated)

- **Contact between Statistics and Thermodynamics**

The actual number of all possible microstates will, of course, be a function of *N*, *V* and *E* and may be denoted by the symbol $\Omega(N, V, E)$. Then the entropy of the system is defined as

$$S = k_B \ell n\, \Omega$$

where k_B is Boltzmann's constant. As it stands, this formula determines the absolute value of the entropy of a given physical system in terms of the total number of microstates (Ω) accessible to it confirmly with the given macrostate. The zero of entropy then corresponds to the special state for which only one microstate ($\Omega = 1$) is accessible the so called *unique configuration*; the statistical approach thus provides a theoretical basis for the third law of thermodynamics.

Further contact is established with the help of laws of thermodynamics as

$$dQ = dE + dW$$

$$\Rightarrow \quad TdS = dE + PdV - \mu dN \qquad \text{...(1)}$$

where μ is chemical potential and N is total number of particles. From equation (1),

$$\left(\frac{\partial S}{\partial E}\right)_{N,V} = \frac{1}{T};\ \left(\frac{\partial S}{\partial V}\right)_{N,E} = \frac{P}{T};\ \left(\frac{\partial S}{\partial N}\right)_{V,E} = -\frac{\mu}{T}$$

Also we can write using the well known relationship in partial differential calculus namely that if three variables x, y and z are mutually related then

$$\left(\frac{\partial x}{\partial y}\right)_z \cdot \left(\frac{\partial y}{\partial z}\right)_x \cdot \left(\frac{\partial z}{\partial x}\right)_y = -1$$

$$P = \left(\frac{\partial S}{\partial V}\right)_{N,E} \Big/ \left(\frac{\partial S}{\partial E}\right)_{N,V} = -\left(\frac{\partial E}{\partial V}\right)_{N,S}$$

$$\text{and } \mu = -\left(\frac{\partial S}{\partial N}\right)_{V,E} \Big/ \left(\frac{\partial S}{\partial E}\right)_{N,V} = \left(\frac{\partial E}{\partial N}\right)_{V,S}$$

$$\text{while } T = \left(\frac{\partial E}{\partial S}\right)_{N,V}$$

The rest of thermodynamics follows straight forwardly. For instance the Helmholtz free energy F, the Gibbs free energy G and the enthalpy H are given by

$$F = E - TS$$

$$G = F + PV = E - TS + PV = \mu N \text{ and } H = E + PV = G + TS$$

The specific heat at constant volume C_V and the one at constant pressure C_P, would be given by

$$C_V = T \cdot \left(\frac{\partial S}{\partial T}\right)_{N,V} = \left(\frac{\partial E}{\partial T}\right)_{N,V} \quad \textit{and}$$

$$C_P = T \cdot \left(\frac{\partial S}{\partial T}\right)_{N,P} = \left(\frac{\partial (E + PV)}{\partial T}\right)_{N,P} = \left(\frac{\partial H}{\partial T}\right)_{N,P}$$

Problem: The chemical potential of an ideal Bose gas at any temperature is : **[JAM-2008]**

(a) Necessarily negative
(b) Either zero or negative
(c) Necessarily positive
(d) Either zero or positive

oln. The chemical potential μ for an ideal Bose gas is either zero or negative.

Correct option is (b)

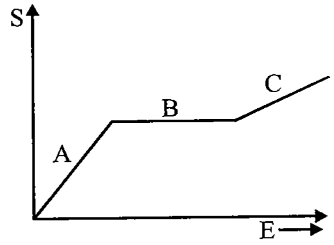
Problem: The entropy of a system, S, is related to the accessible phase space volume Γ by $S = k_B\, \ell n\, \Gamma(E, N, V)$ where E, N and V are the energy, number of particles and volume respectively. From this one can conclude that

(a) does not change during evolution to equilibrium
(b) Oscillates during evolution to equilibrium
(c) Is a maximum in equilibrium
(d) Is a minimum in equilibrium

Soln. In the micrononical ensemble, the phase volume is maximized to obtain the characteristic thermodynamics quantity which is entropy.

Correct option is (c)

Problem: The entropy S of a thermodynamic system as a function of energy E is given by the following graph



The temperatures of the phases A, B and C, denoted by T_A, T_B and T_C, respectively, satisfy the following inequalities:

(a) $T_C > T_B > T_A$ (b) $T_A > T_C > T_B$ (c) $T_B > T_C > T_A$ (d) $T_B > T_A > T_C$

Soln. $\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)$

$$\left(\frac{\partial S}{\partial E}\right)_A > \left(\frac{\partial S}{\partial E}\right)_C > \left(\frac{\partial S}{\partial E}\right)_B$$

$\Rightarrow T_B > T_C > T_A$

Correct answer is (c)

Problem: The internal energy E of a system is given by $E = \frac{bS^3}{VN}$, where b is a constant and other symbols have their usual meaning. The temperature of this system is equal to

(a) $\frac{bS^2}{VN}$ (b) $\frac{3bS^2}{VN}$ (c) $\frac{bS^3}{V^3N}$ (d) $\left(\frac{S}{N}\right)^2$

Soln. $E = \frac{bS^3}{VN}$

$\because \quad dQ = TdS = dE + PdV - \mu dN$

$\Rightarrow \quad T = \left(\frac{\partial E}{\partial S}\right)_{N,V} = \frac{3bS^2}{NV}$

Correct answer is (b)

Problem: A paramagnetic system consisting of N spin-half particles, is placed in an external magnetic field. It is found that $N/2$ spins are aligned parallel and the remaining $N/2$ spins are aligned antiparallel to the magnetic field. The statistical entropy of the system is,

(a) $2Nk_B \ell n\, 2$ (b) $\frac{N}{2}k_B\, \ell n\, 2$ (c) $\frac{3N}{2}k_B\, \ell n\, 2$ (d) $Nk_B\, \ell n\, 2$

Soln. For spin- 1/2 system, number of microstates under equilibrium condition (as the given spin configuration is analogous to equilibrium spin configuration)

$\Omega = (2s+1)^N = 2^N$

Therefore, Entropy $S = k_B \ \ell n\, \Omega = k_B \ \ell n\, 2^N = Nk_B \ \ell n\, 2$

Hence correct option is (d).

Problem: The total energy, E of an ideal non-relativistic Fermi gas in three dimensions is given by $E \propto \dfrac{N^{5/3}}{V^{2/3}}$ where N is the number of particles and V is the volume of the gas. Identify the correct equation of state (P being the pressure),

(a) $PV = \dfrac{1}{3}E$ (b) $PV = \dfrac{2}{3}E$ (c) $PV = E$ (d) $PV = \dfrac{5}{3}E$

Soln. Let, $E = \dfrac{K}{V^{2/3}}$ ...(i)

Now, $P = -\left(\dfrac{\partial E}{\partial V}\right)_{N,S} = -\dfrac{\partial}{\partial V}\left(\dfrac{K}{V^{2/3}}\right) = \dfrac{2}{3}KV^{-5/3} = \dfrac{2}{3}\dfrac{K}{V^{5/3}}$ or $P = \dfrac{2K}{3V \times V^{2/3}}$

$PV = \dfrac{2}{3}\dfrac{K}{V^{2/3}} = \dfrac{2}{3}E$ [Using equation (i)]

Hence correct option is (d).

Problem: The entropy of a gas containing N particles enclosed in a volume V is given by $S = Nk_B \ \ell n\left(\dfrac{aVE^{3/2}}{N^{5/2}}\right)$, where E is the total energy, a is a constant and k_B is the Boltzmann constant. The chemical potential μ of the system at a temperature T is given by

(a) $\mu = -k_B T\left[\ell n\left(\dfrac{aVE^{3/2}}{N^{5/2}}\right) - \dfrac{5}{2}\right]$ (b) $\mu = -k_B T\left[\ell n\left(\dfrac{aVE^{3/2}}{N^{5/2}}\right) - \dfrac{3}{2}\right]$

(c) $\mu = -k_B T\left[\ell n\left(\dfrac{aVE^{3/2}}{N^{3/2}}\right) - \dfrac{5}{2}\right]$ (d) $\mu = -k_B T\left[\ell n\left(\dfrac{aVE^{3/2}}{N^{3/2}}\right) - \dfrac{3}{2}\right]$

Soln. $\mu = -T\left(\dfrac{\partial S}{\partial N}\right)_{V,E} = -T\,k_B \ \ell n\left(\dfrac{aVE^{3/2}}{N^{5/2}}\right) - T\,k_B N \cdot \dfrac{N^{5/2}}{\cancel{aVE^{3/2}}} \cdot \cancel{aVE^{3/2}}\left(-\dfrac{5}{2}N^{-7/2}\right)$

$= -T\,k_B \ \ell n\left(\dfrac{aVE^{3/2}}{N^{5/2}}\right) + \dfrac{5}{2}k_B T = -k_B T\left[\ell n\left(\dfrac{aVE^{3/2}}{N^{5/2}}\right) - \dfrac{5}{2}\right]$

Hence correct option is (a).

Problem: Consider a linear collection of N independent spin 1/2 particles, each at a fixed location. The entropy of this system is (k is the Boltzmann constant)

(a) Zero (b) Nk (c) ½ Nk (d) $Nk\ ln(2)$

Soln. The number of microstates for a particle with spin $S = 2S + 1$.

Therefore, total number of accessible microstates for N independent spins $= (2S+1)^N = \Omega$.

Now, entropy $\equiv k_B\ \ell n\ \Omega = k_B\ \ell n\left[(2S+1)^N\right]$

Putting $S = 1/2$, we have

Entropy $= k_B\ \ell n\left[(1+1)^N\right] = Nk_B\ \ell n\ 2$

Hence correct option is (d).

Problem: The total number of accessible states of N non-interacting particles of spin 1/2 is

(a) 2^N (b) N^2 (c) $2^{N/2}$ (d) N

Soln. The total number of accessible states for N non-interacting spin(S) particles $= (2S+1)^N$.

So, for spin $S = \frac{1}{2}$ particles, number of states $= \left(2\times\frac{1}{2}+1\right)^N = 2^N$

Hence, the correct option is (a).

The Canonical Ensemble

A system in contact with a heat reservoir of infinite heat capacity or in thermal equilibrium at temperature T and having N particles confined to a volume V is considered as canonical ensemble.

The whole thermodynamics of the canonical ensemble is understood in terms of partition function.

Partition Function

Let us consider an assembly of ideal gas molecules obeying quantum statistics. Let the distribution of gas molecules be such that n_i molecules occupy the ith state with energy ε_i and degeneracy (i.e., the number of energy levels with the same energy) g_i. Then from Maxwell-Boltzmann distribution law, we have

$$n_i = g_i e^{-\alpha} e^{-\varepsilon_i/k_B T}$$

Let us substitute $e^{-\alpha} = A$. Then $n_i = g_i A e^{-\varepsilon_i/k_B T}$

If the total number of gas molecules in the assembly is N, then

$$N = \sum_i n_i = \sum_i g_i A e^{-\varepsilon_i/k_B T}$$

$$= A\sum_i g_i e^{-\varepsilon_i/k_B T} \quad ...(1)$$

The term $A(=e^{-\alpha})$ has been taken out of the summation sign because α is a constant.

From equation (1), we get

$$\frac{N}{A} = \sum_i g_i e^{-\varepsilon_i/k_B T} = Z \text{ (say)} \quad ...(2)$$

The quantity Z is called the Boltzmann partition function or simply the partition function. Therefore, the quantity Z indicates how the gas molecules of an assembly are distributed or partitioned among the various energy levels.

If the energy of the ith level is ε_i then keeping in mind that the weight of an individual level is unity equation (2) can be written as

$$Z = \sum_i e^{-\varepsilon_i/k_B T}$$

And if particle obeys classical statistics then $Z = \int g(\varepsilon)\cdot e^{-\varepsilon/k_B T} d\varepsilon$

Partition Function and Thermodynamic Quantities

If $Z_1(V,T)$ is the partition function of a single particle then the partition function of the system having N independent particles is given by $Z_N(V,T)=[Z_1(V,T)]^N$ if particles are distinguishable and $Z_N(V,T)=\dfrac{[Z_1(V,T)]^N}{N!}$ if particles are indistinguishable.

The Helmholtz free energy of the system is given by $F=-k_BT\cdot\ell n\{Z_N(V,T)\}$

As we know $F=U-TS$

or $dF=dU-TdS-SdT$

or $dF=-SdT-PdV+\mu dN \qquad \{\because\ dU=TdS-PdV+\mu dN\}$

So $S=-\left(\dfrac{\partial F}{\partial T}\right)_{N,V}$; $P=-\left(\dfrac{\partial F}{\partial V}\right)_{N,T}$ and $\mu=\left(\dfrac{\partial F}{\partial N}\right)_{V,T}$

Also $U=F+TS=F-T\left(\dfrac{\partial F}{\partial T}\right)_{N,V}=-T^2\left[\dfrac{\partial}{\partial T}\left(\dfrac{F}{T}\right)\right]_{N,V}=\left[\dfrac{\partial(F/T)}{\partial(1/T)}\right]_{N,V}$

$$C_V=\left(\frac{\partial U}{\partial T}\right)_{N,V}=-T\left(\frac{\partial^2 F}{\partial T^2}\right)_{N,V}$$

and the Gibbs free energy is $G=F+PV=F-V\left(\dfrac{\partial F}{\partial T}\right)_{N,T}=N\left(\dfrac{\partial F}{\partial N}\right)_{V,T}=N\mu$

Hence all thermodynamics is derived from partition function $Z=\sum g_i\cdot e^{-\varepsilon_i/kT}$.

Problem: The free energy of a gas N particled in a volume V and at a temperature T is $F=NK_BT\ln\left[a_0V\left(k_BT\right)^{5/2}/N\right]$, where a_0 is a constant and k_B denotes the Boltzmann constant. The internal energy of gas is

(a) $\dfrac{3}{2}Nk_BT$ (b) $\dfrac{5}{2}Nk_BT$

(c) $Nk_BT\ln\left[a_0V\left(k_BT\right)^{5/2}/N\right]-\dfrac{3}{2}Nk_BT$ (d) $Nk_BT\ln\left[a_0V/\left(k_BT\right)^{5/2}\right]$

Soln. The Free energy, $F=Nk_BT\ \ell n\left[a_oV\left(k_BT\right)^{5/2}/N\right]=k_BT\ \ell n\left[a_oV\left(k_BT\right)^{5/2}/N\right]^N$

As $F=-k_BT\ \ell n\ Z$, we have

$$\ell n\ Z=-\ell n\left[a_oV\left(k_BT\right)^{5/2}/N\right]^N$$

$$\Rightarrow U=-\frac{\partial}{\partial\beta}\ell n\ Z=k_BT^2\frac{\partial}{\partial T}\ell n\ Z \qquad \left[\because \beta=\frac{1}{k_BT}\Rightarrow\partial\beta=\frac{-1}{k_BT^2}\partial T\right]$$

$$=k_BT^2\frac{\partial}{\partial T}\left[-\ell n\left\{a_oV\left(k_BT\right)^{5/2}/N\right\}^N\right]=k_BT^2\frac{\partial}{\partial T}\left[-\ell n\,T^{\frac{5N}{2}}-\ell n\left(\frac{a_oVk_B^{5/2}}{N}\right)^N\right]$$

$$=-\frac{5N}{2}k_BT^2\times\frac{1}{T}=-\frac{5}{2}N\,k_BT$$ [Extra -ve sign is due to the wrong sign in the given question]

Correct answer is (b).

Problem: The free energy for a photon gas is given by $F=-\left(\frac{a}{3}\right)VT^4$, where a is a constant. The entropy S and the pressure P of the photon gas are

(a) $S=\frac{4}{3}aVT^3, P=\frac{a}{3}T^4$ (b) $S=\frac{1}{3}aVT^4, P=\frac{4a}{3}T^3$

(c) $S=\frac{4}{3}aVT^4, P=\frac{a}{3}T^3$ (d) $S=\frac{1}{3}aVT^3, P=\frac{4a}{3}T^4$

Soln. $F=-\frac{a}{3}VT^{+4}$

$$P=-\left(\frac{\partial F}{\partial V}\right)_T=\frac{a}{3}T^4$$

$$S=\left(\frac{\partial F}{\partial T}\right)_V=\frac{4}{3}aVT^3$$

Hence correct option is (a).

Problem: The partition function of a single gas molecule is Z_α. The partition function of N such non-interacting gas molecules is then given by

(a) $\frac{(Z_\alpha)^N}{N!}$ (b) $(Z_\alpha)^N$ (c) $N(Z_\alpha)$ (d) $\frac{(Z_\alpha)^N}{N}$

Soln. Gas molecules ⇒ distinguishable (classical particles). If Z_α = partition function for single particle, then for N distinguishable particles the partition function = $(Z_\alpha)^N$. On the other hand, for N non-distinguishable particles, the partition function = $\frac{(Z_\alpha)^N}{N!}$.

Hence, the correct option is (b).

Problem: The free energy of a photon gas enclosed in a volume V is given by $F=-\frac{1}{3}aVT^{-4}$, where a is a constant and T is the temperature of the gas. The chemical potential of the photon gas is

(a) zero (b) $\frac{4}{3}aVT^3$ (c) $\frac{1}{3}aT^{-4}$ (d) aVT^{-4}

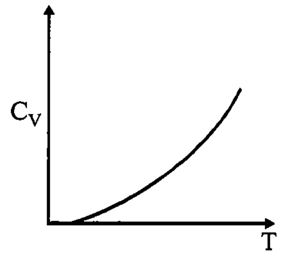
Soln. $\mu=\left(\frac{\partial F}{\partial N}\right)_{V,T}$; for this case $F=-\frac{1}{3}aVT^{-4}$ which is independent of N. Therefore, $\mu=0$. Anyway, for photon gas (in equilibrium) $\mu=0$.

Hence, the correct option is (a).

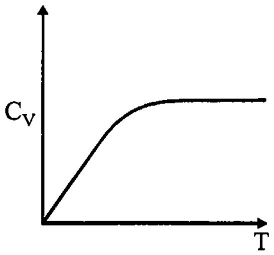
Problem: The partition function for a gas of photons is given by $\ell n\, Z=\frac{\pi^2}{45}\frac{V(k_BT)^3}{\hbar^3c^3}$.

The specific heat of the photon gas veries with temperature as

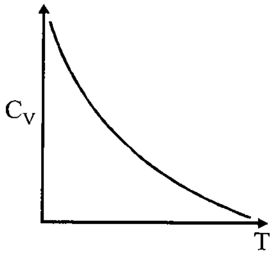
(a)



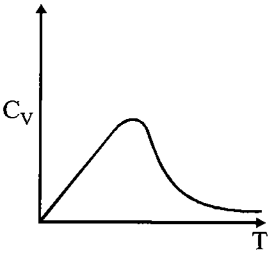
(b)



(c)



(d)



Soln. Here, $\log Z = \frac{\pi^2}{45} V \frac{(k_B T)^3}{\hbar^3 c^3}$

Putting $k_B T = \frac{1}{\beta}$, we get $\log Z = \frac{\pi^2}{45} \frac{V}{\hbar^3 c^3} \frac{1}{\beta^3}$

$$U = -\frac{\partial}{\partial \beta} \log Z = -\frac{\pi^2}{45} \frac{V}{\hbar^3 c^3} \frac{(-3)}{\beta^4} = \frac{\pi^2}{15} \frac{V}{\hbar^3 c^3} (k_B T)^4 \quad \because \beta = \frac{1}{k_B T}$$

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V = \frac{4\pi^2}{15} \frac{V k_B^4}{\hbar^3 c^3} T^3 \Rightarrow C_V \propto T^3$$

Hence correct option is (b).

Problem: Common Data for next two problems :

Consider a two levels quantum system with energies $\varepsilon_1 = 0,\ \varepsilon_2 = \varepsilon$.

(i) The Helmholtz free energy of the system is given by

(a) $-k_B T\, \ell n\, (1 + e^{-\varepsilon/k_B T})$ (b) $k_B T\, \ell n\, (1 + e^{-\varepsilon/k_B T})$

(c) $\frac{3}{2} k_B T$ (d) $\varepsilon - k_B T$

Soln. $F = -k_B T \log\left[e^{-\varepsilon_1/k_B T} + e^{-\varepsilon_2/k_B T}\right]$ $[\because\ \varepsilon_1 = 0,\ \varepsilon_2 = \varepsilon]$

$F = -k_B T \log\left[1 + e^{-\varepsilon/k_B T}\right]$

Hence correct option is (a).

(ii) The specific heat of the system is given by

(a) $\frac{\varepsilon}{k_B T} \frac{e^{-\varepsilon/k_B T}}{(1 + e^{-\varepsilon/k_B T})^2}$ (b) $\frac{\varepsilon^2}{k_B T^2} \frac{e^{-\varepsilon/k_B T}}{(1 + e^{-\varepsilon/k_B T})}$

(c) $-\frac{\varepsilon^2 e^{-\varepsilon/k_B T}}{(1 + e^{-\varepsilon/k_B T})^2}$ (d) $\frac{\varepsilon^2}{k_B T^2} \frac{e^{-\varepsilon/k_B T}}{(1 + e^{-\varepsilon/k_B T})^2}$

Soln. $\frac{\partial F}{\partial T} = \frac{e^{-\varepsilon/k_B T}}{(1 + e^{-\varepsilon/k_B T})} \times \left(\frac{-\varepsilon}{k_B T^2}\right) \times (-k_B T) = \frac{\varepsilon}{T} \frac{e^{-\varepsilon/k_B T}}{(1 + e^{-\varepsilon/k_B T})}$

$$C_V = \frac{\varepsilon}{k_B T} \times \frac{\partial F}{\partial T} = \frac{\varepsilon^2}{k_B T^2} \frac{e^{-\varepsilon/k_B T}}{(1 + e^{-\varepsilon/k_B T})}$$

Hence correct option is (b).

Statistical Average of a Physical Quantity

Statistical average of a physical quantity "f" is given by $\langle f \rangle = \dfrac{\sum\limits_i f_i\, g_i \cdot e^{-\varepsilon_i/k_BT}}{\sum\limits_i g_i \cdot e^{-\varepsilon_i/k_BT}}$

Statistical average of energy and relative root mean square fluctuation in energy:

$$U = \langle E \rangle = \frac{\sum\limits_i \varepsilon_i \cdot g_i\, e^{-\beta\varepsilon_i}}{\sum\limits_i g_i \cdot e^{-\beta\varepsilon_i}}\ ;\ \beta = \frac{1}{k_BT}$$

Differentiate it with respect to parameter β, holding the energy values ε_i constant, we obtain

$$\frac{\partial U}{\partial \beta} = -\frac{\sum\limits_i \varepsilon_i^2 \cdot g_i\, e^{-\beta\varepsilon_i}}{\sum\limits_i g_i \cdot e^{-\beta\varepsilon_i}} + \left[\frac{\sum\limits_i \varepsilon_i\, g_i\, e^{-\beta\varepsilon_i}}{\sum\limits_i g_i \cdot e^{-\beta\varepsilon_i}}\right]^2$$

whence if follow that $\langle(\Delta E)^2\rangle = \langle E^2\rangle - \langle E\rangle^2 = -\dfrac{\partial U}{\partial \beta} = kT^2\left(\dfrac{\partial U}{\partial T}\right) = kT^2 C_V$

and the relative root-mean-square fluctuation in E

$$\frac{\sqrt{[\langle(\Delta E)^2\rangle]}}{\langle E\rangle} = \frac{\sqrt{kT^2C_V}}{U}$$

Problem: Consider a system of two non-interacting classical particles which can occupy any of the three energy levels with energy values $E = 0$, e and 2e having degeneracies $g(E) = 1$, 2 and 4 respectively. The mean energy of the system is

(a) $\varepsilon\left[\dfrac{4\exp\left(\dfrac{-\varepsilon}{k_BT}\right) + 8\exp\left(\dfrac{-2\varepsilon}{k_BT}\right)}{1 + 2\exp\left(\dfrac{-\varepsilon}{k_BT}\right) + 4\exp\left(\dfrac{-2\varepsilon}{k_BT}\right)}\right]$

(b) $\varepsilon\left[\dfrac{2\exp\left(\dfrac{-\varepsilon}{k_BT}\right) + 8\exp\left(\dfrac{-2\varepsilon}{k_BT}\right)}{1 + 2\exp\left(\dfrac{-\varepsilon}{k_BT}\right) + 4\exp\left(\dfrac{-2\varepsilon}{k_BT}\right)}\right]$

(c) $\varepsilon\left[\dfrac{2\exp\left(\dfrac{-\varepsilon}{k_BT}\right) + 4\exp\left(\dfrac{-2\varepsilon}{k_BT}\right)}{1 + 2\exp\left(\dfrac{-\varepsilon}{k_BT}\right) + 4\exp\left(\dfrac{-2\varepsilon}{k_BT}\right)}\right]^2$

(d) $\varepsilon\left[\dfrac{\exp\left(\dfrac{-\varepsilon}{k_BT}\right) + 2\exp\left(\dfrac{-2\varepsilon}{k_BT}\right)}{1 + \exp\left(\dfrac{-\varepsilon}{k_BT}\right) + \exp\left(\dfrac{-2\varepsilon}{k_BT}\right)}\right]$

Soln. Total energy due to a particular level = occupancy of the level × energy.

$$\therefore\ E_1 = 0 \cdot \exp\left[-\frac{0}{k_BT}\right]$$

$$E_2 = \varepsilon \cdot g(E) \cdot \exp\left[-\frac{\varepsilon}{k_BT}\right] = 2\varepsilon \exp\left[-\frac{\varepsilon}{k_BT}\right]$$

Similarly, $E_3 = 2\varepsilon \cdot 4\exp\left[-\dfrac{2\varepsilon}{k_BT}\right]$

Also, mean energy $= \dfrac{\sum_i E_i}{\text{total occupancy}} = \dfrac{E_1+E_2+E_3}{\sum_i g_i \exp\left(-\dfrac{\varepsilon_i}{k_BT}\right)}$

$$= \frac{0+2\varepsilon\exp\left(-\dfrac{\varepsilon}{k_BT}\right)+8\varepsilon\exp\left(-\dfrac{2\varepsilon}{k_BT}\right)}{\exp\left(-\dfrac{0}{k_BT}\right)+2\exp\left(-\dfrac{\varepsilon}{k_BT}\right)+4\exp\left(-\dfrac{2\varepsilon}{k_BT}\right)} = \varepsilon\left[\frac{2\exp\left(-\dfrac{\varepsilon}{k_BT}\right)+4\exp\left(-\dfrac{2\varepsilon}{k_BT}\right)}{1+2\exp\left(-\dfrac{\varepsilon}{k_BT}\right)+4\exp\left(-\dfrac{2\varepsilon}{k_BT}\right)}\right]$$

Hence correct option is (b).

Problem: A system has two energy levels with energies ε and 2ε. The lower level is 4-fold degenerate while the upper level is doubly degenerate. If there are N non-interacting classical particles in the system, which is in thermodynamic equilibrium at a temperature T, the fraction of particles in the upper level is:

(a) $\dfrac{1}{1+e^{-\varepsilon/k_BT}}$ (b) $\dfrac{1}{1+2e^{\varepsilon/k_BT}}$ (c) $\dfrac{1}{2e^{\varepsilon/k_BT}+4e^{2\varepsilon/k_BT}}$ (d) $\dfrac{1}{2e^{\varepsilon/k_BT}-4e^{2\varepsilon/k_BT}}$

Soln. $E_L = E,\ E_U = 2E$

$g_L = 4,\ g_U = 2$

$$N(E_i) \propto g_i \exp\left[-\frac{E_i}{k_BT}\right]$$

$$\therefore\ N(E_L) = C\times g_L \exp\left[-\frac{E_L}{k_BT}\right]$$

Similarly, $N(E_U) = C\times g_U \exp\left[-\dfrac{E_U}{k_BT}\right]$

The fraction of particles in upper level $= \dfrac{N(E_U)}{N(E_U)+N(E_L)}$

$$= \frac{2\cdot\exp\left[-\dfrac{2E}{k_BT}\right]}{4\exp\left[-\dfrac{E}{k_BT}\right]+2\exp\left[-\dfrac{2E}{k_BT}\right]} = \frac{1}{1+2\exp\left[\dfrac{E}{k_BT}\right]}$$

Hence correct option is (b).

Problem: N distinguishable particles are distributed among three states having energies 0, k_BT and $2k_BT$ respectively. If the total equilibrium energy of the system is 151.23 k_BT, the number of particles of the system is about

(a) 152 (b) 264 (c) 356 (d) 635 **[D.U.-2016]**

Soln. The average energy is

$$\langle E\rangle = \frac{N\left[0\,e^{-0/k_BT}+k_BT\,e^{-k_BT/k_BT}+2k_BT\,e^{-2k_BT/k_BT}\right]}{e^{-0/k_BT}+e^{-k_BT/k_BT}+e^{-2k_BT/k_BT}} = 151.23\,k_BT$$

$$\Rightarrow \quad \frac{N\left[k_BT\left(e^{-1}+2e^{-2}\right)\right]}{1+e^{-1}+e^{-2}}=151.23\,k_BT$$

$$\Rightarrow \quad N=\left(\frac{1+e^{-1}+e^{-2}}{e^{-1}+2e^{-2}}\right)151.23=356.01$$

Correct option is (c)

Problem: Consider a Maxwell-Boltzmann system with two states of energies E and 2E. If each state is two-fold degenerate, then the partition function would be (in terms of E, k and T, where k is the Boltzmann constant and T is the absolute temperature):

(a) $z=2e^{-3E/kT}$

(b) $z=e^{-E/kT}\left[1+2e^{-E/kT}\right]$

(c) $z=2e^{-E/kT}\left[1+e^{-2E/kT}\right]$

(d) $z=2e^{-E/kT}\left[1+e^{-E/kT}\right]$

Soln. The partition function is

$$z=\sum_i g_i\, e^{-E_i/kT}$$

where g_i is the degeneracy of the E_i the state.

So, $z=2e^{-E/kT}+2e^{-2E/kT}=2e^{-E/kT}\left[1+e^{-E/kT}\right]$

Correct option is (d)

Problem: A particle in thermal equilibrium has only 3 possible states with energies $-\varepsilon$, 0, ε. If the system is maintained at a temperature $T>>\frac{\varepsilon}{k_B}$, then the average energy of the particle can be approximated to,

(a) $\frac{2\varepsilon^2}{2k_BT}$ (b) $-\frac{2\varepsilon^2}{3k_BT}$ (c) $\frac{\varepsilon^2}{k_BT}$ (d) 0 **[JEST-2015]**

Soln. The average energy is

$$\langle E\rangle=\frac{-\varepsilon\, e^{\varepsilon/k_BT}+0\,e^{-0/k_BT}+\varepsilon\, e^{-\varepsilon/k_BT}}{e^{+\varepsilon/k_BT}+e^{-0/k_BT}+e^{-\varepsilon/k_BT}}$$

$$=\frac{\varepsilon\left(e^{-\varepsilon/k_BT}-e^{\varepsilon/k_BT}\right)}{e^{-\varepsilon/k_BT}+1+e^{\varepsilon/k_BT}}=\frac{\varepsilon\left(1-e^{2\varepsilon/k_BT}\right)}{1+e^{\varepsilon/k_BT}+e^{2\varepsilon/k_BT}}$$

for $T>>\frac{\varepsilon}{k_B}$ or $\varepsilon<<k_BT$, we have

$$\langle E\rangle=\frac{\varepsilon\left[1-\left(1+2\varepsilon/k_BT\right)\right]}{1+1+\varepsilon/k_BT+1+2\varepsilon/k_BT}=\frac{-2\varepsilon^2/k_BT}{3+3\varepsilon/k_BT}$$

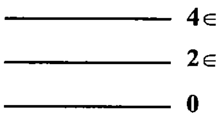
$$\Rightarrow \quad \langle E\rangle=-\frac{2\varepsilon^2}{3k_BT} \qquad \left[\because\ 1+\frac{\varepsilon}{k_BT}\approx 1 \text{ for } \frac{\varepsilon}{k_BT}<<1\right]$$

Correct option is (b)

Problem: A gas of N non-interacting particles is in thermal equilibrium at temperature T. Each particle can be in any of the possible non-degenerate states of energy 0, 2ε and 4ε. The average energy per particle of the gas, when $\beta\varepsilon \ll 1$, is:

(a) 2ε (b) 3ε (c) $2\varepsilon/3$ (d) ε

Soln.



$$\langle E\rangle=\frac{0.e^{-0/k_BT}+2\varepsilon\, e^{-2\varepsilon/k_BT}+4\varepsilon\, e^{-4\varepsilon/k_BT}}{e^{-0/k_BT}+e^{-2\varepsilon/k_BT}+e^{-4\varepsilon/k_BT}}=\frac{2\in\left[e^{-2\varepsilon/k_BT}+2e^{-4\varepsilon/k_BT}\right]}{1+e^{-2\varepsilon/k_BT}+e^{-4\varepsilon/k_BT}}$$

Since, $\beta\varepsilon << 1,\ \dfrac{\varepsilon}{k_BT} << 1 \Rightarrow$ we retain only the first term of exponentials.

Therefore, $\langle E\rangle=\dfrac{2\varepsilon(1+2)}{(1+1+1)}=2\varepsilon$

Correct answer is (a)

Harmonic Oscillator System

For a system of N oscillators, each of mass m and angular frequency ω

(i) In classical mechanics, the Hamiltonian of an oscillator is $H = p^2/2m + m\omega^2q^2/2$. Thus, the classical partition function in 1-D is

$$Z_1=\frac{1}{h}\int e^{-E/k_BT}dq\,dp=\frac{1}{h}\int_{-\infty}^{\infty}e^{-p^2/2mk_BT}dp\int_{-\infty}^{\infty}e^{-m\omega^2q^2/2k_BT}dq=\frac{1}{h}\sqrt{2\pi mk_BT}\sqrt{\frac{2\pi k_BT}{m\omega^2}}=\frac{2\pi k_BT}{h\omega}$$

(ii) In quantum mechanics, the energy of an oscillator is $E_n = (n + 1/2)\hbar\omega$. Thus, the quantum mechanical partition function is

$$Z_1=\sum_n e^{-E_n/k_BT}=\sum_{n=0}^{\infty}e^{-(n+1/2)\hbar\omega/k_BT}=e^{-\hbar\omega/2k_BT}\sum_{n=0}^{\infty}(e^{-\hbar\omega/k_BT})^n$$

$$=\frac{e^{-\hbar\omega/2k_BT}}{1-e^{-\hbar\omega/k_BT}}=\frac{1}{e^{\hbar\omega/2k_BT}-e^{-\hbar\omega/2k_BT}}=\frac{1}{2\sinh(\hbar\omega/2k_BT)}$$

It is obvious that when $(\hbar\omega/k_BT)\to 0$ the case (ii) converges into (i). Therefore, here in after, we shall consider only the case (ii).

Since the partition function Z_N of a system of N independent partitions is equal to the product of the partition function Z_1 of individual particle, we have

$$Z_N=Z_1^N=\left[2\sinh\frac{\hbar\omega}{2k_BT}\right]^{-N}$$

Using $\beta = 1/k_BT$, the internal energy U of the system is

$$U=-\frac{\partial}{\partial\beta}\ell n\,Z_N=N\frac{\partial}{\partial\beta}\ell n\left[\sinh\frac{\beta\hbar\omega}{2}\right]=N\left[\frac{\hbar\omega}{2}\right]\coth\frac{\hbar\omega}{2k_BT}=N\left[\frac{\hbar\omega}{2}+\frac{1}{e^{\hbar\omega/k_BT}-1}\right]$$

Specific heat at constant volume C_V is

$$C_V=\left(\frac{\partial U}{\partial T}\right)_{N,\omega}=Nk\frac{e^{\hbar\omega/kT}}{(e^{\hbar\omega/kT}-1)^2}\left[\frac{\hbar\omega}{kT}\right]^2=N\left[\frac{\hbar\omega}{2}+kT\,\ell n\,(1-e^{-\hbar\omega/kT})\right]$$

The entropy S is

$$S=\frac{U-F}{T}=Nk\left[\frac{\hbar\omega}{2kT}\coth\frac{\hbar\omega}{2kT}-\ell n\left(2\sinh\frac{\hbar\omega}{2kT}\right)\right]$$

Problem: If the partition function of a harmonic oscillator with frequency ω at a temperature T is $\frac{k_BT}{\hbar\omega}$, then the free energy of N such independent oscillators is:

(a) $\frac{3}{2}Nk_BT$ (b) $k_BT\ln\frac{\hbar\omega}{k_BT}$ (c) $Nk_BT\ln\frac{\hbar\omega}{k_BT}$ (d) $Nk_BT\ln\frac{\hbar\omega}{2k_BT}$

Soln. The Helmholtz' free energy of N independent harmonic oscillator in terms of partition function Z is

$F=-Nk_BT\ \ell n\ Z$

In question, $Z=\frac{k_BT}{\hbar\omega}$

So, $F=-Nk_BT\ \ell n\ Z=Nk_BT\ \ell n\frac{1}{Z}=Nk_BT\ \ell n\frac{\hbar\omega}{k_BT}$

Hence correct option is (c).

Problem: A system is in thermal equilibrium has energies 0 and E. Calculate thermodynamic properties of the system.

Soln. Let T be the temperature of the system. The partition function Z of the system is

$$Z=e^{0/k_BT}+e^{-E/k_BT}=1+e^{-E/k_BT}$$

(i) Free energy F of the system is

$$F=-k_BT\ \ell n\ Z=-k_BT\ \ell n\left[1+e^{-E/k_BT}\right]$$

(ii) Entropy S of the system is $S=-\left(\frac{\partial F}{\partial T}\right)_{N,V}=k\ \ell n\left[1+e^{-E/k_BT}\right]+\frac{E}{T^2}\frac{1}{e^{E/k_BT}+1}$

(iii) Internal energy U is

$$U=F+ST$$

$$=-k_BT\ \ell n\left[1+e^{-E/k_BT}\right]+k_BT\ \ell n\left[1+e^{-E/k_BT}\right]+\frac{E}{e^{E/k_BT}+1}$$

$$=\frac{E}{e^{E/k_BT}+1} \qquad ...(1)$$

(iv) Specific heat at constant volume C_V is

$$C_V=\left(\frac{\partial U}{\partial T}\right)_{N,V}=k_B\left(\frac{E}{k_BT}\right)^2\frac{e^{E/k_BT}}{\left(e^{E/k_BT}+1\right)^2} \qquad ...(2)$$

(a) At a low temperature $(E/k_BT)\gg1$, and equation (2) reduces to

$$C_V=k_B\left(\frac{E}{k_BT}\right)^2e^{-E/k_BT}$$

Since with the decreases of T, the function e^{-E/k_BT} reduces much faster than the increases of the function $(E/k_BT)^2$, therefore

$$C_V\to0 \quad \text{when} \quad T\to0$$

(b) At a high temperature $(E/k_BT) \ll 1$, and equation (2) reduces to

$$C_V = k\left(\frac{E}{k_BT}\right)^2 \frac{1}{4}$$

Hence,

$$C_V \to 0 \quad \text{when} \quad T \to \infty$$

Two Level System

In a system of N independent particles, each particle can have only one of the two energies, $-\in$ and $\in$. Suppose N_1 particles are in the state with energy $-\in$ and N_2 in the state with energy $\in$, total energy of the system is $E = M\in = -N_1\in + N_2\in$ so that $M = N_2 - N_1$. Since $N = N_2 + N_1$, we have

$$N_1 = \frac{1}{2}(N-M) \quad N_2 = \frac{1}{2}(N+M)$$

There are $N!/(N_1!\,N_2!)$ possibly ways of choosing N_1 particles out of N to occupy the state $-\in$ and of them gives a different microscopic state energy E. Hence the thermodynamics weight is

$$W_i = {}^NC_{N_1}\ \frac{N!}{\left[\frac{1}{2}(N-M)\right]!\left[\frac{1}{2}(N+M)\right]!}$$

The entropy of the system is

$$\begin{aligned} S &= k_B\ \ell n\ W_i \\ &= k_B\left[N\ \ell n\ N - \frac{1}{2}(N-M)\ \ell n\frac{1}{2}(N-M) - \frac{1}{2}(N+M)\ \ell n\frac{1}{2}(N+M)\right] \\ &= -k_B\left[N_1\ \ell n(N_1/N) + N_2\ \ell n(N_2/N)\right] \end{aligned}$$

Here, we used Stirling's approximation formula i.e.

$$\ell n N! = N\ell nN - N \qquad \text{(for large } N\text{)}$$

The temperature T is

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_{N,V} = \frac{\partial S}{\partial M}\frac{\partial M}{\partial E} = \frac{1}{\in}\frac{\partial S}{\partial M} = \frac{k}{2\in}\ell n\left(\frac{N-M}{N+M}\right)$$

This equation shows that $T < 0$ for $M > 0$ $(E > 0)$. This case is unacceptable on the physical ground. However, the system is normal in the range $M < 0$ $(E < 0)$ and we can discuss the properties of this system of this case. From this equation, we have

$$\frac{N_1}{N_2} = \frac{N-M}{N+M} = \exp(2\in/kT)$$

or

$$\frac{N_1}{N} = \frac{\exp(\in/kT)}{\exp(\in/kT) + \exp(-\in/kT)}$$

$$\frac{N_2}{N} = \frac{\exp(-\in/kT)}{\exp(\in/kT) + \exp(-\in/kT)}$$

These relations give the probabilities of finding any one particle in the states $-\in$ and $\in$ respectively. The energy

$$E = N_2\in - N_1\in = -N\in\tanh(\in/k_BT)$$

The specific heat at constant volume C_V is

$$C_V = \left[\frac{\partial E}{\partial T}\right]_V = Nk(\in/kT)^2\ \text{sech}^2(\in/k_BT)$$

Magnetic System

A system comprising N identical particles is placed in a uniform magnetic field H and is kept at a temperature T. When a particle having spin 1/2 is placed in a magnetic field H, its each energy level splits into two with changes in energies by $-\mu H$ and μH and the particle has a magnetic moment μ or $-\mu$ along the direction of the magnetic field, respectively.

As the spins of particles are independent of each other, the partition function of the total system Z_N is equal to the product of the partition functions for spins of individual particle. The partition function for spins of individual particle is

$$Z_1 = e^{\mu H/k_BT} + e^{-\mu H/k_BT} = 2\cosh(\mu H/k_BT)$$

Thus,

$$Z_N = Z_1^N = [2\cosh(\mu H/k_BT)]^N$$

The Helmholtz free energy is

$$F = -k_BT\,\ell n\,Z_N = -Nk_BT\,\ell n\,[2\cosh(\mu H/k_BT)]$$

The entropy is

$$S = -\frac{\partial F}{\partial T} = Nk_B\left[\ell n\{2\cosh(\mu H/k_BT)\} - (\mu H/k_BT)\tanh(\mu H/k_BT)\right]$$

Total energy is $U = F + TS = -N\mu H\tanh(\mu H/k_BT)$

Total magnetic moment is

$$M = -\frac{\partial F}{\partial H} = N\mu\tanh(\mu H/k_BT)$$

The specific heat at constant volume C_V is

$$C_V = \left[\frac{\partial U}{\partial T}\right]_H = Nk_B(\mu H/k_BT)^2\operatorname{sech}^2(\mu H/k_BT)$$

Problem: Consider a system of N non-interacting spins, each of which has classical magnetic moment of magnitude μ. The Hamiltonian of this system in an external magnetic field $\vec{H}$ is $H = -\sum_{i=1}^{N}\vec{\mu}_i.\vec{H}$, where $\vec{\mu}_i$ is the magnetic moment of the ith spin. The magnetization per spin at temperature T is:

(a) $\dfrac{\mu^2 H}{k_BT}$ (b) $\mu\left[\coth\left(\dfrac{\mu H}{k_BT}\right) - \dfrac{k_BT}{\mu H}\right]$

(c) $\mu\sinh\left(\dfrac{\mu H}{k_BT}\right)$ (d) $\mu\tanh\left(\dfrac{\mu H}{k_BT}\right)$

Soln. Interaction energy $= -\vec{\mu}.\vec{H}$

$$\langle\mu_z\rangle = \frac{\int \mu_z e^{+\vec{\mu}.\vec{H}/k_BT}\,d\Omega}{\int e^{+\vec{\mu}.\vec{H}/k_BT}\,d\Omega}$$

$$\langle\mu_z\rangle = \frac{\mu\int_0^{2\pi}\int_0^{\pi}\cos\theta\, e^{\mu H\cos\theta/k_BT}\sin\theta\, d\theta\, d\varphi}{\int_0^{2\pi}\int_0^{\pi} e^{\mu H\cos\theta/k_BT}\sin\theta\, d\theta\, d\varphi}$$

[Using $\int d\varphi = 2\pi$ and *put* $\cos\theta = x,\ \mu H/k_BT = a$]

$$\langle \mu_z \rangle = \frac{\mu \int_0^{\pi} \cos\theta\, e^{a\cos\theta} \sin\theta d\theta}{\int_0^{\pi} e^{a\cos\theta} \sin\theta d\theta} = \mu\left[\int_{-1}^{+1} x e^{ax} dx \Big/ \int_{-1}^{+1} e^{ax} dx\right]$$

$$\text{First integral} = \mu\left[x\frac{e^{ax}}{a}\Bigg|_{-1}^{+1} - \frac{1}{a}\int_{-1}^{1} e^{ax} dx\right] = \mu\left[\left(\frac{e^a + e^{-a}}{2a}\right) - \frac{1}{a}\left|\frac{e^{ax}}{a}\right|_{-1}^{+1}\right]$$

$$= \mu\left[\frac{2\cosh a}{a} - \frac{1}{a^2}\sinh a\right] = \frac{2\mu}{a}\left[\cosh a - \frac{\sinh a}{a}\right]$$

$$\text{Second integral} = \int_{-1}^{+1} e^{ax} dx = \frac{2\sinh a}{a}$$

$$\langle \mu_z \rangle = \frac{\mu \frac{2}{a}\left(\cosh a - \frac{1}{a}\sinh a\right)}{\frac{2}{a}\sinh a} = \mu\left(\coth a - \frac{1}{a}\right)$$

$$\langle \mu_z \rangle = \mu\left(\coth\frac{\mu H}{kT} - \frac{k_B T}{\mu H}\right)$$

Correct answer is (b)

Problem: Consider a system whose three energy levels are given by 0, ε and 2ε. The energy level ε is two-fold degenerate and the other two are non-degenerate. The partition function of the system with $\beta = \frac{1}{k_B T}$ is given by

(a) $1 + 2e^{-\beta\epsilon}$ (b) $2e^{-\beta\epsilon} + e^{-2\beta\epsilon}$ (c) $\left(1 + e^{-\beta\epsilon}\right)^2$ (d) $1 + e^{-\beta\epsilon} + e^{-2\beta\epsilon}$

Soln. For the given energy levels the partition function can be written as

$z = e^{-\beta\cdot 0} + 2e^{-\beta\varepsilon} + e^{-\beta\cdot 2\varepsilon}$; where the factor 2 is due to two fold degeneracy of energy level ε.

Therefore, $z = 1 + 2e^{-\beta\varepsilon} + (e^{-\beta\varepsilon})^2 = (1 + e^{-\beta\varepsilon})^2$

Hence correct option is (c).

Problem: Common Data for next two problems

There are four energy levels E, $2E$, $3E$ and $4E$ (where $E > 0$). The canonical partition function of two particles is, if these particles are

(i) Two identical fermions

(a) $e^{-2\beta E} + e^{-4\beta E} + e^{-6\beta E} + e^{-8\beta E}$

(b) $e^{-3\beta E} + e^{-4\beta E} + 2e^{-5\beta E} + e^{-6\beta E} + e^{-7\beta E}$

(c) $(e^{-\beta E} + e^{-2\beta E} + e^{-3\beta E} + e^{-4\beta E})^2$

(d) $e^{-2\beta E} - e^{-4\beta E} + e^{-6\beta E} - e^{-8\beta E}$

Soln. For fermions the distribution of particles can be as illustrated below.

Energy Level → **Configuration ↓**	E	$2E$	$3E$	$4E$
I	A	A		
II	A		A	
III	A			A
IV		A	A	
V		A		A
VI			A	A

From the above table it is clear that :-
Number of configuration with total energy $3E = 1$.
Number of configuration with total energy $4E = 1$.
Number of configuration with total energy $5E = 2$.
Number of configuration with total energy $6E = 1$.
Number of configuration with total energy $7E = 1$.
Hence correct option is (b).

(ii) Two distinguishable particles

(a) $e^{-2\beta E}+e^{-4\beta E}+e^{-6\beta E}+e^{-8\beta E}$

(b) $e^{-3\beta E}+e^{-4\beta E}+2e^{-5\beta E}+e^{-6\beta E}+e^{-7\beta E}$

(c) $(e^{-\beta E}+e^{-2\beta E}+e^{-3\beta E}+e^{-4\beta E})^2$

(d) $e^{-2\beta E}-e^{-4\beta E}+e^{-6\beta E}-e^{-8\beta E}$

Soln. Since the particles are indistinguishable they can be treated classically i.e., any number of particles can be present in any level.

Now, the partition function for one particle with the given energy levels can be written as

$Z = e^{-\beta E}+e^{-2\beta E}+e^{-3\beta E}+e^{-4\beta E}$.

Therefore, the partition function for two particles equal to Z^2.

Hence correct option is (c).

Classical and Quantum Statistics

The study of statistical mechanics can be divided into two main classes :
1. Classical Statistics or Maxwell-Boltzmann Statistics.
2. Quantum Statistics (Fermi-Dirac and Bose-Einstein Statistics)

(1) Classical Statistics

Let there be total N particles out of which n_i particles are in ε_ith energy level having g_i states (or having degeneracy g_i). Let us label the g_i cells. The first particle may occupy any of the g_i cells. The second particle may occupy any of the re-meaning $(g_i - 1)$ cells etc. Thus, the total number of ways of accommodating all n_i particles in g_i cells is $g_i^{n_i}$.

Taking into account for the fact that there are n_i particles in g_i cells for each $i = 1, 2, 3,\ldots$. The total number of ways is $\prod_i g_i^{n_i}$. The total number of particles is N. So, preventing all the particles in which n_i are alike ($i = 1, 2, 3,\ldots$) is $\dfrac{N!}{\prod n_i!}$.

So, the total number of microstates is

$$\Omega = \frac{N!}{\prod n_i!}\prod_i g_i^{n_i} \qquad \ldots(1)$$

Having found the number of microstates, we are ready to find the most probable state. We maximise $\ell n\,\Omega$ subject to the two constraints

$$\sum_i n_i = N \qquad \text{...(2)}$$

$$\sum_i n_i\varepsilon_i = E \qquad \text{...(3)}$$

where, N and E are constants.

That gives $\boxed{n_i = g_i e^{-(\beta\varepsilon_i - \alpha)}}$, here $\alpha = -\dfrac{\mu}{k_BT} = -\dfrac{E_F}{k_BT}$ and $\beta = \dfrac{1}{k_BT}$ and g_i is degeneracy factor which is the classical or Maxwell Boltzmann distribution.

(2) Quantum Statistics

- **Bose-Einstein Statistics**

This statistics is obeyed by identical, indistinguishable particles of integral spin (bosons) that have symmetrical wave function and is so named as it was derived by Bose for light quanta and generalized by Einstein.

Consider a system having n indistinguishable particles. Let these particles be divided into quantum groups or levels such that there are $n_1, n_2, \ldots, n_i$ number of particles in group whose approximate constant energies are $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_i \ldots$ respectively. Let g_i be the number of eigen states (i.e. degeneracy of statistical weight) of the i th level (for a non degenerate case $(g_i - 1)$.

For finding Bose-Einstein distribution consider a box divided into g_i sections (or cells) and n_i indistinguishable particles are to be distributed among these sections. The choice that which of the sections will head the sequence can be made in g_i ways. Once this has been done the total number of permutations of n_i particles and the remaining $(g_i = 1)$ sections is $(n_i + g_i - 1)$. Thus, the total number of possible ways in which n_i particles can be distributed in g_i sections is

$$g_i\left[(n_i + g_i + 1)!\right] \qquad \text{...(4)}$$

Since the particles are indistinguishable the permutations of particles among themselves will not give rise to indistinguishable arrangements. Obviously these are $n_i!$ such permutations, hence expression (4) must be divided by $n_i!$. Further, the distributions which arise by mere permutations of sections among themselves do not produce different states. Obviously there are $g_i!$ such permutations hence expressions must also be divided by $g_i!$

Thus, the required number of ways in which n_i particles are to be distributed among g_i sub-levels of ith quantum group is

$$\frac{[g_i(n_i + g_i - 1)!]}{n_i!g_i!} = \frac{(n_i + g_i - 1)!}{n_i(g_i - 1)!}$$

Similar expressions can be found for various other quantum states. Considering all the available quantum groups, with n_1 particles in first group of energy ε_1, n_2 particles in second group of energy ε_2 and so on, the total number of distinguishable arrangements are given by

$$G = \frac{(n_1 + g_1 - 1)}{n_1!(g_1 - 1)!}\cdot\frac{(n_2 - g_2 - 1)!}{n_2!(g_2 - 1)!}\cdot\frac{(n_i + g_i - 1)!}{n_i!(g_i - 1)!} = \prod_i \frac{(n_i + g_i - 1)!}{n_i!(g_i - 1)!} \qquad \text{...(5)}$$

As n_i and g_i are large numbers, we may neglect one in above expression therefore, equation (5) may be written as

$$G = \prod_i \frac{(n_i + g_i)!}{n_i!(g_i)!} \qquad \text{...(6)}$$

In accordance with the postulate of equal a priori probability of states, the probability Ω of the system for occuring with the specified distribution is proportional to the total number of eigen states, i.e.,

$$\Omega = \prod_i \frac{(n_i + g_i)!}{n_i!\,g_i!} \times \text{constant} \qquad \text{...(7)}$$

$$= C\prod_i \frac{(n_i + g_i)!}{n_i!\,g_i!} \text{ where } C \text{ is constant}$$

The distribution law of Bose-Einstein statistics can now be obtained by determining the most probable distribution as in classical statistics.

That gives $\boxed{n_i = \frac{g_i}{e^{\alpha+\beta\varepsilon_i}-1}}$

This equation represents the most probable distribution of the particles among various energy levels for a system obeying Bose-Einstein statistics and is therefore known as *Bose-Einstein's distribution law.*

• Fermi-Dirac Statistics

This statistics is obeyed by indistinguishable particles of half-integral spin that have anti-symmetric wave functions and obey Pauli exclusion principle.

Consider a system having n indistinguishable particles. Let these particles be divided into quantum groups or levels such that there are $n_1, n_2, ..., n_i...$ number of particles in groups whose approximate constant energies are $\varepsilon_1, \varepsilon_2, ... \varepsilon_i...$ respectively. Let g_i denote the degeneracy or statistical weight (i.e., number of sublevels) of ith level.

Since due to Pauli's principle, no cell can occupy more than one particle, therefore among g_i cells only n_i cells are occupied by one particle each and the remaining $(g_i - n_i)$ cells are empty. The possible number of such a distribution is given by $g_i!$ corresponding to the permutations of the g_i cells. But as the particles are indistinguishable, therefore all the occupied n_i cells are similar to each other. Hence $n_i!$ permutations of occupied cells will not give rise to indistinguishable arrangements. Further $(g_i - n_i)!$ permutations of empty cells among themselves also give rise to indistinguishable arrangements. Hence, the number of distinguishable arrangements of n_i particles in g_i cells are

$$\frac{g_i}{n!\,(g_i - n_i)} \quad ...(1)$$

Similar expressions can also be found for various other quantum states. Considering all the available quantum groups, with n_i particles in first group of energy $\varepsilon_1; n_2$ particles in second group of energy ε_2 and so on and keeping in mind that due to indistinguishable of particles, the division of total number of n particles into groups of $n_1, n_2, ..., n_i...$ particles can be done in only one way; the total number of eigen states for the whole system is given by

$$G = \prod_i \frac{g_i!}{n_i!\,(g_i - n_i)!} \quad ...(2)$$

In accordance with the postulate of equal a priori probability of states, the probability Ω of the system for occuring with the specified distribution is proportional to the total number of eigenstates i.e.,

$$\Omega = \prod_i \frac{g_i!}{n_i!\,(g_i - n_i)!} \times \text{constant}$$

$$= C\prod_i \frac{g_i!}{n_i!\,(g_i - n_i)!} \quad ...(3)$$

where, C is a constant.

The Fermi-Dirac distribution law can be obtained by determining the most probable distribution.

The two subsidiary conditions are :

1. The total number of particles of the systems is constant $n = \sum_i n_i = \text{constant}$

or $\delta n = \delta n_i = 0$ ...(4)

2. Total energy of the system is constant

i.e., $E = \sum n_i\,\varepsilon_i = \text{constant}$

i.e., $\delta E = \sum_i \varepsilon_i \delta n_i = 0$ ...(5)

That gives $\boxed{n_i = \frac{g_i}{e^{\alpha+\beta\varepsilon_i}+1}}$

This equation represents the most probable distribution of the particles among various energy levels for a system obeying Fermi-Dirac statistics and is therefore known as *Fermi-Dirac distribution law.*

Comparison of Three Statistics

The expressions for the most probable distribution in the three statistics are

(i) Maxwell-Boltzmann Statistics $n_i = \frac{g_i}{e^{\alpha+\beta\varepsilon_i}} = \frac{g_i}{e^{(\varepsilon_i-\varepsilon_F)/k_BT}}$

(ii) Bose-Einstein Statistics $n_i = \frac{g_i}{e^{\alpha+\beta\varepsilon_i}-1} = \frac{g_i}{e^{(\varepsilon_i-\varepsilon_F)/k_BT}-1}$

(iii) Fermi-Dirac Statistics $n_i = \frac{g_i}{e^{\alpha+\beta\varepsilon_i}+1} = \frac{g_i}{e^{(\varepsilon_i-\varepsilon_F)/k_BT}+1}$

It is to be note that if g_i/n_i is very large in comparison to unity, we may write

$$\frac{g_i}{n_i} \approx \frac{g_i}{n_i}+1 \approx \frac{g_i}{n_i}-1$$

i.e., for large values of g_i/n_i Bose-Einstein and Fermi-Dirac distributions approach the Maxwell-Boltzmann distribution.

Problem: A Maxwell-Boltzmann gas has two particles in the *i*-th state whose degeneracy is three. Find out the number of independent ways selecting the particles in the state.

Soln. Here, we have $n_i = 2, N = 2$ and $g_i = 3$. The number of independent ways W_i of selecting the particles in the *i*-th state is $W_i = N!\frac{g_i^{n_i}}{n_i!} = 2!\frac{3^2}{2!} = 9$

Denoting the two distinguisable particles by the alphabets A and B, 9 independent ways of arranging the particles in 3 levels of the state are given in the table below:

Ways	Levels		
	1	2	3
1	A B	–	–
2	–	A B	–
3	–	–	A B
4	A	B	–
5	B	A	–
6	A	–	B
7	B	–	A
8	–	A	B
9	–	B	A

Problem: A Fermi-Dirac gas has two particles in the *i*-th state whose degeneracy is three. Find out the number of independent ways of selecting the particles in the state.

oln. Here, we have $n_i = 2$ and $g_i = 3$. The number of independent ways W_i of selecting the particles in the level is

$$W_i = \frac{g_i!}{n_i!(g_i-n_i)!} = \frac{3!}{2!(3-2)!} = \frac{3!}{2!1!} = 3$$

Denoting the two indistinguishable particles by the alphabets A and A, 3 independent ways of arranging the particles, obeying Pauli's exclusion principle, in 3 levels of the state are given in the table below:

Ways	Levels		
	1	2	3
1	A	A	–
2	A	–	A
3	–	A	A

Problem: A Bose-Einstein gas has two particles in the *i*-th state whose degeneracy is three. Find out the number of independent ways of selecting the particles in the state.

Soln. Here, we have $n_i = 2$ and $g_i = 3$. The number of independent ways W_i of selecting the particles in the state is

$$W_i = \frac{(n_i + g_i - 1)!}{n_i!\,(g_i - 1)!} = \frac{(2+3-1)!}{2!\,(3-1)!} = \frac{4!}{2!\,2!} = 6$$

Denoting the two indistinguishable particles by the alphabets A and A, 6 independent ways of arranging the particles in 3 levels of the state are given in the table below:

Ways	Levels		
	1	2	3
1	A A	–	–
2	–	A A	–
3	–	–	A A
4	A	A	–
5	A	–	A
6	–	A	A

Problem: Let N_{MB}, N_{BE}, N_{FD} denote the number of way in which two particles can be distributed in two energy states according to Maxwell-Boltzmann, Bose-Einstein and Fermi-Dirac respectively. Then $N_{MB} : N_{BE} : N_{FD}$ is:

(a) 4 : 3 : 1 (b) 4 : 2 : 3 (c) 4 : 3 : 3 (d) 4 : 3 : 2 **[JAM-2014]**

Soln.

MB

E_1	E_2
AB	
	AB
A	B
B	A

$N_{MB} = 4$

BE

E_1	E_2
··	
	··
·	·

$N_{BE} = 3$

FD

E_1	E_2
·	·

$N_{FD} = 1$

So, $N_{MB} : N_{BE} : N_{FD} = 4:3:1$

Correct option is (a)

Problem: Which of the following atoms cannot exhibit Bose-Einstein condensation, even in principle?

(a) $^{1}H_{1}$ (b) $^{4}He_{2}$ (c) $^{23}Na_{11}$ (d) $^{40}K_{19}$

Soln. To exhibit Bose-Einstein condensation, particle must be a Boson $\Rightarrow$ Integral spin.

Since, all proton, neutron and electron have half integral spin, only those atoms whose $n_p + n_n + n_e$ is even will be a boson.

where

$$\left.\begin{array}{l} n_p = \text{number of proton} \\ n_n = \text{number of neutron} \end{array}\right\} = n_n = \text{number of neutron} = \text{mass number.}$$

$$n_e = \text{number of electron}$$

For ${}^1H_1 : n_n + n_e = 1 + 1 = 2$ (even) (a)

For ${}^4He_2 : n_n + n_e = 4 + 2 = 6$ (even) (b)

For ${}^{23}Na_{11} : n_n + n_e = 23 + 11 = 34$ (even) (c)

For ${}^{40}K_{19} : n_n + n_e = 40 + 19 = 59$ (odd) (d)

So, ${}^{40}K_{19}$ cannot exhibit Bose-Einstein condensation.

Hence, the correct option is (d).

Problem: It is necessary to apply quantum statistics to a system of particles, if

(a) there is substantial overlap between the wave functions of the particles

(b) the mean free path of the particles is comparable to the inter-particle separation

(c) the particles have identical mass and charge

(d) the particles are interacting

Soln. The quantum realm begins when de Broglie wavelength of the particles (or associated with the particles) becomes comparable to the size of the particle. Under this condition, the wavelength of the particles starts overlapping.

Hence, the correct option is (a).

Problem: Consider the Fermi-Dirac distribution function f(E) at room temperature (300 K) where E refers to energy. If E_F is the Fermi energy, which of the following is true?

(a) $f(E)$ is a step function

(b) $f(E_F)$ has a value of 1/2

(c) States with $E < E_F$ are filled completely

(d) $f(E)$ is large and tends to infinity as E decreases much below E_F.

Soln. $$f(E) = \frac{1}{\exp[(E - E_F)/kT + 1]}$$

For $E = E_F$,

$$f(E_F) = \frac{1}{\exp[0] + 1} = \frac{1}{2}$$

Hence correct option is (b).

Problem: A photon gas is at thermal equilibrium at temperature T. The mean number of photons in an energy state $\varepsilon = \hbar\omega$ is:

(a) $\exp\left(\frac{\hbar\omega}{k_BT}\right) + 1$ (b) $\exp\left(\frac{\hbar\omega}{k_BT}\right) - 1$ (c) $\left(\exp\left(\frac{\hbar\omega}{k_BT}\right) + 1\right)^{-1}$ (d) $\left(\exp\left(\frac{\hbar\omega}{k_BT}\right) - 1\right)^{-1}$

Soln. The mean occupation number or the average number of particles in a state with energy ε is given by

$$\bar{n} = \frac{1}{\exp\left[(\varepsilon - \mu)/k_BT\right] \pm 1}$$

where, '+' sign is for Fermi-Dirac distribution and '–' sign is for Bose-Einstein distribution.

Since, photon is a Boson, we must use '–' sign.

Also for photon $\mu = 0$.

$$\therefore\ \bar{n}_{\text{photon}} = \frac{1}{\exp\left(\dfrac{\varepsilon}{k_B T}\right)-1} = \left[\exp\left(\frac{\hbar\omega}{k_B T}\right)-1\right]^{-1} \qquad (\because\ \varepsilon = \hbar\omega)$$

Hence correct option is (d).

Problem: The probability that an energy level ε at a temperature T is unoccupied by a fermion of chemical potential μ is given by

(a) $\dfrac{1}{e^{(\varepsilon-\mu)/k_BT}+1}$ (b) $\dfrac{1}{e^{(\varepsilon-\mu)/k_BT}-1}$ (c) $\dfrac{1}{e^{(\mu-\varepsilon)/k_BT}+1}$ (d) $\dfrac{1}{e^{(\mu-\varepsilon)/k_BT}-1}$

Soln. The distribution function for fermions is given by $f(\varepsilon) = \dfrac{1}{\exp\left[(\varepsilon-\mu)/k_BT\right]+1}$.

Also, $f(\varepsilon)$ = probability that an energy level ε at a temperature T is occupied by a fermions of chemical potential μ.

Therefore, probability than an energy level ε at temperature T is unoccupied by a fermions of chemical potential μ is a $1-f(\varepsilon)$.

($\because$ Probability of unoccupancy = 1 − probability of occupancy)

$$\Rightarrow \text{ Required probability } = 1 - f(\varepsilon) = 1 - \frac{1}{\exp\left[(\varepsilon-\mu)/k_BT\right]+1}$$

$$= \frac{\exp\left[(\varepsilon-\mu)/k_BT\right]\cancel{+1}\,\cancel{-1}}{\exp\left[(\varepsilon-\mu)/k_BT\right]+1} = \frac{\exp\left[(\varepsilon-\mu)/k_BT\right]}{\exp\left[(\varepsilon-\mu)/k_BT\right]+1}$$

Dividing numerator and denominator by $\exp\left[(\varepsilon-\mu)/k_BT\right]$, we get the required

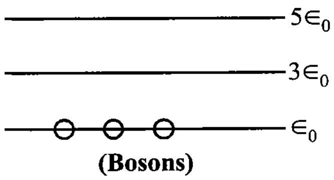
$$\text{probability} = \frac{1}{1+1/\exp\left[(\varepsilon-\mu)/k_BT\right]} = \frac{1}{\exp\left[(\mu-\varepsilon)/k_BT\right]+1}$$

Hence correct option is (c).

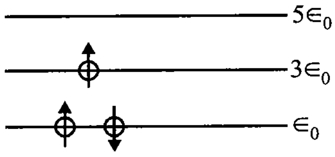
Problem: Consider two different systems each with three identical non-interacting particles. Both have single particle states with energies ε_0, $3\varepsilon_0$ and $5\varepsilon_0$, ($\varepsilon_0 > 0$). One system is populated by spin-$\frac{1}{2}$ fermions and the other by bosons. What is the value of $E_F - E_B$ where E_F and E_B are the ground state energies of the fermionic and bosonic systems respectively?

(a) $6\varepsilon_0$ (b) $2\varepsilon_0$ (c) $4\varepsilon_0$ (d) ε_0

Soln.



(Bosons)



(Fermions)

According to Pauli's exclusion principle

$E_B = 3\epsilon_0$ $E_F = 2\epsilon_0 + 3\epsilon_0 = 5\epsilon_0$

$E_F - E_B = 5\epsilon_0 - 3\epsilon_0 = 2\epsilon_0$

Correct answer is (b)

Problem: Consider a system of 3 fermions, which can occupy any of the 4 available energy states with equal probability. The entropy of the system is

(a) $k_B \ln 2$ (b) $2k_B \ln 2$ (c) $2k_B \ln 2$ (d) $3k_B \ln 4$

Soln. The distribution of 3 fermions (A, A, A) in 4 energy states (I, II, III, IV) is illustrated in the following table :

States → **Microstates ↓**	1	2	3	4
I	A	A	A	
II	A	A		A
III	A		A	A
IV		A	A	A
Total no. of microstates = 4				

Now, entropy is defined as $S = k_B\, \ell n\, \Omega$

where Ω = number of accessible microstates (which is = 4 in present case).

Therefore, $S = k_B\, \ell n\, 4 = k_B\, \ell n\, 2^2 = 2k_B\, \ell n\, 2$.

Hence correct option is (b).

PROBLEMS TO BE PLAYED

1. The probability that an energy level ε at a temperature T is unoccupied by a fermion of chemical potential μ is given by

(a) $\frac{1}{e^{(\varepsilon-\mu)/k_BT}+1}$ (b) $\frac{1}{e^{(\varepsilon-\mu)/k_BT}-1}$ (c) $\frac{1}{e^{(\mu-\varepsilon)/k_BT}+1}$ (d) $\frac{1}{e^{(\mu-\varepsilon)/k_BT}-1}$

2. The density of states between energy ε and $\varepsilon + d\varepsilon$ is

(a) $\frac{4\pi L^2 m}{h^2} d\varepsilon$ (b) $\frac{4\pi L^2 m}{h^2}\frac{1}{\sqrt{\varepsilon}} d\varepsilon$ (c) $\frac{4\pi L^2 m}{h^2}\sqrt{\varepsilon d\varepsilon}$ (d) $\frac{4\pi L^2 m}{h^2}\varepsilon d\varepsilon$

3. The ground state energy E_0 of the system in terms of the Fermi energy E_F and the number of electrons N is given by

(a) $\frac{1}{3}NE_F$ (b) $\frac{1}{2}NE_F$ (c) $\frac{2}{3}NE_F$ (d) $\frac{3}{5}NE_F$

4. Thermodynamic variables of a system can be volume V, pressure P, temperature T, number of particles N, internal energy E and chemical potential μ, etc. For a system to be specified by microcanonical (MC), Canonical (CE) and Grand Canonial (GC) ensembles, the parameters required for the respective ensembles are

(a) $MC:(N,V,T);CE(E,V,N);GC:(V,T,\mu)$

(b) $MC:(E,V,N);CE:(N,V,T);GC:(V,T,\mu)$

(c) $MC:(V,T,\mu);CE:(N,V,T);GC:(E,V,N)$

(d) $MC:(E,V,N);CE:(V,T,\mu);GC:(N,V,T)$

5. Which of the following relations between the particle number density n and temperature T must hold good for a gas consisting of non-interacting particles to be described by quantum statistics?

(a) $n/T^{1/2} << 1$ (b) $n/T^{3/2} << 1$

(c) $n/T^{3/2} >> 1$ (d) $n/T^{1/2}$ and $n/T^{3/2}$ can have any value.

6. In a classical micro-canonical ensemble for a system of N non-interacting particles, the fundamental volume in phase space which is regarded as "equivalent to one micro-state" is

(a) h^{3N} (b) h^{2N} (c) h^{N} (d) h

where h is the Planck's constant.

7. Which of the following statements is true?

(a) In a micro-canonical ensemble the total number of particles N and the energy E are constants while in a canonical ensemble N and temperature T are constants.

(b) In a micro-canonical ensemble the total number of particles N is a constant but the energy E is variable while in a canonical ensemble N and T are constants.

(c) In a macro-canonical ensemble N and E are constants while in a canonical ensemble N and T both vary.

(d) In a micro-canonical ensemble N and E are constants while in a canonical ensemble N is a constant but T varies.

8. A collection of N two-level system with energies 0 and $E > 0$ in the thermal equilibrium at temperature T. For $T \to \infty$, the specific heat approaches

(a) 0 (b) Nk_B (c) $3Nk_B/2$ (d) ∞

9. Two particles are said to be distinguishable when

(a) the average distance between them is large compared to their de-Broglie wavelength.

(b) the average distance between them is small compared to their de-Broglie wavelengths.

(c) they have overlapping wavepackets.

(d) their total wave function is symmetric under particle exchange.

10. In a canonical ensemble
(a) the energy and the temperature are constants.
(b) the entropy and the energy are constans.
(c) the temperature and the density are constants.
(d) the density and the entropy are constants.

11. The entropy of a system in the microcanonical ensemble formalism
(a) Is independent of the number of microslates.
(b) Is directly proportional to the number of microstates.
(c) Is inversely proportional to the number of microstates.
(d) Increases logarithmicaly with increase in number of microstates.

12. The difference between Maxwell-Boltzmann (MB), and Bose-Einstein (BE) distribution is that
(a) Both are identical except that one is for classical particles and the other for quantum particles.
(b) Density of occupied states at lower energies is more for BE than for MB.
(c) Density of occupied states at lower energies is less for BE than for MB.
(d) MB is applicable to particles with zero spin and BE to those of finite spin.

13. In a canonical ensemble description of a system, which of the following quantities remain fixed?
(a) Energy of the system
(b) Square of the energy of the system
(c) Volume of the system
(d) Number of particles in the system.

14. The chemical potential μ is:
(a) Positive in Fermi-Dirac statistics (*FD*) and positive in Bose-Einstein statistics (BE) at room temperature (*RT*)
(b) Positive in *FD* and negative in *BE* at *RT*
(c) Negative in Maxwell Boltzmann statistics and negative in *FD* at *RT*.
(d) Always positive in all the three statistics.

ANSWER KEY

1. (c) **2. (a)** **3. (b)** **4. (b)** **5. (c)** **6. (a)** **7. (a)**

8. (a) **9. (a)** **10. (c)** **11. (d)** **12. (a)** **13. (c, d)** **14. (c)**

Appendix-I

Specific & Latent Heats

Specific Heat Capacity

If $'\Delta Q'$ amount of heat absorbed or rejected by a substance of mass '*m*' when it undergoes a temperature change $'\Delta T'$, then the specific heat capacity of that substance is given by

$$s = \frac{\Delta Q}{m\,\Delta T} \Rightarrow \Delta Q = ms\,\Delta T$$

It is the property of the substance which determines the change in the temperature of the substance (undergoing no phase change) when a given quantity of heat is absorbed (or rejected) by it.

If the amount of substance is specified in terms of moles '*n*' (instead of mass '*m*' in kg), we can define molar specific heat capacity (C) as

$$C = \frac{\Delta Q}{\mu \Delta T}$$

- If the gas is held under constant pressure of volume during the heat transfer, then it is called molar specific heat capacity at constant pressure (or volume) and is denoted by C_P (or C_V)
- The specific heat capacity of water is 4186.0 J kg K^{-1} and that of carbon is 506.5 J kg^{-1} K^{-1}. It means that you need eight times more heat to raise the temperature of water by the same amount as that of carbon.

Latent Heat:

It heat is added to a given quantity of ice at −10°C, the temperature of ice increases until it reaches its melting point (0°C). At this temperature, the addition of more heat does not increase the temperature but causes the ice to melt, of change its state. Once the entire ice melts, adding more heat will cause the temperature of the water to rise. A similar situation occurs during liquid gas change of state at the boiling point. Adding more heat to boiling water causes vaporisation, without increase in temperature.

The heat required during a change of state depends upon the heat of transformation and the mass of the substance under going a change of state. Thus, if mass '*m*' of a substance under goes a change from one state to the other, then the quantity of heat required is given by

$$Q = mL$$

where L is the latent heat and is a characteristic of the substance. Its SI unit is J/kg. The value of L also depends on the pressure. Its value is usually quoted at standard atmospheric pressure. The latent heat for a solid-liquid state change is called the latent heat of fusion (L_f) and that for a liquid-gas state change is called the latent heat of vaporisation (L_v).

Appendix-II

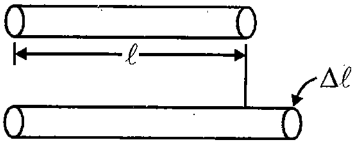
Thermal Expansion

You may have observed that sometimes sealed bottles with metallic lids are so tightly screwed that one has to put the lid in hot water for sometime to open the lid. This would allow the metallic cover to expand, thereby loosening it to unscrew easily. In case of liquids, you may have observed that mercury in a thermometer rises, when the thermometer is put in a slightly warm water. If we take out the thermometer from the water, the level of mercury falls again. Similarly, in the case of gases, a balloon partially inflated in a cool roommay expand to full size when placed in warm water on the other hand, a fully inflated balloon when immersed in cold water would start shrinking due to contraction of the air inside.

It is our common experience that most substances expand on heating and contract on cooling. A change in the temperature of a body causes change in its dimensions. The increase in the dimension of a body due to the increase in its temperature is called *thermal expansion.* The expansion in length is called *linear expansion.* The expansion in area is called *area expansion.* The expansion in volume is called *volume expansion.*

If the substance is in the form of a long rod, then for small change in temperature, ΔT, the fractional change in length, $\Delta \ell / \ell$, is directly proportional to ΔT.

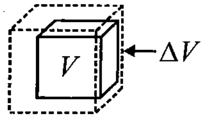
$$\Rightarrow \quad \boxed{\frac{\Delta \ell}{\ell} = \alpha_\ell \, \Delta T} \qquad \ldots (1)$$



where α_ℓ is known as the *coefficient of linear expansion* and is characteristic of the material. The values of α_ℓ for copper and glass are $1.7 \times 10^{-5}\,\mathrm{K}^{-1}$ and $0.32 \times 10^{-5}\,\mathrm{K}^{-1}$. It means that copper expands about five time more than glass for the same rise in temperature. Normally, metals expand more and have relatively high values of α_ℓ.

Similarly, we consider the fractional change in volume, $\Delta V/V$, of a substance for temperature ΔT and define the coefficient of volume expansion α_V, we have then

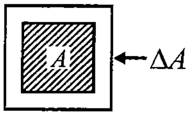
$$\boxed{\frac{\Delta V}{V} = \alpha_V \, \Delta T} \qquad \ldots (2)$$



where α_V is also characteristic of the temperature.

Similarly, for area expansion, we have

$$\boxed{\frac{\Delta A}{A} = \alpha_A \, \Delta T} \qquad \ldots (3)$$



Relation between α_ℓ, α_A and α_V

Imagine a cube of length ℓ, that expands equally in all directions, when its temperature increases by ΔT, we have

$$\Delta \ell = \alpha_\ell \ell \Delta T \qquad \text{... (4)}$$

So change in the volume of the cube,

$$\Delta V = (\ell + \Delta \ell)^3 - \ell^3 = 3\ell^2 \Delta \ell \qquad \text{... (5)}$$

where we have neglected terms in $(\Delta \ell)^2$ and $(\Delta \ell)^3$ as they are small compared to ℓ.

So, $\Delta V = \frac{3V}{\ell} \Delta \ell = 3V \alpha_\ell \Delta T$ ... (6) [Using (4)]

Also by definition, $\Delta V = \alpha_V V \Delta T$ ... (7)

From equations (6) and (7), $\alpha_V = 3\alpha_\ell$ ... (8)

Similarly, we have can obtain, $\boxed{\frac{\alpha_V}{3} = \frac{\alpha_A}{2} = \alpha_\ell}$ ... (9)

Problem: An aluminium flagpole is 33m high. By how much does its length increase as the temperature increases, by 15°C?

(The coefficient of linear expansion for aluminium is $2.5 \times 10^{-5} K^{-1}$).

(a) 1.24 mm (b) 12.4 mm (c) 1.24 cm (d) 12.4 cm

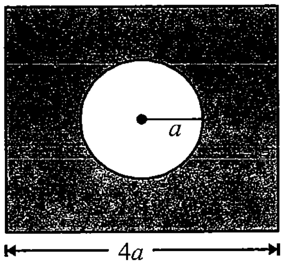
Soln. Given $\ell = 33\,m$; $\Delta T = 15°C = 15\,K$; $\alpha_\ell = 2.5 \times 10^{-5}\,K$

Using, $\frac{\Delta \ell}{\ell} = \alpha_\ell \Delta T$

$\Rightarrow \quad \Delta \ell = \alpha_\ell \Delta T \cdot \ell = 2.5 \times 10^{-5} \times 15 \times 33 = 1.24$ cm

Correct option is (c)

Problem: A thin square sheet of copper (coefficient of linear expansion α) of side $4a$ has a circular hole of radius a cut out symmetrically as shown below :



If the temperature of the sheet is increased by an amount ΔT, the area of the sheet changes by

(a) $(16+\pi)\alpha a^2 \Delta T$ (b) $2(16+\pi)\alpha a^2 \Delta T$ (c) $(16-\pi)\alpha a^2 \Delta T$ (d) $2(16-\pi)\alpha a^2 \Delta T$

Soln. The area of the plate is

$$A = (4a)^2 - \pi(a)^2 = (16-\pi)a^2$$

The coefficient of area expansion, $\alpha_A = 2\alpha$

So the change in area is

$$\Delta_A = \alpha_A A \Delta T$$

$$\Rightarrow \Delta A = 2\alpha(16-\pi)a^2 \Delta T$$

$\Rightarrow \quad \Delta A = 2(16-\pi)\alpha a^2 \Delta T$

Correct option is (d)

Problem: The coefficient of volume expansion of glycerin is $49\times10^{-5}\ K^{-1}$. What is the fractional change in its density for a 30°C rise in temperature?

(a) 1.5×10^{-2} (b) 1.5×10^{-3} (c) 0.15×10^{-2} (d) 0.15×10^{-3}

Soln. Given: $\alpha_V = 49\times10^{-5} K^{-1}$; $\Delta T = 30^\circ C = 30 K$

The density is defined as

$$d = \frac{m}{V}$$

where m is the mass and V is the volume

$$\Rightarrow \quad \Delta d = \frac{-m}{V^2}\Delta V$$

$$\therefore \quad \left|\frac{\Delta d}{d}\right| = \left|-\frac{\Delta V}{V}\right| = \alpha_V\,\Delta T$$

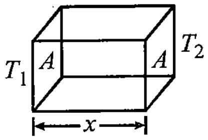
$$= 49\times10^{-5}\times30 = 0.14770 = 1.5\times10^{-2}$$

Correct option is (a)

Appendix-III

Thermal Conduction

If one end of a metal rod is placed in a store, the temperature of the rod gradually increases. Heat starts transfering from one end to the another end of the rod. This process of transfering of heat is called heat conduction and the molecular collisions are responsible for it. Some materials easily transmits the heat from end to the other but some takes a very large amount of time to do so. This ability of a material to conduct heat is measured by "thermal conductivity" of the material.



Consider a slab of uniform cross-section A and length x. Let one face of the slab be at temperature T_1 and the other at T_2. In order to avoid the transfer of heat from the sides, we have covered the remaining surfaces with a non-conducting material.

After sufficient time, steady state is reached and the temperature remains unchanged as the time passes. It is found that in steady state the heat current ($\Delta Q/\Delta t$) is proportional to the temperature difference ($T_1 - T_2$) between the ends, proportional to the area of cross-section and inversely proportional to the length x.

Mathematically, $\dfrac{\Delta Q}{\Delta t} = \dfrac{K\,A(T_1 - T_2)}{x}$

where K is a constant for the material of the slab and is called the *thermal conductivity* of the material.

If the area of the cross-section is not uniform or if the steady state condition is not reached the equation can only be applied to a thin layer of material perpendicular to heat flow. If dT be the temperature difference along the layer of thickness dx, the heat current is given by

$$\frac{dQ}{dt} = -KA\frac{dT}{dx}$$

where the quantity dT/dx is called the *temperature gradient*. The minus sign indicates that dT/dx is negative along the direction of the heat flow.

The unit of thermal conductivity is J $\mathrm{s^{-1}m^{-1}\,K^{-1}}$ or J $\mathrm{s^{-1}m^{-1}\,{}^\circ C^{-1}}$ or W $\mathrm{m^{-1}\,K^{-1}}$ or W $\mathrm{m^{-1}\,{}^\circ C^{-1}}$.

Problem : One face of a copper cube of edge 10 cm is maintained at 100 °C and the opposite face is maintained at 0°C. All other surfaces are covered with an insulating material. Find the amount of heat flowing per second through the cube. (Thermal conductivity of copper is 385 W $\mathrm{m^{-1}\,{}^\circ C^{-1}}$).

Soln. The amount of heat flowing per second is :

$$\frac{\Delta Q}{\Delta t} = \frac{KA(T_1 - T_2)}{x} = \frac{385 \times (10\text{ cm})^2 \times (100 - 0)\,{}^\circ\mathrm{C}}{(10\text{ cm})}$$

$$= 3850\ \dot{\mathrm{W}}\,.$$

Problem : Two bars, both if length ℓ and cross sectional area A, made of different materials with thermal conductivities K_1 and K_2 are joined together to form a composite bar of length 2ℓ. The free ends of the composite bar are kept at temperatures T_1 and T_2. The temperature at the junction of the two bars is:

(a) $\dfrac{\ell^2(T_1+T_2)\sqrt{K_1K_2}}{2A(K_1+K_2)}$

(b) $\dfrac{T_1T_2(K_1+K_2)}{(T_1+T_2)\sqrt{K_1K_2}}$

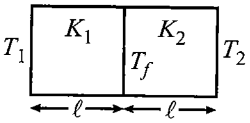
(c) $\dfrac{K_1T_1+K_2T_2}{K_1+K_2}$

(d) $\dfrac{T_1+T_2}{2}$

Soln. Let the final temperature of the junction be T_f. Then,

Since, ends are joined together to form a composite bar, the heat current is same in both the bars. So, we have

$$\left(\frac{\Delta Q}{\Delta t}\right)_1 = -\left(\frac{\Delta Q}{\Delta t}\right)_2$$ [The minus sign indicates the exchange of heat]



$$\Rightarrow \quad \frac{K_1A(T_f-T_1)}{\ell} = -\frac{K_2A(T_f-T_2)}{\ell}$$

$$\Rightarrow \quad K_1(T_f-T_1) = -K_2(T_f-T_2)$$

$$\Rightarrow \quad T_f = \frac{K_1T_1+K_2T_2}{K_1+K_2}$$

Correct option is (c).

Problem : If a metal rod is heated at one end and immersed in ice (at 0°C) at the other end, then the mass of ice that melts in a given time, *t*, is

(a) directly proportional to the length of the rod and inversely proportional to the area of cross-section.
(b) directly proportional to the area of cross-section and inversely proportional to thermal conductivity of the rod.
(c) directly proportional to both the length and thermal conductivity of the rod. **[H.C.U.-2012]**
(d) directly proportional to the thermal conductivity of the rod and inversely proportional to its length.

oln. The heat current is given by

$$\frac{\Delta Q}{\Delta t} = \frac{KA(T_1-T_2)}{x}$$

where $\Delta Q = mL_f$ (as ice is melting)

Since, $\dfrac{mL_f}{\Delta t} = \dfrac{KA(T_1-T_2)}{x}$

$$\Rightarrow \quad m = \frac{KA(T_1-T_2)}{x}\frac{\Delta t}{L_f}$$

Clearly, $m \propto K$ and $m \propto \dfrac{1}{x}$.

Correct option is (d).

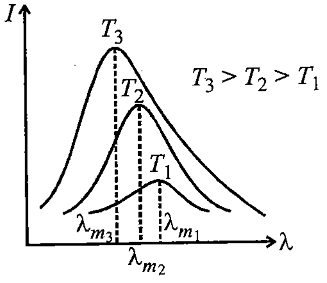
Appendix-IV

Blackbody Radiation

A body that absorbs all the radiation falling on it is called a *blackbody*. Such a body will emit radiation at the fastest rate. The radiation emitted by a blackbody is called *blackbody radiation*. A blackbody is also called an ideal radiator. Among the materials, lampblack is close to a blackbody. If reflects only about 1 % of the radiation falling on it.

Wiens's Displacement Law

The intensity of thermal radiation emitted by a body (not necessarily a blackbody) near a given wavelength is plotted against the wavelength for different temperatures. We see that as the temperature is increased, the wavelength corresponding to the highest intensity decreases. In fact, this wavelength λ_m is inversely proportional to the absolute temperature of the emitter. So, $\lambda m T = b$, where b is a constant known as Wien's constant and is equals to 2.8979×10^{-3} m·K for a blackbody.



Problem : The light from the Sun is found to have a maximum intensity near the wavelength of 470 nm. Assuming that the surface of the Sun emits as a blackbody, calculate the temperature of the surface of the Sun.

Soln. For a blackbody, $\lambda m T = 2.8979 \times 10^{-9}\ m \cdot K$

Thus, $T = \dfrac{2.8979 \times 10^{-3}\ m \cdot K}{470 \times 10^{-9}\ m} \simeq 6165\ K$

Problem : The intensity of radiation emitted by the Sun has its maximum value at a wavelength of 500 nm. The corresponding value for the value the star Polaris is 380 nm. Assuming that these stars behave like blackbodies, the ratio (T_S / T_P) of the surface temperature of the Sun Polaris is **[JNU 2011]**

(a) 1.70 (b) 1.32 (c) 0.76 (d) 0.58

Soln. Given $(\lambda m)_s = 500$ nm ; $(\lambda m)_p = 380$ nm

For a blackbody, $\lambda m T = 2.8979 \times 10^{-3}\ m \cdot K$

$$\Rightarrow \frac{T_s}{T_p} = \frac{(\lambda m)_p}{(\lambda m)_s} = \frac{380}{500} \simeq 0.76$$

Correct option is (c).

Stefan-Boltzmann Law

The energy of thermal radiation emitted per unit time by a body of surface area A and temperature T is given by

$$u = e\sigma A T^4$$

where σ is called the Stefan-Boltzmann's constant and is equals to 5.67×10^{-8} W m^{-2} K^{-4}. Also, the e is called *emissivity* of the surface and its value lies between 0 and 1. It is equal to unity for blackbodies.

Problem : A blackbody of surface area 10 cm^2 is heated to 127 °C and is suspended in a room at temperature 27 °C. Calculate the initial rate of loss of heat from the body to the room.

Soln. For a blackbody at temperature T, the rate of emission is $u = \sigma A T^4$

When it is kept in a room at temperature to, the rate of absorption is $u_0 = \sigma A T_0^4$.

Therefore, the net rate of loss of heat is $u - u_0 = \sigma A(T^4 - T_0^4)$.

Here, $A = 10 \times 10^{-4} m^2$, $T = 400 K$, $T_0 = 300 K$

Therefore, $u - u_0 = (5.67 \times 10^{-8})(10 \times 10^{-4})(400^4 - 300^4) = 0.99\, W$

Problem : The power per unit area emitted by a surface of a blackbody (in Wm^{-2}) at temperature 230 K is about

(a) 112.67 (b) 128.67 (c) 158.67 (d) 178.67 **[DU 2016]**

Soln. The power emitted by a blackbody is

$P = \sigma A T^4$; $\sigma = 5.67 \times 10^{-8}$ W m^{-2} K^{-4}

So, power per unit area is

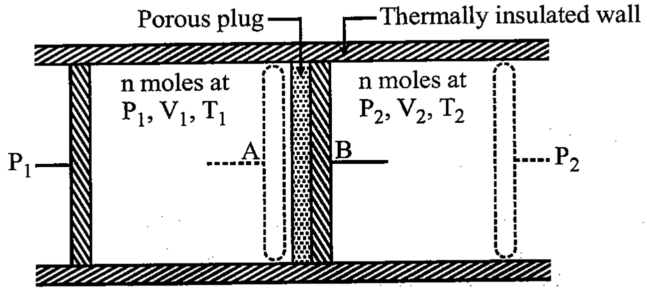
$$\frac{P}{A} = \sigma T^4 = 5.67 \times 10^{-8} \times 230^4 = 158.67 \text{ W m}^{-2}.$$

Correct option is (c).

Appendix-V

Joule-Thomson Experiment

In 1852, Lord Kelvin Thomson in collaboration with Joule devised an experiment, usually known as the *porous plug experiment.* In this experiment, a highly compressed gas is being continuously forced at constant pressure through a porous plug. The gas during its passage through the pores becomes throttled or wire drawn, viz., the molecules of the gas are drawn further apart from the one another. The gas thus suffers a volume expansion.



(*The dotted lines indicate the final position of the piston*)

The apparatus used by Joule and Thomson is shown in the above figure. A continuous flow of gas at a much higher pressure P_1 and higher temperature T_1 is forced through a porous plug in a tube, from which it emerges at a lower pressure P_2 and a lower temperature T_2, since the plug offers great resistance to the flow of the gas. The plug is surrounded by a non-conducting jacket so that the process is adiabatic. Therefore, this process is known as *Joule-Thomson Expansion* or *Adiabatic Throttling.*

Theory of Joule-Thomson experiment

Suppose that the pressure on the two sides of the porous plug are kept constant at values P_1 and P_2, where $P_2 < P_1$. Let us assume that before the process starts the gas on one side has internal energy U_1. After expanding through porous plug let the internal energy of the gas on other side be U_2. If the piston A (shown in figure) is slowly moved towards plug so as to keep the pressure constant at P_1 as the gas escapes, then the piston B will move slowly outward to keep the pressure on the other side constant at P_2. If x_1 and x_2 be the distances through which the piston A and B move respectively and 'a' the area cross-section of the piston, then

The force acting at piston $A = P_1$ a

So, work done by this force upon the gas is $P_1 a x_1 = P_1 V_1$.

Similarly, the work done by the gas in forcing the piston $B = P_2 a x_2 = P_2 V_2$.

Therefore, the net workdone by the gas is $W = P_2 V_2 - P_1 V_1$.

Since, the device is thermally insulated $dQ = 0$, and from the first law of thermodynamics, the work done by the system is equal to the decrease in its internal energy, $U_1 - U_2$,

i.e., $P_2 V_2 - P_1 V_1 = U_1 - U_2$ or $U_1 + P_1 V_1 = U_2 + P_2 V_2$ or $H_1 = H_2$.

Hence, the enthalpy change in the throttling process, i.e., $dH = 0$.

Thus, we conclude that the enthalpy is conserved. So, in the throttling process, enthalpy remains constant. This process is known as Joule Thomson or Joule Kelvin expansion which is '*quasistatic isenthalpic*' in nature.

Joule-Thomson Coefficient

Joule-Thomson coefficient (μ) is the ratio of changes in temperature to the change in pressure at constant enthalpy and calculated to be equal to

$$\mu = \left(\frac{\partial T}{\partial P}\right)_H = \frac{1}{C_p}\left[T\left(\frac{\partial V}{\partial T}\right)_P - V\right] \qquad \ldots (1)$$

For Ideal Gas, $\mu = 0$.

The Joule Thomson coefficient is zero means there is no change in temperature of a perfect gas, if it is expand adiabatically across the porous plug.

For van der Waals' gas,

$$\mu = \left(\frac{\partial T}{\partial P}\right)_H = \frac{1}{C_p}\left(\frac{2a}{RT} - b\right) \qquad \ldots (2)$$

As the pressure is lower on the emergent side of the porous plug therefore, the dP is necessarily a negative quantity.

Estimates of Joule-Thomson Cooling

It is clear from the equation (2) that for a given difference in pressure on two sides of plug,

1. If $T < \frac{2a}{Rb}$ then right side term would be positive, hence dT will be negative or the gas will be cooled down on passing through porous plug.
2. If $T > \frac{2a}{Rb}$, then right side term would be negative, hence dT will be positive or the gas will be heated up on passing through porous plug.
3. If $T = \frac{2a}{Rb}$, then right side term would be zero, hence dT will be zero or the gas will be neither cooled nor warmed on passing through porous plug, i.e., no change in temperature.

The temperature below which the gases shows cooling effect and above which shows heating effect is known as the *temperature of inversion*, and is written as

This shows that Joule-Thomson expansion will necessarily cool a gas if its initial temperature is below its inversion temperature.

Problem: Calculate the drop in temperature produced by adiabatic throttling process in the case of oxygen when the pressure is reduced by 40 atmospheres. Initial temperature of the gas obeys van der Waals' equation. The values of few constants are given as,

$$a = 1.3 \times 10^{12} \text{ cm}^4 \text{ dynes/mole}^2,\ b = 31.2 \text{ cm}^3\text{/mole},\ C_p = 7 \text{ cal/mole-K}$$

Soln. Using equation (2), $\left(\frac{\partial T}{\partial P}\right)_H = \frac{1}{C_p}\left[\frac{2a}{RT} - b\right]$ or $\partial T = \frac{\partial P}{C_p}\left[\frac{2a}{RT} - b\right]$

$$\partial P = 40 \text{ atmosphere} = 40 \times 76 \times 13.6980 \text{ dynes/cm}^2$$

$$C_p = 7\frac{\text{cal}}{\text{mole-K}} = \frac{7 \times 4.2 \times 10^7 \text{ ergs}}{\text{mole-K}};\ T = 30\,^\circ\text{C} = 303\text{ K};\ R = \frac{8.32 \times 10^7 \text{ ergs}}{\text{mole-K}}$$

$$\text{So, } \partial T = \left[\frac{40 \times 76 \times 13.6 \times 980}{7 \times 4.2 \times 10^7}\right]\left[\frac{2 \times 1.32 \times 10^{12}}{9.31 \times 10^7 \times 303} - 31.2\right]$$

$$= 10.16\,^\circ\text{C} = 10.16\text{K}.$$

MODERN PHYSICS

for

M.Sc. Entrance

IIT-JAM | JNU | TIFR | JEST | BHU | DU

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Chapter 1

Particle Properties of Wave

1.1: Blackbody radiation

A significant hint of the failure of classical physics arose from investigations of thermal radiation (Planck, 1900). According to Einstein (1905) electromagnetic radiation is quantized in photons.

- **Photons and Planck's quantum of action:**

Photons are the energy quanta of the electromagnetic field.

Photon energy (E_{Ph}) : It is proportional to the frequency f or the angular frequency $\omega = 2\pi f$. Usually it is given in electron volts (eV),

$$E_{Ph} = hf = \hbar\omega$$

Photon momentum ($\vec{P}_{Ph}$): It is proportional to the wave number vector k $\left(\text{with } |\vec{k}|\right) = \frac{2\pi}{\lambda}$, λ is the wavelength of the electromagnetic radiation),

$$\vec{P}_{Ph} = \hbar\vec{k}, \qquad |\vec{P}_{Ph}| = \hbar k = h/\lambda$$

The vector $\vec{k}$ points along the propagation direction of the electromagnetic radiation.

Planck's quantum of action: An universal constant,

$$h = 6.626\times10^{-34} \text{ Js},$$

$$\hbar = \frac{h}{2\pi} = 1.0545\times10^{-34} \text{ Js} = 6.5821\times10^{-22} \text{ MeV s}.$$

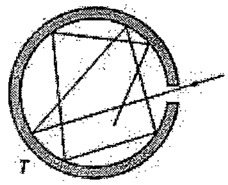
energy of photon			ML^2T^{-2}
	Symbol	Unit	Quantity
$E = \hbar\cdot\omega, \quad \omega = 2\pi f,$ $\hbar = \frac{h}{2\pi}$	E	J	energy
	ω	rad s^{-1}	angular frequency
	f	s^{-1}	frequency
	h	J s	quantum of action

momentum of photon			MLT^{-1}
	Symbol	Unit	Quantity
$\vec{p} = \hbar\cdot\vec{k}, \quad k = \frac{2\pi}{\lambda},$ $\hbar = \frac{h}{2\pi}$	$\vec{k}$	m^{-1}	wave-number vector
	h	J s	quantum of action
	$\vec{p}$	kg m/s	momentum vector

- **Thermal radiation and the blackbody radiator:**

Thermal radiation, temperature radiation, the electromagnetic radiation of a body at finite temperature. The body also absorbs a fraction of the thermal radiation from its environment. There is a permanent exchange of energy between the body and its environment.

In the end, this process leads to temperature equilibrium. Blackbody radiator, a body with the reflectance zero. A blackbody absorbs any incident radiation completely.



(Figure 2.1: Model of the blackbody)

Cavity radiator model of a blackbody radiator (Fig. 2.1): A box with a small aperture in the wall. The wall is impenetrable for radiation from inside (ideally reflecting) and has a definite temperature. The probability that a photon enters the cavity through the aperture and, after multiple reflection by the inner walls, leaves the cavity through the aperture again, is negligible (absorptance $\alpha = 1$). The aperture appears absolutely black.

Cavity radiation, the thermal radiation leaving the aperture of a cavity radiator. The spectral distribution of the radiation energy density of the cavity radiation depends on the temperature of the cavity radiator.

According to Kirchhoff's law (The absorptance is equal to the emittance), the spectral radiance $L_{e,f}$ of an arbitrary thermal radiator may be reduced to that of a black body.

For the radiation field in the interior of the cavity, one defines

radiant energy density			$\mathrm{ML^{-1}T^{-2}}$
	Symbol	Unit	Quantity
$u = \dfrac{Q}{V}$	u Q V	$\mathrm{J/m^3}$ J $\mathrm{m^3}$	radiant energy density radiant energy volume

- **Planck's radiation law:**

This law describes the frequency and temperature dependence of the radiant energy density of the cavity radiation:

spectral radiant energy density			$\mathrm{ML^{-1}T^{-1}}$
	Symbol	Unit	Quantity
$u_f(f,T) = \dfrac{8\pi f^2}{c^3} \cdot \dfrac{hf}{e^{hf/(kT)} - 1}$	$u_f(f,T)$ c f h k T	$\mathrm{J\,s\,m^{-3}}$ $\mathrm{m\,s^{-1}}$ $\mathrm{s^{-1}}$ J s $\mathrm{J\,K^{-1}}$ K	spectral radiant energy density speed of light frequency quantum of action Boltzmann constant temperature

- **Connection between radiant energy density and frequency:**

The dependence of the spectral radiant energy density of the cavity radiation on the angular frequency ω or wavelength λ reads as follows:

$$u_\omega(\omega,T) = u_f(f,T)\frac{df}{d\omega} = \frac{1}{2\pi}u_f(f,T),$$

$$u_\omega(\omega,T)=\frac{\hbar\omega^3}{\pi^2c^3}\frac{1}{e^{\hbar\omega/(kT)}-1},\quad u_\lambda(\lambda,T)=u_f(f,T)\left|\frac{df}{d\lambda}\right|=\frac{f^2}{c}u_f(f,T)$$

$$u_\lambda(\lambda,T)=\frac{8\pi hc}{\lambda^5}\frac{1}{e^{hc/(k\lambda T)}-1}$$

- **Limiting cases of Planck's formula:**

Wien's law: for $hf >> kT$

$$u_f(f,T)=\frac{8\pi f^3h}{c^3}e^{-\frac{hf}{kT}}$$

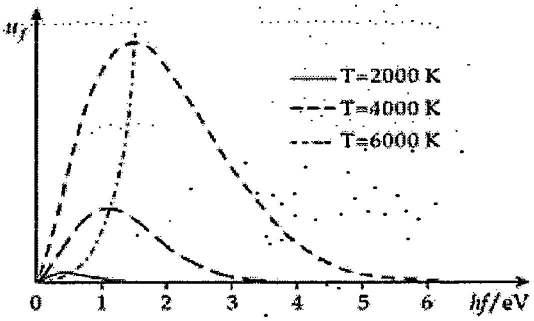
Rayleigh-Jeans law: for $hf << kT$

$$u_f(f,T)=\frac{8\pi f^2}{c^3}kT$$

- **Wien's displacement law:**

With increasing temperature, the maximum of the spectral radiant energy density $u_f(f,T)$ is shifted to higher photon energy, i.e., to higher frequencies (shorter wavelengths) (Fig. 2.2)

Wien's displacement law			L
	Symbol	Unit	Quantity
$\lambda_{\max}=\frac{b}{T}$	$\lambda_{\max}$	m	wavelength at max. $u_f(f,T)$
$b=2.8978\cdot10^{-3}$ m · K	b	m · K	Wien's constant
	T	K	temperature



(Figure 2.2: Radiant energy density $u_f(f;T)$ for various temperatures according to Planck's radiation law. Dashed-dotted line: Rayleigh-Jeans law.)

- **Stefan-Boltzmann law:**

Integration of the spectral radiant energy density over all frequencies yields the total radiant flux φ_{tot} of a radiation emitted by an area A. The total radiant flux φ_{tot} is proportional to T^4.

total radiant flux ~ temperature^4			ML^2T^{-3}
	Symbol	Unit	Quantity
$\Phi_{\text{tot}}=\sigma\cdot A\cdot T^4$	Φ_{tot}	W	total radiant flux
$\sigma=5.67051(19)$	A	m^2	area
$\cdot10^{-8}$ W/(m^{2}K^4)	σ	W/(m^{2}K^4)	Stefan-Boltzmann constant
	T	K	temperature

1.2. Photoelectric effect

• Lenard's experiments

To study the phenomenon, Lenard set up an apparatus similar to that illustrated in Figure given below. It consists of an evacuated tube T having two plane metal plates C and D, C is coated with alkali metal and kept at a negative potential by means of a battery B. A is another metal placed parallel to C inside the tube and is kept at zero potential. A is thus positive relative to C. At the centre of A there is a small hole through which the beam of negative ions emitted from C can pass. There is a side tube covered by a quartz window G through which UV-radiation is incident on the plate C, resulting in the production of negative ions from C photoelectrically. The negative ions are attracted to A and a part of it, passing through the hole in A, is finally collected at the collector D. The feeble electric current can be detected with an electrometer (not shown).

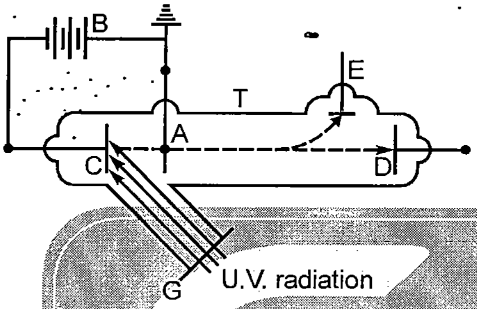


Fig. Lenard's experiments on photoelectricity

The main tube is placed between the pole pieces of a magnet such that the field is perpendicular to the plane of the diagram. The field bends the path of the negative ions emanating from the hole into circular arcs so that the ions are now collected by the collector E, instead of D. Another electrometer (not shown) records the correspondings current.

Let V be the p.d. between C and A. Then the kinetic energy of the negative ions is

$$\frac{1}{2}mv^2 = eV \qquad \text{...(i)}$$

If R be the radius of curvature of the ion track, known from the geometry of the apparatus, and B the magnetic field

$$Bev = \frac{mv^2}{R} \qquad \text{...(ii)}$$

From the relations (i) and (ii),

$$\frac{e}{m} = \frac{2V}{B^2R^2} \qquad \text{...(iii)}$$

Lenard's measurements of e/m of the negative ions established uniquely the identify of the ions to be electrons. Lenard next observed that if a positive potential is applied to C, the electric current recorded decreases with the increase of positive potential; a stage comes when the current becomes zero. This is because electrons being negatively charged experience a repulsive force in going towards A. If V_s be the potential difference between C and A when the current is zero, V_s is called the stopping potential – because electrons of all velocities up to the maximum v_m emitted from C are stopped from reaching A. Under this condition

$$\frac{1}{2}mv_m^2 = eV_s$$

V_s is usually of the order of few volts.

Even if the intensity of the incident radiation is now increased, it will cause no electron to reach A. However, on making C negative w.r.t. A, an increase in the intensity of illumination (the frequency and stopping potential remaining, the same) causes the current to increase.

If the intensity is kept constant, but the frequency of the incident radiation is varied, it is observed that higher the frequency of radiation, greater is the stopping potential and also the energy of the electrons.

Another very important observation that Lenard made during his experiment is that the photoelectric current is independent of the frequency of light. It depends, as already stated, only on the intensity of the light used.

Subsquently, after Lenard, many other workers – notably O.W. Richardson, K.T. Compton – confirmed the observations of Lenard.

Note 1: To explain the findings of Lenard on photoelectricity. Einstein in 1905 proposed his celebrated photoelectric equation based on the quantum ideas of Max Planck.

Note 2: The most outstanding series of experiments on photoelectricity was however conducted in 1916 by R.A. Millikan, which not only re-confirmed Lenard's observations but also verified beyond doubt the validity of Einstein's equation.

• Failure of the electromagnetic wave theory

The above experimental facts cannot be explained on the basis of the electromagnetic wave theory of light.

According to the electromagnetic theory, light radiations consist of electric and magnetic fields and they may exert forces to liberate electrons from the metal surface. On this basis, the light of higher intensity, consisting of stronger fields, should give higher velocity (or energy) to the emitted photoelectrons. So, according to the wave theory, the energy of the emitted electron should depend on the intensity of incident radiation which, however, is contrary to experimental observation.

Also, according to the electromagnetic theory, the velocity (or kinetic energy) of electrons should not depend on the frequency of radiation for whatever be the frequency, the electron would be emitted if it gets sufficient time to gather the necessary energy for emission. Further, if the intensity of the radiation be very weak, classically a longer time should elapse before the electrons gain sufficient energy to leave the emitting surface. Both the conclusions are in disagreement with observed facts – the velocity of photoelectron is frequency dependent and that there is no observable between the incidence of radiation and the emission of photoelectron.

Also, according to electromagnetic wave theory, given sufficient time, all electrons should be able to collect the energy required for emission. There is no reason therefore why the photoelectric current should be intensity dependent, as is observed.

The wave theory thus completely fails to explain the characteristics of the photoelectric emission.

• Einstein's light quantum hypothesis: photoelectric equation

Max Planck, in 1901, put forward a radically new hypothesis to explain the black body radiation that emission and absorption of radiation is not a continuous process but occur discretely as an integral multiple of a basic unit, called the quantum energy. Each quantum carries a definite amount of localised energy E proportional to the frequency ν of the radiation.

Therefore, $E = h\nu$

where h is a universal constant known as Planck's constant of action. Emission and absorption of energy take place in discrete steps of $nh\nu$, $n = 1, 2, 3, \ldots$

Einstein's light quantum hypothesis—Einstein (1905) proposed the correct explanation of the photoelectric phenomenon by an extension of the quantum idea of Planck. This extension is known as Einstein's light quantum hypothesis.

Einstein proposed the following postulates.

1. Radiation is not only emitted and absorbed in discrete quanta, called photons, having energy $E = h\nu$, but also travels in quanta with the speed of light. Each photon is a bundle of localised energy in space.
2. The energy $h\nu$ of a photon is so localised or concentrated that it can transfer the whole of its energy content to one electron (which thereby gains the quantum of energy) and that no fractional measures are allowed in the process.

In explaining the photoelectric effect, Einstein also postulated that electrons do not normally leave a metal surface of their own because they are bound in the metal. A certain minimum energy, called the work function. ϕ, is required to release an electron from the metal surface.

• Photoelectric equation

When photons fall on a metallic surface, they can either (i) be reflected according to the laws of optics or else, (ii) be absorbed completely by an electron in being knocked out from the surface. In the latter case, the emission of electron is possible only when $h\nu > \phi$, and the balance of energy $(h\nu - \phi)$ is taken up by the emitted electron as its kinetic energy in emission. Those electrons which are ejected from some depth below the surface expend some of their acquired energy in collisions with atoms. The energy lost in this way is a variable quantity so that photoelectrons are emitted with a range of kinetic energies (velocities). Thus the maximum kinetic energy of the emitted photoelectrons is given by.

$$\frac{1}{2}mv_m^2 = h\nu - \phi \qquad ...(i)$$

When the full energy of the photon is used up in releasing the electron from the metal and nothing is left as the kinetic energy of the electron,

$$h\nu - \phi = 0$$

Therefore,
$$h\nu_0 = \phi \qquad ...(ii)$$

where ν_0 is the corresponding incident frequency and is called the threshold frequency.

Therefore, from (i),
$$\frac{1}{2}mv_m^2 = h\nu - h\nu_0 = h(\nu - \nu_0) \qquad ...(iii)$$

Equation (i) and equation (ii) are the two alternative expressions of the well known photoelectric equation of Einstein.

The equations (i) to (iii) above clearly demonstrate the following features.

(i) The maximum kinetic energy of the emitted photoelectrons depends only on the frequency and not on the intensity of the incident light, and maximum kinetic energy increases linearly with the incident frequency.

(ii) If $\nu < \nu_0$, the kinetic energy of the photoelectrons would be negative which is impossible. This means that there is a threshold frequency ν_0, depending on the work function ϕ of the particular metal, below which no photoelectric emission occurs.

(iii) The increase in intensity of incident radiation of a given frequency $\nu (\geq \nu_0)$ is equivalent to an increase in the number of photons striking the emitting surface per second, but the energy $h\nu$ of each photon will remain the same. Hence the number of electrons emitted per second, that is, photocurrent will correspondingly increase, but the maximum energy will remain the same.

(iv) However low the intensity may be, whenever a photon with $\nu \geq \nu_0$ falls on the surface of the emitter, emission occurs by the immediate absorption of photon energy by some electron that escapes instantaneously with no observable time lag.

Thus Einstein's photoelectric equation can explain all the photoelectric laws obtained experimentally.

Note: Materials in solid, liquid or gasesous state, other than metal surface, emit photoelectrons if irradiated with e.m. radiation of different frequencies from γ-rays and x-rays to ultraviolet and visible radiation.

• Verification of Einstein's equation: Millikan's experiments

Millikan, in 1914-16, carried out a set of experiments which first verified quantitatively the Einstein's photoelectric equation. These investigations also made the first direct (and accurate) photoelectric determination of Planck's constant h and also the value of work function ϕ.

Apparatus — The principal parts of the apparatus used by Millikan are illustrated in Figure below. B is a small aperture through which light in a narrow frequency range enters the highly evacuated tube T via the quartz window D and falls on the rotation of a wheel W without disturbing the vacuum. By means of the knife N and the rotating wheel W, any one of the metal surfaces could be shaved off so that a clean surface always faces the radiation.

The emitted electrons are collected by C, a collector grid (electrode) very near to D. It is made of oxidised copper wire so that electrons are not emitted from its surface when incident radiation passes through it. An electrometer E connected to the collector measures the photocurrent. Applying a positive potential to the wheel W via F, a retarding field (that can be varied by varying the potential) could be set up between W and C to check the less energetic reaching the collector.

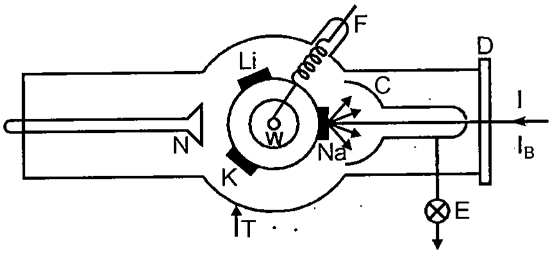


Fig. Millikan's experiments on verification of Einstein's equation

Experiments—Using monochromatic incident radiations of different wavelengths, Millikan made a careful study of the photoelectric current i for the various retarding potentials V for each of the alkali metals, and the behaviour of current as a function of potentials, keeping the wavelength of radiation fixed. The results of such measurements are shown by the curves in Figure. The different curves are representative of different intensities I_1, I_2, I_3 $(I_1 > I_2 > I_3)$. When the potential of C is positive relative to W, the current is almost constant (saturated).

Saturation current—for a given intensity. The saturation current is reached when all the electrons emitted from the metal surface reached the collector C. Interestingly, there is some photoelectric current even when $V = 0$. This shows that some of the electrons must have been emitted with a finite initial velocity.

If the potential of C is made negative relative to W, the current decreases and goes to zero at a particular potential $(-V_s)$ which is the same for all intensities. The negative potential V_s is known as the stopping potential and is connected to the maximum kinetic energy by

$$eV_s = \frac{1}{2}mv_m^2 = h(\nu - \nu_0) \qquad \text{...(i)}$$

$$\text{or } \; V_s = h(\nu - \nu_0)/e \qquad \text{...(ii)}$$

where e is the electronic charge.

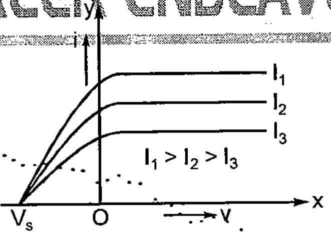


Fig. Photoelectric current vs potential curve at different intensities (same frequency) of radiation.

Millikan thus measured the maximum kinetic energy of the electrons emitted by a given frequency (or wavelength). When the intensity of the incident radiation is increased, the saturation current increases (that is, more electrons are ejected), but none of the electrons is more energetic, since the stopping potential remains unchanged. The stopping potential is independent of the intensity of the incident radiation.

Millikan next measured the variation of photocurrent with potential difference between C and W using incident light of different frequencies (or wavelengths), but of the same intensity. The saturation current is the same indicating that number of electrons emitted in each case is the same. But the striking point is that the stopping potential has larger negative value for higher frequency. This is illustrated in Figure where the stopping potential is higher (more negative) as the incident frequency increases $(\nu_3 > \nu_2 > \nu_1)$. But the saturation current is independent of the frequency, being dependent only on the intensity.

Millikan next measured the variation of photocurrent with potential difference between C and W using incident light of different frequencies (or wavelengths), but of the same intensity. The saturation current is the same indicating that number of electrons emitted in each case is the same. But the striking point is that the stopping potential has larger negative value for highest frequency. This is illustrated in figure where the stopping potential is (more negative) as the incident frequency increases $(\nu_3 > \nu_2 > \nu_1)$. But the saturation current is independent of the frequency, being dependent only on the intensity.

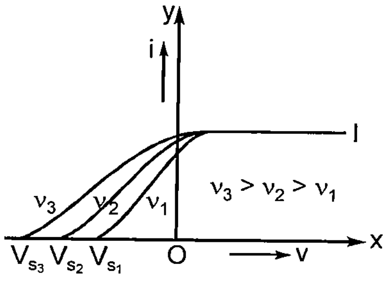


Fig. Photo current vs potential curve at different frequencies (same intensity) of radiation

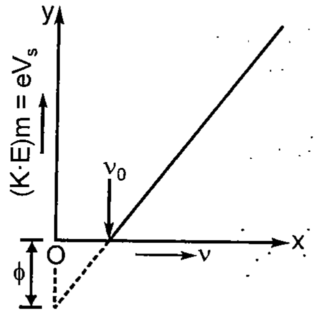


Fig. Graph of $(K \cdot E)_{max}$ as a function of ν.

Millikan repeated the experiments by measuring V_s for a range of frequencies and for all the three surfaces. A graph of $(K \cdot E.)_{max}$ as a function of ν was obtained figure by Millikan. The graph was a straight line showing that the maximum kinetic energy of the photoelectrons increases linearly with ν. The straight line intercepted the ν-axis at some frequency ν_0. This is the threshold frequency. The equation of the straight line is

$$\frac{1}{2} m v_m^2 = \text{constant} \times (\nu - \nu_0) \qquad \text{...(iii)}$$

Whatever be the metal the slope of the line was always the same, although ν_0 differed from one metal to another. These experimental results were thus is agreement with Einstein's equation.

Note 1: While measuring the potential V_s an error arises due to contact potential difference which was corrected by Millikan by introducing an additional electrode in his apparatus.

Note 2: It is to be noted that the Einstein's theory implies that the emission of one electron occurs by the absorption of one photon. But it does not follow that every incident photon causes emission of an electron even if the photon frequency $\nu \geq \nu_0$, the threshold frequency. This is because the photon might be involved in many other alterniative processes to be discussed later. Thus the ratio of the number of emitted electrons and that of the incident photons is much less than unity. This ratio is called the quantum yield and is of importance in the design of photoelectric devices.

SOLVED PROBLEMS

1. Find the wavelength and frequency of a 1.0 keV photon.

Soln. Wavelength of the photon $\lambda = \frac{hc}{E} = \frac{12.4 \text{ keV}}{1.0 \text{ keV}} = 12.4 \text{ Å}$

Frequency of the photon $\nu = \frac{c}{\lambda} = \frac{3\times10^{8} \text{ m/s}}{12.4\times10^{-10} \text{ m}} = 2.42\times10^{14}$ Hz

2. Find the momentum of a 12.0 MeV photon

Soln. Momentum of the photon $p = \frac{E}{c} = 12 \text{ MeV}/c$

3. Monochromatic light of wavelength 3000Å is incident normally on a surface of area 4 cm^2. If the intensity of the light is 15×10^{-2} W/m^2, determine the rate at which photons strike the surface.

Soln. The energy per photon is

$$E = \frac{hc}{\lambda} = \frac{\left(6.63\times10^{-34} \text{ Js}\right)\left(3\times10^{8} \text{ m/s}\right)}{3\times10^{-7} \text{ m}} = 6.63\times10^{-19} \text{ J/s}$$

The total energy flux is:

$$IA = \left(15\times10^{-2} \text{ W/m}^2\right)\left(4\times10^{-4} \text{ m}^2\right) = 6\times10^{-5} \text{ W} = 6\times10^{-5} \text{ J/s}$$

Hence, the rate at which photons strike the surface is:

$$\frac{6\times10^{-5} \text{ J/s}}{6.63\times10^{-19} \text{ J/photon}} = 9.05\times10^{13} \text{ Photon/s}$$

4. The kinetic energies of photoelectrons range from zero at 4.0×10^{-19} J when light of wavelength 3000Å falls on a surface. What is the stopping potential for this light?

Soln. Maximum kinetic energy of the photoelectrons $K_{\text{max}} = 4.0\times10^{-19} \text{ J}\times\frac{1 \text{ eV}}{1.6\times10^{-19} \text{ J}} = 2.5 \text{ eV}$

Then, from $eV_s = K_{\text{max}}$, $V_s = 2.5$ V

5. What is the threshold wavelength for above problem?

Soln. From the photoelectric equation

$$eV_s = h\nu - eW_0 = \frac{hc}{\lambda} - \frac{hc}{\lambda_{\text{th}}} \quad \text{or} \quad 2.5 \text{ eV} = \frac{12.4\times10^{3} \text{ eV.Å}}{3000 \text{ Å}} - \frac{12.4\times10^{3} \text{ eV Å}}{\lambda_{\text{th}}}$$

Solving, $\lambda_{th} = 7590$Å

6. The emitter in a photoelectric tube has a threshold wavelength of 6000Å. Determine the wavelength of the light incident on the tube if the stopping potential for this light is 2.5 V.

Soln. The work function is $W_n = h\nu_{\text{th}} = \frac{hc}{\lambda_{\text{th}}} = \frac{12.4\times10^{3} \text{ eV Å}}{6000 \text{ Å}} = 2.07 \text{ eV}$

The photoelectric equation then gives

$$eV_s = h\nu - eW_0 = \frac{hc}{\lambda} - eW_0 \quad \text{or} \quad 2.5 \text{ eV} = \frac{12.4\times10^{3} \text{ eV Å}}{\lambda} - 2.07 \text{ eV}$$

Solving, $\lambda = 2713$ Å

7. The work function of a particular emitter is 2.0 eV and light of wavelength 3000 Å is used to cause emission. Find the stopping potential and the velocity of most energetic electrons.

Soln. Light wavelength, $\lambda = 3000\ \text{Å} = 3000 \times 10^{-10}\,\text{m}$.

$$\therefore\ h\nu = \frac{hc}{\lambda} = \frac{6.62 \times 10^{-34} \times 3 \times 10^{8}}{3000 \times 10^{-10}}\,\text{J} = 6.62 \times 10^{-19}\,\text{J} = \frac{6.62 \times 10^{-19}}{1.6 \times 10^{-19}}\,\text{eV} = 4.14\,\text{eV}$$

Now, from the relation $V_s e = h\nu - \phi$, we get the stopping potential, $V_s = 4.14 - 2.0 = 2.14\,\text{V}$.

Again, the kinetic energy of the electron, $\frac{1}{2}mv^2 = V_s e$

$$\therefore\ v = \sqrt{\frac{2V_s e}{m}} = \sqrt{2 \times 2.14 \times 1.76 \times 10^{11}} \qquad (\because\ e/m = 1.76 \times 10^{11}\,\text{C/kg})$$

$$= 8.68 \times 10^{5}\,\text{m/s}.$$

8. The wavelength of the photoelectric threshold of tungsten is 2300 Å. Determine the kinetic energy of electrons ejected from the surface by ultraviolet light of wavelegth 1800 Å.

Soln. The work function $\phi = h\nu_0 = \dfrac{hc}{\lambda_0}$

Therefore, $\phi = \dfrac{6.62 \times 10^{-34} \times 3 \times 10^{8}}{2300 \times 10^{-10}} = 0.86 \times 10^{-18}\,\text{J}$

The energy of the incident radiation is $h\nu = \dfrac{hc}{\lambda} = \dfrac{6.62 \times 10^{-34} \times 3 \times 10^{8}}{1800 \times 10^{-10}}\,\text{J} = 1.10 \times 10^{-18}\,\text{J}$

Therefore, kinetic energy of the ejected electron E is given by

$$E = h\nu - \phi = (1.10 - 0.86) \times 10^{-18}\,\text{J} = \frac{0.24 \times 10^{-18}}{1.6 \times 10^{-19}}\,\text{eV} = 1.5\,\text{eV}$$

9. The work function of potassium is 1.9 V. If radiation of wavelength 4.5×10^{-7} m be incident on it, what would be (i) the threshold wavelength, (ii) the maximum kinetic energy of ejected electrons and (iii) the minimum retarding potential ?

Soln. By the problem, $h\nu_0 = \dfrac{hc}{\lambda_0} = 1.9\,\text{eV}$

Therefore, $\lambda_0 = \dfrac{hc}{1.9\,\text{eV}} = \dfrac{6.62 \times 10^{-34} \times 3 \times 10^{8}}{1.9 \times 1.6 \times 10^{-19}}\,\text{m} = 6.53 \times 10^{-7}\,\text{m}$

Now, the maximum K·E., $E_m = h\nu - \phi = \dfrac{hc}{\lambda} - \phi$

Therefore, $E_m = \left\{\dfrac{6.62 \times 10^{-34} \times 3 \times 10^{8}}{4.5 \times 10^{-7}} - 1.9 \times 1.6 \times 10^{-19}\right\}\text{J} = (4.41 - 3.04) \times 10^{-19}\,\text{J} = 1.37 \times 10^{-19}\,\text{J}$

Minimum retarding potential V is given by $eV = E_m$

Therefore, $V = \dfrac{E_m}{e} = \dfrac{1.37 \times 10^{-19}}{1.6 \times 10^{-19}} = 0.86$ volt.

10. A ray of ultraviolet light of wavelength 3000 Å falling on the surface of a material whose work function is 2.28 eV, ejects an electron. What will be the velocity of the emitted electron ?

Soln. The work function, $\phi = 2.28\,\text{eV}$

$$= 2.28 \times 1.6 \times 10^{-19}\,\text{J}$$

$$= 3.648 \times 10^{-19}\,\text{J}$$

Wavelength of light, $\lambda = 3000\,\text{Å} = 3000 \times 10^{-10}\,\text{m}$

$\therefore$ Frequency, $\nu = \dfrac{c}{\lambda} = \dfrac{3 \times 10^{8}}{3000 \times 10^{-10}} = 10^{15}\,Hz$

Now, kinetic energy of emitted electron is given by $\dfrac{1}{2}mv^2 = h\nu - \phi$

$$= 6.62 \times 10^{-34} \times 10^{15} - 3.648 \times 10^{-19}$$

$$= 2.972 \times 10^{-19}\,J$$

$$\therefore \quad v = \sqrt{\frac{2 \times 2.972 \times 10^{-19}}{9 \times 10^{-31}}} = 8.13 \times 10^{5}\,m/s.$$

11. Calculate the value of photoelectric work function for solution metal in electron volt, given that the threshold wavelength is 6800 Å and $h = 6.62 \times 10^{-34}\,Js.$

Soln. Threshold wavelength $\lambda_0 = 6800\,\text{Å} = 6800 \times 10^{-10}\,m$

Threshold frequency $\nu_0 = \dfrac{c}{\lambda_0} = \dfrac{3 \times 10^{8}}{6800 \times 10^{-10}}\,Hz$

Photoelectric work function of sodium

$$\phi = h\nu_0 = \frac{6.62 \times 10^{-34} \times 3 \times 10^{8}}{6800 \times 10^{-10}}\,J = \frac{6.62 \times 10^{-34} \times 3 \times 10^{8}}{6800 \times 10^{-10} \times 1.6 \times 10^{-19}}\,eV = 1.82\,eV.$$

12. The work function of a metal is 3.31×10^{-12} erg. Calculate the maximum wavelength of radiation which would start the emission of photoelectrons ($h = 6.62 \times 10^{-27}$ erg. sec, $c = 3 \times 10^{10}$ cm/s).

Soln. Let ν_0 be the threshold frequency of radiation.

Therefore, work function, $\phi = h\nu_0$

Therefore, $\nu_0 = \dfrac{\phi}{h} = \dfrac{3.31 \times 10^{-12}}{6.62 \times 10^{-27}} = 5.0 \times 10^{14}\,s^{-1}$

Therefore, threshold wavelength, $\lambda_0 = \dfrac{c}{\nu_0} = \dfrac{3 \times 10^{10}}{5 \times 10^{14}} = 6 \times 10^{-5}\,cm = 6000\,Å.$

13. The maximum energy of photoelectrons emitted by a metal surface is $3.62 \times 10^{-19}\,J$ when the incident radiation is 3000 Å. When the incident radiation is 5000 Å, the maximum energy of ejected electrons is 0.972×10^{-19} J. Calculate the Planck's constant and the threshold wavelength of metal.

Soln. From the photoelectric equation $(K.E.)_{\max} = h\nu - \phi$ ...(i)

Here, $\nu_1 = \dfrac{c}{\lambda_1} = \dfrac{3 \times 10^{8}}{3000 \times 10^{-10}} = 1.0 \times 10^{15}\,Hz$; Also, $\nu_2 = \dfrac{c}{\lambda_2} = \dfrac{3 \times 10^{8}}{5000 \times 10^{-10}} = 0.6 \times 10^{15}\,Hz$

From equation (i), we obtain

$$\left.\begin{aligned} 3.62\times10^{-19} &= h(1.0\times10^{15})-\phi \\ \text{and } 0.972\times10^{-19} &= h(0.6\times10^{15})-\phi \end{aligned}\right\}$$

Solving these two simultaneous equations $h = 6.62\times10^{-34}\,Js$ and $\phi = 3.0\times10^{-19}\,J$.

Threshold wavelength, $\lambda_0 = \dfrac{hc}{\phi} = \dfrac{6.62\times10^{-34}\times3\times10^{8}}{3.0\times10^{-19}} = 6.62\times10^{-7}\,m = 6620\,\text{Å}$.

14. The work functions of tungsten and barium are 4.52 and 2.5 eV respectively. Calculate the wavelength of light which can just eject electrons from tungsten and from barium. Which metal would you select for a photocell for use with visible light?

Soln. Since the work function $\phi = h\nu_0$ where ν_0 is the threshold frequency, $\nu_0 = \phi/h$.

$\therefore$ Threshold wavelength, $\lambda_0 = \dfrac{c}{\nu_0} = \dfrac{ch}{\phi}$ (c = velocity of light)

For tungsten, $\phi = 4.52\,eV = 4.52\times1.6\times10^{-19}\,J$

Therefore, $\lambda_0 = \dfrac{3\times10^{8}\times6.62\times10^{-34}}{4.52\times1.6\times10^{-19}} = 2.746\times10^{-7}\,m = 2746\,\text{Å}$

For barium, $\lambda_0 = \dfrac{4.52}{2.5}\times2746\,\text{Å} = 4965\,\text{Å}$

Plainly, barium would be the metal of choice for a photocell for use with visible light.

15. The work function of aluminium is 4.2 eV. Find the kinetic energy of the fastest and the slowest photoelectrons, the stopping potential and the cut-off wavelength when radiation of 2000 Å falls on an aluminium surface.

Soln. From Einstein's photoelectric equation $(K.E.)_{\max} = h\nu - \phi$

Now, $h\nu = h\dfrac{c}{\lambda} = \dfrac{6.62\times10^{-34}\times3\times10^{8}}{2000\times10^{-10}} = 9.93\times10^{-19}\,J = \dfrac{9.93\times10^{-19}}{1.6\times10^{-19}}\,eV = 6.2\,eV$

$\therefore$ $(K.E.)_{\max} = (6.2-4.2)\,eV = 2\,eV$. $\therefore$ Stopping potential = 2 volt

Plainly, $(K.E.)_{\max} = 0$ (zero)

Cut-off frequency (threshold), $\nu_0 = \dfrac{\phi}{h} = \dfrac{4.2\times1.6\times10^{-19}\,J}{6.62\times10^{-34}\,Js} = 1.01\times10^{15}\,Hz$

$\therefore$ Cut-off wavelength, $\lambda_0 = \dfrac{c}{\nu_0} = \dfrac{3\times10^{8}}{1.01\times10^{15}} = 2.97\times10^{-7}\,m = 2970\,A$.

16. Calculate the fractional change in the light intensity of a beam of intensity 10^{-7} Js^{-1}, if one photon is added per second to the beam, the wavelength of light being 6000 A.

Soln. The energy of a photon, $E = h\nu = h\dfrac{c}{\lambda} = \dfrac{6.6\times10^{-34}\times3\times10^{8}}{6000\times10^{-10}} = 3.3\times10^{-19}\,J$

Therefore, the number of photons per sec in the beam is $N = \dfrac{10^{-7}}{3.3\times10^{-19}} = 0.30\times10^{12}$

Therefore, fractional change in the number of photons or the fractional change in intensity on addition of 1 photon per second is

$$\frac{1}{N} = \frac{1}{0.30\times10^{12}} = 3.3\times10^{-12}.$$

17. How many photons of $\lambda = 5\times10^{-7}\,m$ must fall per second on a blackened plate to produce a force of 1 dyne? If the mass of the plate is 1 g and its specific heat $0.42\times10^{3}\,\text{Jkg}^{-1}$, at what rate would its temperature rise ?

Soln. Momentum of a photon, $p = h/\lambda$. Let n be the number of photons falling per second.

Therefore, total momentum change per sec of incident photons $= n(h/\lambda)$.

Since all the photons are completely absorbed by the plate, the change in momentum per sec $= n(h/\lambda)$ = force produced on the plate.

$$\therefore\ n(h/\lambda) = \frac{1}{10^5} = 10^{-5} \qquad (\because\ 1\ \text{Newton} = 10^5\ \text{dyne})$$

$$\Rightarrow\ n = \lambda\times10^{-5}/h = \frac{5\times10^{-7}\times10^{-5}}{6.62\times10^{-34}} = 7.55\times10^{21}$$

Energy absorbed by the plate per sec, $Q = nh\nu = \dfrac{nh\nu}{\lambda}$.

If ΔT be the rise in temperature per sec, then Q = mass × specific heat × $\Delta T = \text{ms}\Delta T$

$$\Rightarrow\ \Delta T = \frac{Q}{ms} = \frac{nhc}{\lambda ms}$$

But, we have just seen that $\dfrac{nh}{\lambda} = 10^{-5}$ $\therefore\ \Delta T = 10^{-5}c/(10^{-3}\times0.42\times10^{3})$

$$\text{or,}\quad \Delta T = \frac{10^{-5}\times3\times10^{8}}{10^{-3}\times0.42\times10^{3}} = \frac{3\times10^{3}}{0.42} = 7.14\times10^{3}\,K.$$

18. The photon energy for light of wavelength 3.1×10^{-7} m is $\left[hc = 1.24\times10^{-6}\ \text{eV m}\right]$ **[BHU 2010]**

(a) 3.1 eV (b) 6.2 eV (c) 2.4 eV (d) 4.0 eV

Soln. The photon energy is

$$E = \frac{hc}{\lambda} = \frac{1.24\times10^{-6}\ \text{eVm}}{3.1\times10^{-7}} = 4\ \text{eV}.$$

Correct option is (d)

19. The photoelectric threshold for tungsten is 2300 Å. The energy of the electrons emitted from the surface by the ultraviolet light of wavelength 1800 Å incident on the tungsten $\left(h = 6.63\times10^{-34}\ \text{J-sec}\right)$ is **[BHU 2011]**

(a) 2.39×10^{-12} joule (b) 2.39×10^{-19} joule (c) 1.48×10^{-19} joule (d) 3.56×10^{-19} joule

Soln. Kinetic energy of the emitted electrons will be

$$E_k = \frac{hc}{\lambda} - \frac{hc}{\lambda_{th}} = \frac{1240\,eV-nm}{180\,nm} - \frac{1240\,eV-nm}{230\,nm} = 1.5\,eV = 2.4\times10^{-19}\,J$$

Correct option is (b)

20. The photoelectric equation is derived under the assumption that **[BHU 2012]**

(a) electrons are associated with waves of wavelength $\lambda = \frac{h}{p}$, where P is the momentum

(b) light is emitted only when electrons jump between the orbits

(c) light is absorbed in quanta of energy $E = h\nu$

(d) light behaves like a wave

Soln. The photoelectric equation considers the quantization of light, i.e., light is absorbed or removed only in integral multiple of $h\nu$ only.

Correct option is (c)

21. The number of electrons emitted increases if the incident light has **[BHU 2012]**

(a) higher frequency (b) higher wavelength (c) higher intensity (d) lower frequency

Soln. If the incident light has higher intensity, number of incident photons increases and hence number of emitted electrons also increases.

Correct option is (c)

22. The photoelectric threshold for a metal is 3000 Å. The kinetic energy of an electron ejected from it by radiation of wavelength 1200 Å is **[BHU 2012]**

(a) 3.1 eV (b) 12.6 eV (c) 6.2 eV (d) 21.4 eV

Soln. The kinetic energy of an ejected electron is

$$\text{K.E.} = h(\nu - \nu_0) = hc\left(\frac{1}{\lambda} - \frac{1}{\lambda_0}\right)$$

$$= 6.6 \times 10^{-34} \times 3 \times 10^{8}\left(\frac{1}{1200 \times 10^{-10}} - \frac{1}{3000 \times 10^{-10}}\right) \text{J}$$

$$= 6.6 \times 3 \times 10^{-16} \times \frac{1800}{1200 \times 3000}$$

$$= 9.9 \times 10^{-19}\ \text{J}$$

$$\simeq 6.2\ \text{eV}.$$

Correct option is (c)

23. A metal surface has work function equivalence to energy of wavelength of light 4500 Å. When a beam of light with wavelength λ 4000 Å falls on it **[BHU 2015]**

(a) photoelectrons will not be emitted

(b) photoelectrons will be emitted with certain speeds

(c) photoelectrons will be emitted with zero speeds

(d) photoelectrons may or may not be emitted

Soln. Since wavelength of the incident light is less than the threshold wavelength, therefore photoelectrons will be emitted with certain speeds.

Correct option is (b)

24. In the phenomenon of the photo electric, the ejected electrons are : **[BHU 2016]**

(a) most of the times relativistic (b) most of the times non-relativistic

(c) always relativistic (d) most of the times stationary

Soln. The ejected electrons have kinetic energy of the order of eV as the incident photon has. This makes the electrons non-relativistic in case of photoelectric effect as their rest mass energy is of order of keV.

Correct option is (b)

5. Threshold wavelength of a photon that is capable of generating photoelectric effect in a metal, is **[BHU 2016]**

(a) the minimal value (b) the maximum value (c) the average value (d) the root square mean value

oln. The threshold wavelength is the maximum wavelength of the incident light to eject photo-electrons from the metals.

Correct option is (b)

6. Photons with energies $p\omega$ and $q\omega$ are incident one after another on a metal having work function ω, where p and q are integers. The ratio of respective maximum velocities is **[HCU 2012]**

(a) $\sqrt{\frac{p}{q}}$ (b) $\sqrt{\frac{q}{p}}$ (c) $\sqrt{\frac{p-1}{q-1}}$ (d) $\sqrt{\frac{q-1}{p-1}}$

oln. According to Einstein's theory of photoelectric effect,

$$\frac{1}{2}mv_{\max}^2 = h\nu - \phi$$

$$\therefore \frac{1}{2}mv_{\max}^2(p) = p\omega - \phi = p\omega - \omega = \omega(p-1) \Rightarrow v_{\max}(p) = \sqrt{\frac{2\omega}{m}(p-1)}$$

Similarly $v_{\max}(q) = \sqrt{\frac{2\omega}{m}(q-1)}$

$$\therefore \frac{v_{\max}(p)}{v_{\max}(q)} = \sqrt{\frac{p-1}{q-1}}$$

Correct option is (c)

7. The threshold wavelength of potassium is 558 nm. If a light of wavelength 400 nm is incident on it the stopping potential will be given by **[HCU 2013]**

(a) 8.88 V (b) 0.88 V (c) 88.8 V (d) 2.22 V

Soln. The stopping potential, $V_s = \frac{hc}{e}\left(\frac{1}{\lambda} - \frac{1}{\lambda_T}\right) = \frac{6.62\times10^{-34}\times3\times10^8}{1.6\times10^{-19}}\times\left(\frac{1}{400} - \frac{1}{558}\right)\times10^9$

$$= \frac{6.62\times3\times158\times10^{-34}\times10^8\times10^9}{1.6\times10^{-19}\times400\times558} = 8.78\times10^{-1}\text{ V} = 0.88\text{ Volt.}$$

Correct option is (b)

28. In the photoelectric effect experiment, if we increase the intensity of the incident light, then **[HCU 2016]**

(a) The kinetic energy of the emitted electrons increase

(b) The kinetic energy of the emitted electrons decreases because the number of electrons emitted increases

(c) The kinetic energy of the emitted electrons does not change but the number of electrons emitted increases

(d) both the kinetics energy and the number of electrons emitted increase

Soln. According to quantum theory of light, intensity of light is proportional number of photons. i.e $I \propto n$.

Since the number of emitted electrons is proportional to the number of incident photon, so number of emitted electrons will increase as intensity increases.

But the kinetic energy of the emitted electrons depend only on the frequency of incident photon.

So kinetic energy will not change with the intensity of light.

Correct option is (c)

PRACTICE SET

1. Find the wavelength and frequency of a 1 MeV photon. $\left[1.24\times10^{-2}\,\text{Å}; 2.42\times10^{20}\text{ Hz}\right]$

2. Find the wavelength and frequency of a photon whose momentum is 0.02 Me V/c. $\left[6.20\times10^{-1}\text{Å};\ 4.84\times10^{18}\times10^{18}\text{ Hz}\right]$

3. Find the momentum of a 4 keV photon. [4 KeV/c]

4. Find the energy of a photon whose momentum is 10 MeV/c. [10 MeV]

5. Find the energy of a photon whose wavelength is 4000 Å. [3.1 eV]

6. Find the energy and momentum of a photon whose frequency is 10^6 Hz. $\left[4.14\times10^{3}\text{ MeV}; 4.14\times10^{3}\text{ MeV/c}\right]$

7. A 300 MHz plane electromagnetic wave is incident normally on a surface of area 50 cm². If the intensity of the wave is 9×10^{-5} W/m², determine the rate at which photons strike the surface. [2.26×10^{18} photons/s]

8. The maximum energy of the emitted electrons when a material is illuminated with 3000 Å light is 1.2 eV. Find the work function. [2.93 eV]

9. Light of wavelength 4500Å is incident on two photoelectric tubes. The emitter in the first tube has a threshold wavelength of 6000Å and the emitter in the second tube has a work function twice as large as in the first tube. Find the stopping potential in each of the tubes. [$V_{s_1} = 0.69$ V; There is no photoelectric emission from the second tube]

10. Find the strength of the traverse magnetic field required to bend all the photoelectrons within a circle of radius 20 cm when light of wavelength 4000Å is incident on a barium emitter. The work function of barium is 2.5 eV. [1.32×10^{-5} T]

11. A 100 keV photon scatters from a free electron initially at rest. Find the recoil velocity of the electron if the photon scattering angle is 180°. (Use energy and momentum conservation) [0.319c]

12. In Compton scattering the scattered photon and electron are detected. It is found that the electron has a kinetic energy of 75 keV and photon an energy of 200 keV. What was the initial wavelength of the photon? [0.045 Å]

13. Calculate the maximum energy, in electron-volts, that can be transferred to an electron in a Compton experiment when the incident quanta are X-rays of wavelength 0.50 Å. [4.7 eV]

14. An electron that undergoes a "head-on" collision with an X-ray photon has a stopping potential of 70 kV. If the electron was initially at rest, what are wavelengths of the initial and scattered X-ray photons? [0.0716Å, 0.1201 Å]

Chapter 2

X-rays and Compton Scattering

2.1. X-rays

• Introduction

X-radiations form a part of the electromagnetic spectrum and were discovered accidentally by Wilhelm Conrad Röntgen in 1895. While carrying out investigations on the discharge of electricity through rarefied gases, he had observed that the portion of the glass of a cathode ray tube hit by the cathode rays emitted only only fluorescent light but also an invisible radiation along with it. As the nature of the new radiation could not be deciphered immediately, it was termed for the time being, the unknown of X-radiation. The name, however, survives till today although many call it ' Röntgen rays' after its discover.

The discovery of X-rays opened up a new era in physics, a period of unusual growth and unprecedented activity. Not only physics, the other branches of knowledge that reaped immediate benefit from the discovery are medicine, geology and industry. Within a year of the first announcement of the discovery, as many as 49 books and pamphlets had been published on the subject, and more than 1000 articles had appeared in various journals. In commemorating the 50th anniversary of the discovery G.E.M. Jauncey rightly observed : 'November 8, 1895 may be taken as the date of a new era in physics. Before this notable date, it may be said people were living in the classical era.... clinging for dear life to the classical ideas of the existance of luminiferous ether and the non-existence of anything smaller in mass than the hydrogen atom or ion. A great jolt was needed to shake the minds of physicists loose from these ideas.

• Properties of X-rays

As a result of the work of Röntgen and others, the wave nature and various other properties of the new radiation were subsequently established. The principal properties are :

1. The X-rays, like light, are electromagnetic radiations of extremely short wavelength (100-0.0001 Å). They show reflection, refraction, interference, diffraction and also polarization under suitable conditions.
2. They are unaffected by electric and magnetic fields, indicating that they are not beams of charged particles unlike cathode rays. They travel in straight lines with the speed of light.
3. They posses extremely high penetrating power. The shorter the wavelength, the greater is the power of penetration. They penetrate more in low density substances and less in high density ones. For instances, while ordinary glass is quite transparent to X-rays, lead glass is almost opaque. Usually they pass easily through matter that are opaque to visible rays. Long wavelength X-rays are absorbed by materials to varying degrees and this forms the basis of radiography.
4. They affect (i.e. blacken) photographic plate even when wrapped in black papers by several folds. The degree of blackening depends on the intensity of the radiation. Photography, in fact, is a very valuable technique for the study of X-radiation.
5. They cause fluorescence in many substances such as barium platinocyanide, zinc-sulphide, calcium salts and almost all the organic substances. The fluorescence is such a distinctive characteristic of the substance that it could be easily identical by it.

6. They ionise gases through which they pass, and the ionising power is dependent on and is a measure of the intensity of X-rays. The intensity of X-rays could thus be measured by their ionising power.
7. X-rays also produce photoelectric effect and thus exhibit their cospuscular nature. This nature is further confirmed by the Compton effect of X-radiation.
8. They are also capable of destroying living cells and cause genetic mutation.

• Application of X-rays

X-radiation find many important and useful applications in diverse fields, e.g. pure science, industry, medicine etc. as has been stated already.

In science—X-ray spectroscopy gives a powerful method for identification of chemical elements and determination of atomic structure. X-rays crystallography enables us to realise that all solids – from rock salt to nucleic acid – are crystalline in their natural state, and that almost every aspect of solid behaviour depends on the arrangement of its atoms and molecules. X-rays studies help to detect this arrangement and explain the associated properties of the solids. Such studies are almost a routine affair with geologists and solid state physicists.

In industry—X-ray radiography has been utilised in solving problems of the constitution and physical behaviour of metals and alloys, textile fibres, wool, silk, rayon, waxes, soaps, cement, ceramics, sugar, paints, plastics and many other materials. They are used for detecting flaws in casts, invisible surface cracks, defects in welding and in industrial materials such as machine tools and high speed automobile parts, presence of foreign materials etc. by non-invasive (non-destructive) testing. They are also used for fluoroscopic examination of food materials, oil painting and art treasures. The detective department and the customs utilise the radiation to examine boxes, suitcases and such other containers without opening them for contraband goods or dangeraous bombs and explosives inside them.

In medicine—In medicine, X-rays are used for both diagnostic and therapeutic purposes. Medical radiography detects and also locates internal defects of human bodies, fractures of bones, presence of such foreign bodies as bullets, stones etc. They are used in diagnosis of tumours, tubercular lungs and bones. Ultrasonography has of late, comp up as a powerful competitor in the field. In the field of therapy, deep X-radiation is used in the treatment of cancer using the property of its selective action in destroying abnormal cells. Fixation of proper dose, and the administration by dosimetry, is extremely important as the normal tissue is also susceptible of being affected by irradiation.

The above enumeration, however, does not exhaust the evergrowing applications of X-rays in almost every field of human activity.

• Production of X-rays

The principle of the production of X-rays is that they are produced whenever electrons with energies in the region of kV strike heavy metallic targets. The process may thus be considered as the inverse of photoelectric effect. The impinging electrons lose their energy on collision with the target and at each collision a part, or extremely rarely, the whole of the lost energy appears as X-rays.

Modern X-rays tube—After his initial experiments, Röntgen designed and developed a new type of discharge tube – gas tube – for the production of X-rays. Such X-ray tubes are now of historical interest only and would not be described here.

One of the modern forms of X-ray tubes is the Coolidge tube (Figure-1) designed and developed by the American physicist W. D. Coolidge in 1913.

It consists essentially of a bulb B with two coaxial projecting tubes T, T in one of which there is a sprial (filament) F of tungsten wire which acts as the cathode. It could be heated to a temperature as high as 2000°C by electric current derived from a low voltage battery (L.T.) or a transformer when it would emit thermoelectrons. A metallic cup C of molybdenum surrounds the filament F in order to focus the thermoelectrons onto the target. The anode A is made of copper with a target of tungsten or molybdenum housed in the other projecting tube. The vacuum of the tube is of the order of 10^{-6} mm of mercury. A high potential, ~ 50 – 200 kV, is maintained between the cathode and anode by the high tension source H.T. This large potential difference across the tube causes the electrons from the heated cathode to impinge onto the target with terrific kinetic energy and it is this impact that produces X-rays. The intensity of the beam is controlled by resulting the filament current.

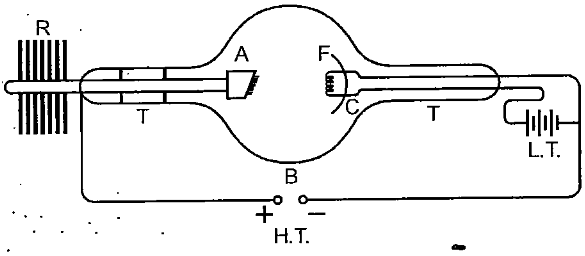


Fig. 1 Coolidge tube

The impact of the electrons heats up the anode and to prevent melting of anode by excessive heating there is an arrangement of cooling it either by a number of radiator fins *R*, as in small tubes, or by circulating water or oil through the anode which is usually tubular.

The Coolidge tube, unlike gas-filled tubes of Röntgen days, has many advantages. The intensity of X-rays depends, (i) on the current and (ii) the quality of anode potential both of which could be separately controlled in a Coolidge tube; the former by the heating current through the filament and the latter by the applied potential difference. Further, the Coolidge tube is, in general, more stable than the gas tube during continued operation.

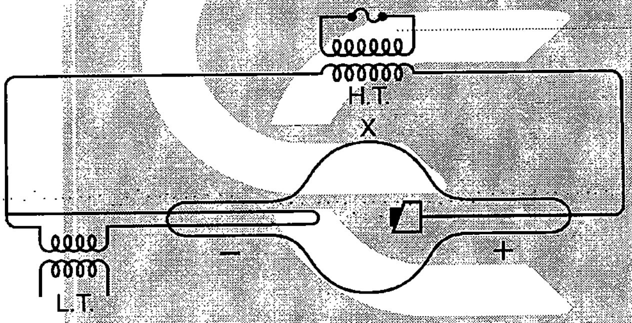


Fig. 2 Small X-ray plant

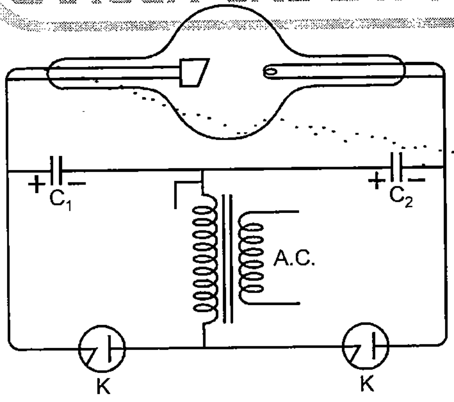


Fig. 3 Large X-ray plant

X-ray installation—In small X-ray plants, where the applied potential difference is realtively small, the cathode is heated by the low tension transformer L.T. (Figure-2) while the accelerating voltage is supplied by high tension transformer H.T. Electrons are attracted towards the anode only during one half of the cycle when the anode is positive. The X-rays are generated during this half cycle. In the other half of the cycle, however, X-rays cannot be produced. The tube thus acts as its own rectifier and suffers from the disadvantage that X-rays are produced only in one half of the cycle thereby shortening the useful life of the bulb.

In large X-ray plants where the applied p.d. is relatively higher, kenotron rectifiers are used. A kenotron is a high voltage diode rectifier. The arrangement for full wave rectification, using two kenotrons *K*, *K*, is shown in (Figure-3). In one half of the cycle, the left kenotron conducts when the condenser C_1 gets charged; in the other half, it is the right kenotron that conducts and the condenser C_2 is charged. Such a circuit is known as a voltage doubler circuit. This is because the total voltage on the anode is nearly double the voltage of the secondary of the transformer.

Betaron — One of the most modern X-ray generators is the betatron devised in 1941 by the American Physicist D.W. Kerst. Here the electrons can be accelerated to a very high energy, ~ 10^8 eV, and consequently X-rays of extremely high energy are produced. Such high energy X-rays are being fruitfully used in investigating the structure of atomic nuclei apart from then increasing utility in the field of medicine and surgery for both diagnostic and therapeutic purpose.

The construction, principle of action and operation of the betatron has been discussed in the Chapter : Particle Accelerators.

• Intensity of X-rays: measurement

The measurement of intensity of X-radiation is generally made by ionisation chamber. The principle of the method is as follows: A suitable gas or vapour is taken inside the chamber and the passage of X-rays through it produces ionisation. The ion-current developed is proportional to the X-ray intensity and is measured by an electrometer.

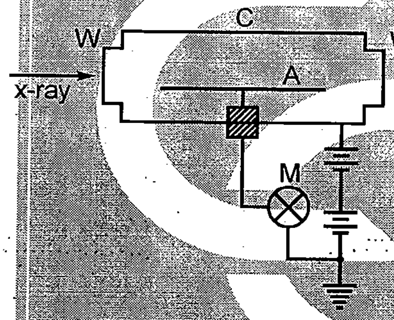


Fig. 4 Measurement of intensity of X-rays

The method is illustrated in Figure-4. *C* is a cylindrical metallic chamber, usually of brass, 0.2 to 1 m long, with the open ends closed by two thin aluminium sheets W, W, known as the windows. X-radiations enter the cylinder through one of the windows and leave it by the other. The chamber is filled with air, ethyl bromide vapour etc. The X-ray beam ionises the gas or vapour inside the chamber. The positive ions are drawn to the tube wall which acts as the cathode and the negative ions move to the anode A, a metallic rod placed inside the chamber parallel to its axis but insulated from the wall. The anode A is connected to an electrometer M and a potential difference of several hundred volts is maintained between A and C. The electrometer M is capable of measuring very small ionisation current produced by X-radiation. The ion-current is found to be linearly related to the X-ray intensity, and thus gives the intensity of the beam.

A scintillation counter (detailed discussion of which is made in Chapter : Nuclear Radiation, Detectors) is especially suitable for measuring the intensity of X-rays from high voltage sources. The X-ray beam is allowed to penetrate a crystal or a liquid and is transformed into the visible fluorescent radiation. This radiation actuates a photomultiplier tube and the resulting photoelectric current is multiplied and measured. The intensity of X-rays is directly proportional to the photocurrent.

• Variation of X-ray intensity with wavelength

X-rays of different wavelengths are emitted from the X-rays tube and if the intensity is plotted against wavelength, then curves of the nature as shown in Figure-5 and Figure-6 are obtained. The different curves in each Figure corresponds to different voltages used. While Figure-5 gives I vs λ graph for a target material of high atomic number, e.g. tungsten ($Z = 74$), Figure-6 gives I-λ curve when a target material of low Z-value, e.g. Mo ($Z = 42$), is used.

The main features of the intensity distribution may be summed up as follows :

1. The intensity increases with increasing potential difference between the cathode and the anode.
2. For the same potential difference the curve, starting from the long wavelength side, attains a maximum and then falls rapidly to zero at the sharp cut-off wavelength λ_0 or $\lambda_{\min}$ which gets shifted to shorter wavelengths with increasing voltages.
3. For a given wavelength, the intensity is higher, higher the potential difference and as shown in Figure-5 I varies continuously with λ up to a voltage difference ~ 50 kV.
4. For the same p.d., the intensity increases with increase of the at. no. Z of the target.

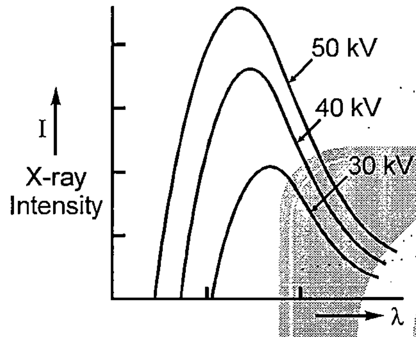


Fig.5-Intensity vs. wavelength curves (high Z-value)

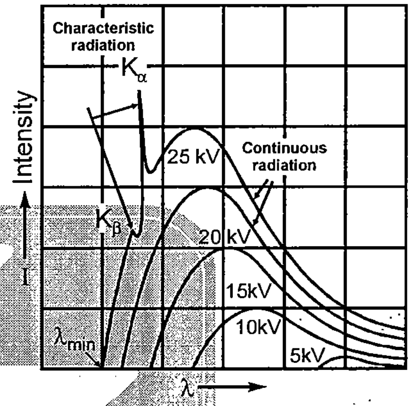


Fig.6-Intensity vs. wavelength curves (low Z-value)

5. At potentials ~ 25 kV or higher, several discrete peaks appear, superimposed on the continuous background, at specific wavelengths with target of $Z = 42$ (Figure-6). For target with still lower Z, e.g. Cu (Z = 29), the peaks begin to appear at still lower voltage ($V \geq 8$ kV), the peak wavelengths for Cu are however different from those of Mo. The heights of the peaks increase with increasing voltage.
6. The wavelengths of the peaks in the intensity distribution curve are characteristic of the target material and are independent of the applied voltage. These are called characteristic X-rays. Higher the Z, shorter is the wavelength of characteristic X-rays.

• Origin of continuous X-radiation

The origin of continuous X-radiation, the existence of minimum wavelength and the calculation of its magnitude can only, and easily, be understood from the quantum concept of radiation.

According to the quantum theory, each X-ray photon is produced as a result of particular collision of one electron. The photon of maximum energy is generated when the electron loses the whole of its energy in a single collision. The electrons are accelerated to an energy Ve by the accelerating voltages V. So the maximum energy of the photon is given by

$$E_m = h\nu_m = \frac{1}{2}mv^2 = eV \quad \text{...(i)}$$

where ν_m is the maximum frequency of the photon.

But the maximum frequency ν_m corresponds to the minimum wavelength λ_0, given by

$$\lambda_0 = \frac{c}{\nu_m} = \frac{hc}{Ve} \quad \text{...(ii)}$$

where c is the velocity of light in free space.

If V is measured in volt, we have $\lambda_0 = \frac{12413}{V(\text{volt})} \mathring{A}$ ...(iii)

So, if $V = 10{,}000$ volts, $\lambda_0 = 1.24\,\mathring{A}$; if $V = 50\,\text{kV}$, $\lambda_0 = 0.248\,\mathring{A}$, etc.

The magnitude of λ_0 can be measured accurately and the equation (ii) may be exploited to find the value of h, the Planck's constant. The value of h thus determined is in good agreement with those obtained from thermal radiation and photoelectric effect. Further, equation (iii) or (ii) shows that higher the V, lower is λ_0; this is also observed experimentally.

Events in which electron gives up the whole of its content are extremely rare and, in general, an electron produces several photons of wavelength $\lambda > \lambda_0$ in a succession of collisions. The electrons also cause some ionisation and much (sometimes 99%) of the kinetic energy gets converted into heat energy.

Note 1. The above mechanism of emission of electromagnetic radiation from an accelerated or decelerated electron is called bremsstrahlung process and the emitted radiation bremsstrahlung (brems = brake, strahlung = radiation, bremsstrahlung = brake-radiation i.e. radiation due to deceleration).

Note 2. X-rays are often classified on the basis of their penetrating power. Most penetrating radiation is called hard X-radiation, the least penetrating one the soft X-rays, the intermediate one the medium radiation.

2.8. Origin of characteristic X-ray lines

In addition to the smooth continuous spectrum just discussed there appear, as we have seen, some sharp, characteristic X-ray lines at sufficiently high voltages. The mechanism of their production is quite different and lies in the interior electronic structure of the atoms.

The wavelength of X-ray lines are usually of the order of few angstrom or even shorter, implying that their energies are ~ 10^4 eV or higher. Consequently, the atomic electrons whose transitions give rise to X-ray lines must have binding energies ~ 10^4 eV or higher. This is possible only if inner atomic orbits close to the nucleus house the electrons.

According to Bohr, an atom of an element of atomic number Z contains in its normal state Z electrons revolving round the nucleus in different shells. The innermost shell corresponds to the first quantum orbit $n = 1$ and is called the K-shell by Kossel. The two electrons that can exist in this shell being closest, to the nucleus require the greatest excitation energy for their removal. Next, in sequence, comes the L-shell with $n = 2$, M-shell with $n = 3$, N-shell with $n = 4$ and so on having electrons that require decreasing excitation energy values for their removal.

Let a target-atom be bombarded by high speed electrons and a K-electron which is tightly bound be knocked out. If E_K be the binding energy of K-electron, then the kinetic energy of the knocked out electron is

$$\frac{1}{2} mv^2 = Ve - E_K$$

where Ve represents the kinetic energy of the incident high speed electron.

So the minimum energy of the incident electron to knock out a K-electron is E_K.

Now, on knocking out a K-electron, a vacancy is created in the lowest-lying K-shell and transition of an electron from the higher (and relatively loosely bound) L-shell fills it up. The energy difference $E_K - E_L$, due to the transition results in the emission of K_α -line of frequency ν_{KL}, given by

$$h\nu_{KL} = E_K - E_L$$

So the frequency of K_α -line (or its wavelength) is a characteristic of the target atom. Since the targets used in X-rays tubes are usually of high Z-values, the low-lying levels have binding energies ~ keV. The emitted radiation is thus in keV region, and not in optical region.

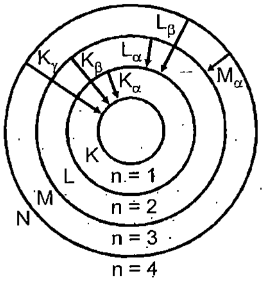


Fig.7-Kossel diagram representing x-radiation series

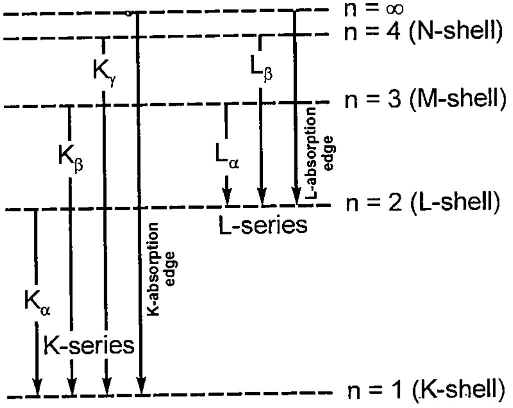


Fig.8-Energy level diagram of x-ray series spectra

If however an electron from the M-shell fills up the said vacancy, the K_β -line is emitted and so on, forming the K-series lines. When an electron from the L-shell is knocked out (or falls into the K-shell), electrons froms the M, N,.... shells may fall into the L-shell to fill up the vacancy and the L-series lines are emitted. Other series lines also have a similar origin. The short wavelength limit for each series is called the corresponding absorption edge.

The origin of the lines of different X-radiation series by orbital transitions is best understood and represented by what is called Kossel diagram as shown in Figure-7. The corresponding energy level diagram is presented in Figure-8.

Summing up:

K_α -line : Transition from $K \to L$ level
K_β -line : Transition from $K \to M$ level
L_α -line : Transition from $L \to M$ level
L_β -line : Transition from $L \to N$ level

The corresponding frequencies of the lines are

$$\nu_{K_\alpha} = \frac{(E_K - E_L)}{h}; \quad \nu_{K_\beta} = \frac{(E_K - E_M)}{h}$$

$$\nu_{L_\alpha} = \frac{(E_L - E_M)}{h}; \quad \nu_{L_\beta} = \frac{(E_L - E_N)}{h}$$

Note 1: The above characteristic radiation may as well be generated by allowing the continuous X-radiation to be scattered from different elements outside the X-ray tube. These X-ray photons can knock out K-electrons, L-electrons etc. of the scatterer if the energy of the X-ray photons $h\nu$ is greater than, E_K, E_L etc. The excess energy is carried away by the characteristic X-ray photon.

Note 2: Since $E_K > E_L > E_M$ etc. it is obvious that $\nu_K > \nu_L > \nu_M$.... or $\lambda_K < \lambda_L < \lambda_M$....

2.9. Fine structure of characteristic X-ray lines

On careful exmination, it is seen that the characteristic X-ray lines are not single but possess fine structure, i.e. each line consists of a number of closely-spaced components. The origin of this fine structure of X-ray lines can be explained on the basis of quantum numbers *n*, *l* and *j* associated with each energy level of the electron. For a given *n*, which determines the electron shell (i.e. $n = 1$ for K-shell, $n = 2$ for L-shell etc.), *l* can assume values $l = 0, 1, 2, ...(n-1)$. For a given *l* again, the total quantum number *j* can have only two values :

$j = l + \frac{1}{2}$ or $j = l - \frac{1}{2}$. For $l = 0$, however, $j = \frac{1}{2}$ only.

The multiplicity of the level corresponding to different *n*, *l*, *j*'s are given below in a tabular form.

Shell	n	l	j	Energy level	Multiplicity
K	1	0	$\frac{1}{2}$	K_1	1
L	2	0	$\frac{1}{2}$	L_1	3
		1	$\frac{1}{2}, \frac{3}{2}$	L_{II}, L_{III}	
M	3	0	$\frac{1}{2}$	M_1	5
		1	$\frac{1}{2}, \frac{3}{2}$	M_{II}, M_{III}	
		2	$\frac{3}{2}, \frac{5}{2}$	M_{IV}, M_V	

Again, the transition between the different levels are determined by certain selection rules.

These are : (i) $\Delta l = \pm 1$ and (ii) $\Delta j = 0, \pm 1$

The fine structure of X-ray levels and the allowed transitions between them are shown in Figure-9. It is observed that each K-line is a doublet : K_{α_1} and K_{α_2}, K_{β_1} and K_{β_2}. While K_{α_1} is due to transition from $K \to L_{III}$, K_{α_2} is due to transition from $K \to L_{II}$. The splitting of L.M. etc. lines are relatively more complex.

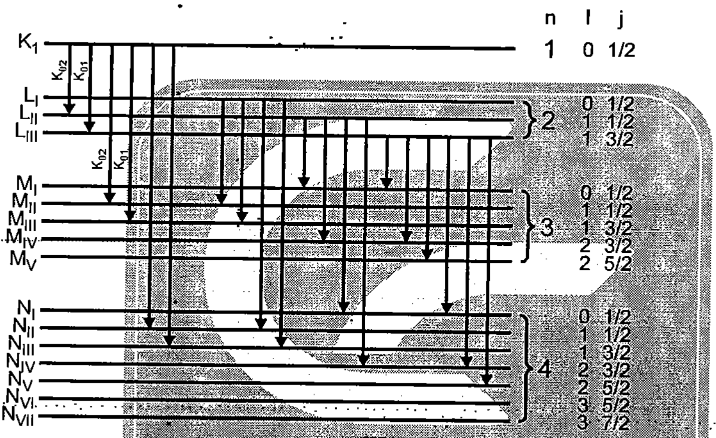


Fig. 9 Fine structure of X-ray levels

2.11. Moseley's work on X-ray spectra

A comprehensive investigation of the characteristic X-ray spectra was made by the English Physicist H.G.J. Moseley (1913-14) who used nearly 40 elements from aluminium to gold as targets in the X-ray tube. The radiation from each target was analysed by the single crystal X-ray spectrograph using a thin crystal of potassium ferrocyanide. The spectrum was recorded on a photographic plate and the entire arrangement was placed in a vacuum chamber in order to avoid the absorption of long wavelength X-rays in air.

The photographic plate, on development, showed traces of dark lines of varying intensities, each corresponding to a characteristic line of the given element, The characteristic spectra could easily be divided into two distinct groups – a short wavelength group, generally less than 1 nm (K-series) and a long wavelength group, greater than 1 nm (L-series). The wavelengths of the L-series were roughly ten times as great as those of the K-series. Subsequent investigations confirmed the existence of two other series lines (M-series and N-series) of longer wavelength in heavier atomic species (Z > 66).

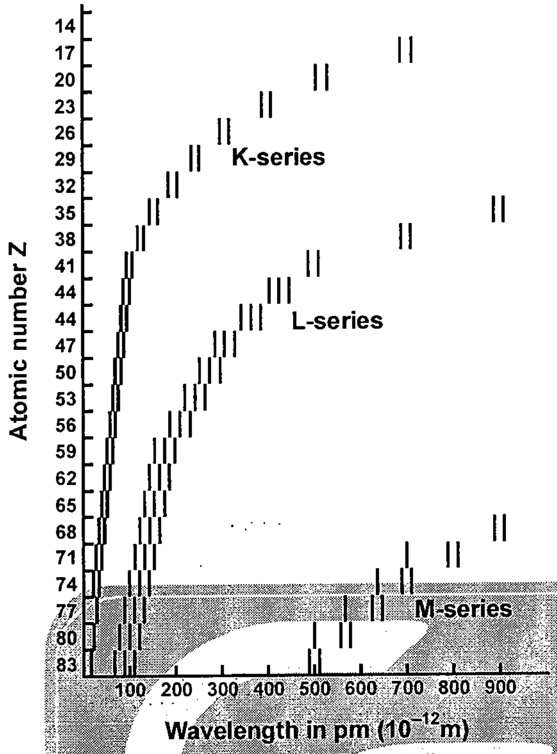


Fig. 10-K, L and M-series spectra of different elements

Moseley demonstrated that, unlike the optical spectra, the characteristic X-ray spectra are much simpler in character consisting, as they are, of comparatively fewer lines. Further, they are the same type irrespective of the elements, except that the wavelengths of the corresponding lines decrease with increasing atomic weight. The K, L and M-series spectra of different elements are shown in Figure-10.

Moseley's law—Moseley from his investigation deduced the following relation between the frequency ν of any particular line emitted by an element and its atomic number Z.

$\sqrt{\nu} = a(Z-b)$ where a, b are constants characteristic of the element.

This is known as the Moseley's law. By plotting the square root of the frequency of the line (say, the most intense of the K-series) against the atomic number Z of the element concerned, Moseley obtained (Figure-11) a straight line given by the above equation. The same linear relation was found to hold good for any line in any series, of course with different slopes. From Moseley's diagram, b is the small intercept on the Z-axis.

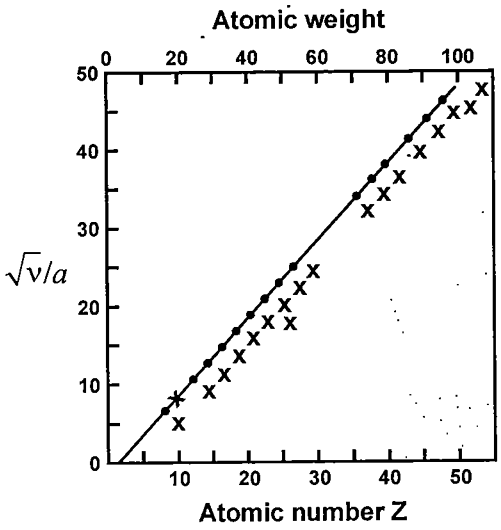


Fig. 11 Moseley diagram : $\sqrt{\nu}/a$ vs. Z-plot (also at. wt. plot)

In Figure-11 ‘crosses’ correspond to $\sqrt{\nu}/a$ vs atomic weight, while the ‘dots’ correspond to $\sqrt{\nu}/a$ vs atomic number Z. It is apparent from the plot that in so far as the frequency of the characteristic line is concerned, the atomic number is more fundametal than the atomic weight. That is to say, the chemical identify of an element depends on the atomic number Z and not on the atomic weight or mass number A.

Importance of Moseley’s work— It was Moseley’s investigation which for the first time showed that it would be more logical to arrange elements in the periodic table in the order on increasing atomic number instead of atomic weight. When this was done, it removed certain anomalies in the order of some elements in the old Periodic table of Mendeleev. For instance, argon of atomic weight 40 comes before potassium of atomic weight 39 from the consideration of chemical properties of these two elements. The atomic numbers of argon and potassium are however 18 and 19 respectively. So the arrangement is correct if made in the order of atomic number. Other such examples are cobalt nickel and tellurium iodine.

Moseley’s work also helped (i) in the discovery of new elements in the missing gaps of atomic number 43, 61, 72 and 75 in the Periodic table and (ii) in the determination of atomic numbers of rare-earth elements and fixing their positions in the periodic table.

Derivation of Moseley’s law — The general form of Moseley’s law can be easily derived from Bohr’s theory.

Let a transition occur from state n_1 to state n_2. So the energy of the emitted X-ray photon is

$$h\nu = RhcZ^2\left(\frac{1}{n_1^2}-\frac{1}{n_2^2}\right) \text{ or } \nu = RcZ^2\left(\frac{1}{n_1^2}-\frac{1}{n_2^2}\right) = a^2Z^2 \qquad \text{...(i)}$$

where R is Rydberg constant, $a^2 = Rc\left(\frac{1}{n_1^2}-\frac{1}{n_2^2}\right)$ and c the velocity of light in free space

$$\therefore \sqrt{\nu} = aZ$$

For K_α-line, $n_1 = 1$, $n_2 = 2$, so that $a^2 = \frac{3}{4}Rc$.

The presence of the constant b in Moseley equation, as deduced from experimental data, has been interpreted as the shielding effect of nucleus by the remaining electrons in the lower level and b is known as the screening constant. For instance, for K_α-emission one of the two K-electrons is first emitted leaving a single electron in K-shell. The negative charge of this residual electron partially screens the nuclear charge $+Ze$ of the atom. The effective Coulomb force on the L-electron is thus due approximately to a net positive charge $+(Z-1)e$. So instead of Z^2 in equation (i), we should rather write $(Z-1)^2$, that is to say,

$$\nu = \frac{3}{4}Rc(Z-1)^2 \text{ or } \sqrt{\nu} = a(Z-1)$$

where $a^2 = \frac{3}{4}Rc$, as before and $b = 1$.

With this interpretation, we have the Moseley’s law $\sqrt{\nu} = a(Z-b)$.

Note 1. For K_α-line the screening constant $b = 1$. A plot of the frequency $\sqrt{\nu}/a$ against Z (Figure-11) shows from the intercept of the Z-axis that as a matter of fact $b \sim 1$.

Note 2. We know that K_α, K_β-lines have doublet structure, a fact unknown at the time of Moseley’s discovery. So in Moseley’s diagram the mean frequencies of the doublet were plotted. Subsequent precise measurements showed that the Moseley diagram of K_α-lines actually consists of two very close straight lines with slightly different slopes.

2.12. Absorption of X-rays

When a parallel beam of X-rays impinges on a slab of matter, most of it is transmitted through the material, but a part gets absorbed and the intensity of the beam emerging from the opposite side of the slab decreases. This absorption of X-rays leads to several phenomena. But they are collectively accounted for by what is called the absorption coefficients μ and μ/ρ.

Absorption coefficients—Let I be the intensity of a monochromatic X-ray beam that penetrates normally a thickness dx of a material. Then $-dI/dx$ is the rate of change of intensity with thickness and negative sign indicate that intensity decrease with thickness.

But, experimentally, the rate of diminution of intensity is found to be proportional to the intensity I.

$$\therefore \quad -\frac{dI}{dx} \propto I$$

Assuming the material under discussion to be homogeneous, we get $\frac{dI}{dx} = -\mu I$ ...(i)

where μ is a characteristic constant of the material concerned. The constant μ is known as the linear absorption coefficient of the material.

From equation (i), we get $\frac{dI}{I} = -\mu dx$

Integrating, we get $\ell n\, I = -\mu x + A$

where A is the constant of integration.

At $x = 0$, $I = I_0$, the original intensity of the incident beam. Therefore, $A = \ell n\, I_0$

$$\therefore \quad \ell n\,(I/I_0) = -\mu x \Rightarrow I = I_0 e^{-\mu x} \qquad \text{...(ii)}$$

Determination of μ —The linear absorption co-efficient μ of a material is usually measured by finding the thickness x_h required to reduce the intensity to half its initial value.

From equation (ii), therefore, $\frac{I}{I_0} = e^{-\mu x_h} = \frac{1}{2}$

$$\therefore \quad \mu = \frac{\ell n 2}{x_h} = \frac{0.693}{x_h}$$

Thus if x_h is known, the co-efficients μ becomes known and x_h is called half-value thickness.

Further, a plot of $\ell n\, I$ against x (Figure-b) would give a straight line the slope of which is $-\mu$. From a measurement of this slope also, μ can be evaluated. The dimension of μ is plainly the inverse of length, m^{-1}.

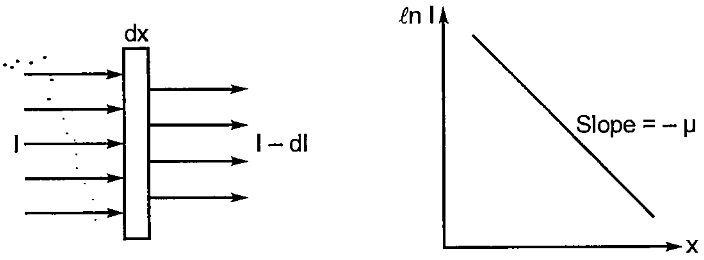


Fig. (a) Absorption of x-rays (b) ℓn I vs. x plot

It is by the value of μ that the quality of an X-ray beam is fixed. If μ is small and x is large, the X-rays are said to be hard, if however μ is large, they are called soft.

Mass absorption co-efficients — It is found that μ varies rather widely for the same substance in their different physical states, e.g. ice, water and steam, due to the density factor. So μ is not a true constant for the substance. A true material constant, independent of its physical state, in this respect is the concept of mass absorption co-efficients as defined below.

Let ρ be the density, s the surface area, x the thickness and m the mass of the absorbing material. Then,

$$\rho = \frac{m}{sx}$$

$$\therefore \quad x = \frac{m}{\rho s}$$

Therefore, from (ii), we obtain : $I = I_0 e^{-\mu m/s\rho}$ ...(iii)

Plainly, m/s is the mass per unit area of the absorber and the quantity $\mu/\rho \; (= \mu_m)$ is called the mass absorption co-efficients of the absorber.

Therefore, Mass absorption co-efficients $\mu_m = \dfrac{\text{linear absorption coefficient}}{\text{density of the material}}$

The value of $\mu_m (= \mu/\rho)$ increases rapidly with atomic weight and thus explains why the Pb-screens shield x-radiations better than screens aluminium of other material of low atomic weights. X-rays or γ-ray radiography makes practical use of equation (iii), which accounts for the images of cavities in metal casting and so forth.

Secondary radiations — As already stated, the absorption of X-rays by matter leads to several phenomena. The part of the radiation removed from the incident beam (i.e. amount absorbed) produces, apart from heat, secondary radiations consisting mainly of the following.

1. **Photoelectrons** — The X-ray photons may be absorbed by electrons in inner shells (e.g. $K, L, M,...$) which are thereby ejected photoelectrically with very high speed. The process is usually accompanied by the emission of characteristic fluorescent radiation due to electronic transition from outer shells. The net effect of the process is gradual diminution of the energy of the beam as it passes through the absorber.
2. **Scattered X-rays** — The incident X-rays may be scattered by the atomic electrons, and result in scattered X-rays of practically the same nature as the original radiation.
3. **Characteristic X-rays** — which depend only on the nature of the material but independent of the quality of the incident beam of X-radiation.

At higher energies, there could be photo-nuclear and other effects to attenuate the energy of X-radiation in its passage through the absorbers.

2.13. Variation of μ_m with wavelength : Absorption edges

The mass absorption coefficients, $\mu_m \propto Z^4$ where Z is the atomic number of the material. So as Z increases μ_m increases rapidly, as has already been stated.

μ_m also depends on the wavelength of X-radiation and the relation is $\mu_m \propto \lambda^3$. So as λ increases, μ_m also increases rapidly. The variation of μ_m with λ is represented graphically in (Figure-13), where in it is observed that as λ increases, μ_m increases continuously up to a certain λ-value, followed by a sudden decrease in μ_m at a definite wavelength $\lambda_1 = \lambda_K$. The μ_m again rises continuously with λ followed again by a sudden decrease at another definite wavelength $\lambda_2 = \lambda_{L_I}$, and immediately thereafter two more discontinuous – one at $\lambda_3 = \lambda_{L_{II}}$ and another at $\lambda_4 = \lambda_{L_{III}}$. The pattern gets more or less similarly repeated as λ increases.

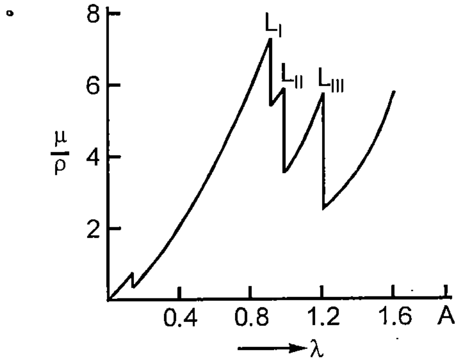


Fig.13 Variation of μ_m with λ

The wavelength λ_K corresponding to the first discontinuity in μ_m-λ curve is known as the *K*-absorption edge. The following discontinuities at λ_{L_I}, $\lambda_{L_{II}}$ and $\lambda_{L_{III}}$ are respectively called λ_{L_I}, $\lambda_{L_{II}}$ and $\lambda_{L_{III}}$ -absorption edges. The wavelengths of the different absorption edges are dependent on the absorber. Obviously, $\lambda_K < \lambda_{L_I} < \lambda_{L_{II}} < \lambda_{L_{III}}$. The wavelength λ_K is slightly less than the maximum wavelength of the characteristic X-rays emitted by the absorber.

Explanation for discontinuous— The appearance of the discontinuous can be readily explained. So long the energy E_v of the incident X-ray photons is greater than E_K, the binding energy of electron in the K-shell of absorbing the incident radiation with a kinetic energy.

$$\frac{1}{2}mv^2 = E_v - E_K$$

and the process may be termed photoelectric emission from the K-shell. As E_v decreases, μ_m increases continuously till $E_v = E_K$. When E_v becomes less than E_K, the incident radiation is not capable of knocking out K-electron any longer photoelectrically. So μ_m suddenly decreases giving rise to a discontinuity at $E_v = E_K$. However, even if $E_v < E_K$, the incident radiation can eject an L-electron as $E_L < E_K$. So with further increase of λ, μ_m rises again due to greater probability of absorption of the radiation by L-electrons. At $E_v = E_K$, this probability is maximum. Thereafter μ_m drops again since E_v becomes less than E_L and a second discontinuity is resulted. The pattern gets repeated in the same fashion.

Absorption edges—Unlike in the visible region, no discrete line appears in X-ray absorption spectrum. If an inner shell electron absorbs X-ray photon incident on it, it will be emitted as long as E_v is greater than its binding energy. No transition of the ejected electron to an outer shell is possible as they are all filled up. Necessarily therefore incident radiation of all energies greater than the binding energy of the electron will be absorbed. This implies that x-radiations of all λ's less than a definite limiting value will be absorbed. The absorption spectrum thus shows a continuum with a sharp cur-off value on the long wavelength-end. It is this cur-off wavelength that corresponds to an absorption edge.

Note 1. Since the energies of different absorption edges (*K*, *L*, *M*,..., etc.) are equal to different binding energies (E_K, E_L, E_M... etc.) of the electrons in corresponding shells, the energy difference between different absorption edges will provide the energy differences of X-ray levels.

Note 2. Taking photoelectric emission and scattering into consideration, we may write $\mu = \mu_p + \mu_s$ where μ_p is the part contributed to μ due to photoelectric effect and μ_s that due to scattering. So the mass absorption coefficient, $\mu_m = \mu/\rho = (\mu_p + \mu_s)/\rho$.

The atomic absorption coefficient, μ_a is obtained by dividing the mass absorption coefficients by the number of atoms in unit mass of the absorber.

Therefore, atomic absorption coefficient, $\mu_a = \dfrac{\mu/\rho}{N_A/A}$ where N_A is Avogadro number and A the atomic weight of the absorber.

2.14. Scattering of X-rays : Thomson's theory

Let E be the electric field of the incident unpolarised X-radiation. So the acceleration of the scatterer (electron) of charge e and mass m is

$$a = \frac{eE}{m} \qquad ...(i)$$

Under the influence of the incident radiation, the electron is set into forced vibration. According to the electromagnetic theory, it would emit electromagnetic waves of the same frequency, constituting the scattered radiation, in different directions from the scattering centre. If the velocity of the vibrating electron is small, compared to the light velocity, then it can be shown that the electric field of the scattered wave at a distance r from the scattering centre in a direction β with the incident electric vector $\vec{E}$ is

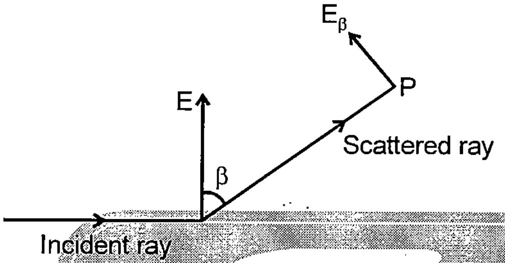


Fig.14-Scattering of x-rays

$$E'_\beta = \frac{ea}{4\pi\varepsilon_0 c^2 r}\sin\beta = \frac{e^2E}{4\pi\varepsilon_0 c^2 m}\frac{\sin\beta}{r}, \text{ using equation (i)}$$

where ε_0 is the permitivity of free space.

By Poynting's theorem, the intensity of incident beam is given by $I_0 = \varepsilon_0 cE^2$ ...(ii)

and that of the scattered beam is $I'_\beta = \varepsilon_0 cE'^2_\beta = \dfrac{e^4E^2}{16\pi^2 m^2 c^3 \varepsilon_0}\dfrac{\sin^2\beta}{r^2}$ ...(iii)

$$\text{Therefore, } \frac{I'_\beta}{I_0} = \frac{e^4}{16\pi^2\varepsilon_0^2 m^2 c^4}\frac{\sin^2\beta}{r^2}, \text{ using (ii) and (iii).}$$

The incident radiation being unpolarised, we must take time-average of $\sin^2\beta$ in (iii). Let the incident radiation be along x-axis so that the electric vector $\vec{E}$ is in yz-plane (Figure-14a). Let E_y and E_z be the components of $\vec{E}$ along y-axis and z-axis respectively. Let the scattering centre be at O and the direction of the scattered beam OP be in xy-plane, making an angle θ with the incident beam. Then the electric field E'_1 and E'_2 of the scattered beam associated with components E_y and E_z of the incident electric field vector $\vec{E}$ will be given by

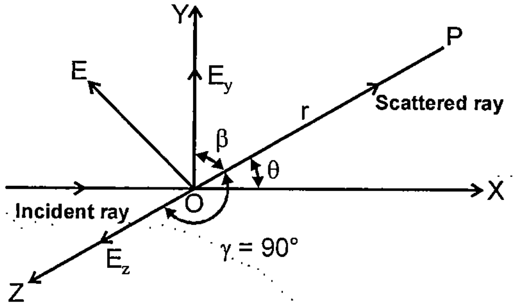


Fig.14(a)- Resolution of electric vector of incident beam

$$E'_1 = \frac{e^2E_y\sin\left(\frac{\pi}{2}-\theta\right)}{4\pi\varepsilon_0 mc^2 r} = \frac{e^2E_y}{4\pi\varepsilon_0 mc^2}\frac{\cos\theta}{r} \quad \left(\because\ \beta = \frac{\pi}{2}-\theta\right) \text{ and } E'_2 = \frac{e^2E_z}{4\pi\varepsilon_0 mc^2 r}$$

Therefore, resultant electric intensity of the scattered wave E'_θ is $E'^2_\theta = E'^2_1 + E'^2_2$, since E'_1, E'_2 are mutually perpendicular.

Therefore, intensity of the scattered beam I'_θ is given by $I'_\theta = \varepsilon_0 c E'^2_\theta = \varepsilon_0 c\left(\frac{e^2}{4\pi\varepsilon_0 mc^2 r}\right)^2 (E_y^2\cos^2\theta + E_z^2)$

whose time-average is $\overline{I'_\theta} = \frac{e^4}{16\pi^2\varepsilon_0 m^2 c^3 r^2}(\vec{E}_y^2\cos^2\theta + \vec{E}_z^2)$.

The time-averaged value of the incident intensity $\overline{I_0} = \varepsilon_0 c\vec{E}^2 = \varepsilon_0 c\left(\vec{E}_y^2 + \vec{E}_z^2\right)$.

Since the electric vector in the incident unpolarised beam is randomly oriented, $\vec{E}_y^2 = \vec{E}_z^2 = \frac{1}{2}\vec{E}^2 = \frac{1}{2\varepsilon_0 c}\overline{I_0}$.

Therefore, $I'_\theta = \frac{e^4}{32\pi^2\varepsilon_0^2 m^2 c^4 r^2}(1+\cos^2\theta)\, I_0$, omitting the bars.

This is the scattered intensity for a single electron. In a scatterer of thickness Δx and unit cross-section, the number of scatterers (electrons) is $n\Delta x$ where n is the number of electrons per unit volume of scatterer, each scattering independent of the other (incoherent scattering). So the resultant intensity of the scattered beam is

$$I_s = \frac{ne^4\Delta x}{32\pi^2\varepsilon_0^2 m^2 c^4 r^2}(1+\cos^2\theta)\, I_0 \qquad \text{...(iv)}$$

In deriving the above expression, the binding energy of electrons in the atom has been neglected (free electrons). Note from (iv) that the intensity is different in different directions, being minimum when $\theta = \pi/2$, that is, in a direction normal to incident radiation, when the scattered wave is plane polarized with electric vector along z.

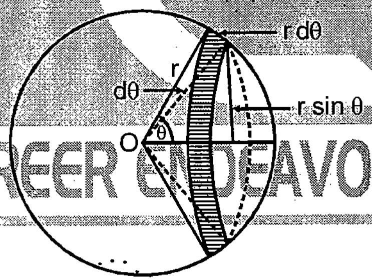


Fig.15-Computation of total scattered energy

To compute the total scattered energy, we are to integrate the above expression for θ. Taking r to be large compared to the dimension of the scatterer and assuming it to be concentrated at the origin O, let us draw (Figure-15) a sphere of radius r about O. The area of the ring-shaped strip of the sphere between two cones of semi-vertical angles θ and $\theta + d\theta$ is

$$dS = (2\pi r\sin\theta)\, r d\theta$$
$$= 2\pi r^2 \sin\theta\, d\theta$$

$\therefore$ Total scattered energy,

$$W_s = \int I_s dS = I_0\frac{ne^4\Delta x}{32\pi^2 r^2 m^2 c^4\varepsilon_0^2} 2\pi r^2\int_0^\pi (1+\cos^2\theta)\sin\theta\, d\theta = \frac{8\pi}{3} n\left(\frac{e^2}{4\pi\varepsilon_0 mc^2}\right)^2 I_0\Delta x$$

$\therefore$ Scattering coefficient, $\sigma = \frac{W_s}{I_0\Delta x} = \frac{8\pi}{3} n\left(\frac{e^2}{4\pi\varepsilon_0 mc^2}\right)^2$

This is the Thomson's scattering formula and this type of scattering is known as Thomson's scattering or unmodified scattering since there is no change in the frequency (or wavelength) of the rays.

The ratio σ/n is called the classical scattering cross-section, σ_T, for a free electron.

$$\therefore \quad \sigma_T = \frac{8\pi}{3}\left(\frac{e^2}{4\pi\varepsilon_0 mc^2}\right)^2 = \frac{8\pi}{3} r_0^2 \text{ where } r_0 = \frac{e^2}{4\pi\varepsilon_0 mc^2} \text{ and is referred to as classical electron radius.}$$

Since both r_0 and σ_T depend on the three fundamental constants e, m and c, they are universal constants. It is easy to see that the dimension of σ_T is of area and that of r_0 $(=2.81\times10^{-15}\,\text{m})$ is of length.

Note. Thomson's theory of X-ray scattering has been verified by a number of workers, notably by Barkla, Hewlet, Coade, Compton, Hagenow and others.

Application of Thomson's formula—Thomson's formula was applied to compute the value of Z and the number of electrons per atom from an experimental evaluation of σ, by the British physicist Barkla.

$$\text{We have : } n = \frac{3\sigma}{8\pi r_0^2} = \frac{3\sigma}{8\pi\times(2.81\times10^{-15})^2} = \frac{\sigma}{6.62\times10^{-29}}$$

Barkla obtained the value of σ/ρ for carbon equal to 0.02 m²/kg (σ expressed in m^{-1} and ρ in kg/m³)

$$\therefore \quad n = \frac{0.02\rho}{6.62\times10^{-29}} = 3\times10^{26}\rho$$

The number of carbons atoms per unit volume is $n_a = \frac{N_A\rho}{12} = \frac{6.025\times10^{23}\times\rho\times10^3}{12} = 5.02\times10^{25}\rho.$

$\therefore$ Number of electrons per atom in carbon is given by $Z = \frac{n}{n_a} = \frac{3\times10^{26}\rho}{5.02\times10^{25}\rho} = 6.$

This is the same as the atomic number of the element carbon in the periodic table. This implies that atomic number of carbon is equal to the number of electrons per atom. This was found to be valid for other elements as well. Later experiments of Chadwick of England with scattering of α-particles vindicated that the number of positive charge in the atomic nucleus is also equal to the atomic number of the element. So, it follows that the nucleus charge of an atom is equal to the total negative charge of the atomic electrons.

2.15. Compton effect or Compton scattering

When a beam of monochromatic X-rays is scattered by a light element such as carbon, the scattered radiation in a particular direction consists not only of the unmodified incident radiation, as it excepted from the classical theory, but also of another new modified radiation of slightly shorter frequency. This type of X-ray scattering is called Compton scattering or Compton effect after the name of its discoverer A. H. Compton of the United States of America.

It was found that the wavelength shift, $d\lambda$, between the two radiations varies with the angle of scattering, but is independent of the wavelength of the primary beam and also the nature of the scatterer.

Explanation—On the basis of the wave theory, the phenomenon is not intelligible. The result however can be explained readily if, as suggested by Compton, taking the corpuscular view, the process is regarded as a collision between a photon and an electron with the laws of conservation of momentum and energy holding good. For, if a photon of energy $h\nu$ strikes an electron it will communicate its kinetic energy to the electron and thereby will itself lose energy. The scattered photon would then have a smaller energy $h\nu'$ and so smaller frequency ν'.

Calculation of wavelength-shift—Let the energy E of the photon before collision be $h\nu$. So its momentum M before collision is $h\nu/c$ where c is the velocity of light in vacuo. Let the energy E' of the photon after collision be $h\nu'$. So its momentum M' after collision is $h\nu'/c$.

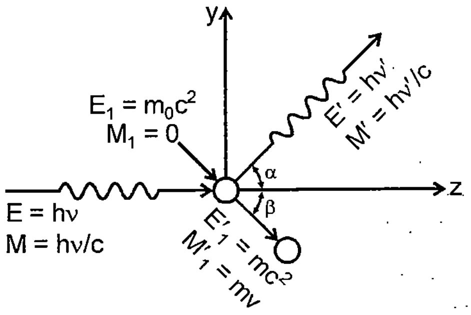


Fig.16-Compton effect

For simplicity, we regard the electron at rest previous to collision so that its energy E_1 is m_0c^2 corresponding to the rest mass m_0 and its momentum M_1 is zero. Let v be the velocity of the electron after collision so that its momentum M_1' is mv and its energy $E_1' = mc^2$.

If α be the angle of scattering and β the angle of derivation of the electron, then from the laws of conservation of momentum and energy, we have

$$\left.\begin{aligned}\frac{h\nu}{c} &= \frac{h\nu'}{c}\cos\alpha + mv\cos\beta \\ 0 &= \frac{h\nu'}{c}\sin\alpha - mv\sin\beta\end{aligned}\right\} \quad \ldots \text{ Momentum conservation} \qquad \ldots(\text{i})$$

$$\text{and} \quad h\nu + m_0c^2 = h\nu' + mc^2 \quad \ldots \text{ Energy conservation} \qquad \ldots(\text{ii})$$

From equation (i)
$$\left.\begin{aligned} mvc\cos\beta &= h(\nu - \nu'\cos\alpha) \\ mvc\sin\beta &= h\nu'\sin\alpha\end{aligned}\right\} \qquad \ldots(\text{iii})$$

Squaring the equations in (iii) and adding $m^2v^2c^2 = h^2(\nu^2 + \nu'^2 - 2\nu\nu'\cos\alpha)$ ...(iv)

From equation (ii) again $mc^2 = h(\nu - \nu') + m_0c^2$

$$\Rightarrow \quad m^2c^4 = h^2(\nu^2 + \nu'^2 - 2\nu\nu') + 2m_0c^2h(\nu - \nu') + m_0^2c^4 \qquad \ldots(\text{v})$$

Substracting (iv) from (v),

$$m^2c^4\left(1 - \frac{v^2}{c^2}\right) = -2h^2\nu\nu'(1 - \cos\alpha) + 2m_0c^2h(\nu - \nu') + m_0^2c^4$$

or, $m_0^2c^4 = -2h^2\nu\nu'(1 - \cos\alpha) + 2m_0c^2h(\nu - \nu') + m_0^2c^4 \quad \left(\because\ m = m_0\Big/\sqrt{1 - v^2/c^2}\right)$

$$\therefore \quad 2h^2\nu\nu'(1 - \cos\alpha) = 2m_0c^2h(\nu - \nu') \text{ or } (1 - \cos\alpha) = \frac{m_0c^2}{h}\frac{(\nu - \nu')}{\nu\nu'} = \frac{m_0c^2}{h}\left(\frac{1}{\nu'} - \frac{1}{\nu}\right) \qquad \ldots(\text{vi})$$

But $d\lambda = \lambda' - \lambda = c\left(\frac{1}{\nu'} - \frac{1}{\nu}\right)$

$$\therefore \quad d\lambda = \frac{h}{m_0c}(1 - \cos\alpha) = \frac{h}{m_0c}2\sin^2\alpha/2, \quad \text{using (vi)},$$

$$= 2\lambda_0\sin^2\alpha/2 \qquad \ldots(\text{vii})$$

where $\lambda_0 = h/m_0c$ is a universal constant as it involves the fundamental constants h, c and m_0. It has the dimension of length and is known as the Compton wavelength of the electron having a value 0.024 Å.

Dicussion—The equation (vii) shows that the Compton shift $d\lambda$ is a function of α, the angle of scattering and is independent of the wavelength λ of the incident radiation and also the nature of the scatterer. The main features of the phenomenon are thus established.

If $\alpha = 90°$, the expression gives $d\lambda = h/m_0c = 0.024$ Å in agreement with the experimental value obtained by Compton.

When $\alpha = 0$, $d\lambda = 0$; and when $\alpha = 180°$, $d\lambda = 2\lambda_0 = 2\times$ Compton wavelength and that is the maximum possible shift.

It therefore follows from (vi) that the Compton shift is most readily detected for radiation where λ is not greater than few angstrom. For $\lambda = 5$Å, $d\lambda = 1\%$; for $\lambda = 1$Å, $d\lambda = 5\%$. Compton effect is also shown by γ-rays.

Unmodified line—The above treatment holds for a free electron, not bound to an atom. If the X-radiation collides with a bound electrons, the whole atom recoils as a result of Compton scattering and m_0 in the formula is to be replaced by M_A, the mass of the atom when it is seen from (vii) that $d\lambda \to 0$ for all values of α since $M_A \gg m_0$. This gives rise to the unmodified line observed by Compton. This conclusion has been verified by Ross using different elements as scatterer.

Note. We assumed in developing the relation (vii) that the electron is not only free but is also at rest. If, however, the electron be not at rest but in motion, the Compton radiation gets broadened.

- **Energy of scattered radiation and recoil electron**

We shall now calculate the energy of the scattered photon and the kinetic energy of the recoil electron.

Energy of scattered photon—From equation (vi) of Art. 2.15

$$\frac{1}{\nu'} - \frac{1}{\nu} = \frac{h}{m_0c^2}(1 - \cos\alpha)$$

$$\therefore \quad \nu' = \frac{1}{\frac{1}{\nu} + \frac{h}{m_0c^2}(1-\cos\alpha)} \text{ or } h\nu' = \frac{h\nu}{\nu\left\{\frac{1}{\nu} + \frac{h}{m_0c^2}(1-\cos\alpha)\right\}}$$

$$= \frac{h\nu}{1 + \frac{h\nu}{m_0c^2}(1-\cos\alpha)} \qquad \text{...(i)}$$

This gives the energy of the scattered photon in terms of the angle of scattering α and the initial energy $h\nu$ of the incident photon.

Kinetic energy of recoil electron—Since the gain in electron energy is equal to the loss in photon energy, the kinetic energy of the recoil electron is given by

$$E_K = h\nu - h\nu'$$

$$= h\nu - \frac{h\nu}{1 + \frac{h\nu}{m_0c^2}(1-\cos\alpha)}, \quad \text{using equation (i)}$$

$$= h\nu = \left\{1 - \frac{1}{1 + \frac{h\nu}{m_0c^2}(1-\cos\alpha)}\right\} = h\nu = \frac{\frac{h\nu}{m_0c^2}(1-\cos\alpha)}{1 + \frac{h\nu}{m_0c^2}(1-\cos\alpha)}$$

which is the expression for the kinetic energy of recoil electron.

$$\therefore\ (E_K)_{\max} = \frac{h\nu}{\left(1+\dfrac{m_0c^2}{2h\nu}\right)},\ \text{when } \alpha = 180°.$$

Relation between angle of scattering and angle of deviation — From the set of equations (iii)

$$\tan\beta = \frac{\nu'\sin\alpha}{\nu-\nu'\cos\alpha} = \frac{\sin\alpha}{\nu/\nu'-\cos\alpha} \qquad \text{...(ii)}$$

$$\text{Again,}\ \frac{1}{\nu'} = \frac{1}{\nu} + \frac{h}{m_0c^2}(1-\cos\alpha)$$

$$= \frac{1}{\nu}\left\{1+\frac{h\nu}{m_0c^2}(1-\cos\alpha)\right\} = \frac{1}{\nu}(1+2a\sin^2\alpha/2),$$

$$\text{where } a = \frac{h\nu}{m_0c^2} \quad \therefore\ \nu' = \frac{\nu}{1+2a\sin^2\alpha/2}$$

Substituting this value of ν' in (ii) above,

$$\tan\beta = \frac{2\sin\dfrac{\alpha}{2}\cos\dfrac{\alpha}{2}}{1+2a\sin^2\dfrac{\alpha}{2}-\cos\alpha} = \frac{2\sin\dfrac{\alpha}{2}\cos\dfrac{\alpha}{2}}{2a\sin^2\dfrac{\alpha}{2}+2\sin^2\dfrac{\alpha}{2}}$$

$$\therefore\ \cot\beta = (1+a)\tan\frac{\alpha}{2} = \left(1+\frac{h\nu}{m_0c^2}\right)\tan\frac{\alpha}{2}$$

which is the relation between the angle of scattering, α of the photon and the angle of deviation β of the recoil electron.

SOLVED PROBLEMS

1. An X-ray tube operates at 30 kV. Calculate the shortest wavelength ofthe emitted X-rays.

Soln. Energy of electrons generating the X-rays = Ve. When the above energy is completely converted to X-rays, maximum frequency v_m or minimum wavelength λ_0 is obtained.

$$\therefore \qquad Ve = hv_m = \frac{hc}{\lambda_0} \qquad \therefore\ \lambda_0 = \frac{hc}{Ve}$$

Now, $V = 30\times10^3\ volt, h = 6.62\times10^{-34} Js, e = 1.602\times10^{-19} C$

$$\therefore \qquad \lambda_0 = \frac{6.62\times10^{-34}\times3\times10^8}{1.602\times10^{-19}\times30\times10^3} = 4.132\times10^{-11} m = 0.4132A$$

2. Electrons bombarding the anode of Coolidge tube produce X-rays of wavelength 1A. Find the energy of each electron at the moment of impact.

Soln. Let v be the electron velocity at the time of impact with the anode.

Therefore, energy of electron $= \frac{1}{2}mv^2 = Ve = hv = \frac{hc}{\lambda}$

where, V is the freqency and λ the wavelength of X-rays, V the operation voltage of the tube, e the eelctronic charge.

Therefore, energy $= \frac{hc}{\lambda} = \frac{6.62\times10^{-34}\times3\times10^8}{1\times10^{-10}} = 19.86\times10^{-16} J = 1.24\times10^4\ eV$

3. The intensity of a beam of X-rays is reduced to 90% of its initial value in passing through a slab of beryllium 3 mm thick and of density $1.84\times10^3\ kg/m^3$. Calculate the mass absorption coefficient.

Soln. Here, $\frac{I}{I_0} = 90\% = 0.9$; x = 3 mm = 3×10^{-3} m and $\rho = 1.84\times10^3\ kg/m^3$

From, $\frac{I}{I_0} = e^{-\mu x}$, we get

$$0.9 = e^{-3\times10^{-3}\mu}$$

$$\therefore \qquad 3\times10^{-3}\mu = -\ln 0.9 = 0.105 \qquad \therefore\ \mu = 35$$

Therefore, mass absorption coefficient, $\mu_m = \frac{\mu}{\rho} = \frac{35}{1.84\times10^3} = 19\times10^{-3} m^2kg^{-1}$

4. An X-ray photon of wavelength 0.1 A is reflected at an angle 90° with its original direction after collision with an electron at rest. Find the energy it loses on collision.

Soln. Incident wavelength, $d\lambda = \frac{h}{m_0c} = 0.024$ Å

Therefore, wavelength of scattered radiation, $\lambda' = 0.1 + 0.024 = 0.124Å = 0.124\times10^{-10} m$

Incident frequency, $v = \frac{c}{\lambda} = 3\times10^8 / 0.1\times10^{-10} = 3\times10^{19} Hz$

Scattered frequency, $v' = \frac{c}{\lambda'} = 3\times10^8 / 0.124\times10^{-10} = 2.419\times10^{19} Hz$

Therefore, energy loss in collision is $\Delta E = hv - hv' = h(v - v') = 6.6\times10^{-34}(3-2.419)\times10^{19}$

$$= 6.6\times0.581\times10^{-15}J = 2.39\times10^{4}eV$$

5. Find the critical voltage that must be applied to an X-ray tube to excite the K-series of copper, given that the K-absorption limit is 1.380Å

Soln. Here, $\lambda_{\min} = 1.380\times10^{-10}m$. From the relation, $Ve = \frac{hc}{\lambda}$

$$Ve = \frac{hc}{e\lambda} = \frac{6.62\times10^{-34}\times3\times10^{8}}{1.602\times10^{-19}\times1.38\times10^{-10}} = 8.98\ kV$$

6. Calculate the fractional change in the wavelength of an X-ray of wavelength 0.400 Å that undergoes a 90° Compton scattering from an electron.

Soln. $\lambda' - \lambda = \frac{h}{M_0c}(1-\cos\theta) = (0.0243\text{ Å})(1-\cos 90^\circ) = 0.0243\text{Å}$

$$\frac{\lambda'-\lambda}{\lambda} = \frac{0.0243\text{ Å}}{0.400\text{ Å}} = 0.0608$$

7. An X-ray of wavelength 0.300 Å undergoes a 60° Compton scattering. Find the wavelength of the scattered photon and the energy of the electron after the scattering.

Soln. $\lambda' = \lambda + \frac{h}{m_0c}(1-\cos\theta) = 0.30\text{ Å} + (0.0243\text{ Å})(1-\cos 60^\circ) = 0.312\text{ Å}$

From energy conservation,

$$\frac{hc}{\lambda} + m_0c^2 = \frac{hc}{\lambda} + K_e + m_0c^2 \quad or \quad \frac{12.4\text{ keV Å}}{0.3\text{ Å}} = \frac{12.4\text{ keV Å}}{0.312\text{ Å}} + K_e \Rightarrow K_e = 1.59\text{ keV}$$

8. In Compton scattering the angle of scattering ϕ of the photon and the change in wave length $\Delta\lambda$ are related by $\Delta\lambda = \frac{h}{m_0c}(1-\cos\phi)$

If the incident photon have wavelength 2.0×10^{-10}m and $\phi = 90^\circ$, deduce

(i) the wavelength of the scattered photon

(ii) the energy of the recoil electron and

(iii) the angle at which the recoil electron appears.

Soln. If λ' and λ are the wavelengths of scattered and incident photons, we have

$$\Delta\lambda = \lambda' - \lambda = \frac{h}{m_0c}(1-\cos\phi) = \frac{6.6\times10^{-34}}{9.1\times10^{-31}\times3\times10^{8}}(1-\cos 90^\circ)$$

$$= \frac{6.6\times10^{-34}}{9.1\times10^{-31}\times3\times10^{8}}(1-0) = 2.4\times10^{-12}\text{m.} = 0.024\text{ Å}$$

(ii) Neglecting the binding energy of the recoil electron the energy of the recoil electron is given by

$$T = (m - m_0)c^2 = h\nu - h\nu' = h(\nu - \nu') = hc\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right)$$

$$= hc\frac{(\lambda' - \lambda)}{\lambda\lambda'} = 6.6 \times 10^{-34} \times 3 \times 10^{8}\left[\frac{2.024 \times 10^{-10} - 2 \times 10^{-10}}{2.024 \times 10^{-10} \times 2 \times 10^{-10}}\right] J$$

$$= \frac{6.6 \times 10^{-34} \times 3 \times 10^{8} \times 0.024 \times 10^{-10}}{2.024 \times 10^{-10} \times 2 \times 10^{-10}} J$$

$$= 1.17 \times 10^{-17} J$$

(iii) The angle θ at which the recoil electron appears is given by

$$\tan\theta = \frac{\lambda \sin\phi}{\lambda' - \lambda\cos\phi} = \frac{2.0 \times 10^{-10} \sin 90°}{2.024 \times 10^{-10} - 2.0 \times 10^{-10} \cos 90} = \frac{2.0 \times 10^{-10}}{2.204 \times 10^{-10}} = \frac{2}{2.024}$$

$$\tan\theta = 0.9980; \quad \theta = \tan^{-1}(0.9980); \quad \theta = 44°57'$$

9. A beam of gamma radiation having photon energy of 510 keV is incident on a foil of aluminium. Calculate the wavelength of the radiation incident at 90° and also the energy and direction of the emission of the corresponding electron.

Soln. Energy of each photon $= h\nu = \frac{hc}{\lambda} = 510 \text{ keV}$

i.e. $\frac{hc}{\lambda} = 510 \times 10^{3} \times 1.6 \times 10^{-19}$ joule; $\Rightarrow \lambda = \frac{hc}{510 \times 10^{3} \times 1.6 \times 10^{-19}}$

$$= \frac{6.6 \times 10^{-34} \times 3 \times 10^{8}}{510 \times 5.6 \times 10^{-16}} = 2.426 \times 10^{-12} m$$

According to Compton scattering

$$\lambda' - \lambda = \frac{h}{m_0 c}(1 - \cos\phi) \quad \Rightarrow \lambda' = \lambda + \frac{h}{m_0 c}(1 - \cos\phi)$$

$$= 2.426 \times 10^{-12} + \frac{6.6 \times 10^{-34}}{9.1 \times 10^{-31} \times 3 \times 10^{8}}(1 - \cos 90°)$$

$$= 2.426 \times 10^{-12} + 2.417 \times 10^{-12} = 4.843 \times 10^{-12} m$$

The energy of recoil electron is given by

$$E = h\nu - h\nu' = h(\nu - \nu') = hc\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right) = \frac{6.6 \times 10^{-34} \times 3 \times 10^{8} \times 2417 \times 10^{-12}}{2.426 \times 4.843 \times 10^{-24}} = 4.072 \times 10^{-14} \text{ joules.}$$

The direction of recoil electron θ is given by

$$\tan\theta = \frac{\lambda \sin\phi}{\lambda' - \lambda\cos\phi} = \frac{2.426 \times 10^{-12} \sin 90°}{4.843 \times 10^{-12} - 2.426 \cos 90°} = \frac{2.426 \times 10^{-12}}{4.843 \times 10^{-12}} = \frac{2.426}{4.843}$$

$$\tan\theta = 0.5008; \qquad \theta = \tan^{-1}(0.5008); \qquad \theta = 26°36'$$

0. In Compton scattering, wavelength shift is towards **[BHU 2010]**
   (a) shorter wavelength
   (b) longer wavelength
   (c) shorter or longer depending on the angle of scattering
   (d) shorter or longer side depending on photon energy

Soln. The incident photon loses its energy to the target particle and hence its wavelength increases.

Correct option is (b)

1. X-rays of 10.0 p.m. are scattered from a target. The maximum energy of the recoil electron (h/mc for electron is 2.426×10^{-12} m) is **[BHU 2011]**

(a) 3.27×10^{-15} joules
(b) 6.54×10^{-15} joules
(c) 5.64×10^{-15} joules
(d) 2.37×10^{-15} joules

Soln. Energy of the incident x-ray photon $h\nu=\frac{hc}{\lambda}=\frac{1240\ eV-nm}{10\times10^{-3}\ nm}=124\ keV$

Maximum energy of the recoil electron will be

$$E_{k,\max}=\frac{h\nu}{1+\frac{m_0c^2}{2h\nu}}=\frac{124}{1+\frac{511}{248}}keV=40.52\ keV=6.48\times10^{-15}J$$

12. The atomic number Z of the element which has a K_α X-ray line of wavelength 0.180 nm (given that Rydberg constant $R=1.097\times10^7\ m^{-1}$) is **[B.H.U-2011]**

(a) 27 (b) 26 (c) 52 (d) 54

Soln. Using Moseley's law,

$$\bar{\nu}=\frac{3}{4}R(Z-\sigma)^2$$

where $\bar{\nu}$ is the wave number and σ is screening constant. For K_α X-ray line $\sigma=1$.

$$\Rightarrow\ \frac{1}{\lambda}=\frac{3}{4}\times1.097\times10^7\times(Z-1)^2$$

$$\Rightarrow\ \frac{1}{0.180\times10^{-9}}=\frac{3}{4}\times1.097\times10^7(Z-1)^2$$

$$\Rightarrow\ (Z-1)^2=\frac{4\times10^9}{0.180\times3\times1.097\times10^7}$$

$$\Rightarrow\ (Z-1)^2=675.24$$

$$\Rightarrow\ (Z-1)=25.98\approx26\ \Rightarrow\ Z=27$$

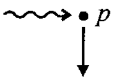
Correct option is (a)

13. Photons of wavelength λ scatter elastically from free protons initially at rest. The wavelength of the protons scattered at 90° is increased by **[BHU 2012]**

(a) $\frac{\lambda}{137}$ (b) $\frac{h}{m_eC}$ (c) $\frac{h}{m_pC}$ (d) 0

where m_e and m_p are the rest mass of the electron and proton.

Soln. The shift in the wavelength is $\Delta\lambda = \frac{h}{m_p c}(1-\cos\theta)$, when $\theta = 90°$, $\Delta\lambda = \frac{h}{m_p c}$.



Correct option is (c)

14. According to Moseley's law, the relation between the atomic number Z and frequency ν is given by

(a) $\nu \propto (Z-b)$ (b) $\nu \propto Z$ (c) $\nu \propto (Z-b)^3$ (d) $\nu \propto (Z-b)^2$ **[B.H.U-2013]**

Soln. According to Moseley's law,

$$\nu = \frac{3}{4} Rc(Z-\sigma)^2$$

where ν is the frequency, R is the Rydberg constant, c is the speed of light, σ is the screening constant and Z is the atom number.

$$\therefore \ \nu \propto (Z-b)^2$$

Correct option is (d)

15. The Compton shift in X-ray wavelength depends on :

(a) Scattering angle only (b) Scattering angle and wavelength of X-ray both

(c) Wavelength of X-ray only (d) None of the (a), (b) and (c) **[B.H.U-2014]**

Soln. The Compton shift in X-ray wavelength is given by

$$\Delta\lambda = \frac{2h}{m_0 c}\sin^2\frac{\alpha}{2} \qquad \text{... (i)}$$

where *h* is Planck's constant, *c* is speed of light, α is the scattering angle, m_0 is the electron rest mass,

From (i) one can conclude that the Compton shift in X-ray wavelength depends only on the scattering angle.

Correct option is (a)

6. X-ray spectrum of a cobalt larget ($Z = 27$) contains strong k_α line of wavelength 0.1785 nm and a weak line having wavelength of 0.1537 nm due to impurities. The atomic number of impurity element is

(a) 29 (b) 24 (c) 23 (d) 30 **[B.H.U-2015]**

Soln. According to Moseley's law :

$$\bar{\nu} \propto (Z-\sigma)^2$$

where $\bar{\nu}$ is the wave number and σ is screening constant

$$\frac{1}{\lambda} \propto (Z-\sigma)^2$$

For strong K_α line

$$\frac{1}{0.1785\,nm} \propto (27-1)^2 \qquad \text{... (i)}$$

For weak K_α line

$$\frac{1}{0.1537\,nm} \propto (Z-1)^2 \qquad \text{... (ii)}$$

For K_α lines $\sigma = 1$

Dividing (i) by (ii),

$$\frac{1}{0.1785\,nm} \div \frac{1}{0.1537\,nm} = \frac{26^2}{(Z-1)^2}$$

$$\Rightarrow \ (Z-1)^2 = \frac{26^2 \times 0.1785}{0.1537} = 785.075 \Rightarrow \ Z-1 \approx 28 \ \Rightarrow \ Z = 29$$

Correct option is (a)

7. Due to Compton scattering, the change in wavelength is **[BHU 2015]**
(a) dependent on electron charge
(b) dependent on electron mass and charge
(c) dependent on electron mass and scattering angle both
(d) dependent on scattering angle only

oln. The change in wavelength in Compton scattering is

$$\Delta\lambda = \frac{h}{m_0 c}(1-\cos\theta)$$

where m_0 is the rest mass of the electron and θ is the scattering angle.

Correct option is (c)

8. The change in the wavelength of the scattered light in the Compton scattering is equal to the Compton wavelength of the electron when: **[BHU 2016]**
(a) the deflection angle of photon is $\pi/2$ (b) the deflection angle of photon is 0
(c) the deflection angle of photon is $\pi/3$ (d) the deflection angle of photon is π

oln. The change in the wavelength of the scattered light is

$$\Delta\lambda = \frac{h}{m_0 c}(1-\cos\theta)$$, where θ is the deflection angle.

Since, wavelength shift is equal to compton wavelength, then

$$\Delta\lambda = \frac{h}{m_0 c} \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$$

Correct option is (a)

9. The maximum value of the change in the wavelength of the scattered light in the case of Compton scattering is:
(a) 0.024 Å (b) 0.048 Å (c) 0.0 Å (d) 0.072 Å **[BHU 2016]**

oln. The change in the wavelength of the scattered light is $\Delta\lambda = \frac{h}{m_0 c}(1-\cos\theta)$.

The change would be maximum when $\cos\theta = -1$.

Therefore, $(\Delta\lambda)_{\max} = \frac{2h}{m_0 c} = 0.048$ Å

Correct option is (b)

0. In a Compton scattering experiment, the recoil electron has to be :
(a) relativistic (b) non-relativistic **[BHU 2016]**
(c) Sometimes relativistic (d) Sometimes non-relativistic

oln. In Compton scattering experiment, the incident photon has energy of order of keV and hence the recoil electron also has the kinetic energy of order of keV. Moreover, the rest mass energy of the electron is of order of keV. So the rest mess energy of the electron is comparable to its kinetic energy. This is why the recoil electron is relativistic in nature.

Correct option is (a)

21. If the incident photon has a wavelength equal is 2Å and its angle of deflection is 90°, the kinetic energy of the recoil electron in the Compton scattering would be **[BHU 2016]**

(a) 1.17×10^{-17} ergs (b) 1.17×10^{-17} joules

(c) 1.17×10^{-17} KeV (d) 1.17×10^{-17} eV

Soln. Energy of the incident photon $h\nu=\dfrac{hc}{\lambda}=\dfrac{1240\,eV-nm}{0.2\,nm}=6.2\,keV$

Kinetic energy of the recoil electron

$$E_k=\frac{\dfrac{h^2\nu^2}{m_0c^2}(1-\cos\alpha)}{1+\dfrac{h\nu}{m_0c^2}(1-\cos\alpha)}=\frac{\dfrac{(6.2)^2}{(511)}(1-0)}{1+\dfrac{6.2}{511}(1-0)}keV=74.11\,eV=1.18\times10^{-17}\,J$$

Correct option is (b)

22. In the Compton scattering there is an **[BHU 2016]**
(a) inelastic collision of a photon with a free e^-
(b) inelastic collision of a photon with a proton
(c) inelastic collision of a photon with a neutron
(d) inelastic collision of a photon with a bound electron of an atom

Soln. A photon collide inelastically with a free electron in the Compton scattering.

Correct option is (a)

23. In the case of Rutherford scattering, if the impact parameter is zero, the deflection of the α-particle would be by the : **[B.H.U-2016]**

(a) Angle $\dfrac{\pi}{2}$ (b) Angle $\dfrac{\pi}{4}$ (c) Angle π (d) Angle 2π

Soln. In case of Rutherford scattering, impact parameter b is given by, $b=\dfrac{Zke^2}{2(K.E)}\cot\dfrac{\theta}{2}$

where Z is the atomic number of target nuclei, $k=\dfrac{1}{4\pi\epsilon_0}$, e is the electron charge, K.E, is the kinetic energy of α-particle and θ is the scattering angle.

Now, $b=0\Rightarrow\cot\dfrac{\theta}{2}=0\Rightarrow\dfrac{\theta}{2}=\dfrac{\pi}{2}\Rightarrow\theta=\pi$

Correct option is (c)

24. Rutherford scattering is important because it sheds light on **[B.H.U-2016]**
(a) the structure of the molecule (b) the size of the molecule
(c) the quantum nature of the molecule (d) the structure of the atom

Soln. Conclusions drawn from Rutherford scattering experiment:
(i) an atom was much more than just empty space and scattered electron.
(ii) an atom must have a positively charged center that contains most of its mass.
Therefore, Rutherford scattering is important because it sheds light on the structure of the atom.
Correct option is (d)

25. Separate Compton effect experiments are carried out using visible light and X-rays. If the scattered radiation is observed at the same scattering angle, then **[HCU 2012]**

(a) the X-rays have greater shift in wavelength and greater change in photon energy.

(b) the two radiations have the same shift in wavelength and X-rays have a greater change in photon energy.

(c) the two radiations have the same shift in wavelength and visible light has greater change in photon energy

(d) the two radiations have the same shift in wavelength and the same change in photon energy.

Soln. The change in the wavelength of the incident radiation is

$$\Delta\lambda = \frac{h}{m_0 c}(1-\cos\theta)$$

Since the change depends only on the scattering angle, the change in the wavelength of both the radiation would be same.

$$\Rightarrow \lambda_f - \lambda_i = \frac{h}{m_0 c}(1-\cos\theta)$$

$$\Rightarrow \lambda_f = \lambda_i - \frac{h}{m_0 c}(1-\cos\theta) \quad \text{... (i)}$$

The change in the photon energy is $\Delta E = hc\left(\frac{1}{\lambda_i} - \frac{1}{\lambda_f}\right) = \frac{hc}{\lambda_i}\frac{\Delta\lambda}{\lambda_f} = \frac{hc\,\Delta\lambda}{\lambda_i(\lambda_i + \Delta\lambda)}$

Since, $(\lambda_i)_{\text{visible}} > (\lambda_i)_{\text{X-ray}}$

$(\Delta E)_{\text{X-ray}} > (\Delta E)_{\text{visible}}$

Correct option is (b)

26. A scientist performing the Compton scattering experiment finds that the incident wavelength λ_1 is shifted by 1.5 percent for the scattering angle of 120°. The waelength λ_1 is given by **[HCU 2013]**

(a) 0.242 nm (b) 2.420 nm (c) 24.200 nm (d) 0.024 nm

Soln. The according to compton scattering, the wavelength of the scattering light is given by

$$\lambda' = \lambda + \frac{h}{m_e c}(1-\cos\theta) \Rightarrow \lambda' = \lambda_1 + \frac{h}{m_e c}(1-\cos 120^\circ)$$

$$\Rightarrow \lambda' = \lambda_1 + \frac{h}{m_e c}1.5 \qquad \Rightarrow \lambda' - \lambda_1 = \frac{h}{m_e c}1.5$$

$$\text{Given : } \Delta\lambda = \lambda' - \lambda_1 = \frac{15}{100}\times\lambda_1 = \frac{h}{m_e c}\times 1.5$$

$$\Rightarrow \lambda_1 = \frac{h}{m_e c}\times 10 = \frac{6.62\times10^{-3}}{9.1\times10^{-31}\times3\times10^{8}} = 0.24\times10^{-10}\text{ m} = 0.024\text{ nm}.$$

Correct option is (d)

PRACTICE SET

1. What is the wavelength of X-ray emitted when 100 kV electrons strike a target? What is the frequency?

[0.124Å; 24.2×10^{18}]

2. The mass absorption coefficient of silver is 3.8 m^2kg^{-1} for X-rays of $\lambda = 0.4\,\text{Å}$. find the atomic mass absorption coefficient of silver. [6.087×10^{-25} m^2/atom]

3. Determine the maximum speed attained by electrons in an X-ray tube operating at 10 kV.

[5.964×10^7 m/s]

4. A photon with energy 0.46 MeV is scattered at an angle 120° by a free electron at rest. Find (i) the energy of the scattered photon and (ii) the energy transferred to the electron.

[0.20 MeV; 0.26 MeV]

5. When photons of wavelength 0.024Å are incident on a target, scattered photons are detected at an angle 60°. Calculate the wavelength of the scattered photon and the angle at which it is scattered.

[0.036 Å; 40.9°]

Chapter 3

Wave Properties of Particle

3.1 Wave particle Duality:

According to classical mechanics, the particle can behave as a particle and wave can behave like a wave. Particle is characterized by energy and momentum and wave is characterized byamplitude and wave vector. In 1923, De-broglie postulated that a moving particle can behave as a wave. The de-broglie wavelength associated with a particle of rest mass 'm_0' moving with a velocity 'v' is given by

$$\lambda = \frac{h}{p} = \frac{h}{\gamma m_0 v}, \qquad \text{where } \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \text{ is Lorentz factor}$$

This hypothesis is experimentally verified by Davission-Germer experiment in 1927. If the De-broglie wavelength is comparable to size of the particle, then the wave nature of the particle can be experimentally detected. Otherwise, if the De-broglie wavelength is very small, the wave nature cannot be detected experimentally.

Schrodinger modified the concept wave nature of matter i.e. particle is not like a single plane wave but is like a group of waves i.e Wave packet. It is just the superposition of waves different wavelengths and frequencies.

Phase Velocity:

A plane wave travelling along positive direction of x-axis can be represented as,

$$\psi(x,t) = A\sin(\omega t - kx) = A\sin\omega\left(t - \frac{x}{v}\right)$$

where, ω is the angular frequency, v is the velocity and A is the amplitude of the wave.

The phase ϕ of the wave at any time 't' at a distance 'x' from the origin is $\phi(x,t) = (\omega t - kx)$. Phase velocity v_p of the wave is the velocity with which displacement of a given phase, moves in the forward direction. It is also the velocity of a monochromatic wave with which crest or trough of the wave travels in a medium. In a dispersive medium, v_p is different for different wavelengths. The phase velocity of the propagating monochromatic wave is given by, $v_p = \frac{\omega}{k} = \frac{E}{p}$.

From the relativistic mass energy relation, $v_p = \frac{E}{p} = \frac{\gamma m_0 c^2}{\gamma m_0 v} = \frac{c^2}{v}$

Since, the phase velocity of a de-broglie wave is always greater than the velocity of light in free space. But according to the special theory of relaticvity, no energy can be transmitted with velocity greater than c.

This suggests that the phase velocity has no physical significance and we can not respresent the wave nature of a moving particle by a single plane wave.

Group velocity:

A wave group or wave packet is described as the superposition of individual waves of different wavelengths. These waves interfere with each other and results in variation of amplitude. These defines the wave group shape. If the medium is non-dispersive, then phase velocity of the individual waves will not change with wavelength. As a result, wave packet travels with common phase velocity. But, in a dispersive medium, phase velocity of individual waves will be different. As a result, wave packet moves with a different velocity than phase velocity which is known as group velocity 'v_g'.

Let, the wave group is formed by the superposition of two monochromatic waves having same amplitude but differing in angular frequency by an amount '$\Delta\omega$' and in wave number 'Δk' i.e

$$\psi_1 = A\sin(\omega t - kx)$$

$$\psi_2 = A\sin\left[(\omega+\Delta\omega)t - (k+\Delta k)x\right]$$

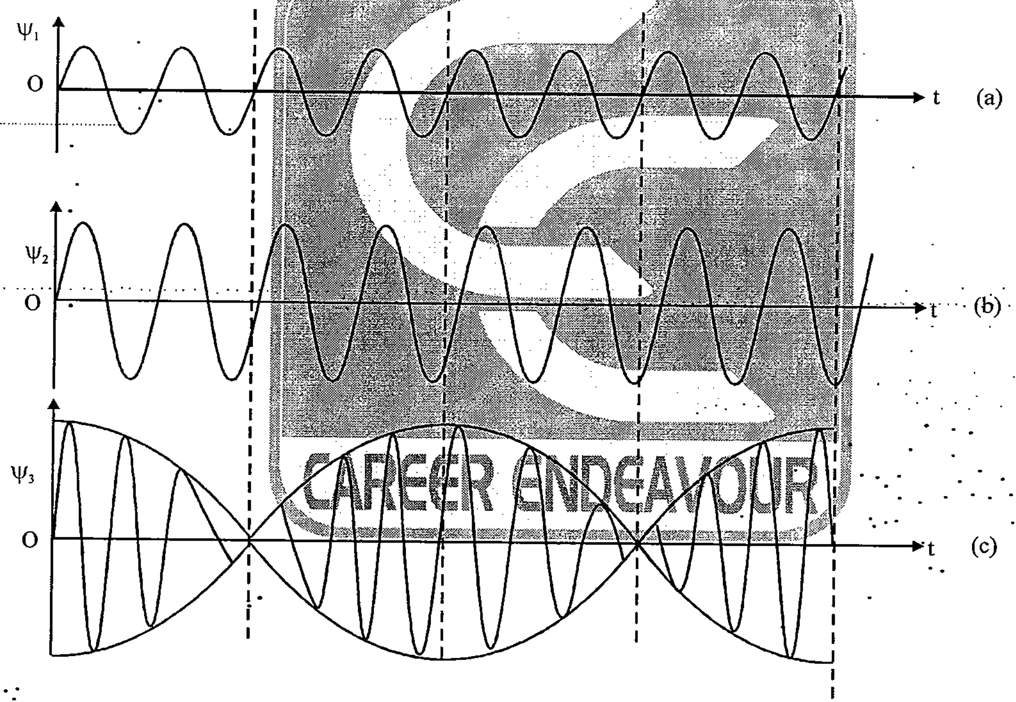


Figure : Superposition of two sinusoidal waves of slightly different frequncies [(a) and (b)] Curve (c) represents the resultant motion.

The resultant wave i.e. wave packet can be written as,

$$\psi = 2A\cos\left[\frac{1}{2}(\Delta\omega t - \Delta kx)\right]\sin(\omega t - kx)$$

Now, group velocity is the velocity with which maximum amplitude or the envelope of the wave packet moves and it is given as: $v_g = \frac{d\omega}{dk} = \frac{dE}{dp}$.

From the relativistic mass-energy relation :

$$E^2 = p^2c^2 + m_0^2c^4 \qquad \Rightarrow 2E\frac{dE}{dp} = 2pc^2$$

Therefore, $$v_g = \frac{dE}{dp} = \frac{pc^2}{E} = \frac{\gamma m_0 vc^2}{\gamma m_0 c^2} = v$$

Relation between Phase velocity and Group velocity: $v_g = v_p - \lambda\frac{dv_p}{d\lambda}$

For, non-dispersive medium, all waves travel with same phase velocity: $\frac{dv_p}{d\lambda} = 0 \Rightarrow v_g = v_p$

Dispersive medium are of two types:

(i) Normal dispersive medium: $\frac{dv_p}{d\lambda} = +ve \Rightarrow v_g < v_p$

(ii) Anamolous dispersive medium: $\frac{dv_p}{d\lambda} = -ve \Rightarrow v_g > v_p$

3.2 Heisenberg's Uncertainty principle:

According to classical mechanics, a moving particle has a definite momentum and position in space. Both the position and momentum of such a particle can be definitely determined. Recently, the classical view has been proved to be an approximation and is inadquate for describing the dynamical behaviour of the macroscopic objects. It cannot describe satisfactorily the behaviour of sub-atomic particles.

In quantum mechanics, a particle is described by a wave packet of linear extension (Δx) which surrounds the position of the classical particle and moves with group velocity. According to Max Born's probability interpretation, the particle may be found everywhere within the wave packet. This means that the position of the particle is uncertain within the limits (Δx) of the wave packet. So, we may say that the particle may be found anywhere within (Δx). Again, a wave packet is formed by the superposition of large number of waves having different wavelengths, the wavelength the wave packet cannot be said definitely. What we can say that the wavelength of the wave packet lies within certain range λ to $\lambda + \Delta\lambda$. As the momentum of a particle is related to wavelength by

$$p = \frac{h}{\lambda}, \Rightarrow \Delta p = -\frac{h}{\lambda^2}\Delta\lambda$$

the particle momentum is also within p to $p + \Delta p$. So, we cannot definitely say what is the momentum of the particle. Hence, because of the wave nature of partice there is uncertainty (Δx) in position of the particle as well as uncertainty in corresponding momentum (Δp).

If the number of superposed wave as well as their range of wavelength $(\Delta\lambda)$ be increased, then the linear speed (Δx) of the wave packet decreases. But simultaneously the uncertainty (Δp) in momentum increases as $\Delta p \propto \Delta\lambda$. When $\Delta\lambda = \infty$, $\Delta p = \infty$ and $\Delta x = 0$, i.e, the wave pocket reduces to a point. In this position, there is no uncertainty in the position of the particle.

Now for a definite $\lambda = 0, \Delta\lambda = 0$, then $\Delta p = 0\ (\because \Delta p \propto \Delta\lambda)$, then there is no uncertainty in the momentum of the particle but in this case $\Delta x = \infty$, i.e., the position uncertainty is infinite since the associated wave with the particle may now extend up to ∞.

Statement of the uncertainty principle:

The uncertainty principle states that, it is fundamentally impossible to determine sumultaneously the position and the momentum of a particle to an accuracy better than one quantum of action h.

The mathematical expression of the particle is,

$$\Delta x \Delta p_x \geq \hbar,$$

where, $\hbar = \frac{h}{2\pi} = 1.0545 \times 10^{-31} Js$, h being the Planck's constant $= 6.62 \times 10^{-34} Js$. Here Δx is the error (uncertainty) in the determination of position of the particle and Δp_x is the error (uncertainty) in the determination of x-component of momentum. If the position be determined accurately, then Δx becomes smaller and consequently larger is Δp_x and vice versa. It is to be kept in mind that the uncertainties are not in the appratus used for the determination but in nature itself.

In three dimensions the uncertainty relations are expressed as follows:

$$\Delta x \Delta p_x \geq \hbar; \quad \Delta y \Delta p_y \geq \hbar; \quad \Delta z \Delta p_z \geq \hbar$$

(a) If $\Delta x = 0$, then, $\Delta p_x = \frac{\hbar}{\Delta x} \Rightarrow \Delta p_x = \frac{\hbar}{0} = \infty$.

This means that, if the uncertainty in the position measurement of a moving particle be zero, then at the same moment the uncertainty in the momentum measurement will be infinity.

(b) If the momentum of a moving particle at any instant be determined accurately (i.e., $\Delta p_x = 0$), then the uncertainty (error) in the determination of position of the particle will be infinite.

(c) For a particle of mass 'm' moving with velocity v_x along the x-direction, its momentum is $p_x = mv_x$,

$$\Delta x \Delta p_x = \hbar, \quad \Delta x \Delta v_x = \frac{\hbar}{m}$$

For a heavy particle $m \to \infty$ and in that case, $\Delta x \Delta v_x = \frac{\hbar}{\infty} = 0$

For such a particle both the position and momentum can be determined accurately. This is true for macroscopic bodies whose motions are discussed in classical mechanics. Thus the uncertainty relation has no meaning in classical mechanics. But for microscopic objects such as electron, proton, atom, etc, quantum mechanics is applicable and as such the uncertainty relation is a reality in describing the dynamical behaviour of such objects.

Time-Energy Uncertainty Relation:

The time-energy uncertainty principle is stated as follows:

In any simultaneous determination of time and energy of a moving particle, the product of the uncertainties is equal to the Planck's constant, $\Delta E \Delta t \geq \hbar$

Here, ΔE is the uncertainty in the measurement of energy of Δt, the corresponding uncertainty in the measurement of time.

If ΔE be the maximum uncertainty in the determination energy of a system in a particular state, then Δt is the minimum, i.e., the system remains for a minimum time in the state. On the other hand, if a system remains in a particular state for a maximum interval of time, then the uncertainty in the measurement of energy is the minimum.

Angular momentum- angular position Uncertainty Relation:

$$\Delta L.\Delta\phi \geq \hbar$$

Where, ΔL is the uncertainty in the angular momentum of the particle and $\Delta\phi$ is the uncertainty in the angular position of the particle.

Causality in Quantum Mechanics:
According to the principle of causality, knowledge of the behaviour of a system at one instant is sufficient for the prediction of its future behaviour. Thus, it is the quantum analogue of the classical principle that, the future behaviour of a particle can be determined if its position and velocity are known at a given instant. The law of causality, however, holds strictly in classical mechanics. As no observation can be made without disturbing a microsystem, it does not hold good for atomic systems.
According to Max Born, the principle of causality can be viewed from two alternate angles. In wave-particle picture the principle of causality breaks down. But if the system be described by a wave function $\psi(\vec{r},t)$ which satisfies a differential equation of first order in time, then the behaviour of the system at a point $P(\vec{r})$ at any time 't' can be described uniquely if ψ is known at t = 0 at the same point $P(\vec{r})$. This shows that even for microsystems, the principle of causality is true. But the measurements on a microsystem cannot be carried out without seriously disturbing it. Hence it is impossible to determine the state of the system unambiguousy at t = 0. Hence it is impossible to determine the state of the system unambiguously at t = 0. Thus the principle of causality breaks down. Therefore, the probabilistic interpretation of the observed events are to be adopted in accordance with the uncertainty principle.

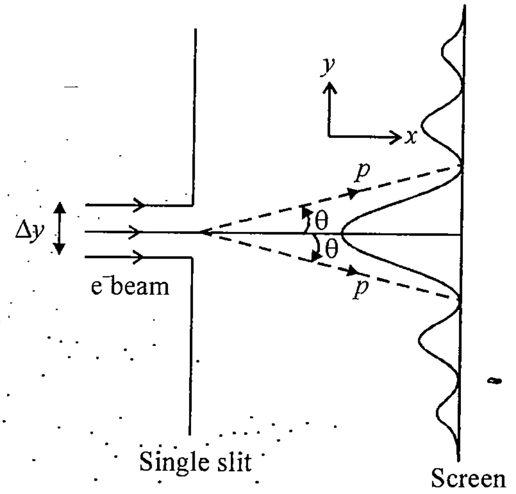
Concept of Bohr Orbit in view of uncertainty principle:
We have the energy-time uncertainty relation, $\Delta E.\Delta t \geq \hbar$
According to the concept of Bohr, each electron revolves around the nucleus in quantised stationary orbits with definite energy E. So, the uncertainty is $\Delta E = 0$.

So, $$\Delta t = \frac{\hbar}{\Delta E} = \frac{\hbar}{0} = \infty$$

Hence, all energy states of the Bohr atom must have infinite lifetime $(\tau = \infty)$. This implied that radiation would not occur due to the transition of electrons. But actually $\Delta t \sim 10^{-8}$ sec leading to the occurrence of emission of radiation due to electronic transition. Thus the concept of Bohr orbit violets the Heisenberg's uncertainty principle.

Again, if Δt is finite, then $\Delta E = \frac{\hbar}{\Delta t}$ is also finite. This means that the energy levels in the excited state must have a finite spread. As a result the radiation emitted due to the transition of an electron from the excited state to the ground state is not monochromatic. This leads to the conclusion that the spectral lines can never be infinitely sharp but must have a natural broadening. This is contradiction with Bohr's theory of spectral series.

Electron difraction through a single slit:
Let a monoenergetic parallel beam of electrons moving along the x-direction be incident on a single split of width Δy. As the electron in motion is associated with de-Broglie wave, the electron beam after passing through the slit undergoes diffraction producing a fraunhofer diffraction pattern on the photoplate perpendicular to the x-axis.



The first minimum of the diffraction pattern satisfies the relation $\Delta y \sin\theta = \lambda$, where θ is the angle of diffraction for the first order and λ is the de-Broglie wavelength. Now the uncertainty in the position of an electron in the slit is Δy as it is very difficult to locte exactly the position of the electron in the slit,

$$\Delta y = \frac{\lambda}{\sin\theta}$$

Before incidence, let the momentum of the electron along the x-direction be p. After diffraction if the electron moves along OC, then the component of momentum p along the y-direction is $p\sin\theta$. Since the electron may be anywhere within $-\theta$ to $+\theta$, the y-component of momentum must be anywhere between $-p\sin\theta$ to $p\sin\theta$.

Thus the uncertainty in the momentum measurement of the electron is,

$$\Delta p_y = p\sin\theta - (-p\sin\theta) = 2p\sin\theta = 2\frac{h}{\lambda}\sin\theta$$

Therefore, $\Delta y \cdot \Delta py = \dfrac{\lambda}{\sin\theta} \times \dfrac{2h}{\lambda}\sin\theta = 2h \geq h$

To decrease the uncertainty in the determination position of the electron, Δy must be decreased. But the above equation shows that a decrease in Δy leads to the increase in θ and the diffraction pattern will be wider. This will cause a larger uncertainty in Δp. Thus the electron diffraction experiment established the truth of uncertainty principle.

Consequences of Heisenberg's Uncertainty principle:

(1) Non-existance of electrons within the nucleus:

Suppose electron exist within the nucleus, then uncertainty in the position of nucleus will be of the order of twice of radius of the nucleus i.e. $\Delta x = 2 \times 10^{-15}\,m$.

Then the minimum energy of the electron is given by

$$E_{\min}^2 = p_{\min}^2 c^2 + m_0^2 c^4 \Rightarrow E_{\min} = \sqrt{(\Delta p)^2 c^2 + m_0^2 c^4}$$

$$\Rightarrow E_{\min} = \sqrt{\left(\frac{\hbar}{2\Delta x}\right)^2 c^2 + m_0^2 c^4} \approx 10 MeV$$

Therefore, if electron exist within the nucleus, then minimum energy of the electron will be approximately 10 MeV. But, experimentally energy comes out to be of the order of 4 MeV. Thus, we can say that electron does not exist within the nucleus.

(2) **Ground state energy of a linear harmonic oscillator:**

Let a particle of mass m is executing SHM along x-axis and let at any time 't' its position is 'x' from the mean position. Therefore, $\Delta x = x$

Total energy of LHO $= E = E_k + E_P = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \approx \frac{(\Delta p)^2}{2m} + \frac{1}{2}m\omega^2(\Delta x)^2$

$$= \left(\frac{\hbar}{2x}\right)^2 \frac{1}{2m} + \frac{1}{2}m\omega^2 x^2 = \frac{\hbar^2}{8mx^2} + \frac{1}{2}m\omega^2 x^2$$

For minimum energy, $\frac{dE}{dx} = -\frac{\hbar^2}{4mx^3} + m\omega^2 x = 0 \Rightarrow x^2 = \frac{\hbar^2}{2m\omega}$

So, the minimum energy of the oscillator $= E_{\min.} = \frac{\hbar^2}{8m}\frac{2m\omega}{\hbar} + \frac{1}{2}m\omega^2\frac{\hbar^2}{2m\omega} = \frac{1}{2}\hbar\omega$

SOLVED PROBLEMS

1. What voltage must be applied to an electron microscope to produce electrons of wavelength 0.40Å?

Soln. De-Broglie wavelength of an electron of mass m, charge 'e' and moving under a potential of V volts is,

$$\lambda = \frac{h}{\sqrt{2meV}}$$

From the above relation, $V = 936.3$ volts

2. What is the de-Broglie wavelength of an electron moving with velocity, $v = \frac{3}{5}c$?

Soln. The de-Broglie wavelength of a particle of mass 'm' moving with velocity 'v' is

$$\lambda = \frac{h}{mv} = \frac{h}{m_0 v}\sqrt{1 - v^2/c^2} = 0.0323\ A^0$$

3. You have 10 eV photon and 10 eV electron. Which one has shorter wave-length?

Soln. Wavelength of photon $\lambda = \frac{h}{p} = \frac{hc}{E} = \frac{6.62 \times 10^{-34} \times 3 \times 10^8}{10 \times 1.6 \times 10^{-19}} = 1240\text{Å}$

De-Broglie wavelength of the electron

$$\lambda = \frac{h}{m_0 v} = \frac{h}{\sqrt{2m_0 E}} = 3.88\text{A}^0$$

The de-Broglie wavelength λ of a particle is $\lambda \propto \frac{1}{\sqrt{E}}$ and for a photon $\lambda \propto \frac{1}{E}$. This shows that de-Broglie wavelength of an electron is much smaller as compared to that of a photon for the same energy E.

4. Energy of a particle at absolute temperature T is of the order of kT. Calculate the wavelength of thermal neutrons at 27°C. [Given, the mass of the neutron $= 1.67 \times 10^{-24}\ gm$. Planck's constant, $k = 6.67 \times 10^{-27}\ erg\sec$ and Boltzman constang, $k = 8.6 \times 10^{-5}\ eVk$]

Soln. De-Broglie wavelength of the neutron is

$$\lambda = \frac{h}{\sqrt{2mE}} = \frac{h}{\sqrt{2mkT}} = 1.78\text{Å}$$

5. A beam of monenergetic neutrons corresponding to 27°C is allowed to fall on a crystal. A first order reflection is observed at a glancing angle 30°. Calculate the interplanar spacing of the crystal.

Given, Planck constant, $h = 6.62\times10^{-34}\,Js$

Mass of neutron, $m = 1.67\times10^{-27}\,kg$

Boltzmann constant, $k = 1.38\times10^{-23}\,J/K$

Soln. From Bragg law of diffraction we have,

$$2d\sin\theta = n\lambda$$

where, the glancing angle, $\theta = 30°$, order number of diffraction, n = 1

$$\therefore \quad 2d\sin 30° = \lambda,\ or,\ 2d\times\frac{1}{2} = \lambda,\ or,\ \lambda = d$$

De-Broglie wavelength of the neutron is

$$\lambda = \frac{h}{\sqrt{2mE}} = \frac{h}{\sqrt{2mkT}} = 1.78\text{Å}.$$

Therefore, the interplanar spacing is 1.78 Å.

6. What is the de-Broglie wavelength of electrons of energy 0.1 MeV using both relativistic and non-relativistic expressions for λ? What is the percentage error in using non-relativistic expression for λ?

Soln. The non-relativistic de-Broglie wavelength of an electron of mass 'm' and energy eV is,

$$\lambda_{nr} = \frac{h}{\sqrt{2meV}} = \frac{12.26}{\sqrt{V}} = \frac{12.26}{\sqrt{0.1\times10^6}} = 0.0388\text{Å}$$

The relativistic expression for the de-Broglie wavelength of the same electron is,

$$\lambda_r = \frac{h}{\sqrt{2meV\left(1+\frac{eV}{2mc^2}\right)}} = 0.0371\text{Å}$$

The percentage error in using non-relativistic expression, is

$$\frac{\lambda_{nr}-\lambda_r}{\lambda_{nr}}\times100\% = \frac{(0.0388-0.0371)}{0.0388}\times100\% = 4.38\%$$

7. Assume that the uncertainty in the location of a particle is equal to its de-Broglie wavelength. Calculate the uncertainty in the velocity of the particle.

Soln. According to the uncertainty principle,

$$\Delta x\Delta p_x = \hbar \quad \text{[where } \Delta x \text{ and } \Delta p_x \text{ are the uncertainties in position and momentum]}$$

$$\Rightarrow \Delta v_x = \frac{\hbar}{m\Delta x} = \frac{\hbar}{m\lambda} \quad \left[\because \Delta x = \lambda = \frac{h}{mv}\right]$$

$$= \frac{\hbar}{m\times\frac{h}{mv}} = \frac{v}{2\pi}$$

Suppose that the momentum of a certain particle can be measured to an accuracy of one part in 1000. Determine the minimum uncertainty in the position of the particle if the particle is (i) a $5\times10^{-3}\,kg$ mass moving with a speed of 2 m/s; (ii) an elecron moving with a speed of $1.8\times10^{8}\ m/s$

Soln. Given: $\dfrac{\Delta p}{p}=10^{-3}\ \Rightarrow \Delta p=10^{-3}p=10^{-3}mv$

Therefore, the minimum uncertainty in position of the particle will be

$$(\Delta x)_{\min}=\frac{h}{4\pi mv\times10^{-3}}$$

(i) $(\Delta x)_{\min}=\dfrac{h}{4\pi m_0 v\times10^{-3}}=5.28\times10^{-20}\,\text{Å}$

(ii) $(\Delta x)_{\min}=\dfrac{h\sqrt{1-v^2/c^2}}{4\pi m_0 v\times10^{-3}}=2.57\text{Å}$

The average time interval that elapses between the excitation of an atom and the time it radiates energy is 10^{-8} sec. Calculate the uncertainty in the energy of emitted photons and the limit of accuracy with which the frequency of the emitted radiation may be determined.

Soln. Uncertainty in energy of the emitted photons will be

$$\Delta E=\frac{h}{\Delta t}=4.14\times10^{-7}eV$$

limit of accuracy with which the frequency of the emitted radiation may be determined will be

$$\Delta\nu=\frac{\Delta E}{h}=\frac{6.62\times10^{-26}}{1.6\times10^{-34}}=10^{8}\ cycle/\text{sec}$$

Hence, the limit of accuracy with which the frequency of the emitted radiation may be determined is $\Delta\nu=10^{8}\,Hz$.

10. Find the average time that the atomic system remains in the energy state whose spectral line width is $10^{-4}\,\text{Å}$ for $\lambda=4000\text{Å}$.

Soln. From energy-time uncertainty relation, the average lifetime of the state will be

$$\Delta t=\frac{h}{4\pi\Delta E}=\frac{h}{4\pi\left(\dfrac{hc}{\lambda^2}\Delta\lambda\right)}=\frac{\lambda^2}{4\pi c\Delta\lambda}=4.24\ ns$$

11. In a Stern-Gerlach experiment, a collimated beam silver atom is produced by vapourising silver salt in a furnace at 1200°C. If the beam travels 100cm, use uncertainty principle to find the order of magnitude of the smallest spot that can be recorded on the detector.

[Given: Planck's constant, $h=6.6\times10^{-34}\,Js$; Boltzmann constant, $k=1.4\times10^{-23}\ J/K$. Mass of hydrogen atom $=1.66\times10^{-27}\,kg$]

Soln. If the silver atom of mass 'm' moves with velocity 'v' along the x-axis from the furnace at temperature T, then its kinetic energy is,

$$E_K=\frac{1}{2}mv^2=\frac{3}{2}kT. \qquad \therefore v=\sqrt{\frac{3kT}{m}}$$

Time required by the silver atom to travel a distance of 100 cm is,

$$t=\frac{100}{v}=\frac{100}{\sqrt{3kT/m}}$$

Now time-energy uncertainty relation is,

$$\Delta E\Delta t\approx\hbar\ \Rightarrow(\Delta E)_{\min}(\Delta t)_{\max}\approx\hbar\quad\Rightarrow(\Delta E)_{\min}\approx\frac{\hbar}{(\Delta t)_{\max}}=\frac{\hbar}{t}$$

Now, $$(\Delta E_{\min})^2=\frac{(\Delta p_{\min})^2}{2m}\Rightarrow\Delta p_{\min}=\sqrt{2mE_{\min}}=\sqrt{\frac{2m\hbar}{t}}$$

Maximum uncertainty in the position of the particle

$$(\Delta x)_{\max}\approx\frac{\hbar}{(\Delta p)_{\min}}=\sqrt{\frac{\hbar t}{2m}}=\sqrt{\frac{\hbar}{2m}}\frac{10}{\left(\frac{3kT}{m}\right)^{1/4}}=7\times10^{-7}\,m=0.7\,\mu m$$

The smallest spot on the detector corresponds to the maximum uncertainty in position of the silver atom. Hence, size of the smallest spot = 0.7 µm.

12. The speedometer of a 180 kg vehicle reads 72 km/h at a particular instant. The readings are known to have an accuracy of ±2 km/h. What would be the minimum uncertainty in the position of the vehicle at that instant according to the Heisenberg uncertainty perinciple?

Soln. The uncertainty in the momentum is

$$\Delta p_x=\Delta(mv_x)=m\Delta v_x=(180kg)\times(2km/hr)=100\,kg.m/s$$

Thus the minimum uncertainty in position is

$$(\Delta x)_{\min}=\frac{\hbar}{2\Delta p_x}=0.5\times10^{-36}\,m$$

Remember the size of an atom is of the order of 10^{-10} m and that of a nucleus is 10^{-15} m. A size of 10^{-36} m is too small to detect with our present day technology and hence we can safely ignore it.

13. Gamma rays of energy $E_0=14.4\,keV$ are emitted when ^{57}Fe nuclei make transition from the first excited state to the ground state. The lifetime of this excited state is about 3×10^{-8} s. If the spread in the energy of the excited state is written as $\Delta E=\frac{v}{c}E_0$ find the value of v. Here c is the speed of light in vacuum.

Soln. From energy-time uncertainity principle,

$$\Delta E\Delta t\approx\hbar\ .\ \Rightarrow\Delta E\approx\frac{\hbar}{\Delta t}\ \Rightarrow\frac{v}{c}E_0\approx\frac{\hbar}{\Delta t}$$

$$\Rightarrow v\approx\frac{\hbar c}{E_0\tau}=\frac{hc}{2\pi E_0\tau}=\frac{1240\,eV\,nm}{2\pi\times14.4\times10^3\,eV}\approx0.12\,mm/s\ \text{nature.}$$

14. A proton and an alpha particle have the same kinetic energy, then the relation between their de Broglie wavelength is **[BHU 2010]**

(a) $\lambda_p=\frac{\lambda_a}{2}$ (b) $\lambda_p=\frac{\lambda_a}{4}$ (c) $\lambda_p=2\lambda_a$ (d) $\lambda_p=4\lambda_a$

Soln. The ratio of de-Broglie wavelengths of a proton and an alpha particle is

$$\frac{\lambda_p}{\lambda_a}=\frac{h/\sqrt{2m_pK_p}}{h/\sqrt{2m_aK_a}}$$

where $K_p = K_a$ (given)

$$\therefore \quad \frac{\lambda_p}{\lambda_a} = \sqrt{\frac{m_a}{m_p}}$$

We know that, $m_a = 4\,m_p$

$$\therefore \quad \frac{\lambda_p}{\lambda_a} = \frac{2}{1} \Rightarrow \lambda_p = 2\lambda_a$$

Correct option is (c)

15. A neutron is confined within a nucleus of radius 5×10^{-15} m. The minimum uncertainty in the momentum of nucleon [$m = 1.67 \times 10^{-27}$ kg, $\hbar = 1.05 \times 10^{-34}$ J sec] is **[BHU 2010]**

(a) 1.25×10^{-15} kg-m/sec (b) 4.12×10^{-18} kg-m/sec

(c) 1.05×10^{-28} kg-m/sec (d) 1.65×10^{-21} kg-m/sec

Soln. The minimum uncertainity in the momentum is $\Delta p = \dfrac{\hbar}{2\Delta r} = \dfrac{1.05 \times 10^{-34}}{2 \times 5 \times 10^{-15}} = 1.05 \times 10^{-20}$ kg-m/sec.

Correct option is (c)

16. If the uncertainty in the location of a particle is equal to its de-Broglie wavelength, then the uncertainty in its velocity V is **[BHU 2011]**

(a) $\dfrac{V}{2}$ (b) V (c) $\dfrac{V}{4}$ (d) $\dfrac{V}{3}$

Soln. $\Delta x \Delta p \approx \hbar \Rightarrow \Delta v \approx \dfrac{\hbar}{m\,\Delta x} = \dfrac{\hbar}{m\dfrac{h}{mV}} = \dfrac{V}{2\pi}$

17. An electron has a speed of 300 metres/sec accurate up to 0.01%. With what fundamental accuracy can locate the position of this electron ($m_e = 9.1 \times 10^{-31}$ kg, charge $e = 1.6 \times 10^{-19}$ coulomb and Planck's constant $h = 6.64 \times 10^{-34}$ joules / sec)? **[BHU 2011]**

(a) 2.4 cm (b) 1.2 cm (c) 1.8 cm (d) 3.6 cm

Soln. $\Delta x \Delta p \approx h \Rightarrow \Delta x \approx \dfrac{h}{m\,\Delta v} = \dfrac{6.64 \times 10^{-34}}{\left(9.1 \times 10^{-31}\right) \times \left(300 \times 0.01\%\right)} m = 2.4\ cm$

Correct option (a)

18. The dispersion law for a certain type of wave motion is $\omega = (c^2k^2 + m^2)^{1/2}$, where ω is the angular frequency, k is the magnitude of the propagation vector and c and m are constants. The group velocity of these waves approaches **[BHU 2012]**

(a) infinity as $k \to 0$ and c as $k \to \infty$ (b) c as $k \to 0$ and zero as $k \to \infty$

(c) zero as $k \to 0$ and infinity as $k \to \infty$ (d) zero as $k \to 0$ and c as $k \to \infty$

Soln. Given : $\omega = (c^2k^2 + m^2)^{1/2}$

Group velocity $v_g = \dfrac{d\omega}{dk} = \dfrac{2c^2k}{2(c^2k^2 + m^2)^{1/2}} = \dfrac{c^2k}{\sqrt{c^2k^2 + m^2}}$

So, when $k \to 0$, $v_g = 0$

Now, $v_g = \dfrac{c^2k}{\sqrt{c^2k^2+m^2}} = \dfrac{c^2}{\sqrt{c^2+m^2/k^2}}$

So, when $k \to \infty$, $v_g = c$

Correct option is (d)

19. The uncertainty relation holds for the following situation **[BHU 2012]**
(a) holds for particles only
(b) does not hold for microscopic particles
(c) holds for both microscopic and macroscopic particles
(d) depends on the nature of the particle

Soln. The uncertainity relation holds for all particles whether microscopic or macroscopic.

Correct option is (c)

20. If the Broglie wavelength of an electron and proton are equal, then **[BHU 2012]**
(a) the velocity of proton is less than that of the electron
(b) the velocities of proton and electron are equal
(c) the velocity of proton is greater than that of the electron
(d) the energy of electron is less than that of the proton

Soln. Given : $\lambda_p = \lambda_e$

$$\Rightarrow \frac{h}{m_p v_p} = \frac{h}{m_e v_e} \Rightarrow \frac{v_e}{v_p} = \frac{m_p}{m_e} > 1 \Rightarrow v_e > v_p$$

Correct option is (a)

21. The position and momentum of a 1 keV electron are simultaneously determined. If its position is located to within 1 Å, what is the percentage of uncertainty in its momentum? **[BHU 2012]**
(a) 0.61 (b) 6.17 (c) 61.73 (d) 0.061

Soln. Given : $E = 1\,\text{keV}$

$$\Rightarrow \quad p = \sqrt{2mE} = \sqrt{2\times 9.1\times 10^{-31}\times 10^{3}\times 1.6\times 10^{-19}}$$
$$= 1.70\times 10^{-24}\ \text{kg-m/s.}$$

The minimum uncertainity in the momentum is

$$\Delta p = \frac{\hbar}{\Delta x} = \frac{6.6\times 10^{-34}}{2\pi\times 10^{-10}} = 10.5\times 10^{-25}\ \text{kg-m/s.}$$

So the percentage in uncertainity of the momentum is

$$\frac{\Delta p}{p}\times 100 = \frac{10.5\times 10^{-25}}{1.70\times 10^{-24}}\times 100 \simeq 61.76$$

Correct option is (c)

22. A quantum particle in a three-dimensional space with momenta $\vec{p} = (p_x, p_y, p_z)$ and position $\vec{r} = (x, y, z)$. The uncertainty in the measurement of position and momentum along z-axis is **[BHU 2013]**
(a) $\Delta z\,\Delta p_y \ge \hbar$ (b) $\Delta y\,\Delta p_z \ge \hbar$ (c) $\Delta x\,\Delta p_z \ge \hbar$ (d) $\Delta z\,\Delta p_z \ge \hbar$

Soln. The uncertainity relation is given by $\Delta x_i\ \Delta p_i \ge \hbar$.

So for $i = 3$, i.e., $x_3 = z$ and $p_3 = p_z$, we have $\Delta z\ \Delta p_z \ge \hbar$.

Correct option is (d)

3. An electron is confined to a box of length 10^{-8} m. Calculation of the minimum uncertainty in its velocity, with $m_e = 9\times10^{-31}$ kg, $\hbar = 1.05\times10^{-34}$ J-sec is **[BHU 2013]**

(a) 1.17×10^{4} m/sec (b) 1.17×10^{6} m/sec (c) 1.17×10^{2} m/sec (d) 1.17×10^{8} m/sec

oln. Given: $\Delta x = 10^{-8}$ m

$$\text{Using, } \Delta x\,\Delta p_x \geq \hbar \textbf{ or } m\Delta x\,\Delta v_x \geq \hbar \textbf{ or } \Delta v_x \geq \frac{\hbar}{m\Delta x} \textbf{ or } \Delta v_x \geq \frac{1.05\times10^{-34}}{9.1\times10^{-31}\times10^{-8}}$$

$$\textbf{or } \Delta v_x \geq 0.115\times10^{-15} \textbf{ or } \Delta v_x \geq 1.15\times10^{4}$$

So $(\Delta v_x)_{\min} \approx 1.15\times10^{4}$ m/s.

Correct option is (a)

4. An electron is confined to a box of length L. If the length of the box changes to $2L$, how would the uncertainty the uncertainty of momentum charge? **[BHU 2014]**

(a) uncertainty of momentum will be twice (b) uncertainty of momentum will be half

(c) uncertainty of momentum will be one fourth (d) uncertainty of momentum will be four times

oln. The uncertainity of momentum in particle in a box problem is $\Delta p = \dfrac{n\pi\hbar}{L}$

So if the length of the box changes to $2L$, the uncertainity in momentum gets half.

Correct option is (b)

5. The de-Broglie wavelength of an electron with energy 100 eV is equal to : **[BHU 2014]**

(a) 12.3 Å (b) 1.23 Å (c) 123 Å (d) 0.123 Å

oln. The de-Broglie wavelength is

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE_k}}$$

$$= \frac{6.6\times10^{-34}}{\sqrt{2\times9.1\times10^{-31}\times100\times1.6\times10^{-19}}} \simeq 1.22\times10^{-10} = 1.22\text{ Å}$$

Correct option is (b)

26. The de Broglie wavelength, associated with a light of wavelength (λ) is : **[BHU 2016]**

(a) 2λ (b) $\frac{3}{2}\lambda$ (c) $\frac{2}{3}\lambda$ (d) λ

Soln. The concept of de-Broglie wavelength is valid only for moving particles not for light. So, the wavelength of light of wavelength λ is remained to be λ.

Correct option is (d)

27. The de Broglie wavelength, associated with a neutron at temperature 27°C, is nearly equal to **[BHU 2016]**

(a) 2.77 Meter (b) 1.77 Angstrom (c) 2.77 Micrometer (d) 1.77 Nanometer

Soln. The thermal wavelength of a particle is given by $\lambda = \dfrac{h}{\sqrt{3m\,k_BT}}$, where m is the mass of the particle and k_B is the Boltzmann's constant.

$$\therefore\ \lambda = \frac{6.6\times10^{-34}}{(3\times1.67\times10^{-27}\times1.38\times10^{-23}\times300)^{1/2}} \simeq 1.45\text{ Å}.$$

Correct option is (b)

28. The de Broglie wavelengths associated with 2eV photon and electron are : **[BHU 2016]**

(a) ≈ 6200 Å and ≈ 20 Å, respectively (b) ≈ 7200 Å and ≈ 9 Å, respectively

(c) ≈ 6200 Å and ≈ 9 Å, respectively (d) ≈ 7200 Å and ≈ 20 Å, respectively

Soln. For electron, de-Broglie wavelength is

$$\lambda = \frac{hc}{E} = \frac{6.6\times10^{-34}\times3\times10^{8}}{2\times1.6\times10^{-19}} \Rightarrow \lambda \simeq 6.18\times10^{-7} \simeq 6200\text{ Å}.$$

Now, for electron, the de-Broglie wavelength is

$$\lambda = \frac{h}{\sqrt{2mE}} = \frac{6.6\times10^{-34}}{\sqrt{2\times9.1\times10^{-31}\times2\times1.6\times10^{-9}}} \simeq 0.86\times10^{-19} \simeq 9\text{ Å}.$$

Correct option is (c)

29. An electron of mass m and charge e, initially at rest, gets accelerated by a constant electric field E. The rate of change of de-Broglie wavelength of this electron with time, (ignoring relativistic effects) is given by

(a) $-\frac{h}{eEt^2}$ (b) $-\frac{eht}{E}$ (c) $-\frac{mh}{eEt^2}$ (d) $-\frac{h}{eE}$ **[HCU 2010]**

Soln. The de-Broglie wavelength is $\lambda = \frac{h}{p}$

Therefore, $\frac{d\lambda}{dt} = -\frac{h}{p^2}\frac{dp}{dt}$, where $\frac{dp}{dt} = eE$ and $p = eEt$

Therefore, $\frac{d\lambda}{dt} = -\frac{h}{eEt^2}$.

Correct option is (a)

30. The uncertainty relation holds for the following situation **[HCU 2010]**

(a) holds for particles only

(b) does not hold for macroscopic particles

(c) holds for both microscopic and macroscopic particles

(d) depends on the nature of the particle

Soln. The uncertainity relation holds for all particles whether they are microscopic or macroscopic.

Correct option is (c)

1. The phase velocity of pressure waves v_p is related to the wave vector k by $v_p = Ck^{-1/2}$. (C is a constant). The group velocity is given by **[HCU 2010]**

(a) $\frac{v_p}{2}$ (b) $2v_p$ (c) kv_p (d) zero

oln. Phase velocity $v_p = \frac{\omega}{k} = \frac{C}{\sqrt{k}} \Rightarrow \omega = C\sqrt{k}$

Group velocity $v_g = \frac{d\omega}{dk} = \frac{C}{2\sqrt{k}} = \frac{v_p}{2}$

Correct option is (a)

2. The de-Broglie wavelength of an electron with a kinetic energy of 120 eV is nearly **[HCU 2010]**

(a) 1000 pm (b) 100 pm (c) 10 pm (d) 1 pm

oln. The de-Broglie wavelength is

$$\lambda = \frac{h}{\sqrt{2mE_k}} = \frac{6.6\times10^{-34}}{\sqrt{2\times9.1\times10^{-31}\times120\times1.6\times10^{-19}}} \simeq 100 \text{ pm}.$$

Correct option is (b)

3. The de-Broglie wavelength of a neutron (kinetic energy 3/2 kT) at 0 °K is
(a) 1.5 Å (b) 2 Å (c) 0 Å (d) 273 Å **[HCU 2011]**

oln. The de-Broglie wavelength is $\lambda = \frac{h}{\sqrt{2mE_k}}$, where $E_k = \frac{3}{2}kT$

$$\therefore\ \lambda = \frac{h}{\sqrt{3mkT}} = \frac{6.6\times10^{-34}}{\sqrt{3\times1.67\times10^{-27}\times1.38\times10^{-23}\times273}} \simeq 1.5 \text{ Å}.$$

Correct option is (a)

4. If surface waves in a lake have a phase velocity given by $v = a\sqrt{\lambda/2\pi}$, where a is constant, then their group velocity is **[HCU 2012]**

(a) v (b) $2v$ (c) $\frac{v}{2}$ (d) $\frac{v}{a}$

oln. Group velocity $v_g = v_p - \lambda\frac{dv_p}{d\lambda} = a\sqrt{\frac{\lambda}{2\pi}} - \lambda a\frac{1}{\sqrt{2\pi}}\frac{1}{2\sqrt{\lambda}} = \frac{a}{2}\sqrt{\frac{\lambda}{2\pi}} = \frac{v}{2}$

Correct option is (c)

5. De-Broglie hypothesized that the linear momentum and wavelength of a free massive particle are related to one of the following constants. **[HCU 2014]**

(a) Rydberg's constant (b) Speed of light
(c) Planck's constant (d) Boltzmann's constant

Soln. According to de-Broglie hypothesized $\lambda = \frac{h}{p}$; (where λ is planck's constant).

Correct option is (c)

36. The de-Broglie wavelength of electrons $\left(m_e = 9.1\times10^{-31}kg \text{ and } \in = 1.6\times10^{-19}C\right)$ accelerated from rest through a potential difference of V (volts) is **[HCU 2016]**

(a) $\sqrt{\frac{15}{V}}\,nm$ (b) $\sqrt{\frac{1.5}{V}}\,nm$ (c) $\frac{15}{\sqrt{V}}\,nm$ (d) $\sqrt{\frac{V}{1.5}}\,nm$

Soln. According to energy conservation $\frac{1}{2}mv_e^1 = \text{eV} \Rightarrow p_e = \sqrt{2m_e\,\text{eV}}$

So the de-Broglie wavelength of electrons is

$$\lambda_e = \frac{h}{p_e} = \frac{h}{\sqrt{2m_e\,\text{eV}}} = \frac{6.62\times10^{-34}}{\sqrt{2\times9.1\times10^{-31}\times1.6\times10^{-19}\times\text{V}}} = \sqrt{\frac{1.5}{\text{V}}} \text{ nm}.$$

Correct option is (b)

Chapter 4

Wave Functions & Operators

4.1 Introduction :

There is always a quantity asscociated with any type of waves, which varies periodically with space and time. In water waves, the quantity that varies periodically is the height of the water surface and in light waves, electric field and magnetic field vary with space and time. For De-broglie waves or matter waves asscociated with a moving particle, the quantity that vary with space and time, is called wave function of the particle.

The temporal and spatial evolution of a quantum mechanical particle is described by a wave function $\psi(x,t)$ for 1-D motion and $\psi(\vec{r},t)$ for 3-D motion. . It contains all possible information about the state of the system. These are known as configuration space wave function.

Condition for a physically accepted, well behaved, realistic wave function :

(i) $\psi(x,t)$ should be finite, single-valued and continuous everywhere in space.

(ii) $\frac{d\psi}{dx}$ should be continuous everywhere in space. But, $\frac{d\psi}{dx}$ may be discontinuous in some cases as follows:

(a) If the potential under which the particle is moving, has an infinite amount of discontinuity at some points,

(b) If the potential under which the particle is moving, is of dirac delta nature.

(iii) $\psi(x,t)$ should be square integrable i.e. $\int_{-\infty}^{\infty}|\psi(x,t)|^2 dx$ = finite quantity

Example 1: Which of the following wave function is acceptable as the solution of the Schrodinger equation for all values of x?

(a) $\psi(x) = A\sec x$ (b) $\psi(x) = A\tan x$ (c) $\psi(x) = Ae^{x^2}$ (d) $\psi(x) = Ae^{-x^2}$

Soln. $\psi(x) = A\sec x$ and $\psi(x) = A\tan x$ is not finite at $x = \frac{\pi}{2}$

$\psi(x) = Ae^{x^2}$ is not finite at $x = \pm\infty$; $\psi(x) = Ae^{-x^2}$ is finite everywhere in space.

Correct option is (d)

Example 2: A ball bounces back off earth. You are asked to solve this quantum mechanically assuming the earth is an infinitely hard sphere. Consider surface of earth as the origin implying $V(0) = \infty$ and a linear potential eleswhere (i.e. $V(x) = -mgx \text{ for } x > 0$). Which of the following wave function is physically admissible for this problem (with $k > 0$):

(a) $\psi(x) = \frac{e^{-kx}}{x}$ (b) $\psi(x) = Ae^{-kx^2}$ (c) $\psi(x) = -Axe^{kx}$ (d) $\psi(x) = Axe^{-kx^2}$

Soln. Since the earth has been taken as infinitely hard sphere, therefore the wave function of the ball will be zero for $x < 0$, non-zero for $x > 0$ and zero at infinity.

Since $\psi(x)$ should be continous everywhere, the wavefunction of the ball will be zero at $x = 0$.

Correct option is (d).

4.2 Physical significance of wave function :

Generally, $\psi(x,t)$ is a complex quantity. It can be multipliedd by any complex number without affecting its physical significance. In general, $\psi(x,t)$ has no direct physical significance. But the quantity $\psi^*(x,t)\psi(x,t) = |\psi(x,t)|^2$ is real, physically significant and is defined as position probability density i.e. probability of finding the particle per unit length at time 't'. Therefore, for 1-D motion the probability of finding the particle between x to $x + dx$ at time 't' is given by

$$\psi^*(x,t)\psi(x,t)\,dx = |\psi(x,t)|^2\,dx$$

and for 3-D motion the probability of finding the particle within the volume element $d\tau$ located between $\vec{r}$ and $\vec{r} + d\vec{r}$ at time 't' is given by

$$\psi^*(\vec{r},t)\psi(\vec{r},t)\,d\tau = |\psi(\vec{r},t)|^2\,d\tau$$

Now, $\psi(x,t)$ or $\psi(\vec{r},t)$ should be chosen such that total probability of finding the particle in the entire space should be equal to unity i.e.

$$\int_{-\infty}^{\infty} \psi^*(x,t)\psi(\vec{r},t)\,d\tau = \int_{-\infty}^{\infty} |\psi(x,t)|^2\,d\tau = 1 \text{ (for 1-D motion)}$$

$$\int_{\substack{all\\space}} \psi^*(\vec{r},t)\psi(\vec{r},t)\,d\tau = \int_{\substack{all\\space}} |\psi(\vec{r},t)|^2\,d\tau = 1 \text{ (for 3-D motion)}$$

This is called the normalization condition of the wave function.

If $\psi(x,t)$ or $\psi(\vec{r},t)$ is normalized at some time it remains normalized forever. This can be understood as a conservation of probability or conservation of normalization.

Method of Normalization:

Consider $\varphi(x,t)$ is an unnormalized wave function. We can construct a normalized wave function as $\psi(x,t) = N\varphi(x,t)$ where N is the normalization constant. Therefore,

$$\int_{-\infty}^{\infty} \psi^*(x,t)\psi(x,t)\,dx = |N^2| \int_{-\infty}^{\infty} \varphi^*(x,t)\varphi(x,t)\,dx = 1$$

$$\Rightarrow N = \frac{1}{\sqrt{\int_{-\infty}^{\infty} \varphi^*(x,t)\varphi(x,t)\,dx}}$$

Example 3: Normalize the wave function given by

$$\psi(x) = Ne^{-\alpha x} \quad (x > 0)$$
$$= Ne^{\alpha x} \quad (x < 0)$$

Soln. Normalization condition is

$$\int_{-\infty}^{\infty} \psi^* \psi dx = 1 \Rightarrow |N|^2 \left[\int_{-\infty}^{0} e^{2\alpha x} dx + \int_{0}^{\infty} e^{-2\alpha x} dx \right] = 1$$

$$\Rightarrow |N|^2 \left[\frac{1}{2\alpha} + \frac{1}{2\alpha} \right] = 1 \quad \Rightarrow N = \sqrt{\alpha}$$

Orthogonality condition of wave functions :

Two wavefunctions $\psi_m(x)$ and $\psi_n(x)$ are said to be orthogonal to each other, if

$$\int_{-\infty}^{\infty} \psi_m^*(x) \psi_n(x) dx = 0 \; (m \neq n)$$

i.e. if a particle is in the state $\psi_m(x)$, then the particle cannot be in the state $\psi_n(x)$ simultaneously together.

Orthonormality condition of wave functions :

Two wavefunctions $\psi_m(x)$ and $\psi_n(x)$ are said to be orthonormal to each other, if

$$\int_{-\infty}^{\infty} \psi_m^*(x) \psi_n(x) dx = 0 \; (m \neq n)$$

4.3 Operator formalism :

An operator is a mathematical rule (or procedure) which operating on one function transforms it into another function i.e. $\hat{A}\psi(x) = \phi(x)$. Every dynamical variable in Quantum mechanics represented by a operator.

Operations	**Symbol**	**Result of the operation**
Taking the square root	$\sqrt{}$	$\sqrt{x^m} = x^{m/2}$
Differentiation w.r.t. x	$\frac{d}{dx}$	$\frac{d}{dx}(x^m) = mx^{m-1}$

Position Operator : $\hat{x}, \hat{y}, \hat{z}$

Momentum Operator : $\hat{p}_x = -i\hbar \frac{\partial}{\partial x}, \hat{p}_y = -i\hbar \frac{\partial}{\partial y}, \hat{p}_z = -i\hbar \frac{\partial}{\partial z} \Rightarrow \hat{p} = -i\hbar \vec{\nabla}$

Potential energy Operator : $\hat{V}$

Kinetic energy Operator : $\hat{K} = \frac{\hat{p}^2}{2m} = -\frac{\hbar^2}{2m} \vec{\nabla}^2$

Commutator Bracket:

The commutator bracket of two operators $\hat{A}$ and $\hat{B}$ is defined as $\left[\hat{A}, \hat{B}\right] = \hat{A}\hat{B} - \hat{B}\hat{A}$.

Example:

(1) $\left[\hat{x}, \frac{d}{dx}\right]\psi = \hat{x}\frac{d\psi}{dx} - \frac{d}{dx}(\hat{x}\psi) = x\frac{d\psi}{dx} - x\frac{d\psi}{dx} - \psi = -\psi \Rightarrow \left[\hat{x}, \frac{d}{dx}\right] = -1$

(2) $[\hat{x},\hat{p}_x]\psi=\hat{x}\hat{p}_x\psi-\hat{p}_x\hat{x}\psi=-i\hbar x\frac{\partial\psi}{\partial x}+i\hbar\frac{\partial}{\partial x}(x\psi)=i\hbar\psi\Rightarrow[\hat{x},\hat{p}_x]=i\hbar$

Similarly, $\left[\hat{y},\hat{p}_y\right]=i\hbar$ and $\left[\hat{z},\hat{p}_z\right]=i\hbar$

Linear Operator:

If an operator $\hat{A}$ is said to be a linear if

(i) $\hat{A}\left[c\psi(x)\right]=c\hat{A}\psi(x)$ and (ii) $\hat{A}\left[\psi_1(x)+\psi_2(x)\right]=\hat{A}\psi_1(x)+\hat{A}\psi_2(x)$

All Quantum mechanical Operators are linear in nature.

Properties of the commutator bracket:

1. $\left[\hat{A},\hat{B}\right]=-\left[\hat{B},\hat{A}\right]$
2. $\left[\hat{A},\hat{B}+\hat{C}+\hat{D}+....\right]=\left[\hat{A},\hat{B}\right]+\left[\hat{A},\hat{C}\right]+........$
3. $\left[\hat{A},\hat{B}\right]^\dagger=\left[\hat{B}^\dagger,\hat{A}^\dagger\right]$
4. $\left[\hat{A},\hat{B}\hat{C}\right]=\left[\hat{A},\hat{B}\right]\hat{C}+\hat{B}\left[\hat{A},\hat{C}\right]$
5. $\left[\hat{A},\left[\hat{B}\hat{C}\right]\right]+\left[\hat{B},\left[\hat{C},\hat{A}\right]\right]+\left[\hat{C},\left[\hat{A},\hat{B}\right]\right]=0$
6. $\left[\hat{A},f\left(\hat{A}\right)\right]=0$
7. $\left[f\left(\hat{A}\right),G\left(\hat{A}\right)\right]=0$
8. $\left[f\left(\hat{A}\right),G\left(\hat{B}\right)\right]=0$ only if $\left[\hat{A},\hat{B}\right]=0$
9. $\left[\hat{A},\hat{B}^n\right]=n\hat{B}^{n-1}\left[\hat{A},\hat{B}\right]$
10. $\left[\hat{A}^n,\hat{B}\right]=n\hat{A}^{n-1}\left[\hat{A},\hat{B}\right]$

Example 4: Find the commutator bracket $\left[\left[\hat{A},\hat{B}\right],\left[\hat{B},\hat{A}\right]\right]$

oln: $\left[\left[\hat{A},\hat{B}\right],\left[\hat{B},\hat{A}\right]\right]=\left[\hat{A},\hat{B}\right]\left[\hat{B},\hat{A}\right]-\left[\hat{B},\hat{A}\right]\left[\hat{A},\hat{B}\right]$

$$=\left(\hat{A}\hat{B}-\hat{B}\hat{A}\right)\left(\hat{B}\hat{A}-\hat{A}\hat{B}\right)-\left(\hat{B}\hat{A}-\hat{A}\hat{B}\right)\left(\hat{A}\hat{B}-\hat{B}\hat{A}\right)$$

$$=\hat{A}\hat{B}\hat{B}\hat{A}-\hat{A}\hat{B}\hat{A}\hat{B}-\hat{B}\hat{A}\hat{B}\hat{A}+\hat{B}\hat{A}\hat{A}\hat{B}-\hat{B}\hat{A}\hat{A}\hat{B}+\hat{A}\hat{B}\hat{A}\hat{B}+\hat{B}\hat{A}\hat{B}\hat{A}-\hat{A}\hat{B}\hat{B}\hat{A}=0$$

Example 5: Find the commutator bracket $\left[x^2,p^2\right]$.

oln: $\left[x^2,p^2\right]=\left[x^2,pp\right]=\left[x^2,p\right]p+p\left[x^2,p\right]=2x[x,p](p)+(p)2x[x,p]=2i\hbar(xp+px)$

Example 6: If $\hat{H}=\frac{p^2}{2m}+V(x)$, then calculate $\left[x,\left[x,\hat{H}\right]\right]$.

oln: $\left[x,\hat{H}\right]=\left[x,\frac{p^2}{2m}\right]+\left[x,V(x)\right]=\frac{1}{2m}\left[x,p^2\right]+0=\frac{1}{2m}2p[x,p]=\frac{i\hbar}{m}p$

$$\left[x,\left[x,\hat{H}\right]\right]=\left[x,\frac{i\hbar}{m}p\right]=\frac{i\hbar}{m}[x,p]=-\frac{\hbar^2}{m}$$

Eigenvalues and Eigenfunctions:

If an operator $\hat{\mathrm{A}}$ operating on a function $\psi_n(x)$ gives $\hat{A}\psi_n(x)=\lambda\psi_n(x)$, then $\psi_n(x)$ is called the eigenfunction of $\hat{\mathrm{A}}$ corresponding to eigenvalue λ and the above equation is known as eigenvalue equation of operator $\hat{\mathrm{A}}$.

Example: $\hat{A}=\frac{d^2}{dx^2}$ and $\psi_n(x)=ae^{-2x} \Rightarrow \hat{A}\psi_n(x)=\frac{d^2}{dx^2}\left(ae^{-2x}\right)=4ae^{-2x}=4\psi_n(x)$

$\psi_n(x)$ is an eigenfunction of $\hat{\mathrm{A}}$ corresponding to eigenvalue 4.

If the commutator bracket of two operators $\hat{\mathrm{A}}$ and $\hat{\mathrm{B}}$ is equal to zero i.e. $\left[\hat{A},\hat{B}\right]=0$, then the physical observables corresponding to these operators are simultaneously accurately measurable and they have a complete set of simultaneous eigenfunctions.

Example 7: The operator $\left(x+\frac{d}{dx}\right)$ has the eigenvalue α. Determine the corresponding wavefunction.

Soln: Eigenvalue equation: $\left(x+\frac{d}{dx}\right)\psi=\alpha\psi \Rightarrow \frac{d\psi}{dx}=(\alpha-x)\psi \Rightarrow \int\frac{d\psi}{\psi}=\int(\alpha-x)\,dx$

$$\Rightarrow \ln\psi=\left(\alpha x-\frac{x^2}{2}\right)+\ln\psi_0 \Rightarrow \psi=\psi_0\exp\left(\alpha x-\frac{x^2}{2}\right)$$

Example 8: Find the constant B which makes e^{-ax^2} an eigenfunction of the operator $\left(\frac{d^2}{dx^2}-Bx^2\right)$.

What is the corresponding eigenvalue?

Soln: $\left(\frac{d^2}{dx^2}-Bx^2\right)e^{-ax^2}=\left(4a^2x^2-2a+Bx^2\right)e^{-ax^2}$

For e^{-ax^2} to be eigenfunction of the operator $\left(\frac{d^2}{dx^2}-Bx^2\right)$, then the eigenvalue

$\left(4a^2x^2-2a-Bx^2\right)$ must be independent of x i.e. $\left(4a^2-B\right)=0 \Rightarrow B=4a^2$

Therefore, $\left(\frac{d^2}{dx^2}-Bx^2\right)e^{-ax^2}=-2ae^{-ax^2}$; Thus the eigenvalue of the operator is (-2a).

Hermitian operator:

A operator $\hat{\mathrm{A}}$ is said to be hermitian if $\hat{A}^\dagger=\hat{A}$ and its should satisfy the following relation:

Dirac representation: $\left(\langle\phi|\hat{A}|\psi\rangle\right)^\dagger=\langle\psi|\hat{A}^\dagger|\phi\rangle=\langle\psi|\hat{A}|\phi\rangle$

Schrodinger representation: $\left(\phi,\hat{A}\psi\right)=\left(\hat{A}\phi,\psi\right) \Rightarrow \int\phi^*\hat{A}\psi\,d\tau=\int\left(\hat{A}\phi\right)^*\psi\,d\tau$

Properties:

(1) Eigenvalues of hermitian operators are real.
(2) Eigenfunctions corresponding to different eigenvalues of a hermitian operator are orthogonal.
(3) All physical observable in quantum mechanics are represented by hermitian operators.

Example: $\left(\phi,\hat{p}_x\psi\right)=\int_{-\infty}^{\infty}\phi^*\left(-i\hbar\frac{\partial}{\partial x}\right)\psi\,dx=\int_{-\infty}^{\infty}\phi^*\left(-i\hbar\frac{\partial\psi}{\partial x}\right)dx$

$$=-i\hbar\left[\phi^*\psi\right]_{-\infty}^{\infty}+i\hbar\int_{-\infty}^{\infty}\left(\frac{d\phi^*}{dx}\right)\psi\,dx \quad =-i\hbar\int_{-\infty}^{\infty}\left(\frac{d\phi^*}{dx}\right)\psi\,dx=\int_{-\infty}^{\infty}\hat{p}_x^*\phi^*\psi\,dx$$

Hence, the momentum operator $\hat{p}_x$ is hermitian in nature.

Similarly, we can see that $\frac{\partial}{\partial x}$ is anti-hermitian i.e. $\left[\frac{\partial}{\partial x}\right]^{\dagger}=-\frac{\partial}{\partial x}$

Unitary Operator:
An operator $\hat{A}$ is said to be unitary if $\hat{A}^{\dagger}=\hat{A}^{-1}$

Example 9: Let, $N=\hat{b}^{\dagger}\hat{b}$ where the operator $\hat{b}$ satisfies the relation $\hat{b}\hat{b}^{\dagger}+\hat{b}^{\dagger}\hat{b}=1$ and $\hat{b}^2=\left(\hat{b}^{\dagger}\right)^2=0$;

then eigenvalues of N are

(a) +ve and –ve integers (b) all +ve integers (c) ±1 and 0 only (d) 0 and 1 only

oln: $\hat{N}|\psi\rangle=\lambda|\psi\rangle \Rightarrow \hat{b}^{\dagger}\hat{b}|\psi\rangle=\lambda|\psi\rangle \Rightarrow \left(\hat{b}^{\dagger}\hat{b}\right)^2|\psi\rangle=\lambda^2|\psi\rangle$

$$\Rightarrow \hat{b}^{\dagger}\hat{b}\hat{b}^{\dagger}\hat{b}|\psi\rangle=\lambda^2|\psi\rangle \Rightarrow \hat{b}^{\dagger}\left(1-\hat{b}^{\dagger}\hat{b}\right)\hat{b}|\psi\rangle=\lambda^2|\psi\rangle$$

$$\Rightarrow \hat{b}^{\dagger}\hat{b}-\hat{b}^{\dagger}\hat{b}^{\dagger}\hat{b}\hat{b}|\psi\rangle=\lambda^2|\psi\rangle \Rightarrow \hat{b}^{\dagger}\hat{b}|\psi\rangle=\lambda^2|\psi\rangle \Rightarrow \lambda|\psi\rangle=\lambda^2|\psi\rangle \Rightarrow \lambda=0,1$$

Example 10: Let the wave function of a particle of mass 'm' at a given instant be $\psi(x)=Ae^{-x^2/a^2}$. What will be the function $K\psi(x)$ where 'K' is the kinetic energy? Is this wave function an eigenfunction of kinetic energy?

oln. The kinetic energy operator is $K=-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$. Thus,

$$K\psi(x)=-\frac{\hbar^2}{2m}\frac{d^2}{dx}\left(Ae^{-x^2/a^2}\right)=-\frac{\hbar^2}{2m}\frac{d}{dx}\left[-\frac{2A}{a^2}xe^{-x^2/a^2}\right]$$

$$=-\frac{\hbar^2}{2m}\frac{d}{dx}\left[-\frac{2A}{a^2}xe^{-x^2/a^2}\right]=\frac{2A\hbar^2}{2ma^2}\left[x\left(\frac{-2x}{a^2}e^{-x^2/a^2}\right)+e^{-x^2/a^2}\right]=\frac{A\hbar^2}{ma^2}\left(1-\frac{2x^2}{a^2}\right)e^{-x^2/a^2}$$

This is not an eigenfunction of kinetic energy.

Example 11: An operator A is defined as $A\psi(x)=\psi^*(x)\psi(x)$. Is this a linear operator?

Soln. For the given operator,

$$A\left[\psi_1(x)+\psi_2(x)\right]=\left[\psi_1(x)+\psi_2(x)\right]^*\left[\psi_1(x)+\psi_2(x)\right]$$

$$=\left[\psi^*{}_1(x)+\psi^*{}_2(x)\right]\left[\psi_1(x)+\psi_2(x)\right]$$

$$=\psi^*{}_1(x)\psi_1(x)+\psi^*{}_2(x)\psi_2(x)+\psi^*{}_1(x)\psi_2(x)+\psi^*{}_2(x)\psi_1(x)$$

and $A\psi_1(x)+A\psi_2(x)=\psi^*{}_1(x)\psi_1(x)+\psi^*{}_2(x)\psi_1(x)$

Thus, $A\left[\psi_1(x)+_2(x)\right]=\psi^*{}_1(x)+A\psi_2(x)$ and hence A is not a linear operator.

Example 12. Prove that $[Xp_x,H]=\frac{i\hbar}{m}p_x^2+X[p_x,V]$ where $H=\frac{p_x^2}{2m}+V$, and V is potential energy operator.

Soln. $[Xp_x,H]=X[p_x,H]+[X,H]p_x=X\left[p_x,\frac{p_x^2}{2m}+V\right]+\left[X,\frac{p_x^2}{2m}+V\right]p_x$

$$= X\left[p_x, V\right] + \frac{1}{2m}\left\{p_x\left[X, p_x\right] + \left[X, p_x\right]p_x\right\}p_x$$

$$= X\left[p_x, V\right] + \frac{1}{2m}\left(2i\hbar p_x\right)p_x = \frac{i\hbar}{m}p_x^2 + X\left[p_x, V\right]$$

Example 13. Prove that the operators $i(d/dx)$ and d^2/dx^2 are Hermitian.

Soln. $$\int_{-\infty}^{\infty}\psi_m^*\left(i\frac{d}{dx}\right)\psi_n dx = i[\psi_m^*\psi_n]_{-\infty}^{\infty} - i\int_{-\infty}^{\infty}\psi_n\frac{d}{dx}\psi_m^* dx \quad = \int_{-\infty}^{\infty}\left(i\frac{d}{dx}\psi_m\right)^*\psi_n dx$$

Therefore, id/dx is Hermitian.

$$\int_{-\infty}^{\infty}\psi_m^*\frac{d^2\psi_n}{dx^2}dx = \left[\psi_m^*\frac{d\psi_n}{dx}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{d\psi_n}{dx}\frac{d\psi_m^*}{dx}dx$$

$$= \left[\frac{d\psi_m^*}{dx}\psi_n\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty}\psi_n\frac{d^2\psi_m^*}{dx^2}dx = \int_{-\infty}^{\infty}\frac{d^2\psi_m^*}{dx^2}\psi_n dx$$

Thus, d^2/dx^2 is Hermitian.

Example 14: Find the following commutation relations:

(i) $\left[\frac{\partial}{\partial x}, \frac{\partial^2}{\partial x^2}\right]$ (ii) $\left[\frac{\partial}{\partial x}, F(x)\right]$

Soln. (i) $$\left[\frac{\partial}{\partial x}, \frac{\partial^2}{\partial x^2}\right]\psi = \left(\frac{\partial}{\partial x}\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial x^2}\frac{\partial}{\partial x}\right)\psi = \left(\frac{\partial^3}{\partial x^3} - \frac{\partial^3}{\partial x^3}\right)\psi = 0$$

(iii) $$\left[\frac{\partial}{\partial x}, F(x)\right]\psi = \frac{\partial}{\partial x}(F\psi) - F\frac{\partial}{\partial x}\psi = \frac{\partial F}{\partial x}\psi + F\frac{\partial\psi}{\partial x} - F\frac{\partial\psi}{\partial x} = \frac{\partial F}{\partial x}\psi$$

Thus, $\left[\frac{\partial}{\partial x}, F(x)\right] = \frac{\partial F}{\partial x}$

Example 15: Find the equivalence of the following operators:

(i) $\left(\frac{d}{dx} + x\right)\left(\frac{d}{dx} + x\right)$ (ii) $\left(\frac{d}{dx} + x\right)\left(\frac{d}{dx} - x\right)$

Soln. (i) $$\left(\frac{d}{dx} + x\right)\left(\frac{d}{dx} + x\right)\psi(x) = \left(\frac{d}{dx} + x\right)\left(\frac{d\psi}{dx} + x\psi\right)$$

$$= \frac{d^2\psi}{dx^2} + x\frac{d\psi}{dx} + \psi + x\frac{d\psi}{dx} + x^2\psi = \left(\frac{d^2}{dx^2} + 2x\frac{d}{dx} + x^2 + 1\right)\psi$$

Therefore, $\left(\frac{d}{dx} + x\right)\left(\frac{d}{dx} + x\right) = \frac{d^2}{dx^2} + 2x\frac{d}{dx} + x^2 + 1$

(ii) Similarly, $\left(\frac{d}{dx} + x\right)\left(\frac{d}{dx} - x\right) = \frac{d^2}{dx^2} - x^2 - 1$

Example 16. By what factors do the operators $(x^2p_x^2 + p_x^2x^2)$ and $1/2(xp_x + p_xx)^2$ differ?

oln. $$(x^2p_x^2 + p_x^2x^2)f = -\hbar^2\left[x^2\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2(x^2 f)}{\partial x^2}\right]$$

$$= -\hbar^2 x^2\frac{\partial^2 f}{\partial x^2} - \hbar^2\frac{\partial}{\partial x}\frac{\partial(x^2 f)}{\partial x} \quad = -\hbar^2 x^2\frac{\partial^2 f}{\partial x^2} - \hbar^2\frac{\partial}{\partial x}\left(2xf + x^2\frac{\partial f}{\partial x}\right)$$

$$= -\hbar^2\left(x^2\frac{\partial^2 f}{\partial x^2} + 2f + 2x\frac{\partial f}{\partial x} + x^2\frac{\partial^2 f}{\partial x^2} + 2x\frac{\partial f}{\partial x}\right) \quad = -\hbar^2\left(2x^2\frac{\partial^2}{\partial x^2} + 4x\frac{\partial}{\partial x} + 2\right)f$$

$$\frac{1}{2}(xp_x + p_xx)^2 f = -\frac{i\hbar}{2}(xp_x + p_xx)\left[x\frac{\partial f}{\partial x} + \frac{\partial(xf)}{\partial x}\right]$$

$$= -\frac{i\hbar}{2}(xp_x + p_xx)\left(2x\frac{\partial f}{\partial x} + f\right)$$

$$= -\frac{\hbar^2}{2}\left[x\frac{\partial}{\partial x}\left(2x\frac{\partial f}{\partial x}\right) + x\frac{\partial f}{\partial x} + \frac{\partial}{dx}\left(2x^2\frac{\partial f}{\partial x}\right) + \frac{\partial(xf)}{\partial x}\right]$$

$$= -\frac{\hbar^2}{2}\left(2x^2\frac{\partial^2 f}{\partial x^2} + 2x\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial x} + 2x^2\frac{\partial^2 f}{\partial x^2} + 4x\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial x} + f\right)$$

$$= -\frac{\hbar^2}{2}\left(8x\frac{\partial f}{\partial x} + 2x^2\frac{\partial^2 f}{\partial x^2} + f\right) = -\hbar^2\left(2x^2\frac{\partial^2}{\partial x^2} + 4x\frac{\partial}{\partial x} + \frac{1}{2}\right)f$$

The two operators differ by a term $-(3/2)\hbar^2$

Example 17. Find the eigenvalues and eigenfunctions of the operator d/dx.

oln. Eigenvalue equation: $\frac{d}{dx}\psi(x) = k\psi(x)$ (where k is the eigenvalue and $\psi(x)$ is the eigenfunction)

$$\Rightarrow \frac{d\psi}{\psi} = kdx \quad \Rightarrow \psi = Ce^{kx}$$

Case 1: k is a real positive quantity, ψ is not an acceptable function since it tends to ∞ or $-\infty$ as $x \to \infty$ or $-\infty$.

Case 2: k is purely imaginary (say ia), then $\psi = Ce^{iax}$ which will be finite for all values of x.

Hence, $y = ce^{kx}$ is the eigenfunction of the operator d/dx with eigenvalues $k = ia$, where a is real.

Expectation value of dynamic variables:

It is defined as the average of the result of a large number of independent measurements of a physical observable on the same system.

$$\langle \hat{A} \rangle = \frac{\int_{-\infty}^{\infty} \psi^* \hat{A} \psi \, dx}{\int_{-\infty}^{\infty} \psi^* \psi \, dx} = \frac{\langle \psi | \hat{A} | \psi \rangle}{\langle \psi | \psi \rangle}$$

Note:

(1) If the state of the particle is an eigenfunction of the operator $\hat{A}$, then the expectation value of the physical observable corresponding to $\hat{A}$ will be equal to the eigenvalue of $\hat{A}$ corresponding to the state of the particle.

(2) The state of the particle is given as:

$$|\psi\rangle = C_1|\phi_1\rangle + C_2|\phi_2\rangle + C_3|\phi_3\rangle + \dots\dots\dots\dots = \sum_n C_n|\phi_n\rangle$$

where $|\phi_1\rangle, |\phi_2\rangle$.......... are the eigenfunction of the operator $\hat{A}$, then expectation value of the the physical observable corresponding to $\hat{A}$ will be

$$\langle\hat{A}\rangle = \sum_n |C_n|^2 \lambda_n$$

where λ_n are the eigenvalues of operator $\hat{A}$ corresponding to $|\phi_n\rangle$.

Example 18: A particle of mass 'm' is confined in a 1-D box from $x = -2L$ to $x = 2L$. The wave function of the particle in this state is $\psi(x) = \psi_0 \cos\left(\frac{\pi x}{4L}\right)$

(i) Find the normalization factor.

(ii) The expectation value of P^2 in this state.

Soln. (i) $\int_{-\infty}^{\infty} \psi^* \psi \, dx = 1 \quad \Rightarrow |\psi_0|^2 \int_{-2L}^{2L} \cos^2\left(\frac{\pi x}{4L}\right) dx = 1 \quad \Rightarrow |\psi_0|^2 \int_{-2L}^{2L} \frac{1}{2}\left(1 + \cos\frac{2\pi x}{4L}\right) dx$

$$\Rightarrow \frac{|\psi_0|^2}{2}\left[x - \sin\left(\frac{2\pi x}{4L}\right) \times \frac{4L}{2\pi}\right]_{-2L}^{+2L} = 1 \Rightarrow \frac{|\psi_0|^2}{2}\left[(2L + 2L)\right] = 1 \Rightarrow |\psi_0| = \frac{1}{\sqrt{2L}}$$

$$\langle P^2\rangle = \int_{-2L}^{2L} \psi^* \left[-\hbar^2 \frac{d^2}{dx^2}\right] \psi \, dx = \frac{\hbar^2}{2L} \int_{-2L}^{2L} \cos^2\left(\frac{\pi x}{4L}\right)\left(\frac{\pi}{4L}\right)^2 dx$$

$$= \frac{\hbar^2}{2L} \cdot \frac{\pi^2}{16L^2} \frac{1}{2} \int_{-2L}^{+2L} \left(1 + \cos\frac{\pi x}{2L}\right) dx = \frac{\pi^2 \hbar^2}{32L^3} \int_0^{2L} \left(1 + \cos\frac{\pi x}{2L}\right) dx = \frac{\pi^2 \hbar^2}{16L^2}$$

Example 19: Find the expected value of position and momentum of a particle whose wave function is $\psi(x) = e^{-x^2/a^2 + ikx}$ in all space

Soln. Expected position of the particle $\langle x\rangle = \dfrac{\int_{-\infty}^{\infty} \psi^* x \psi \, dx}{\int_{-\infty}^{\infty} \psi^* \psi \, dx} = \dfrac{\int_{-\infty}^{\infty} e^{-2x^2/a^2} . x dx}{\int_{-\infty}^{\infty} e^{-2x^2/a^2} dx} = 0$

Expected momentum of the particle

$$\langle p\rangle = \frac{\int_{-\infty}^{\infty} e^{-x^2/a^2 - ikx}\left(-i\hbar \frac{\partial}{\partial x}\right) e^{-x^2/a^2 + ikx} dx}{\int_{-\infty}^{\infty} e^{-2x^2/a^2} dx} = -i\hbar \frac{\int_{-\infty}^{\infty} e^{-2x^2/a^2}\left(-\frac{2x}{a^2} + ik\right) dx}{e^{-2x^2/a^2} dx} = \hbar k \frac{\int_{-\infty}^{\infty} e^{-2x^2/a^2} dx}{\int_{-\infty}^{\infty} e^{-2x^2/a^2} dx} = \hbar k$$

Example 20. The wavefunction of a particle in a state is $\psi = N\exp(-x^2/2\alpha)$, where $N = (1/\pi\alpha)^{1/4}$. Evaluate $(\Delta x)(\Delta p)$.

oln. Since ψ is symmetrical about $x = 0, \langle x\rangle = 0$

$$\langle x^2\rangle = N^2\int_{-\infty}^{\infty} x^2\exp\left(\frac{-x^2}{\alpha}\right)dx = \frac{\alpha}{2}$$

Since, ψ is a real wave function, $\langle p\rangle = 0$

$$\langle p^2\rangle = (-i\hbar)^2N^2\int_{-\infty}^{\infty}\exp\left(\frac{-x^2}{2\alpha}\right)\frac{d^2}{dx^2}\exp\left(\frac{-x^2}{2\alpha}\right)dx$$

$$= \frac{\hbar^2N^2}{\alpha}\int_{-\infty}^{\infty}\exp\left(\frac{-x^2}{\alpha}\right)dx - \frac{\hbar^2N^2}{\alpha^2}\int_{-\infty}^{\infty}x^2\exp\left(\frac{-x^2}{\alpha}\right)dx$$

$$= \frac{\hbar^2}{\alpha} - \frac{\hbar^2}{2\alpha} = \frac{\hbar^2}{2\alpha}$$

Therefore, $\Delta x\Delta p = \left[\sqrt{\langle x^2\rangle - \langle x\rangle^2}\right]\left[\sqrt{\langle p^2\rangle - \langle p\rangle^2}\right] = \sqrt{\frac{\alpha}{2}}\sqrt{\frac{\hbar^2}{2\alpha}} = \frac{\hbar}{2}$

Example 21. The Hamiltonian operator of a system is $H = -(d^2/dx^2) + x^2$. Show that $Nx\exp(-x^2/2)$ is an eigenfunction of H and determine the eigenvalue. Also evaluate N by normalization of the function.

Soln. $\psi = Nx\exp(-x^2/2), N$ being a constant

$$H\psi = \left(-\frac{d^2}{dx^2} + x^2\right)Nx\exp\left(-\frac{x^2}{2}\right) = Nx^3\exp\left(-\frac{x^2}{2}\right)\frac{d}{dx}\left[\exp\left(-\frac{x^2}{2}\right) - x^2\exp\left(-\frac{x^2}{2}\right)\right]$$

$$= 3Nx\exp\left(-\frac{x^2}{2}\right) = 3\psi$$

Thus, the eigenvalue of H is 3. The normalization condition gives

$$N^2\int_{-\infty}^{\infty}x^2e^{-x^2}dx = 1 \Rightarrow N^2\frac{\sqrt{\pi}}{2} = 1 \Rightarrow N = \left(\frac{2}{\sqrt{\pi}}\right)^{1/2}$$

The normalized function $\psi = \left(\frac{2}{\sqrt{\pi}}\right)^{1/2}x\exp\left(-\frac{x^2}{2}\right)$

Example 22. Consider the wave function $\psi(x) = A\exp\left(-\frac{x^2}{a^2}\right)\exp(ikx)$, where A is a real constant.

(i) Find the value of A, (ii) calculate $\langle p\rangle$ for this wave function.

Soln. (i) Normalization condition: $A^2\int_{-\infty}^{\infty}\exp\left(-\frac{2x^2}{a^2}\right)dx = 1$

$$\Rightarrow A^2\left(\frac{\pi}{2/a^2}\right)^{1/2}=1 \quad \Rightarrow A=\sqrt{\frac{2}{\pi}}\,a$$

(ii) $\langle p\rangle=\int\psi^*\left(-i\hbar\frac{d}{dx}\right)\psi dx \;=(-i\hbar)A^2\int\limits_{-\infty}^{\infty}\exp\left(-\frac{x^2}{a^2}\right)e^{-ikx}\left(-\frac{2x}{a^2}+ik\right)\exp\left(-\frac{x^2}{a^2}\right)e^{-ikx}dx$

$$=(-i\hbar)\left(-\frac{2}{a^2}\right)\int\limits_{-\infty}^{\infty}\exp\left(\frac{-2x^2}{a^2}\right)xdx+(-i\hbar)(ik)A^2\int\limits_{-\infty}^{\infty}\exp\left(\frac{-2x^2}{a^2}\right)dx$$

In the first term, the integrand is odd and the integral is from $-\infty$ to ∞. Hence the integral vanishes.

Therefore, $\langle p\rangle=\hbar k$, since $A^2\int\limits_{-\infty}^{\infty}\exp\left(\frac{-2x^2}{a^2}\right)dx=1$

Example 23. The normalized wave function of a particle is $\psi(x)=A\exp(iax-ibt)$, where A, a and b are constants. Evaluate the uncertainty in its momentum.

Soln. $\langle p\rangle=-i\hbar\int\psi^*\frac{d}{dx}\psi dx=\hbar a\int\psi^*\psi dx=\hbar a$

$$\langle p^2\rangle=-\hbar^2\int\psi^*\frac{d^2}{dx^2}\psi dx=-\hbar^2(ia)^2\int\psi^*\psi\,dx=\hbar^2a^2$$

Uncertainty in mometum $(\Delta p)=\sqrt{\langle p^2\rangle-\langle p\rangle^2}=\sqrt{\hbar^2a^2-\hbar^2a^2}=0$

Example 24: A particle of mass 'm' is subjected to the potential energy $V(x)=\frac{1}{2}kx^2$. At a particular time, it has the wave function $\psi(x)=Axe^{-x^2/a^2}$. If the particle has a definite total mechanical energy, find the value of 'a'.

Soln. The operator corresponding to the total mechanical energy is $H=-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V(x)$. If the particle has definite vlaue of the total mechanical energy, its wave function should be an eigenfunction of H, that is,

$H\psi(x)=\lambda\psi(x)$ where λ is independent of x.

$$H\psi(x)=\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+\frac{1}{2}kx^2\right]\left[Axe^{-x^2/a^2}\right]$$

$$=-\frac{A\hbar^2}{2m}\frac{d^2}{dx^2}\left(xe^{-x^2/a^2}\right)+\frac{1}{2}kAx^3e^{-x^2/a^2}$$

$$=-\frac{A\hbar^2}{2m}\frac{d}{dx}\left[\left(1-\frac{2x^2}{a^2}\right)e^{-x^2/a^2}\right]+\frac{1}{2}kAx^3e^{-x^2/a^2}$$

$$=-\frac{A\hbar^2}{2m}\left[\left(1-\frac{2x^2}{a^2}\right)\left(-\frac{2x}{a^2}\right)-\frac{4x}{a^2}\right]e^{-x^2/a^2}+\frac{1}{2}kAx^3e^{-x^2/a^2}$$

$$= -\frac{A\hbar^2}{2m}\left[-\frac{6x}{a^2}+\frac{4x^3}{a^4}\right]e^{-x^2/a^2}+\frac{1}{2}kAx^3e^{-x^2/a^2}$$

$$= \frac{3A\hbar^2}{ma^2}xe^{-x^2/a^2}-\left(\frac{2A\hbar^2}{ma^4}-\frac{1}{2}kA\right)x^3e^{-x^2/a^2}$$

If $H\psi(x)$ has to have the same functional form as $\psi(x)$, one should not have the $x^3e^{-x^2/a^2}$ term. So,

$$\frac{2A\hbar^2}{ma^4}-\frac{1}{2}kA=0 \quad \Rightarrow a=\left[\frac{4\hbar^2}{mk}\right]^{1/4}$$

4.4 Schrodinger Equation :

Consider a particle of mass 'm' is moving along x axis, under the potential field $V(x)$ which is independent of time. The total energy of the system can be written as

$$E = E_k + E_p = \frac{p_x^2}{2m}+V(x)$$

Now, the 1-D time-dependent Schrodinger equation for the system can be written as

$$\hat{H}\psi = \hat{E}\psi \qquad \Rightarrow -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}+V(x)\psi = i\hbar\frac{\partial\psi}{\partial t}$$

Solution of 1-D time-dependent Schrodinger equation:

Let, $\psi(x,t)=X(x)T(t)$

Replecing this in the 1-D time-dependent Schrodinger equation

$$\Rightarrow -\frac{\hbar^2}{2m}T\frac{\partial^2 X}{\partial x^2}+VXT=i\hbar X\frac{\partial T}{\partial t}$$

$$\Rightarrow -\frac{\hbar^2}{2m}\frac{1}{X}\frac{d^2X}{dx^2}+V=\frac{i\hbar}{T}\frac{dT}{dt}=E\ (\text{constant})$$

Time dependent part:

$$i\hbar\frac{1}{T}\frac{\partial T}{\partial t}=E \quad \Rightarrow \frac{\partial T}{T}=\frac{E}{i\hbar}\partial t=-\frac{i}{\hbar}Edt$$

$$\Rightarrow T(t)=Ce^{-i\frac{E}{\hbar}t}$$

Therefore, the time dependent part of the solution of the 1-D Schrodinger equation is independent of $V(x)$.

Time independent part:

$$-\frac{\hbar^2}{2m}\frac{1}{X}\frac{\partial^2 X}{\partial x^2}+V=E \Rightarrow -\frac{\hbar^2}{2m}\frac{d^2X}{dx^2}+VX=EX$$

$$\Rightarrow \frac{d^2X}{dx^2}+\frac{2m}{\hbar^2}\left[E-V(x)\right]X=0$$

Therefore, the time independent part of the solution of the 1-D schrodinger equation depends on the nature of $V(x)$.

So, the solution of the 1-D time dependent Schrodinger equation can be written as

$$\psi(x,t)=X(x)T(t)=X(x)e^{-iEt/\hbar}$$

Now, if $\psi_1(x,t), \psi_2(x,t)......\psi_n(x,t)$ are solutions of 1-D time dependent Schrodinger equation corresponding to energies $E_1, E_2, E_3,.....E_n$, then the general solution is

$$\psi(x,t)=\psi_1(x,t)+\psi_2(x,t)+........$$

$$=C_1X_1(x)e^{-iE_1t/\hbar}+C_2X_2e^{-iE_2t/\hbar}+C_3X_3e^{-iE_3t/\hbar}+.......=\sum_n C_nX_n(x)e^{-iE_nt/\hbar}$$

Example 25: The wave function of a spinless particle of mass 'm' is 1-D potential is $\psi(x)=A\exp(-\alpha^2x^2)$, corresponding to eigenvalue $E_0=\frac{\hbar^2\alpha^2}{m}$. The potential $V(x)$ is

(a) $2E_0(1-\alpha^2x^2)$ (b) $2E_0(1+\alpha^2x^2)$ (c) $2E_0\alpha^2x^2$ (d) $2E_0\alpha^2x^2(1+2\alpha^2x^2)$

Soln. Given: $\psi(x)=A\exp(-\alpha^2x^2) \Rightarrow \frac{\partial\psi}{\partial x}=A\exp(-\alpha^2x^2)(-2\alpha^2x)$

$$\Rightarrow \frac{\partial^2\psi}{\partial x^2}=A\exp(-\alpha^2x^2)(4\alpha^4x^2)+A\exp(-\alpha^2x^2)(-2\alpha^2)$$

Replacing all these in the 1-D time independent Schrodinger equation i.e

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}+V(x)\psi=E\psi$$

we get

$$\Rightarrow -\frac{\hbar^2}{2m}\left[A\exp(-\alpha^2x^2)(4\alpha^4x^2)\right]+A\exp(-\alpha^2x^2)V(x)+A\exp(-\alpha^2x^2)(-2\alpha^2)=E.A\exp(-\alpha^2x^2)$$

$$\Rightarrow -\frac{\hbar^2}{2m}\left[4\alpha^4x^2-2\alpha^2\right]+V(x)=E \quad \Rightarrow V(x)-\frac{\hbar^2}{2m}4\alpha^4x^2+\frac{\hbar^2\alpha^2}{m}=\frac{\hbar^2\alpha^2}{m}$$

Therefore, $V(x)=\frac{2\alpha^4\hbar^2x^2}{m}=2E_0\alpha^2x^2$

Correct option is (c).

Example 26: The wave function of a particle moving in a 1-D potential $V(x)$ is $\psi(x)=\exp(-iax+b)$. The potential $V(x)$ will be the form:

(a) $V(x)\propto x^2$ (b) $V(x)\propto x$ (c) $V(x)=0$ (d) $V(x)\propto e^{-ax}$

Soln. $-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}+V\psi=E\psi \Rightarrow -\frac{\hbar^2}{2m}a^2+V(x)=E \qquad \Rightarrow V(x)=0$

Correct option is (c).

Example 27: The ground state (a part from normalization) of a particle moving in a 1-D potential given by, $\psi(x)=\exp\left(-\frac{x^2}{2}\right)\cosh(\sqrt{2}x)$. The potential $V(x)$ is (taking $m=1,\ \hbar=1$)

(a) $\frac{x^2}{2}$ (b) $\frac{x^2}{2}-\sqrt{2}\tanh(\sqrt{2}x)$ (c) $\frac{x^2}{2}-\sqrt{2}\tan(\sqrt{2}x)$ (d) $\frac{x^2}{2}-\coth(\sqrt{2}x)$

oln. Given: $\psi = \exp\left(-\frac{x^2}{2}\right)\cosh\left(\sqrt{2}x\right)$

$$\psi' = \exp\left(-\frac{x^2}{2}\right)(-x)\cosh\left(\sqrt{2}x\right) + \exp\left(-\frac{x^2}{2}\right)\sqrt{2}\sinh\sqrt{2}x$$

$$\psi'' = \exp\left(-\frac{x^2}{2}\right)\left(x^2\right)\cosh\left(\sqrt{2}x\right) - \exp\left(-\frac{x^2}{2}\right)\cosh\sqrt{2}x - \sqrt{2}x\exp\left(-\frac{x^2}{2}\right)\sinh\left(\sqrt{2}x\right)$$

$$+\exp\left(-\frac{x^2}{2}\right)(-x)\sqrt{2}\sinh\sqrt{2}x + \exp\left(-\frac{x^2}{2}\right)2\cosh\sqrt{2}x$$

Replacing all these in the 1-D time independent Schrodinger equation, i.e

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi = E\psi$$

we get

$$\Rightarrow \; -\frac{\hbar^2}{2m}\left[x^2\cosh\left(\sqrt{2x}\right) - \cosh\sqrt{2}x - \sqrt{2}x\right]\sinh\sqrt{2}x - x\sqrt{2}\sinh\sqrt{2}x$$

$$+2\cosh\sqrt{2}x + V\cosh\left(\sqrt{2}x\right) = E\cosh\left(\sqrt{2}x\right)$$

$$\Rightarrow \; -\frac{\hbar^2}{2m}\left[x^2 - 1 - 2\sqrt{2}\tanh\left(\sqrt{2}x\right) + 2\right] + V = E$$

Since, energy of the particle E is independent of x, then

$$\Rightarrow \; V(x) - \frac{\hbar^2}{2m}\left(x^2 - 2\sqrt{2}\tanh\left(\sqrt{2}x\right)\right) = 0 \;\; \Rightarrow \;\; V(x) = \frac{\hbar^2}{2m}\left(x^2 - 2\sqrt{2}\tanh\left(\sqrt{2}x\right)\right)$$

Taking $m = 1, \hbar = 1$ we get, $\Rightarrow V(x) = \frac{x^2}{2} - \sqrt{2}\tanh\left(\sqrt{2}x\right)$

Superposition of States:

If $|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle ...... |\phi_n\rangle$ are orthonormal state vectors/wave functions and are representing state of quantum mechanics system. then any linear combination will also represent state of a quantum mechanics system. In other words, if $\phi_1, \phi_2, ...\phi_n$ are solutions of Schrodinger equation, then linear combination/superposition of the states i.e.

$$|\psi\rangle = C_1|\phi_1\rangle + C_2|\phi_2\rangle + C_3|\phi_3\rangle + ... = \sum_n C_n|\phi_n\rangle$$

will also be a solution of Schrodinger equation.

Note:

(1) $|\psi\rangle$ will be normalized if $|C_1|^2 + |C_2|^2 + |C_3|^2 .... = 1$

(2) Probability of finding a particle in $|\phi_1\rangle$ state $= |C_1|^2 = |\langle\phi_1|\psi\rangle|^2$

Example 28: Consider a system whose initial state at t = 0 is given in terms of 3 orthonormal state vectors

$|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle$ as

$$|\psi(x,0)\rangle = \frac{1}{\sqrt{2}}|\phi_1\rangle + A|\phi_2\rangle + \frac{1}{\sqrt{5}}|\phi_3\rangle \qquad \text{(A is a real constant)}$$

(i) Find the value of A so that $|\psi(x,0)\rangle$ is normalized.

(ii) What are probabilities regarding the system to be in $|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle$ states.

(iii) Find the state of the system at any time 't'.

Soln. (i) Normalization condition:

$$\langle\psi|\psi\rangle = 1 \quad \Rightarrow \sum |C_n|^2 = 1 \quad \Rightarrow \frac{1}{2} + A^2 + \frac{1}{5} = 1 \Rightarrow A^2 = \frac{1}{2} - \frac{1}{5} = \frac{3}{10} \quad \Rightarrow A = \sqrt{\frac{3}{10}}$$

Therefore, $|\psi(x,0)\rangle = \frac{1}{\sqrt{2}}|\phi_1\rangle + \sqrt{\frac{3}{10}}|\phi_1\rangle + \frac{1}{\sqrt{5}}|\phi_3\rangle$

Probabilities regarding the system to be in $|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle$ states are

$$P_1 = |C_1|^2 = \frac{1}{2}, \; P_2 = |C_2|^2 = \frac{3}{10}, \; P_3 = |C_3|^2 = \frac{1}{5}$$

The state of the system at any time 't' will be

$$|\psi(x,t)\rangle = \frac{1}{\sqrt{2}}e^{-iE_1t/\hbar}|\phi_1\rangle + \sqrt{\frac{3}{10}}e^{-iE_2t/\hbar}|\phi_2\rangle + \frac{1}{\sqrt{5}}e^{-iE_3t/\hbar}$$

Example 29: The wavefunction of a particle is given by $|\psi_n\rangle = \frac{1}{\sqrt{2}}|\phi_0\rangle + i|\phi_1\rangle$ where $|\phi_0\rangle$ and $|\phi_1\rangle$ are the normalized eigenstates with energies E_0 and E_1 respectively. The hamiltonian in the state ψ is

(a) $\frac{E_0}{2} + E_1$ (b) $\frac{E_0}{2} - E_1$ (c) $\frac{E_0 - 2E_1}{3}$ (d) $\frac{E_0 + 2E_1}{3}$

Soln. Suppose normalized wavefunction: $|\psi\rangle = C\left[\frac{1}{\sqrt{2}}|\phi_0\rangle + |\phi_1\rangle\right]$ (where C is normalization constnat)

Normalization condition: $\langle\psi|\psi\rangle = 1 \quad \Rightarrow \quad \frac{C^3}{2} + C^2 = 1 \quad \Rightarrow C = \sqrt{\frac{2}{3}}$

Therefore, the normalized wavefunction will be

$$|\psi\rangle = \frac{1}{\sqrt{3}}|\phi_0\rangle + \sqrt{\frac{2}{3}}|\phi_2\rangle$$

Energy of the particle in the given state will be

$$\langle E\rangle = P(\phi_0)E_0 + P(\phi_1)E_1 = \frac{E_0 + 2E_1}{3}$$

4.5 Probability Current density:

As the time changes from t = 0, the probability of finding the particle in some region of space may increase or decrease. If the probability increases in some region, then it should decrease in some other region such that total probability of finding the particle in the entire space should be equal to one. We can assume this as a flow of probability from one region to another region, like a fluid or current. Therefore, the probability flow satisfies the equation of continuity i.e.

$$\vec{\nabla}.\vec{J} + \frac{\partial\rho}{\partial t} = 0$$

where ρ = probability density $= \psi^*\psi$

and $\vec{J}$ = probability current density $= -\frac{i\hbar}{2m}\left[\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right]$

The magnitude of probabilty current density represents the flux of the particles i.e number of particles passing through per unit area per unit time and the direction of probability current density is along the direction of flow of the particles.

Example 30. A particle of mass 'm' is reprensented by wave function $\psi(x) = Ae^{ikx}$ (k = wave number, A = constant). The magnitude of probability current density is

(a) $|A|^2\frac{\hbar k}{m}$ (b) $|A|^2\frac{\hbar k}{2m}$ (c) $|A|^2\frac{(\hbar k)^2}{m}$ (d) $|A|^2\frac{(\hbar k)^2}{2m}$

Soln. Probability current density =

$$\vec{J} = -\frac{i\hbar}{2m}\left[\psi^*\frac{\partial\psi}{\partial x} - \psi\frac{\partial\psi^*}{\partial x}\right]\hat{x} = -\frac{i\hbar}{2m}\left[|A|^2 ik + |A|^2 ik\right]\hat{x} = |A|^2\frac{\hbar k}{m}\hat{x}$$

Correct answer is (a)

Example 31. The wave function of particle moving in free space is $\psi = e^{ikx} + 2e^{-ikx}$

(A) The probability current density for the real part of the wave function

(a) 1 (b) $\frac{\hbar k}{m}$ (c) $\frac{\hbar k}{2m}$ (d) 0

(B) The energy of the particle is

(a) $\frac{5\hbar^2k^2}{2m}$ (b) $\frac{3\hbar^2k^2}{2m}$ (c) $\frac{\hbar^2k^2}{2m}$ (d) $\frac{\hbar^2k^2}{m}$

Soln. **(A)** The real part of the wave function $\psi = e^{ikx} + 2e^{-ikx}$ is $3\cos kx$

Therefore, the corresponding probability density will be zero.

Correct answer is (d)

(B) $$\hat{H}\psi = \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\right)\psi = -\frac{\hbar^2}{2m}\left[\frac{d}{dx}\left\{ik\,e^{+ikx} - 2ik\,e^{-ikx}\right\}\right]$$

$$= -\frac{\hbar^2}{2m}\left[-k^2e^{ikx} - 2k^2e^{-ikx}\right] = \frac{\hbar^2k^2}{2m}\psi$$

Correct answer is (c)

Example 32. Calculate the probability current density for a Gaussian wave pocket

$$\psi(x) = A\exp\left[-\frac{x^2}{a^2} + \frac{ip_0x}{\hbar}\right]$$

Soln. $$\vec{J} = -\frac{i\hbar}{2m}\left[\psi^*\frac{\partial\psi}{\partial x} - \psi\frac{\partial\psi^*}{\partial x}\right]\hat{x}$$

$$= -\frac{i\hbar}{2m}\left[A^*\exp\left(-\frac{x^2}{a^2} - i\frac{p_0x}{\hbar}\right)\left(-\frac{2x}{a^2} + \frac{ip_0}{\hbar}\right)A\exp\left(-\frac{x^2}{a^2} + i\frac{p_0x}{\hbar}\right)\right.$$

$$\left. - A\exp\left(-\frac{x^2}{a^2} + i\frac{p_0x}{\hbar}\right)\left(-\frac{2x}{a^2} - \frac{ip_0}{\hbar}\right)A^*\exp\left(-\frac{x^2}{a^2} - i\frac{p_0x}{\hbar}\right)\right]\hat{x}$$

$$= -\frac{i\hbar}{2m}\left[|A|^2 \exp\left(-\frac{2x^2}{a^2}\right)\left\{-\frac{2x}{a^2}+\frac{ip_0}{\hbar}+\frac{2x}{a^2}+\frac{ip_0}{\hbar}\right\}\right]\hat{x}$$

$$= \left(-\frac{i\hbar}{2m}\right)\left(\frac{2ip_0}{\hbar}\right)|A|^2 \exp\left(-\frac{2x^2}{a^2}\right)\hat{x} = \frac{p_0}{m}|A|^2 \exp\left(-\frac{2x^2}{a^2}\right)\hat{x}$$

Example 33. Calculate the probability current density for a 3-D wave function $\psi(r) = \frac{A}{r}e^{ikr}$.

Soln. $$\vec{J} = -\frac{i\hbar}{2m}\left[\psi^*\frac{\partial \psi}{\partial r} - \psi\frac{\partial \psi^*}{\partial r}\right]\hat{r}$$

$$\vec{J} = -\frac{i\hbar}{2m}\left[\frac{A}{r}e^{-ikr}A\left\{\frac{ik}{r}e^{ikr}-\frac{e^{ikr}}{r^2}\right\}-\frac{A}{r}e^{ikr}.A\left\{-\frac{ik}{r}e^{-ikr}-\frac{1}{r^2}e^{-ikr}\right\}\right]\hat{r}$$

$$= -\frac{i\hbar}{2m}\left[\frac{A^2}{r^2}\left(ik-\frac{1}{r}\right)+\frac{A^2}{r^2}\left(ik+\frac{1}{r}\right)\right] = -\frac{i\hbar}{2m}2\frac{A^2}{r^2}ik = \frac{\hbar k}{m}\frac{|A|^2}{r^2}$$

Example 34. Show that for any normalized wave packet in one dimesion $\int_{-\infty}^{\infty} J(x)\,dx = \langle p_x\rangle/m$.

Soln. $$\int_{-\infty}^{\infty} J(x)\,dx = \frac{\hbar}{2im}\left[\int_{-\infty}^{\infty}\psi^*\frac{\partial}{\partial x}\psi\,dx - \int_{-\infty}^{\infty}\psi\frac{\partial}{\partial x}\psi^*\,dx\right]$$

$$= \frac{\hbar}{2im}\left[\frac{1}{-i\hbar}\int_{-\infty}^{\infty}\psi^*\,dx\left(-i\hbar\frac{d}{dx}\right)\psi(x) - c.c.\right]$$

where c.c. is the complex conjugate of the first term. Now, $\langle p_x\rangle$ must be real. So, the first term in the square bracket is $\frac{1}{-i\hbar}\langle p_x\rangle$ and hence its complex conjugate is $\frac{1}{i\hbar}\langle p_x\rangle$.

Thus, $$\int_{-\infty}^{\infty} J(x)\,dx = \frac{\hbar}{2im}\left[\frac{2}{-i\hbar}\langle p_x\rangle\right] = \frac{\langle p_x\rangle}{m}$$

Example 35. The solution of the Schrodinger equation for a free particle of mass m in one dimension is $\psi(x,t)$. At $t=0$, $\psi(x,0) = A\exp\left(\frac{-x^2}{a^2}\right)$. Find the probability amplitude in momentum space at $t=0$ and at time t.

Soln. $$\Phi(p,0) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\psi(x,0)\exp\left(-\frac{ipx}{\hbar}\right)dx$$

$$= \frac{A}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\exp\left(-\frac{x^2}{a^2}-\frac{ipx}{\hbar}\right)dx = \frac{Aa}{\sqrt{2\hbar}}\exp\left(-\frac{p^2a^2}{4\hbar^2}\right)$$

The Schrodinger equation for the momentum space equation is

$$i\hbar\frac{\partial\Phi}{\partial t}=\frac{p^2}{2m}\Phi\Rightarrow\frac{d\Phi}{\Phi}=\left(-\frac{ip^2}{2m\hbar}\right)dt\Rightarrow\Phi(p,t)=C\exp\left(-\frac{ip^2t}{2m\hbar}\right)$$

At $t=0,\Phi(p,0)=C$. Hence,

$$\Phi(p,t)=\frac{Aa}{\sqrt{2\hbar}}\exp\left(-\frac{p^2a^2}{4\hbar^2}-\frac{ip^2t}{2m\hbar}\right)$$

Example 46. If u_1 and u_2 are eigenstates corresponding to energy eigenvalues E_1 and $E_2(E_1\neq E_2)$ of a Hamiltonian, then in the state space

(a) u_1 and u_2 are parallel **[BHU 2010, 2011]**

(b) u_1 and u_2 are orthogonal

(c) u_1 and u_2 are at an angle θ such that $\cos\theta=\frac{E_1}{E_2}$

(d) u_1 and u_2 are degenerate states

Soln. The eigenstates corresponding to different eigenvalues $(E_1\neq E_2)$ of a Hermitian operator, are orthogonal.

Correct option is (b)

Example 47. If an operator satisfies the equation $\hat{O}\psi=a\psi$, then measurement of $\hat{O}$ will give

(a) a in the state ψ (b) the value $=\int\psi^*o\psi\,d^3x$ in the state ψ

(c) any value of several values greater than a (d) zero **[BHU 2010]**

Soln. The measurement of $\hat{O}$ will give the eigenstate ψ with the corresponding eigenvalue a.

Correct option is (a)

Example 48. The value of the commutation relation $[\hat p,(\hat x^2+\hat p^2)]$ is

(a) $-i\hbar$ (b) $-i\hbar\hat x$ (c) $-i\hbar(\hat x+\hat p)$ (d) $-2i\hbar\hat x$ **[BHU 2010]**

Soln. $\left[\hat p,(\hat x^2+\hat p^2)\right]=\left[\hat p,\hat x^2\right]+\left[\hat p,\hat p^2\right]$

$$=[p,\hat x]\hat x+\hat x[\hat p,\hat x]\qquad\left[\because\left[\hat p,\hat p^2\right]=0\right]$$

$$=-i\hbar\hat x+\hat x(-i\hbar)\qquad\left[\because[\hat x,\hat p]=i\hbar\right]$$

$$=-2i\hbar\hat x$$

Correct option is (d)

Example 49. If ψ_1,ψ_2 are eigenvectors of an operator $\hat A$ with eigenvalues as ½ and ½ respectively, then $a\psi_1+b\psi_2$ (where a,b are complex numbers) is an **[BHU 2010]**

(a) eigenvector of $\hat A$ with eigenvalue $\frac{a-b}{2}$ (b) eigenvector of $\hat A$ with eigenvalue zero

(c) eigenvector of $\hat A^2$ with eigenvalue $\frac{a+b}{4}$ (d) eigenvector of $\hat A^2$ with eigenvalue $\frac{1}{4}$

Soln. Given : $\hat{A}\psi_1 = \frac{1}{2}\psi_1$ and $\hat{A}\psi_2 = -\frac{1}{2}\psi_2$

$$\text{Now, } \hat{A}^2(a\psi_1 + b\psi_2) = a\hat{A}^2\psi_1 + b\hat{A}^2\psi_2$$
$$= \frac{a}{4}\psi_1 + \frac{b}{4}\psi_2$$
$$= \frac{1}{4}(a\psi_1 + b\psi_2)$$

$\Rightarrow$ $a\psi_1 + b\psi_2$ is an eigenvector of $\hat{A}^2$ with eigenvalue 1/4.

Correct option is (d)

Example 50. If wave function in coordinate space is $\psi(x) = \delta(x)$, then momentum space wave function is

(a) $\frac{1}{\sqrt{2\pi}}$ (b) $\delta(p)$ (c) $\frac{1}{\sqrt{2\pi}}\delta(p)$ (d) zero **[BHU 2010]**

Soln. Given : $\psi(x) = \delta(x)$. The momentum space wave function is

$$\psi(p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi(x)\,e^{\frac{ipx}{\hbar}}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(x)\,e^{\frac{ipx}{\hbar}}\,dx = \frac{1}{\sqrt{2\pi}}.$$

Correct option is (a)

Example 51. The expectation value of the momentum p in one-dimensional motion is **[BHU 2011]**

(a) $\int\psi^*\left(-i\hbar\frac{\partial\psi}{\partial t}\right)dx$ (b) $\int\psi^*\left(-\hbar^2\frac{\partial^2\psi}{\partial t^2}\right)dx$ (c) $\int\psi^*\left(-\hbar^2\frac{\partial^2\psi}{\partial x^2}\right)dx$ (d) $\int\psi^*\left(-i\hbar\frac{\partial\psi}{\partial x}\right)dx$

Soln. $\langle p_x\rangle = \int_{-\infty}^{\infty}\psi^*\hat{p}_x\psi\,dx = \int_{-\infty}^{\infty}\psi^*\left(-i\hbar\right)\frac{\partial\psi}{\partial x}dx$

Correct option is (d)

Example 52. If $\psi(r,t)$ represents the wave function of a particle, then to which function it would collapse just after a precise measurement of momentum? **[BHU 2011]**

(a) $\delta(\vec{r}-\vec{r}\,')$

(b) $\delta(\vec{P}-\vec{P}\,')$

(c) $\psi'(r,t)$ with smaller Δp

(d) $\psi'(r,t)$ with smaller Δx

Soln. For the wavefunction $\delta(\vec{P}-\vec{P}\,')$, uncertainty in momentum will be zero i.e. precise measurement of momentum is possible.

Correct option is (b)

Example 53. The wave function of a particle is given by $\psi = Ce^{-\alpha^2x^2}$ in the range $-\infty < x < \infty$, where C and α are constants. The probability of finding the particle in the range $0 < x < \infty$ is **[BHU 2011]**

(a) $\frac{1}{5}$ (b) $\frac{2}{4}$ (c) $\frac{1}{4}$ (d) $\frac{1}{3}$

oln. $$P(0<x<\infty)=\frac{\int_0^\infty \psi^*\psi\,dx}{\int_{-\infty}^\infty \psi^*\psi\,dx}=\frac{C^2\int_0^\infty e^{-2\alpha^2x^2}\,dx}{C^2\int_{-\infty}^\infty e^{-2\alpha^2x^2}\,dx}=\frac{\int_0^\infty e^{-2\alpha^2x^2}\,dx}{2\int_0^\infty e^{-2\alpha^2x^2}\,dx}=\frac{1}{2}$$

Correct option is (b)

Example 54. Which of the following quantum mechanical operators is Hermitian? **[BHU 2011]**

(a) $i\frac{d}{dx}$ (b) $\frac{d}{dx}$ (c) $\frac{d^2}{dx^2}$ (d) $-\frac{d}{dx}$

oln. $\left(\frac{d}{dx}\right)^\dagger=-\frac{d}{dx}$; $\left(-\frac{d}{dx}\right)^\dagger=\frac{d}{dx}$ i.e. these operators are skew-hermitian.

$\left(i\frac{d}{dx}\right)^\dagger=(-i)\left(-\frac{d}{dx}\right)=i\frac{d}{dx}$; $\left(\frac{d^2}{dx^2}\right)^\dagger=\left(\frac{d}{dx}\right)^\dagger\left(\frac{d}{dx}\right)^\dagger=\left(-\frac{d}{dx}\right)\left(-\frac{d}{dx}\right)=\frac{d^2}{dx^2}$ i.e these operators are hermitian.

Correct option is (a, c).

Example 55. The value of the commutation relation $\left[\hat{p},\left(\hat{x}^2+\hat{p}^2\right)\right]$ is **[BHU 2011]**

(a) $-i\hbar\hat{p}$ (b) $-i\hbar\hat{x}$ (c) $-i\hbar(\hat{x}+\hat{p})$ (d) $-2i\hbar\hat{x}$

oln. $$\begin{aligned}\left[\hat{p},(\hat{x}^2+\hat{p}^2)\right]&=\left[\hat{p},\hat{x}^2\right]+\left[\hat{p},\hat{p}^2\right]\\&=[p,\hat{x}]\hat{x}+\hat{x}[\hat{p},\hat{x}] && \left[\because\left[\hat{p},\hat{p}^2\right]=0\right]\\&=-i\hbar\hat{x}+\hat{x}(-i\hbar) && \left[\because[\hat{x},\hat{p}]=i\hbar\right]\\&=-2i\hbar\hat{x}\end{aligned}$$

Correct option is (d)

Example 56. If the wave function for a particle in coordinate space is $\psi(x)=\delta(x)$, then that wave function in the momentum space will be **[BHU 2011]**

(a) $\psi(p)=\frac{1}{\sqrt{2\pi}}$ (b) $\psi(p)=\delta(p)$ (c) $\psi(p)=\frac{1}{\sqrt{2\pi}}\delta(p)$ (d) $\psi(p)=0$

oln. Given : $\psi(x)=\delta(x)$. The momentum space wave function is

$$\psi(p)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi(x)\,e^{\frac{ipx}{\hbar}}\,dx=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(x)\,e^{\frac{ipx}{\hbar}}\,dx=\frac{1}{\sqrt{2\pi}}.$$

Correct option is (a)

Example 57. The eigenvalues of a Hermitian operator are always **[BHU 2012]**

(a) real (b) imaginary (c) degenerate (d) linear

oln. The eigenvalues of a Hermitian operator are always real.

Correct option is (a)

Example 58. The state $\psi=\frac{1}{\sqrt{6}}\psi_{-1}+\frac{1}{\sqrt{2}}\psi_1+\frac{1}{\sqrt{3}}\psi_2$ is a linear combination of three orthonormal eigenstates of the operator $\hat{O}$ corresponding to eigenvalues −1, 1 and 2. What is the expectation value of $\hat{O}$ for this state?

(a) $\frac{2}{3}$ (b) $\sqrt{\frac{7}{6}}$ (c) 1 (d) $\frac{4}{3}$ **[BHU 2012]**

Soln. Given : $\psi=\frac{1}{\sqrt{6}}\psi_{-1}+\frac{1}{\sqrt{2}}\psi_1+\frac{1}{\sqrt{3}}\psi_2$

$$\therefore\ \langle\hat{O}\rangle=\langle\psi|\hat{O}|\psi\rangle$$

$$=\left(\frac{1}{\sqrt{6}}\psi_{-1}^*\frac{1}{\sqrt{2}}\psi_1^*+\frac{1}{\sqrt{3}}\psi_2^*\right)\hat{O}\left(\frac{1}{\sqrt{6}}\psi_{-1}+\frac{1}{\sqrt{2}}\psi_1+\frac{1}{\sqrt{3}}\psi_2\right)$$

$$=\left(\frac{1}{\sqrt{6}}\psi_{-1}^*+\frac{1}{\sqrt{2}}\psi_1^*+\frac{1}{\sqrt{3}}\psi_2^*\right)\left(\frac{-1}{\sqrt{6}}\psi_{-1}+\frac{1}{\sqrt{2}}\psi_1+\frac{2}{\sqrt{3}}\psi_2\right)$$

$$=-\frac{1}{6}+\frac{1}{2}+\frac{2}{3}=-\frac{1+3+4}{6}=1.$$

Correct option is (c)

Example 59. Which of the following functions could represent the radial wave function for an electron in an atom ? (r is the distance of the electron from the nucleus, A and b are constants). **[BHU 2012]**

(1) Ae^{-br} (2) $A\sin(br)$ (3) $\frac{A}{r}$

(a) 1 only (b) 2 only (c) 1 and 2 only (d) 1 and 3 only

Soln. The radial wave function should goes to zero as $r\to\infty$ and has a finite or zero value as $r\to 0$. These conditions are satisfied by only option (1). i.e., Ae^{-br}

Correct option is (a)

Example 60. If ψ is a normalized solution of the Schrödinger equation and Q is the operator corresponding to a physical observable x, the quantity $\psi^*Q\psi$ may be integrated in order to obtain the **[BHU 2012]**

(a) normalization constant for ψ (b) spatial overlap of Q with ψ

(c) mean value of x (d) uncertainty in x

Soln. The quantity $\psi^*Q\psi$ gives the mean value of x.

Correct option is (c)

Example 61. Which of the following is an eigenfunction of the linear momentum operator $-i\hbar\frac{\partial}{\partial x}$ with a positive eigenvalue $\hbar k$, i.e., an eigenfunction that describes a particle that is moving in free space in the direction of positive x with a precise value of linear momentum ? **[BHU 2012]**

(a) $\cos kx$ (b) $\sin kx$ (c) e^{-ikx} (d) e^{ikx}

Soln. The given operator is $\hat{p}=-i\hbar\frac{d}{dx}$

For the state $\psi(x)=e^{ikx}$, we have $\hat{p}\,\psi(x)=-i\hbar\frac{d}{dx}e^{ikx}=\hbar k\,\psi(x)$

Correct option is (d)

Example 62. Which one of the following operators is Hermitian ? **[BHU 2012]**

(a) e^{ix} (b) xp_y (c) $\frac{\partial}{\partial x}$ (d) $\frac{i\partial}{\partial x}$

Soln. $\left(e^{ix}\right)^{\dagger} = e^{-ix^{\dagger}} = e^{-ix}$ i.e. not hermitian.

$\left(xp_y\right)^{\dagger} = \left(p_y\right)^{\dagger}\left(x\right)^{\dagger} = p_y x = xp_y$ i.e hermitian.

$\left(\frac{\partial}{\partial x}\right)^{\dagger} = -\frac{\partial}{\partial x}$ i.e. not hermitian.

$\left(i\frac{\partial}{\partial x}\right)^{\dagger} = (-i)\left(-\frac{\partial}{\partial x}\right) = i\frac{\partial}{\partial x}$ i.e. hermitian.

Example 63. According to wave mechanics, a free particle in space can possess energies of the following type **[BHU 2012]**

(a) Discrete energies (b) Continuous energies

(c) Both types of energies (d) Only one simple value of energy

Soln. A free particle is unbounded and hence possesses continuous energies.

Correct option is (b)

Example 64. Which of the following operators is not linear operator : **[BHU 2014]**

(a) Momentum (b) Parity (c) Time reversal (d) Angular momentum

Soln. Time reversal is not a linear operator.

Correct option is (c)

Example 65. Which of the following statements is incorrect about complex conjugation operator ? **[BHU 2014]**

(a) It is an Hermitian operator (b) It has eigen values ±1

(c) It is not a linear operator (d) It has eigen values ±1

Soln. The complex conjugation operators can take any eigenvalue.

Correct option is (b, d)

Example 66. Which of the following is a eigenstate of momentum operator ? **[BHU 2014]**

(a) $\cos kx + i\sin kx$ (b) $i\sin kx$ (c) $\cos kx$ (d) $A\cos kx + B\sin kx$

Soln. The momentum operator is $\hat{p} \equiv -i\hbar\frac{d}{dx}$

$$\hat{p}\psi = -i\hbar\frac{d}{dx}(\cos kx + i\sin kx) \quad = -\hbar k(-i\sin kx - \cos kx) \quad = \hbar k\psi$$

Correct option is (a)

Example 67. Probabilities current density is represented by the operator : **[BHU 2014]**

(a) $\frac{i\hbar}{2m}\left(\psi\vec{\nabla}\psi^* - \psi^*\vec{\nabla}\psi\right)$ (b) $\frac{\hbar}{2im}\left(\psi\vec{\nabla}\psi^* - \psi^*\vec{\nabla}\psi\right)$

(c) $\frac{i\hbar}{2m}\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right)$ (d) $\frac{\hbar}{2im}\left(\psi\vec{\nabla}\psi^* - \psi\vec{\nabla}\psi^*\right)$

oln. The probability current density is defined as

$$\vec{J} = \frac{i\hbar}{2m}\left(\psi \vec{\nabla}\psi^* - \psi^* \vec{\nabla}\psi\right)$$

Correct option is (a)

Example 68. Consider a system in a state $|\psi\rangle$ given by $|\psi\rangle = \frac{1}{\sqrt{3}}\left[|\psi_1\rangle + \sqrt{2}|\psi_2\rangle\right] \cdot |\psi\rangle$ and $|\psi_2\rangle$ are orthogonal and normalized eigen vectors of the system, with energy eigen values E_1 and E_2, respectively. If a measurement of energy is performed on the system, what is the probability of getting a value E_2 ?

(a) $\sqrt{\frac{2}{3}}$ (b) $\frac{2}{3}$ (c) $\sqrt{\frac{1}{3}}$ (d) $\frac{1}{3}$ **[BHU 2014]**

ln. Given : $|\psi\rangle = \frac{1}{\sqrt{3}}\left[|\psi_1\rangle + \sqrt{2}|\psi_2\rangle\right]$

Therefore, probability of getting a value E_2 is

$$P(E_2) = \frac{|\langle\psi_2 | \psi\rangle|^2}{\langle\psi | \psi\rangle} = \frac{\left(\sqrt{2}/\sqrt{3}\right)^2}{\left(1/\sqrt{3}\right)^2 + \left(\sqrt{2}/\sqrt{3}\right)^2} = \frac{2/3}{1} = \frac{2}{3}$$

Correct option is (b)

Example 69. If a system is invariant under parity, which of the following statements is incorrect ?

(a) The wave functions must have definite parity **[BHU 2014]**

(b) $\langle x\rangle$ is always zero

(c) $\langle p_x^2\rangle$ is always zero

(d) half of wave functions are positive under parity

ln. The invariance of system under parity does not say that $\langle p_x^2\rangle = 0$ always.

Correct option is (c)

Example 70. The wave function of a particle at $t = 0$ is given by

$$|\psi|(0)\rangle = \frac{1}{\sqrt{2}}\left[|u_1\rangle + |u_2\rangle\right], \text{ where } |u_1\rangle \text{ and } |u_2\rangle$$

are the normalized eigen states with eigen values E_1 and E_2, respectively, $(E_2 > E_1)$. The shortest time after which $|\psi(t)\rangle$ will become orthonormal to $|\psi(0)\rangle$ is : **[BHU 2014]**

(a) $\frac{-\hbar\pi}{2(E_2 - E_1)}$ (b) $\frac{\hbar\pi}{E_2 - E_1}$ (c) $\frac{\sqrt{2}\hbar\pi}{E_2 - E_1}$ (d) $\frac{2\hbar\pi}{E_2 - E_1}$

n. Given : $|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left[|u_1\rangle + |u_2\rangle\right]$

The eigenstate after time t is

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left[|u_1\rangle e^{-iE_1t/\hbar} + |u_2\rangle e^{-iE_2t/\hbar}\right]$$

If $|\psi(0)\rangle$ is orthogonal to $|\psi(t)\rangle$, then $\langle\psi(0) | \psi(t)\rangle = 0$

$$\Rightarrow \frac{1}{2}\left[\langle u_1| + \langle u_2|\right]\left[|u_1\rangle e^{-iE_1t/\hbar} + |u_2\rangle e^{-iE_2t/\hbar}\right] = 0$$

$$\Rightarrow e^{-iE_1t/\hbar} + e^{-iE_2t/\hbar} = 0 \qquad \Rightarrow e^{-iE_1t/\hbar} = -e^{-iE_2t/\hbar}$$

$$\Rightarrow e^{i(E_2-E_1)t/\hbar} = -1 = e^{i(2n+1)\pi}\ ;\ n = 0,1,2,3$$

$$\Rightarrow \frac{(E_2-E_1)}{\hbar} = (2n+1)\pi \qquad \Rightarrow t = \frac{(2n+1)\pi\hbar}{(E_2-E_1)}$$

So, for t to be minimum, $n = 0$

Therefore, $t = \dfrac{\hbar\pi}{E_2 - E_1}$.

Correct option is (b)

Example 71. In a quantum system an observable is represented by an operator A. If $|\psi\rangle$ is a state of the system which is not a eigen state of A, then $r = \langle\psi|A|\psi\rangle^2 - \langle\psi|A^2|\psi\rangle$ must be : **[BHU 2014]**

(a) equal to zero (b) greater than zero (c) less than zero (d) greater than or equal to zero

Soln. Given : $r = \langle\psi|A|\psi\rangle^2 - \langle\psi|A^2\psi\rangle$

The uncertainity value is

$$(\Delta A)^2 = \langle\psi|A^2\psi\rangle - \langle\psi|A|\psi\rangle^2 = -r$$

$$\Rightarrow \quad r = -(\Delta A)^2 < 0$$

Correct option is (c)

Example 72. A wave function has the form $\psi_1 = Ae^{-\alpha x}$ for $x > 0$ for the positive value of α. However, it has the form $\psi_2 = e^{+\alpha x}$ for $x < 0$. The normalized form of this function is : **[BHU 2015]**

(a) $\psi_1(x) = \sqrt{\alpha}\, e^{\alpha x}$ for $x > 0$
$\psi_2(x) = \sqrt{\alpha}\, e^{-\alpha x}$ for $x < 0$

(b) $\psi_1(x) = \sqrt{\alpha}\, e^{+\alpha x}$ for $x < 0$
$\psi_2(x) = \sqrt{\alpha}\, e^{-\alpha x}$ for $x > 0$

(c) $\psi_1(x) = \dfrac{1}{\sqrt{\alpha}} e^{\alpha x}$ for $x > 0$
$\psi_2(x) = \dfrac{1}{\sqrt{\alpha}} e^{-\alpha x}$ for $x < 0$

(d) $\psi_1(x) = \dfrac{1}{\sqrt{\alpha}} e^{-\alpha x}$ for $x > 0$
$\psi_2(x) = \dfrac{1}{\sqrt{\alpha}} e^{+\alpha x}$ for $x < 0$

Soln. Given : $\psi(x) = \begin{cases} Ae^{-\alpha x} & ;\ x > 0 \\ Ae^{+\alpha x} & ;\ x < 0 \end{cases}$

The normalization is done by $\displaystyle\int_{-\infty}^{\infty} \psi^*(x)\psi(x)\,dx = 1$

$$\Rightarrow |A|^2 \int_0^{\infty} e^{-2\alpha x}\,dx + |A|^2 \int_{-\infty}^{0} e^{2\alpha x}\,dx = 1 \Rightarrow |A|^2\left(\frac{1}{2\alpha} + \frac{1}{2\alpha}\right) = 1 \Rightarrow |A| = \sqrt{\alpha}$$

$$\therefore \quad \psi(x) = \begin{cases} \sqrt{\alpha}\, e^{-\alpha x} & ;\ x > 0 \\ \sqrt{\alpha}\, e^{\alpha x} & ;\ x < 0 \end{cases}$$

Correct option is (b)

Example 73. If the Hamiltonian is Hermitian, two degenerate eigenstates **[BHU 2015]**

(a) would be orthogonal to each other (b) would be non-orthogonal to each other

(c) may not be orthogonal (d) may not be non-orthogonal

Soln. The eigenstates of a Hermitian operator are othogonal if they have different eigenvalues. So the eigenstates, if degenerate, are non-orthogonal to each other.

Correct option is (b)

Example 74. An anti-Hermitian operator corresponds to the eigenvalue that is

(a) complex (b) real (c) imaginary (d) zero **[BHU 2015]**

Soln. The eigenvalue corresponds to an anti-Hermitian operator is imaginary.

Correct option is (c)

Example 75. One of the following operator is hermitian : **[BHU 2016]**

(a) $\hat{x}\hat{p}_x$ (b) $\left[\hat{x}, \hat{p}_x^2\right]$ (c) $-i\dfrac{d^2}{dx^2}$ (d) $\dfrac{d^2}{dx^2}$

Soln. $\left(\dfrac{d^2}{dx^2}\right)^\dagger = \left(\dfrac{d}{dx}\right)^\dagger\left(\dfrac{d}{dx}\right)^\dagger = \left(-\dfrac{d}{dx}\right)\left(-\dfrac{d}{dx}\right) = \dfrac{d^2}{dx^2}$

Correct option is (d)

Example 76. One of the following is not an eigen state of the parity operator ? **[BHU 2016]**

(a) $\cos x + x\sin x$ (b) $x(\cos x + \sin x)$ (c) $x\cos x + \sin x$ (d) $x\cos x\sin x$

Soln. Now for $\psi(x) = x(\cos x + \sin x)$, we have $\hat{p}\psi(x) = -x(\cos x - \sin x) \neq \psi(-x)$;

So, $\psi(x) = x(\cos x + \sin x)$ is not an eigen state of the parity operator.

Correct option is (b)

Example 77. The wave function of a particle $\psi = \frac{1}{\sqrt{2}}\varphi_0 + i\varphi_1$ is given in terms of the eigen states φ_0 and φ_1 corresponding to the ground state energy and the first excited state energy E_0 and E_1 respectively. The expectation value of the Hamiltonian in the state ψ is given by: **[BHU 2016]**

(a) $\dfrac{1}{2}E_0 + E_1$ (b) $\dfrac{E_0}{2} - E_1$ (c) $\dfrac{E_0 - 2E_1}{3}$ (d) $\dfrac{E_0 + 2E_1}{3}$

Soln. Given : $\psi = \dfrac{1}{\sqrt{2}}\phi_0 + i\phi_1$

$$\therefore\ \langle H\rangle = \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{\left\langle \frac{1}{\sqrt{2}}\phi_0 + i\phi_1 \middle| H \middle| \frac{1}{\sqrt{2}}\phi_0 + i\phi_1 \right\rangle}{\left\langle \frac{1}{\sqrt{2}}\phi_0 + i\phi_1 \middle| \frac{1}{\sqrt{2}}\phi_0 + i\phi_1 \right\rangle}$$

$$= \frac{\frac{1}{2}\langle\phi_0|H|\phi_0\rangle + \langle\phi_1|H|\phi_1\rangle}{\frac{1}{2}\langle\phi_0|\phi_0\rangle + \langle\phi_1 + \phi_1\rangle} = \frac{\frac{1}{2}E_0 + E_1}{\frac{1}{2} + 1} = \frac{E_0 + 2E_1}{3}$$

Correct option is (d)

Example 78. If H is a Hermitian matrix, then e^{iH} is one of the following : **[BHU 2016]**

(a) Hermitian matrix (b) Unitary matrix (c) Orthogonal matrix (d) Null matrix

Soln. $\left(e^{iH}\right)^{\dagger} = e^{-iH^{\dagger}} = e^{-iH}$

Therefore, $\left(e^{iH}\right)\left(e^{iH}\right)^{\dagger} = e^{iH}e^{-iH} = I$ i.e it is a unitary matrix.

Correct option is (b)

Example 79. According to wave mechanism, a free particle in space can possess energies of the following type : **[HCU 2010]**

(a) Discrete energies (b) Continuous energies

(c) Both types of energies (d) Only one single value of energy

Soln. A free particle can possess any value of the energy.

Correct option is (b)

Example 80. The commutator of two hermitian matrices is

(a) hermitian (b) unitary (c) anti-hermitian (d) orthogonal **[HCU 2011]**

Soln. If A and B are two Hermitian matrices, then $A^{\dagger} = A$ and $B^{\dagger} = B$.

Now, $[A, B]^{\dagger} = (AB - BA)^{\dagger} = B^{\dagger}A^{\dagger} - A^{\dagger}B^{\dagger} = BA - AB$

$\Rightarrow \quad [A, B]^{\dagger} = -(AB - BA) = -[A, B]^{\dagger}$

So the commutator of two Hermitian matrices is an anti-Hermitian matrix.

Correct option is (c)

Example 81. The energy eigenstate of a system must necessarily be a **[HCU 2013]**

(a) A stationary state (b) A state having explicit time dependence

(c) A state having implicit time dependence (d) A state with no time dependence

Soln. Energy eigenstate of a system will have an explicit time dependency.

Correct option is (b)

Example 82. If A and B are hermitian matrices then $[A,B]$ is **[HCU 2014]**

(a) hermitian (b) anti-hermitian (c) unitary (d) symmetric

Soln. If A and B are two Hermitian matrices, then $A^{\dagger} = A$ and $B^{\dagger} = B$.

Now, $[A, B]^{\dagger} = (AB - BA)^{\dagger} = B^{\dagger}A^{\dagger} - A^{\dagger}B^{\dagger} = BA - AB$

$\Rightarrow \quad [A, B]^{\dagger} = -(AB - BA) = -[A, B]^{\dagger}$

So the commutator of two Hermitian matrices is an anti-Hermitian matrix.

Correct option is (b)

Example 83. Which of the following functions can be an eigen-function of the operator $i\hbar\dfrac{d}{dx}$?

(a) e^{ikx} (b) $\cos(kx)$ (c) $\sin(kx)$ (d) $\cos(2kx)$ **[HCU 2014]**

Soln. If ψ is the eigen function of the operator $\hat{A}$ with eigenvalue a, then $\hat{A}|\psi\rangle = a|\psi\rangle$

Here, $i\hbar\frac{d}{dx}e^{ikx} = -k\hbar\, e^{ikx} = a\, e^{ikx}$

Correct option is (a)

Example 84. If ψ_1 and ψ_2 are two independent solutions of the time-independent Schrodinger equation, which of the following is also a solution of the same Schrodinger equation? **[HCU 2015]**

(a) $\psi_1 \times \psi_2$ (b) ψ_1 / ψ_2 (c) $\psi_1 \pm \psi_2$ (d) $(\psi_1\psi_2)^{1/2}$

Soln. If ψ_1 and ψ_2 are the wave function then according to superposition theory, the wave function of the composite system will be $\psi = (\psi_1\psi_2)$ and energy will be $E = E_1 + E_2$.

Correct option is (a)

Example 85. Which of the following functions is the eigen-function of the operator, $-i\hbar\frac{d}{dx}$?

(a) $\exp(ikx)$ (b) $\cos(kx)$ (c) $\sin(kx)$ (d) $(\cos(kx)+\sin(kx))$

Soln. If ψ is the eigen function of the operator $\hat{A}$ with eigenvalue a, then $\hat{A}|\psi\rangle = a|\psi\rangle$ **[HCU 2015]**

Here, $-i\hbar\frac{d}{dx}e^{ikx} = k\hbar\, e^{ikx} = a\, e^{ikx}$

Correct option is (a)

Example 86. In a one dimensional problem, the normalized wave function for the ground state is given by $\psi = N e^{-ax^2}$; where N is the normalization constant. What is the value of N? **[HCU 2015]**

(a) $\sqrt{\frac{a}{\pi}}$ (b) $\sqrt{\frac{a}{2\pi}}$ (c) $\sqrt{\frac{2a}{\pi}}$ (d) $\frac{a}{\pi}$

Soln. $\psi = N e^{-ax^2}$

According to normalization condition

$$\int_{-\infty}^{\infty} |\psi|^2\, dx = 1$$

$$\Rightarrow |N|^2 \int_{-\infty}^{\infty} e^{-2ax^2} dx = 1 \Rightarrow 2|N|^2 \int_{0}^{\infty} e^{-2ax^2} dx = 1 \qquad \left[2ax^2 = z \Rightarrow dx = \frac{dz}{2\sqrt{2az}}\right]$$

$$\Rightarrow \frac{1}{2}\sqrt{\frac{2}{a}}\,|N|^2 \int_{0}^{\infty} e^{-z} z^{-1/2}\, dz = 1 \Rightarrow \frac{1}{2}\sqrt{\frac{2}{a}}\,|N|^2 \sqrt{\pi} = 1$$

$$\Rightarrow |N|^2 = \sqrt{\frac{2a}{\pi}} \Rightarrow |N| = \left(\frac{2a}{\pi}\right)^{1/4}$$

Correct option is ()

Example 87. Commutator of two Hermitian operators has to be **[HCU 2015]**

(a) Hermitian (b) anti-Hermitian (c) unitary (d) orthogonal

Soln. If A and B are two Hermitian matrices, then $A^\dagger = A$ and $B^\dagger = B$.

$$\text{Now, } [A, B]^\dagger = (AB - BA)^\dagger = B^\dagger A^\dagger - A^\dagger B^\dagger = BA - AB$$

$$\Rightarrow \quad [A, B]^\dagger = -(AB - BA) = -[A, B]^\dagger$$

So the commutator of two Hermitian matrices is an anti-Hermitian matrix.

Correct option is (b)

Example 88. If $\psi_0(x) = u_0(x)$ and $\psi_1(x) = u_1(x) + \alpha\psi_0(x)$ for $a \le x \le b$ then $\psi_1(x)$ will be orthogonal to $\psi_0(x)$ if **[HCU 2016]**

(a) $\alpha = -\dfrac{1}{|\psi_0|^2}\displaystyle\int_a^b \psi_0^* u_1 dx$ (b) $\alpha = \dfrac{1}{|\psi_0|^2}\displaystyle\int_a^b \psi_0^* u_1 dx$

(c) $\alpha = -\dfrac{1}{|\psi_0|^2}\displaystyle\int_a^b \psi_0^* u_0 dx$ (d) $\alpha = \dfrac{1}{|\psi_0|^2}\displaystyle\int_a^b \psi_0^* u_0 dx$

Soln. $\psi_0(x) = u_0(x)$

$\psi_1(x) = u_1(x) + \alpha\psi_0(x) = u_1(x) + \alpha u_0(x)$

$\psi_0(x)$ and $\psi_1(x)$ will be orthogonal when

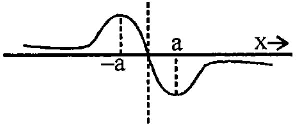
$$\langle \psi_0 | \psi_1 \rangle = 0 \Rightarrow \int_a^b \psi_0(x)\, \psi_1(x)\, dx = 0$$

$$\Rightarrow \quad \int_a^b \left(u_0^*(x) u_1(x) + |u_0(x)|^2\right) dx = 0 \Rightarrow \int_a^b u_0^*(x) u_1(x) dx + \alpha \int_a^b |u_0(x)|^2 dx = 0$$

$$\Rightarrow \quad \alpha = -\frac{1}{\int_a^b |\psi_0|^2 dx} \int_a^b \psi_0^*(x) u_1(x)\, dx$$

Correct option is ( )

Example 89. The wave function of the first excited state of a particle in a simple harmonic oscillator potential may be represented as shown in the figure below: **[DU 2014]**



The probability of finding the particle at a particular position is maximum for:

(a) only x = a (b) only x = –a

(c) Both x = a and x = –a (d) x tending to ±∞

Soln. The probability would be maximum where $|\psi(x)|^2$ is maximum and $|\psi(x)|^2$ is maximum makes $|\psi(x)|$ also either maximum or minimum. So the probability in our case is maximum at $x = \pm a$.

Correct option is (c)

Example 90. Which of the following is an admissible wave function in three dimensions? (Here N_1, N_2, N_3, N_4 and r_0 are constants and r is the radial coordinate): **[DU 2014]**

(a) $\psi = \frac{N_1}{(r^2 + r_0^2)^{1/4}}$ (b) $\psi = \frac{N_2}{(r^2 + r_0^2)^{1/2}}$

(c) $\psi = \frac{N_3}{(r^2 + r_0^2)^{3/2}}$ (d) $\psi = \frac{N_4}{(r^2 + r_0^2)^{2}}$

Soln. Wave function should be finite anf sigle valued everywhere.

Correct option is (d)

Example 91. The wave function of a particle is given by $\psi = \frac{1}{\sqrt{2}} e^{-|x|/2}$ with $-\infty < x < +\infty$. The minimum uncertainty in the momentum is: **[DU 2014]**

(a) $10h/(2\pi)$ (b) $h/(2\pi\sqrt{2})$ (c) $2/h$ (d) $\sqrt{h}$

Soln. Given : $\psi(x) = \frac{1}{\sqrt{2}} e^{-|x|/2} = \begin{cases} \frac{1}{\sqrt{2}} e^{-x/2} & ; \quad x \ge 0 \\ \frac{1}{\sqrt{2}} e^{x/2} & ; \quad x < 0 \end{cases}$

$$\langle x \rangle = \int_{-\infty}^{\infty} \psi(x) x\, \psi(x)\, dx \qquad = \frac{1}{2}\left[\int_{-\infty}^{0} x e^{+x/2}\, dx + \int_{0}^{\infty} x e^{-x}\, dx\right]$$

$$= \frac{1}{2}\left[\left\{x\left(\frac{e^{+x}}{+1}\right) - \left(\frac{e^{+x}}{+1}\right)\right\}_{-\infty}^{0} + \left\{x\left(\frac{e^{-x}}{-1}\right) - \left(\frac{e^{-x}}{1}\right)\right\}_{0}^{\infty}\right] = \frac{1}{2}[(-1) - (0) + (0) - (-1)] \qquad = 0.$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} \psi(x) x^2\, \psi(x)\, dx \qquad = \frac{1}{2}\left[\int_{-\infty}^{0} x^2 e^{x}\, dx + \int_{0}^{\infty} x^2 e^{-x}\, dx\right]$$

$$= \frac{1}{2}\left[\left\{x^2(e^x) - 2xe^x + 2e^x\right\}_{-\infty}^{0} + \left\{x^2\left(\frac{e^{-x}}{-1}\right) - 2x\left(\frac{e^{-x}}{1}\right) + 2\left(\frac{e^{-x}}{-1}\right)\right\}_{0}^{\infty}\right]$$

$$= \frac{1}{2}[(2) - (0) + (0) - (-2)] \qquad = 2.$$

$$\therefore \quad \Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{2}$$

$$\therefore \quad (\Delta p)_{\min} = \frac{h}{4\pi\, \Delta x} = \frac{h}{4\sqrt{2}\pi}.$$

Correct option is ( ).

Example 92. The wave function of the particle is given by a linear combination of two of its energy states ψ_1 and ψ_2 (with energy eigen values E_1 and E_2 respectively) as $\psi = \psi_1 / a + \psi_2 / b$. If a measurement of energy is made, the probability of finding particle energy to be E_1 is **[DU 2015]**

(a) $|b|^2 / [|a|^2 + |b|^2]$ (b) $|a|^2 / [|a|^2 + |b|^2]$

(c) $b/(a+b)$ (d) $a/(a+b)$

Soln. Probability of finding particle energy to be E_1 is

$$P(E_1) = \frac{|\langle \psi_1 | \psi \rangle|^2}{\langle \psi | \psi \rangle} = \frac{\left| \left\langle \psi_1 \middle| \frac{\psi_1}{a} + \frac{\psi_2}{b} \right\rangle \right|^2}{\left\langle \frac{\psi_1}{a} + \frac{\psi_2}{b} \middle| \frac{\psi_1}{a} + \frac{\psi_2}{b} \right\rangle} = \frac{1/|a|^2}{1/|a|^2 + 1/|b|^2} = \frac{|b|^2}{|a|^2 + |b|^2}$$

Correct option is (a)

Example 93. The wave function for a particle is given by $\psi = Ae^{-|x|}$, where A is a constant. This implies that the potential

(a) is differentiable everywhere **[DU 2015]**

(b) is continuous everywhere but not differentiable at $x = 0$

(c) has discontinuity at $x = 0$

(d) has discontinuities at $x = \pm 1$ but differentiable every else

Soln. Given : $\psi = Ae^{-|x|} = \begin{cases} Ae^{-x} & ; \quad x > 0 \\ Ae^{x} & ; \quad x < 0 \end{cases}$

$$\Rightarrow \frac{d\psi}{dx} = \begin{cases} -Ae^{-x} & ; \quad x > 0 \\ Ae^{x} & ; \quad x < 0 \end{cases}$$

So at $x = 0$, $\frac{d\psi}{dx}$ is not continuous and hence not differentiable as well. This is because the potential at $x = 0$ is not continuous as well.

Correct option is (c)

Chapter 5

Bound State problems

5.1. What is a Bound State?

If the motion of a particle is confined to a limited region of space by potential energy so that the particle can move back and forth in the region, then the particle is in a bound state. The simplest example of all motions in bound state is the motion of a particle in a one-dimensional box with zero potential energy and the potential energy is assumed to be infinite at he walls of the box.

The motion of a particle in three-dimensional box, one-dimensional box, motion of a simple harmonic oscillator, motion of a particle in a one-dimensional square well potential with ($E < V_0$), motion of a particle in a one-dimensional square well potential with ($E < V_0$), motion of a particle in one-dimensional (also in three-dimensional) potential well, motion of a particle in spherically symmetric potential, etc. are the common examples of bound state. Application of Schrodinger time-independent equal to such problems leads to discrete energy values of the particles.

5.2. Particle in a One-Dimensional Box:

Consider a particle of mass '*m*' and energy '*E*' is moving along x-axis, in the region from $x = 0$ to $x = a$ under the following potential i.e.

$$\begin{aligned} V(x) &= 0 \qquad 0 < x < a \\ &= \infty \qquad \text{otherwise} \end{aligned}$$

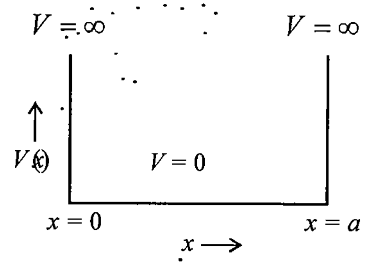


Figure : Schematic diagram of 1-D infinite potential well.

The particle experiences no force within the potential well but feels a sudden large force directed towards the origin as it reaches the points $x = 0, a$. Since, the potential is infinite outside, the particle cannot penetrate outside of the region $0 < x < a$ i.e. $\psi = 0$ for the particle outside the box.

According to 1-D time-independent Schrodinger equaion,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V\psi(x) = E\psi(x) \qquad (1)$$

For the region $0 < x < a$, the equation becomes,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}=E\psi \qquad \Rightarrow \frac{d^2\psi}{dx^2}+\frac{2mE}{\hbar^2}\psi=0$$

$$\Rightarrow \frac{d^2\psi}{dx^2}+k^2x=0 \text{ where } k^2=\frac{2mE}{\hbar^2} \qquad (2)$$

Solution of eqn. (2) can be written as,

$$\psi(x)=A\sin kx+B\cos kx \qquad (3)$$

Since, the particle remains confined within the box, then the probablity of finding the particle at $x = 0$ and $x = a$ must be zero.

Boundary conditions: $\psi(x)$ will be continuous at $x = 0$ and $x = a$

(i) Applying $\psi(x)\big|_{x=0}=0 \qquad \Rightarrow B=0$

(ii) Applying $\psi(x)\big|_{x=a}=0 \qquad \Rightarrow A\sin ka=0 \qquad \Rightarrow ka=n\pi \ \Rightarrow k=\frac{n\pi}{a}$

$$\Rightarrow \psi(x)=A\sin\left(\frac{n\pi x}{a}\right)(n=1,2,3,\ldots\ldots\ldots) \text{ for } 0<x<a \qquad (4)$$

Note: If $n=0 \ \Rightarrow \ k=0 \Rightarrow E=0 \Rightarrow \ \psi(x)=0$ everywhere inside the box. Thereofore, there will be no admissible particle with zero energy within the box.

Normalization of the wave function:

The wave function ψ_n corresponding to n^th^ quantum state is

$$\psi_n(x)=A\sin\frac{n\pi x}{a} \qquad 0\le x\le a$$
$$=0 \qquad \text{otherwise}$$

Since, the particle must be somewhere within the box, the total probability of finding the particle inside the box is unity i.e.

$$\int_0^a \psi_n^*(x)\psi_n(x)\,dx=1 \qquad \Rightarrow \int_0^a A^2\sin^2\frac{n\pi x}{a}dx=1$$

$$\Rightarrow \ A^2\int_0^a\frac{1}{2}\left[1-\cos\frac{2\pi nx}{a}\right]dx=1 \ \Rightarrow \ A=\sqrt{\frac{2}{a}}$$

Therefore, the normallized wave function will be

$$\psi_n(x)=\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right) \qquad (n=1,2,3,\ldots\ldots\ldots\ldots\ldots)$$

Energy eigenvalues:

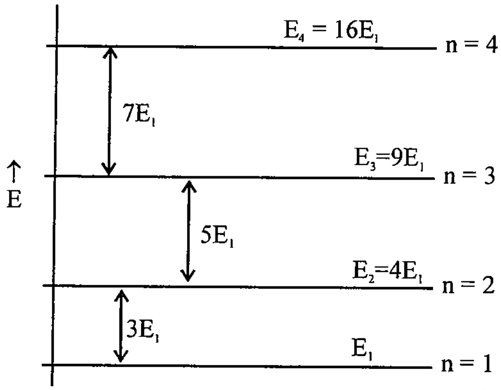
We know that, $\quad k=\sqrt{\frac{2mE}{\hbar^2}} \quad \Rightarrow \frac{n^2\pi^2}{a^2}=\frac{2mE}{\hbar^2} \quad \Rightarrow E_n=\frac{n^2\pi^2\hbar^2}{2ma^2} \quad (n=1,2,3,\ldots..)$

So, we get an infinite sequence of descrete energy levels that corresponds to all integral values of n, where n is called the quantum number representing the different states of the particle.

Ground state energy: $n=1 \quad \Rightarrow E_1=\frac{\pi^2\hbar^2}{2ma^2}$

First excited state energy: $n = 2 \Rightarrow E_2 = \dfrac{4\pi^2\hbar^2}{2ma^2}$

In general, $E_n = n^2 E_1$



$E_4 = 16E_1$
n = 4
$7E_1$
$E_3 = 9E_1$
n = 3
E
$5E_1$
$E_2 = 4E_1$
n = 2
$3E_1$
E_1
n = 1

Figure : Energy levels of a particle in a 1-D infinite potential well

Difference between two consecutive energy levels

$$\Delta E_n = E_{n+1} - E_n = \frac{\pi^2\hbar^2}{2ma^2}\left[(n+1)^2 - n^2\right] = \frac{\pi^2\hbar^2}{2ma^2}(2n+1)$$

Spacing between two energy levels increases with ‘n’, decreases with size of the box ‘a’ and the mass of the particle ‘m’. As $a \to \infty \Rightarrow \Delta E_n = 0$ i.e. the energy levels gets closer, forming a continuum of energy.

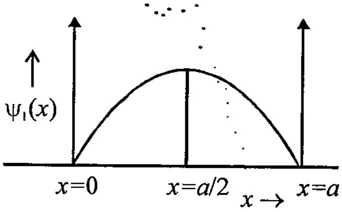
Momentum eigenvalues:

$$E_n = \frac{P_n^2}{2m} \quad \Rightarrow P_n^2 = 2mE_n = 2m\frac{n^2\pi^2\hbar^2}{2ma^2} = \frac{n^2\pi^2\hbar^2}{a^2} \quad \Rightarrow P_n = \pm\frac{n\pi\hbar}{a}$$

where ‘ ± ’ sign is due to the back and forth motion of the particle within the box.
Difference of momentum between two consecutive energy levels

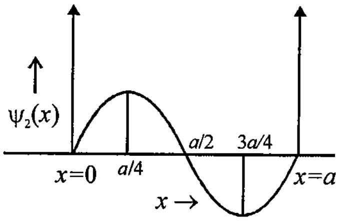
$$\Delta P_n = P_{n+1} - P_n = \pm\left[\frac{(n+1)\pi\hbar}{a} - \frac{n\pi\hbar}{a}\right] = \pm\frac{\pi\hbar}{a}$$

Various eigenstates of the particle inside a 1-D box :



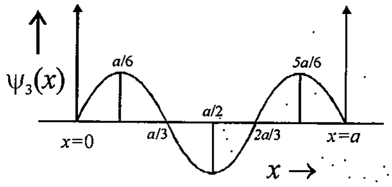
$\psi_1(x)$
$x=0$
$x=a/2$
$x \to$
$x=a$

Ground state



$\psi_2(x)$
$a/2$
$3a/4$
$x=0$
$a/4$
$x=a$
$x \to$

First excited state



$\psi_3(x)$
$a/6$
$5a/6$
$a/2$
$x=0$
$a/3$
$2a/3$
$x=a$
$x \to$

Note: $\psi_n(x)$ has $(n-1)$ nodes (excluding the boundary at $x = 0$ and $x = a$)

Variation of probability density corresponding to various eigenstates of the particle:

Probability density of finding the particle in the n^{th} quantum state is given by,

$$P_n(x)=|\psi_n(x)|^2=\frac{2}{a}\sin^2\frac{n\pi x}{a}$$

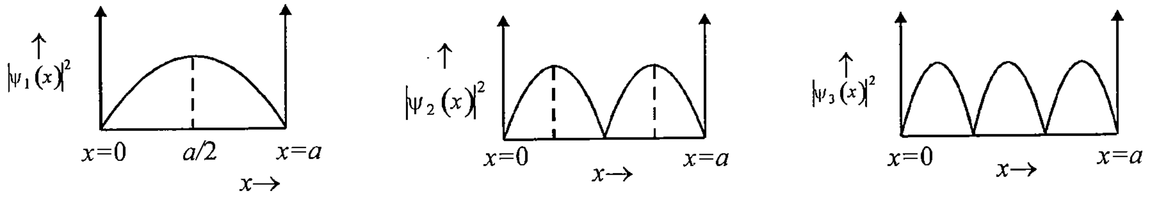


Figure : Variation of probability density corresponding to ground, first excited and second excited state respectively

Note:

(1) Probability of finding the particle in the left half of the well = Probability of finding the particle in the left half of the well = 1/2, for any state n.

(2) Probability of finding the particle between $\frac{a}{4}<x<\frac{3a}{4}$ is 1/2 for $n = 2, 4, 6, ...$

- The energy eigenfunctions of the particle confined in a 1-D box, satisfies the orthonormality condition i.e.

$$\int_0^a\psi_m^*(x)\psi_n(x)\,dx=\delta_{mn}$$

- As the energy eigenfunctions form a complete set, an arbitary function $\psi(x)$ (which is well behaved and satisfy the same boundary conditions) can be expanded in terms of these eigenfunctions as

$$\psi(x)=\sum_{n=1}^{\infty}c_n\,\psi_n(x)=\sum_{n=1}^{\infty}c_n\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$$

Expectation values of $\hat{x},\hat{p}_x,\hat{x}^2,\hat{p}_x^2$:

$$\langle x\rangle=\int_0^a\psi^*x\psi dx=\int_0^a\frac{2}{a}x\sin^2\left(\frac{n\pi x}{a}\right)dx=\frac{2}{a}\int_0^a\frac{x}{2}\left[1-\cos\frac{(2n\pi x)}{a}\right]dx$$

$$=\frac{2}{a}\frac{1}{2}\left[\frac{x^2}{2}-x\frac{\sin\frac{2\pi nx}{a}}{\frac{2\pi n}{a}}-\cos\frac{2n\pi x}{a}\right]_0^a=\frac{1}{a}\left[\frac{a^2}{2}-1+1\right]=\frac{a}{2}$$

$$\langle p\rangle=\int_0^a\psi^*\left(-i\hbar\frac{\partial}{\partial x}\right)\psi dx=\frac{2}{a}(-i\hbar)\int_0^a\sin\left(\frac{n\pi x}{a}\right)\frac{n\pi}{a}\cos\left(\frac{n\pi x}{a}\right)dx=0$$

Note: Expectation value of momentum for a real wave function is always zero.

$$\langle x^2\rangle=\int_0^a\psi^*x^2\,\psi dx=\frac{2}{a}\int_0^a x^2\,\sin^2\frac{n\pi x}{a}dx$$

$$=\frac{1}{a}\left[\left(\frac{x^3}{3}\right)_0^a-\left(x^2\frac{\sin 2n\pi x/a}{2n\pi/a}\right)_0^a+\int_0^a 2x.\sin\frac{2n\pi x/a}{2n\pi/a}dx\right]$$

$$= \frac{1}{a}\left[\frac{a^3}{3} - 0 + 2\left\{-x\frac{\cos\frac{2n\pi x}{a}}{\left(\frac{2n\pi}{a}\right)^2} + \frac{\sin\frac{2n\pi x}{a}}{\left(\frac{2n\pi}{a}\right)^2}\right\}_0^a\right] = \frac{1}{a}\left[\frac{a^3}{3} - \frac{a^3}{2(n\pi)^2}\right] = \frac{a^2}{3} - \frac{a^2}{2n^2\pi^2}$$

$$\langle p^2\rangle = \frac{2}{a}\int_0^a \sin\left(\frac{n\pi x}{a}\right)\left(-\hbar^2\frac{\partial^2}{\partial x^2}\right)\sin\left(\frac{n\pi x}{a}\right)dx = \frac{2}{a}\left(-\hbar^2\right)\left(\frac{n\pi}{a}\right)^2(-1)\int_0^a \sin^2\left(\frac{n\pi x}{a}\right)dx$$

$$= \frac{2}{a}\left(-\hbar^2\right)\left(\frac{n\pi}{a}\right)^2(-1)\int_0^a \sin^2\left(\frac{n\pi x}{a}\right)dx.$$

$$= \frac{n^2\pi^2\hbar^2}{a^2}\int_0^a\left[1-\cos\left(\frac{2n\pi x}{a}\right)\right]dx = \frac{n^2\pi^2\hbar^2}{a^3}\left[x - \frac{\sin\frac{2n\pi x}{a}}{\frac{2n\pi}{a}}\right]_0^a = \frac{n^2\pi^2\hbar^2}{a^2}$$

Uncertainity in the position of the particle is

$$\Delta x = \left[\langle x^2\rangle - \langle x\rangle^2\right]^{1/2} = \left[\frac{a^2}{3} - \frac{a^2}{2n^2\pi^2} - \frac{a^2}{4}\right]^{1/2} = \left[\frac{a^2}{12} - \frac{a^2}{2n^2\pi^2}\right]^{1/2}$$

Uncertainity in the momentum of the particle is

$$\Delta p = \left[\langle p^2\rangle - \langle p\rangle^2\right]^{1/2} = \frac{n\pi\hbar}{a}$$

Therefore, the uncertainity product $\Delta x\Delta p = n\pi\hbar\left[\frac{1}{12} - \frac{1}{2n^2\pi^2}\right]^{1/2}$

Example 1: A particle in a deep square well potential extending from $x = 0$ to $x = L$ has a wave function $\psi(x) = \frac{1}{\sqrt{5}}\left[|\phi_1\rangle + 2|\phi_2\rangle\right]$ where $|\phi_1\rangle$ and $|\phi_2\rangle$ denote the ground state and the first excited state wave functions. Find the expectation of 'x' in this state.

Soln. $\langle x\rangle = \langle\psi|x|\psi\rangle = \frac{1}{\sqrt{5}}\left(\langle\phi_1| + 2\langle\phi_2|\right)x\frac{1}{\sqrt{5}}\left(|\phi_1\rangle + 2|\phi_2\rangle\right)$

$$= \frac{1}{5}\left[\langle\phi_1|x|\phi_1\rangle + 2\langle\phi_1|x|\phi_2\rangle + 2\langle\phi_2|x|\phi_1\rangle + 4\langle\phi_2|x|\phi_2\rangle\right]$$

Now, $\langle\phi_1|X|\phi_2\rangle = \int_0^L\left(\sqrt{\frac{2}{L}}\sin\frac{\pi x}{L}\right)x\left(\sqrt{\frac{2}{L}}\sin\frac{2\pi x}{L}\right)dx$

$$= \frac{1}{L}\int_0^L x\left(\cos\frac{\pi x}{L} - \cos\frac{3\pi x}{L}\right)dx = \frac{1}{L}\left(-\frac{2L^2}{\pi^2} + \frac{2L^2}{9\pi^2}\right) = -\frac{16L}{9\pi^2}$$

Therefore, $\langle x\rangle = \frac{1}{5}\left[\frac{L}{2} - \frac{32L}{9\pi^2} - \frac{32L}{9\pi^2} + 4.\frac{L}{2}\right] = \frac{L}{2} - \frac{64L}{45\pi^2}$

Example 2: Calculate $\langle p_x\rangle$ for $\psi(x)=\frac{1}{\sqrt{2}}(|\phi_1\rangle+|\phi_2\rangle)$ where $|\phi_1\rangle$ and $|\phi_2\rangle$ are the ground state and the first excited state wave functions of a particle in a deep square well potential.

Soln. $\langle p_x\rangle=\langle\psi|p_x|\psi\rangle=\frac{1}{2}\left[\langle\phi_1|p_x|\phi_1\rangle+\langle\phi_2|p_x|\phi_1\rangle+\langle\phi_1|p_x|\phi_2\rangle+\langle\phi_2|p_x|\phi_2\rangle\right]$

$$\langle\phi_2|p_x|\phi_1\rangle=\int_{-\infty}^{\infty}\phi_2^*\left(-i\hbar\frac{d}{dx}\right)\phi_1(x)\,dx=\frac{2}{L}(-i\hbar)\int_0^L\sin\frac{2\pi x}{L}\frac{d}{dx}\left(\sin\frac{\pi x}{L}\right)dx$$

$$=\frac{-i\hbar\pi}{L^2}\int_0^L 2\sin\frac{2\pi x}{L}\cos\frac{\pi x}{L}\,dx=\frac{-i\hbar\pi}{L^2}\int_0^L\left(\sin\frac{3\pi x}{L}+\sin\frac{\pi x}{L}\right)dx=-\frac{8i\hbar}{3L}$$

Similarly, $\langle\phi_1|p_x|\phi_2\rangle=\frac{8i\hbar}{3L}$

Therefore, $\langle p_x\rangle=0$

Example 3: The wave function of a particle in a deep square well potential extending from $x=0$ to $x=L$ is $\psi(x)=\sqrt{30}\,x(x-L)/L^i$ (a) Find the value of i, (b) Find the average value of 'x' in this state.

Soln. (a) Any wave function of a particle in one dimension has the dimensions of $\frac{1}{\sqrt{L}}$. Thus $i=\frac{5}{2}$

(b) $\langle x\rangle=\int_{-\infty}^{\infty}\psi^*(x)x\psi(x)\,dx=\frac{30}{L^5}\int_0^L x^3(x-L)^2\,dx=\frac{L}{2}$

Example 4: A particle is in the ground state of a deep square well in the range $0<x<L$. Suddenly the wall $x=L$ of the well is shifted to $x=2L$, so as to make it a deep square well in the range $0<x<2L$. Immediately after this, the energy of the particle is measured. What will be the probability of getting the energy as the new ground state energy?

Soln. The wave function just before the shifting of the wall is

$$\psi(x)=\sqrt{\frac{2}{L}}\sin\frac{\pi x}{L}\quad\text{for } 0<x<L$$

$$=0\qquad\text{otherwise}$$

This is the energy eigenfunction corresponding to the ground state in the original well. When the wall is shifted, the new ground state wave function becomes,

$$\phi(x)=\sqrt{\frac{2}{2L}}\sin\frac{\pi x}{2L}\quad\text{for } 0<x<2L$$

$$=0\qquad\text{otherwise}$$

The probability of getting the ground state energy in measurements is $|\langle\phi|\psi\rangle|^2$

$$\langle\phi|\psi\rangle=\int_{-\infty}^{\infty}\phi^*(x)\psi(x)\,dx=\int_0^L\sqrt{\frac{2}{2L}}\sin\frac{\pi x}{2L}\sqrt{\frac{2}{L}}\sin\frac{\pi x}{L}\,dx$$

$$=\frac{1}{L\sqrt{2}}\int_0^L\left(\cos\frac{\pi x}{2L}-\cos\frac{3\pi x}{2L}\right)dx=\frac{1}{L\sqrt{2}}\left|\frac{\sin\frac{\pi x}{2L}}{\frac{\pi}{2L}}-\frac{\sin\frac{3\pi x}{2L}}{\frac{3\pi}{2L}}\right|_0^L=\frac{8}{3\sqrt{2}\,\pi}$$

Therefore, the probability is $|\langle\phi|\psi\rangle|^2=\dfrac{32}{9\pi^2}$

Example 5: Consider a particle of mass 10^{-30} kg trapped in an infinitely deep potential well of width 0.5 nm. (a) Find the value of the wave number 'k' in the ground state. (b) How many states are there having value of 'k' between 10 nm^{-1} and 100 nm^{-1} ?

Soln. (a) $k=\dfrac{n\pi}{L}$. In ground state, $k=\dfrac{\pi}{L}=\dfrac{3.14}{0.5nm}=6.28\ nm^{-1}$

(b) For every interval $\dfrac{\pi}{L}$ of k, there is one quantum state.

The total k-interval is, $\Delta k=100\ nm^{-1}-10\ nm^{-1}=90\ nm^{-1}$

So, the number of states with 'k' value in the given interval is, $\dfrac{90\ nm^{-1}}{6.28\ nm^{-1}}=14.33$

There are 14 states in the given interval.

Example 6: A particle of mass 'm' which moves freely inside an infinite potential well of width 'a' has the following wave fn (at $t=0$)

$$\psi(x,0)=\frac{A}{\sqrt{a}}\sin\left(\frac{\pi x}{a}\right)+\sqrt{\frac{3}{5a}}\sin\left(\frac{3\pi x}{a}\right)+\frac{1}{\sqrt{5a}}\sin\left(\frac{5\pi x}{a}\right)\quad [A=\text{real constant}]$$

(a) Find the value of A such that $\psi(x,0)$ is normalized to unity.

(b) If the measurement is carried out, what values will be found and with what probability? and also calculate $\langle E\rangle$

(c) Find the wavefunction $\psi(x,t)$

(d) Determine the probability of finding the system at time 't' in the state $\phi(x,t)=\sqrt{\dfrac{2}{a}}\sin\left(\dfrac{5\pi x}{a}\right)e^{-iE_5\frac{t}{\hbar}}$

Soln. (a) $$\psi(x,0)=\frac{A}{\sqrt{2}}\sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right)+\sqrt{\frac{3}{10}}\sqrt{\frac{2}{a}}\sin\left(\frac{3\pi x}{a}\right)+\frac{1}{\sqrt{10}}\sqrt{\frac{2}{a}}\sin\left(\frac{5\pi x}{a}\right)$$

$$\Rightarrow\quad \psi(x,0)=\frac{A}{\sqrt{2}}|\phi_1\rangle+\sqrt{\frac{3}{10}}|\phi_3\rangle+\frac{1}{\sqrt{10}}|\phi_5\rangle$$

Normalization condition: $\langle\psi(x,0)|\psi(x,0)\rangle=1\ \Rightarrow\ \sum\limits_{n=1,3,5}|C_n|^2=1$

$$\Rightarrow\frac{A^2}{2}+\frac{3}{10}+\frac{1}{10}=1\ \Rightarrow\ A^2=\frac{6}{5}\ \Rightarrow A=\sqrt{\frac{6}{5}}$$

Therefore, $$|\psi(x,0)\rangle=\sqrt{\frac{6}{10}}|\phi_1\rangle+\sqrt{\frac{3}{10}}|\phi_3\rangle+\sqrt{\frac{1}{10}}|\phi_5\rangle$$

(b) The state of the particle is a linear combination of ground state $|\phi_1\rangle$, 2nd excited state $|\phi_3\rangle$ and 4th excited state $|\phi_5\rangle$ respectively. Therefore, the energies we will get are

$$E_1=\frac{\pi^2\hbar^2}{2ma^2},E_3=\frac{9\pi^2\hbar^2}{2ma^2},E_5=\frac{25\pi^2\hbar^2}{2ma^2}$$

Now, the respective probabilties are

$$P(E_1)=\left|\langle\phi_1|\psi\rangle\right|^2=\frac{3}{5};\qquad P(E_2)=\left|\langle\phi_2|\psi\rangle\right|^2=\frac{3}{10};\qquad P(E_1)=\left|\langle\phi_3|\psi\rangle\right|^2=\frac{1}{10}$$

$$\langle E\rangle=P(E_1).E_1+P(E_3).E_3+P(E_5).E_5$$

$$=\frac{3}{5}.\frac{\pi^2\hbar^2}{2ma^2}+\frac{3}{10}\frac{9\pi^2\hbar^2}{2ma^2}+\frac{1}{10}\frac{25\pi^2\hbar^2}{2ma^2}$$

$$=\frac{\pi^2\hbar^2}{2ma^2}\left[\frac{3}{5}+\frac{27}{10}+\frac{5}{2}\right]=\frac{29\pi^2\hbar^2}{10ma^2}$$

(c) $|\psi(x,t)\rangle=\sqrt{\frac{6}{10}}|\phi_1\rangle e^{-iE_1 t/\hbar}+\sqrt{\frac{3}{10}}|\phi_3\rangle e^{-iE_3 t/\hbar}+\sqrt{\frac{1}{10}}|\phi_5\rangle e^{-iE_5 t/\hbar}$

(d) $|\phi(x,t)\rangle=|\phi_5\rangle e^{-iE_5 t/\hbar}\quad\Rightarrow\quad P=\left|\langle\phi(x,t)|\psi\rangle\right|^2=\frac{1}{10}$

Example 7: A particle of mass m confined to move in a potential $V(x)=0$ for $0\le x\le a$ and $V(x)=\infty$ otherwise. The wave function of the particle at time $t=0$ is

$\psi(x,0)=A\left(2\sin\frac{\pi x}{a}+\sin\frac{3\pi x}{a}\right)$ (i) Normalize $\psi(x,0)$; (ii) find $\psi(x,t)$.

Soln. For a particle, in the potential given, the energy eigenvalues and eigenfunctions are given by

$$E_n=\frac{n^2\pi^2\hbar^2}{2ma^2},\ \psi_n(x)=\sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a},\qquad n=1,\ 2,\ 3,\ ....$$

(i) Normalization condition: $\int_0^a\psi^*\psi dx=1\Rightarrow A^2\int_0^a\left(2\sin\frac{\pi x}{a}+\sin\frac{3\pi x}{a}\right)^2dx=1$

$$\Rightarrow A^2\left(4\frac{a}{2}+\frac{a}{2}\right)=1\Rightarrow A=\sqrt{\frac{2}{5a}}$$

Therefore, $\psi(x,0)=\frac{1}{\sqrt{5}}\left(2\sqrt{\frac{2}{a}}\sin\frac{\pi x}{a}+\sqrt{\frac{2}{a}}\sin\frac{3\pi x}{a}\right)=\frac{1}{\sqrt{5}}(2\psi_1+\psi_3)$

(ii) $\psi(x,t)=\frac{1}{\sqrt{5}}(2\psi_1e^{-iE_1t/\hbar}+\psi_3e^{-iE_3t/\hbar})$

Example 8: A particle in a box is in a superposition state and is described by the wave function

$$\psi(x,t)=\frac{1}{\sqrt{a}}\left[\exp\left(\frac{-iE_1t}{\hbar}\right)\cos\frac{\pi x}{2a}+\exp\left(\frac{-iE_2t}{\hbar}\right)\sin\frac{2\pi x}{2a}\right],\qquad -a<x<a$$

where E_1 and E_2 are the energy eigenvalues of the first two states. Find the expectation value of x.

Soln. $\langle x\rangle=\int_{-\infty}^{\infty}\psi^*(x,t)x\psi(x,t)dx=\frac{1}{a}\int_{-a}^{a}x\cos^2\frac{\pi x}{2a}dx+\frac{1}{a}\int_{-a}^{a}x\sin^2\frac{2\pi x}{2a}dx$

$$+\frac{1}{a}\{\exp[i(E_1-E_2)t/\hbar)]+\exp[i(E_2-E_1)t/\hbar)]\}\int_{-a}^{a} x\cos\frac{\pi x}{2a}\sin\frac{2\pi x}{2a}dx$$

The integrands in the first two terms are odd and hence will not contribute.

$$\int_{-a}^{a} x\cos\frac{\pi x}{2a}\sin\frac{2\pi x}{2a}dx=\int_{-a}^{a}\frac{x}{2}\left(\sin\frac{3\pi x}{2a}+\sin\frac{\pi x}{2a}\right)dx=\frac{1}{2}\left[-\frac{8a^2}{9\pi^2}+\frac{8a^2}{\pi^2}\right]=\frac{32a^2}{9\pi^2}$$

Therefore,

$$\langle x\rangle=\frac{2}{a}\cos\frac{(E_2-E_1)t}{\hbar}\left(\frac{32a^2}{9\pi^2}\right)=\frac{64a}{9\pi^2}\cos\frac{(E_2-E_1)t}{\hbar}$$

Example 9: For an electron in a one-dimensional infinite potential well of width 1Å, calculate (i) the separation between the two lowest energy levels; (ii) the frequency and wavelength of the photon corresponding to a transition between these two levels; and (iii) in what region of the electromagnetic spectrum is this frequency / wavelength?

Soln. (i) Energy of the particle in the n^th^ state $E_n=\frac{\pi^2\hbar^2n^2}{2ma^2}$

Separation between two lowest enegy states will be

$$E_2-E_1=\frac{3\pi^2\hbar^2}{2ma^2}=113.27\text{eV}$$

(ii) Frequency of the photon $\nu=\frac{E}{h}=2.7\times10^{16}\ Hz$

Wavelength of the photon $\lambda=\frac{c}{\nu}=\frac{3\times10^8\text{ms}^{-1}}{2.7\times10^{16}\text{s}^{-1}}=1.1\times10^{-8}\text{m}$

(iii) This frequency falls in the vacuum ultraviolet region.

Example 10: A particle of mass m trapped in the potential $V(x)=0$ for $-a\le x\le a$ and $V(x)=\infty$ otherwise. Evaluate the probability of finding the trapped particle between $x=0$ and $x=a/n$ when it is in the nth state.

oln. Wave function $\psi_n(x)=\sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}$

Required probability $P=\int_0^{a/n}\psi_n^*\psi_n dx=\frac{2}{a}\int_0^{a/n}\sin^2\frac{n\pi x}{a}dx\,\frac{1}{a}\int_0^{a/n}\left(1-\cos\frac{2n\pi x}{a}\right)dx=\frac{1}{n}$

Example 11: An electron is in the ground state of a one-dimensional infinite square well with $a=10^{-10}$ m. Compute the force that the electron exerts on the wall during an impact on either wall.

oln. The force on the wall $F=-\frac{dE_n}{da}$

The energy of the ground state $E_1=\frac{\pi^2\hbar^2}{2ma^2}$

and hence the force on the wall $F = -\left.\frac{dE_1}{da}\right|_{a=10^{-10}m} = \left.\frac{\pi^2\hbar^2}{ma^3}\right|_{a=10^{-10}m} = 1.21\times10^{-7}N$

5.3. Particle in a 1-D Symmetric Infinite Potential Well

If a particle be constrained to move in a potential as below,

$$V(x) = \begin{cases} 0, & \text{for} -a/2 < x < a/2 \\ +\infty, & \text{for} \mid x \mid > a/2 \end{cases}$$

having perfectly rigid and impenetrable walls at $x = -a/2$ and $x = +a/2$, then also we can show that the energy eigenvalues are discrete similar to the previous case.

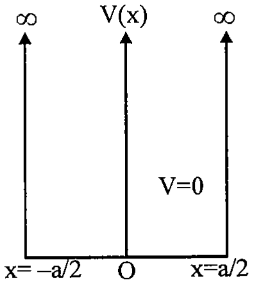


Fig.

The Schrodinger time independent equation for the particle in the region, $-a/2 < x < a/2$ is,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \text{ or, } \frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}E\psi = 0$$

where m is the mass of the particle. The general solution of equation is, $\psi(x) = A\cos\alpha x + B\sin\alpha x$

where, $\alpha = \left(\frac{2mE}{\hbar^2}\right)^{1/2}$.

Here, A and B are two constants of integration which can be found out by applying the following boundary conditions:

1. $\psi = 0$ at $x = a/2$ and **2.** $\psi = 0$ at $x = -a/2$

Applying these boundary conditions to equation we obtain,

$$A\cos\frac{\alpha a}{2} + B\sin\frac{\alpha a}{2} = 0, \quad A\cos\frac{\alpha a}{2} - B\sin\frac{\alpha a}{2} = 0 \quad \therefore \quad A\cos\frac{\alpha a}{2} = 0 \text{ and } B\sin\frac{\alpha a}{2} = 0$$

Now since both A and B cannot be zero (as in that case $\psi(x) = 0$ everywhere). Also for given value of α or E, cos $\frac{\alpha a}{2}$ and sin $\frac{\alpha a}{2}$ cannot be zero.

There are two possible solutions, viz., either $B = 0$ and cos $\frac{\alpha a}{2} = 0$,

$\therefore \quad \frac{\alpha a}{2} = \frac{n\pi}{2}$ (n being an odd integer) or, A = 0 and $\sin\frac{\alpha a}{2} = 0$,

$\therefore \quad \frac{\alpha a}{2} = \frac{n\pi}{2}$ (n being an even integer)

So, $\psi(x) = A\cos\frac{n\pi x}{a}$ (when n is odd) and $\psi(x) = B\sin\frac{n\pi x}{a}$ (when n is even)

The energy eigenvalue for the nth quantum state is given by,

$$\frac{\alpha a}{2} = \frac{n\pi}{2} \text{ or, } \sqrt{\frac{2mE_n}{\hbar^2}} = \frac{n\pi}{a} \text{ or, } \frac{2mE_n}{\hbar^2} = \frac{n^2\pi^2}{a^2} \text{ or, } E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

This shows that $E_n \propto 1/a^2$ for a particular value of n and m. There is thus an infinite sequence of discrete energy levels.

Similar results are obtained if the potential box is given by, $V(x) = \begin{cases} 0, & \text{for} -a < x < a \\ +\infty, & \text{for } x < -a \text{ and } x > a \end{cases}$

5.4 Particle in a 1-D symmetric finite potential well:

Consider a particle of mass 'm' and energy $E\,(< V_0)$ is moving along x-axis, in the region from $x = 0$ to $x = a$ under the following potential i.e.

$$\begin{aligned} V(x) &= 0 \qquad -\frac{a}{2} < x < \frac{a}{2} \\ &= V_0 \qquad \text{otherwise} \end{aligned}$$

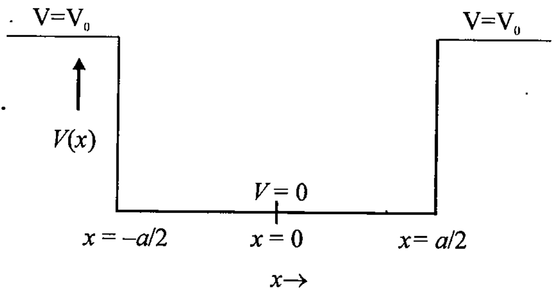


Figure: Schematic diagram of 1-D finite potential well.

The time independent 1-D schrodinger equation in three regions can be written as

Region-1: $-\frac{\hbar^2}{2m}\frac{d^2\psi_1}{dx^2} + V_0\psi_1 = E\psi_1$ (5a)

Region-2: $-\frac{\hbar^2}{2m}\frac{d^2\psi_2}{dx^2} = E\psi_2$ (5b)

Region-3: $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_0\psi_3 = E\psi_3$ (5c)

Aasuming $\alpha^2 = \frac{2m(V_0 - E)}{\hbar^2}$ and $\beta^2 = \frac{2mE}{\hbar^2}$, equations (5a), (5b) and (5c) can be written as

Region-1: $\frac{d^2\psi_1}{dx^2}-\alpha^2\psi_1=0$ (6a)

Region-2: $\frac{d^2\psi_2}{dx^2}+\beta^2\psi_2=0$ (6b)

Region-3: $\frac{d^2\psi_3}{dx^2}-\alpha^2\psi_3=0$ (6c)

Solution of equations (6a), (6b) and (6c) will be of the following form:

$$\psi_1(x)=Ae^{\alpha x}+Be^{-\alpha x} \qquad \left(x\le -\frac{a}{2}\right)$$

$$\psi_2(x)=C\cos\beta x+D\sin\beta x \qquad \left(-\frac{a}{2}<x<\frac{a}{2}\right)$$

$$\psi_3(x)=F\,e^{-\alpha x}+G\,e^{\alpha x} \qquad \left(x\ge\frac{a}{2}\right)$$

Boundary conditions:

(i) $\psi(x)\to 0$ at $x\to\pm\infty$ (ii) $\psi(x)$ and $\frac{d\psi}{dx}$ are continuous at $x=\pm\frac{a}{2}$

Using the first condition, we get B = 0 and G = 0.

Since the particle is moving under a symmetric potential, then the wavefunction of the particle should have a definite parity i.e. either even or odd.

(1) Even bound states:

The wave function of the particle will be of the form:

$$\psi_1(x)=Ae^{\alpha x} \qquad \left(x\le -\frac{a}{2}\right)$$

$$\psi_2(x)=C\cos\beta x \qquad \left(-\frac{a}{2}<x<\frac{a}{2}\right)$$

$$\psi_3(x)=F\,e^{-\alpha x} \qquad \left(x\ge\frac{a}{2}\right)$$

Now, boundary conditions are

$$\psi_1\left(x=-\frac{a}{2}\right)=\psi_2\left(x=-\frac{a}{2}\right)\Rightarrow Ae^{-\alpha\frac{a}{2}}=C\cos\beta\frac{a}{2} \tag{9}$$

$$\frac{d\psi_1}{dx}\left(x=-\frac{a}{2}\right)=\frac{d\psi_2}{dx}\left(x=-\frac{a}{2}\right)\Rightarrow \alpha Ae^{-\alpha\frac{a}{2}}=\beta C\sin\beta\frac{a}{2} \tag{10}$$

Dividing equation (10) by (9), we get, $\beta\tan\beta\frac{a}{2}=\alpha$ (11)

We know that, $\alpha^2+\beta^2=\frac{2mV_0}{\hbar^2}$ (12)

Defining $\beta\frac{a}{2}=p$ and $\alpha\frac{a}{2}=q$, we get

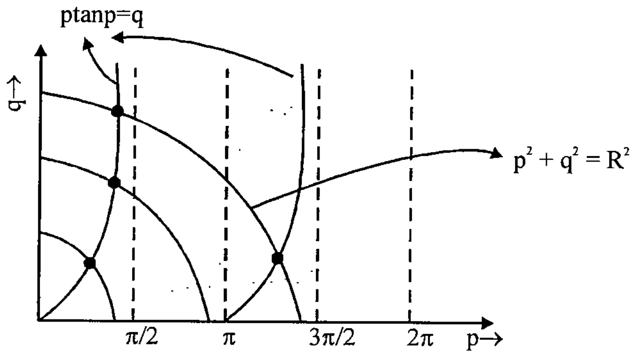
$$p\tan p=q \text{ and } p^2+q^2=\frac{mV_0a^2}{2\hbar^2} \tag{13}$$

The second equation represents a circle in the p-q plane with radius $R=\sqrt{\frac{mV_0a^2}{2\hbar^2}}$ which depends upon the the height V_0 and width a of the finite potential well. Both of these can be solved graphically and the points of intersection of the curves $p\tan p=q$ and $p^2+q^2=\frac{mV_0a^2}{2\hbar^2}$ gives different p values i.e.

$$p_n=p_0,p_2,p_4,\ldots\ldots\ldots$$

which also gives the discrete spectrum of energy values for the even eigenstates as

$$E_n=\frac{2\hbar^2p^2}{ma^2}\quad(n=0,2,4,6\ldots\ldots\ldots)$$



Number of even bound states:

$0<R<\pi\Rightarrow$ one even bound state

$\pi<R<2\pi\Rightarrow$ two even bound state

In general, $n\pi<R<(n+1)\pi\Rightarrow(n+1)$ even bound state

(2) Odd bound states:

The wave function of the particle will be of the form:

$$\psi_1(x)=Ae^{\alpha x}\qquad\left(x\le-\frac{a}{2}\right)$$

$$\psi_2(x)=D\sin\beta x\qquad\left(-\frac{a}{2}<x<\frac{a}{2}\right)$$

$$\psi_3(x)=F\,e^{-\alpha x}\qquad\left(x\ge\frac{a}{2}\right)$$

Now, boundary conditions are

$$\psi_1\left(x=-\frac{a}{2}\right)=\psi_2\left(x=-\frac{a}{2}\right)\Rightarrow Ae^{-\alpha\frac{a}{2}}=-D\sin\beta\frac{a}{2}\qquad(15)$$

$$\frac{d\psi_1}{dx}\left(x=-\frac{a}{2}\right)=\frac{d\psi_2}{dx}\left(x=-\frac{a}{2}\right)\Rightarrow\alpha Ae^{-\alpha\frac{a}{2}}=\beta D\cos\beta\frac{a}{2}\qquad(16)$$

Dividing equation (10) by (9), we get, $\beta\cot\beta\frac{a}{2}=-\alpha$ (17)

We know that, $\alpha^2+\beta^2=\frac{2mV_0}{\hbar^2}$ (18)

Defining $\beta\frac{a}{2}=p$ and $\alpha\frac{a}{2}=q$, we get

$$p\cot p = -q \text{ and } p^2 + q^2 = \frac{mV_0a^2}{2\hbar^2} \tag{19}$$

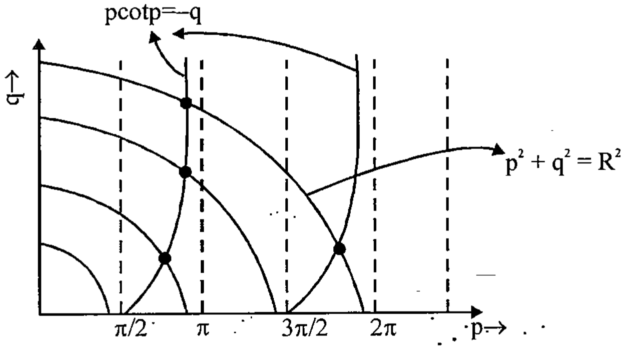
Both of these can be solved graphically and the points of intersection of the curves

$p\cot p = -q$ and $p^2 + q^2 = \dfrac{mV_0a^2}{2\hbar^2}$ gives different p values i.e.

$$p_n = p_1, p_3, p_5, \ldots\ldots\ldots$$

which also gives the discrete spectrum of energy values for the even eigenstates as

$$E_n = \frac{2\hbar^2 p^2}{ma^2} \quad (n = 1,3,5,7\ldots\ldots\ldots)$$



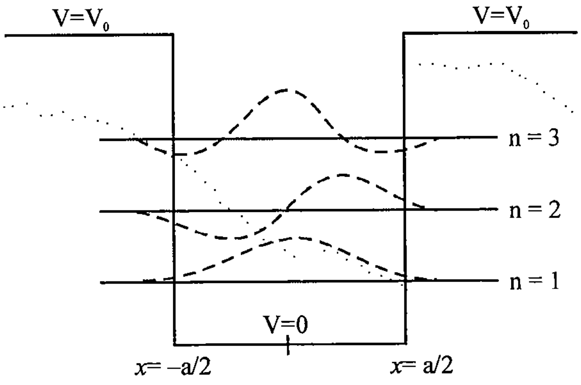
Number of Odd bound states:

$\dfrac{\pi}{2} < R < \dfrac{3\pi}{2} \Rightarrow$ one odd bound state

$\dfrac{3\pi}{2} < R < \dfrac{5\pi}{2} \Rightarrow$ two odd bound state

In general, $(2n-1)\dfrac{\pi}{2} < R < (2n+1)\dfrac{\pi}{2} \Rightarrow n$ odd bound state

Range of R	No. of bound states	Even	Odd
$0 < R < \frac{\pi}{2}$	1	1	0
$\frac{\pi}{2} < R < \pi$	2	1	1
$\pi < R < \frac{3\pi}{2}$	3	2	1
$\frac{3\pi}{2} < R < 2\pi$	4	2	2



Example 12: There are only 3 bound states for a particle of mass 'm' in a 1-D potential well of the form:

$$V(x) = 0 \qquad -\frac{a}{2} < x < \frac{a}{2}$$
$$= V_0 \qquad \text{otherwise}$$

The depth of the potential well satisfies the condition.

(a) $\frac{2\pi^2\hbar^2}{ma^2} < V_0 < \frac{9\pi^2\hbar^2}{2ma^2}$ (b) $\frac{2\pi^2\hbar^2}{2ma^2} < V_0 < \frac{4\pi^2\hbar^2}{2ma^2}$

(c) $\frac{2\pi^2\hbar^2}{2ma^2} < V_0 < \frac{8\pi^2\hbar^2}{ma^2}$ (d) $\frac{2\pi^2\hbar^2}{2ma^2} < V_0 < \frac{50\pi^2\hbar^2}{ma^2}$

Soln. For three bound states of the particle, $\pi < R < \frac{3\pi}{2}$

$$\Rightarrow \pi^2 < R^2 < \frac{9\pi^2}{4} \Rightarrow \pi^2 < \frac{mV_0a^2}{2\hbar^2} < \frac{9\pi^2}{4} \qquad \Rightarrow \frac{2\pi^2\hbar^2}{ma^2} < V_0 < \frac{9\pi^2\hbar^2}{2ma^2}$$

Example 13. Consider a particle of mass 'm' is confined to a finite potential will

$$V(x) = 0 \qquad \text{if } |x| < L$$
$$= V_0 \qquad \text{if } |x| \geq L$$

The particle is found to be in a bound state with energy $\frac{\pi^2\hbar^2}{32ma^2}$. The depth of the well is equal to

(a) $\frac{\pi^2\hbar^2}{2mL^2}$ (b) $\frac{\pi^2\hbar^2}{4mL^2}$ (c) $\frac{\pi^2\hbar^2}{8mL^2}$ (d) $\frac{\pi^2\hbar^2}{16mL^2}$

Soln. $\beta^2 = \frac{2mE}{\hbar^2} = \frac{\pi^2}{16L^2} \qquad \Rightarrow \beta = \frac{\pi}{4L}$

Since, the bound state is the ground state of the particle, therefore

$$\alpha = \beta\tan(\beta L) \Rightarrow \beta = \frac{\pi}{4L}\tan\left(\frac{\pi}{4}\right) = \frac{\pi}{4L}$$

Now, $\alpha^2 + \beta^2 = \frac{2mV_0}{\hbar^2} \qquad \Rightarrow V_0 = 2\frac{\pi^2}{16L^2} \times \frac{\hbar^2}{2m} = \frac{\pi^2\hbar^2}{16mL^2}$

Correct answer is (d)

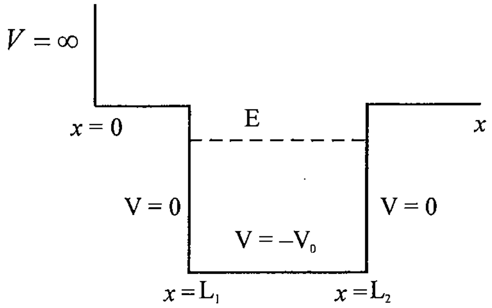
Example 14. Show that the ground state energy of a particle of mass 'm' in a finite square well potential of width L is less than that in infinite square well potential of the same width.

Soln. The ground state energy eigenvalue can be obtained from the intersection of the first branch of the curve $p\tan p = q$ and $p^2 + q^2 = \frac{mV_0L^2}{2\hbar^2}$. This point of intersection must corresponds to a value of p less than $\pi/2$.

This becuase at $p = \frac{\pi}{2}$, the function $q = p\tan p$ goes to infinity and for $p > \frac{\pi}{2}$, the first branch does not exists.

Therefore, $\frac{2mE}{\hbar^2}\frac{L^2}{4} < \frac{\pi^2}{4} \Rightarrow E < \frac{\pi^2\hbar^2}{2mL^2}$

Example 15. Consider a particle in a potential shown in figure below.



Write down the wave functions of the particle (having energy eigenvalue E) in the different regions for bound states.

Soln. As is clear from the figure, the energy E should be negative for bound states. We can define four regions.

Region-I: $x<0$, **Region-II**: $0<x<L_1$, **Region-III:** $L_1<x<L_2$ and **Region-IV:** $x>L_2$. The wave function for a bound state at energy E can be written as, (solving 1-D time independent Schrodinger equation)

$$\phi_I(x)=0;\ \ \phi_{II}(x)=Ae^{\alpha x}+Be^{-\alpha x}$$

$$\phi_{III}(x)=C\sin\beta x+D\cos\beta x$$

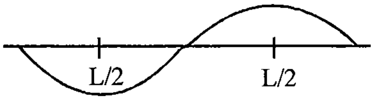
$$\phi_{IV}(x)=F\,e^{-\alpha x}$$

where, $\alpha=\sqrt{\dfrac{-2mE}{\hbar^2}}$ and $\beta=\sqrt{\dfrac{2m(V_0+E)}{\hbar^2}}$

Example 16. A particle is in its first excited state in a finite square well potential extended from $x=-\dfrac{L}{2}$ to $x=\dfrac{L}{2}$. The maximum value of the wave function, at a particular instant, is $\psi=A$ and it occurs at $x=\dfrac{L}{3}$. Find the value of the function at $x=\dfrac{L}{2}$.

Soln. In the first excited state, the wave function is of the form $\psi=A\sin kx$. The maximum value of the wave function is A. According to the question, this occurs at $x=\dfrac{L}{3}$. This means,

$$A=A\sin\frac{kL}{3}\quad\Rightarrow\frac{kL}{3}=\frac{\pi}{2}\ \Rightarrow k=\frac{3\pi}{2L}$$



The wave function at $x=\dfrac{L}{2}$ is $\psi=A\sin\dfrac{kL}{2}=A\sin\left(\dfrac{3\pi}{2L}\times\dfrac{L}{2}\right)=\dfrac{A}{\sqrt{2}}$

Example 17. A finite square well potential has length L and depth V$_0$. A particle of mass $\dfrac{2\hbar^2}{V_0L^2}$ is trapped in this potential. Show that it has only one bound state and the energy corresponding to this states satisfies $\cos\left(\sqrt{E/V_0}\right)=\sqrt{E/V_0}$

Soln. $p^2+q^2=\left(\frac{L}{2}\right)^2\left[\frac{2mV_0}{\hbar^2}\right]=\frac{V_0L^2}{2\hbar^2}\times\frac{2\hbar^2}{V_0L^2}=1$

Thus, $\sqrt{p^2+q^2}<\frac{\pi}{2}$ and hence only one bound state occurs.

For the energy E, $q=p\tan p\Rightarrow\cos p=\frac{1}{\sqrt{1+\tan^2 p}}=\frac{p}{\sqrt{p^2+q^2}}=p$

Using, $p=\beta\frac{L}{2}=\frac{L}{2}\times\sqrt{\frac{2mE}{\hbar^2}}=\frac{L}{2}\sqrt{\frac{2E}{\hbar^2}\times\frac{2\hbar^2}{V_0L^2}}=\sqrt{\frac{E}{V_0}}$

Therefore, $\cos\left(\sqrt{\frac{E}{V_0}}\right)=\sqrt{\frac{E}{V_0}}$

5.5. Particle in 3-D rectangular box :

We consider a free particle of mass m confined to move non-relativistically in a three-dimensional potential box of sides a, b and c parallel to the x, y and z-axes respectively. Since the particle is free, the potential inside the box is $V(x,y,z)=0$. So, the total energy of the particle inside the box is equal to its kinetic energy which is always a positive quantity.

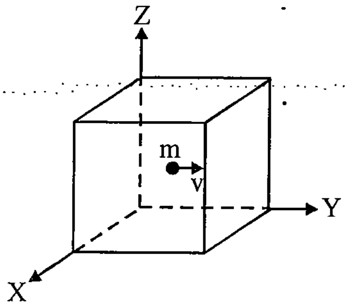


Figure : Schematic diagram of 3-D infinite potential well or 3-D box

If v be the velocity, then the momentum of the particle $p=mv$ and its energy

$$E=\frac{1}{2}mv^2=\frac{p^2}{2m}$$

Since the potential energy $V(x,y,z)$ is independent of time, we here apply Schrodinger time -independent 3-D wave equation to obtain the wave functiuon $\psi(x,y,z)$ and the eigenvalue of the energy of the particle. The three-dimensional potential box has the following characteristics:

$$V(x,y,z)=\begin{cases}0, & \text{for } 0<x<a, 0<y<b, 0<z<c\\ \infty & \text{for} \quad x>a, \quad y>b, \quad z>c\end{cases} \qquad (1)$$

The wave equation for the particle:

The wave equation describing the motion of the particle can be written as

$$\hat{H}\psi = -\frac{\hbar^2}{2m}\nabla^2\psi = E\psi \Rightarrow \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} + \frac{2mE}{\hbar^2}\psi = 0 \qquad (2)$$

As $V(x,y,z) = \infty$ at the boundries and outside the box $\psi(x,y,z) = 0$ at the boundaries and outside the potential box. The boundary conditions can be written as,

$$\left.\begin{aligned} \psi(x,y,z) = 0 \text{ at } x = 0 \text{ and } x = a \\ y = 0 \text{ and } y = b \\ z = 0 \text{ and } z = c \end{aligned}\right\}$$

Using the separation of variable technique, we caan assume the solution of equation (2) to be

$$\psi(x,y,z) = X(x)Y(y)Z(z) \qquad (3)$$

Substituting equation (3) in equation (2), we get

$$YZ\frac{d^2X}{dx^2} + ZX\frac{d^2Y}{dy^2} + XY\frac{d^2Z}{dz^2} + \frac{2mE}{\hbar^2}XYZ = 0$$

Dividing throughout by XYZ, we get

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} + \frac{2mE}{\hbar^2} = 0 \qquad (4)$$

If v_x, y_y and v_z be the component of velocity of the particle along the x, y and z-axes respectively, then the corresponding kinetic energies of the particle are

$$E_x = \frac{1}{2}mv_x^2, \quad E_y = \frac{1}{2}mv_y^2 \text{ and } E_z = \frac{1}{2}mv_z^2 \text{ such that } E = E_x + E_y + E_z$$

Therefore, equation (4) can be written as

$$\left[\frac{1}{X}\frac{d^2X}{dx^2} + \frac{2mE_x}{\hbar^2}\right] + \left[\frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{2mE_y}{\hbar^2}\right] + \left[\frac{1}{Z}\frac{d^2Z}{dz^2} + \frac{2mE_z}{\hbar^2}\right] = 0 \qquad (5)$$

So,

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{2mE_x}{\hbar^2}X = 0 \qquad (6a)$$

$$\frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{2mE_y}{\hbar^2}Y = 0 \qquad (6b)$$

$$\frac{1}{Z}\frac{d^2Z}{dz^2} + \frac{2mE_z}{\hbar^2}Z = 0 \qquad (6c)$$

Solutions of equations (6a), (6b) and (6c) are

$$X(x) = A_1 \sin k_x x + B_1 \cos k_x x \text{ where } k_x = \frac{\sqrt{2mE_x}}{\hbar}$$

$$Y(y) = A_2 \sin k_y y + B_2 \cos k_y y \text{ where } k_y = \frac{\sqrt{2mE_y}}{\hbar}$$

$$Z(z) = A_3 \sin k_z z + B_3 \cos k_z z \text{ where } k_z = \frac{\sqrt{2mE_z}}{\hbar}$$

Boundary Conditions:

(i) $X(x=0)=0 \Rightarrow B_1=0$

$$X(x=a)=0 \Rightarrow A\sin k_x a=0 \quad \Rightarrow \sin k_x a=0 \quad \Rightarrow k_x=\frac{n_x\pi}{a} \quad (n_x=1,2,3.......)$$

(ii) $Y(y=0)=0 \Rightarrow B_2=0$

$$Y(y=b)=0 \Rightarrow A_2\sin k_y a=0 \quad \Rightarrow \sin k_y a=0 \quad \Rightarrow k_y=\frac{n_y\pi}{a} \quad (n_y=1,2,3.......)$$

(iii) $Z(z=0)=0 \Rightarrow B_3=0$

$$Z(z=c)=0 \Rightarrow A_3\sin k_z a=0 \quad \Rightarrow \sin k_z a=0 \quad \Rightarrow k_z=\frac{n_z\pi}{a} \quad (n_z=1,2,3.......)$$

Therefore, the solution will become

$$\psi(x,y,z)=A_1A_2A_3\sin\frac{n_x\pi}{a}x\sin\frac{n_y\pi}{b}y\sin\frac{n_z\pi}{c}z$$

$$\Rightarrow \psi(x,y,z)=A\sin\frac{n_x\pi}{a}x\sin\frac{n_y\pi}{b}y\sin\frac{n_z\pi}{c}z \tag{7}$$

Normalization Condition:

$$\int_0^a\int_0^b\int_0^c |\psi|^2\,dxdydz=1$$

$$\Rightarrow A^2\int_0^a\int_0^b\int_0^c \sin^2\frac{n_x\pi}{a}x\sin^2\frac{n_y\pi}{b}y\sin^2\frac{n_z\pi}{c}z\,dxdydz=1$$

$$\Rightarrow A^2.\frac{a}{2}.\frac{b}{2}.\frac{c}{2}=1 \Rightarrow A=\sqrt{\frac{8}{abc}}$$

Now, the eigenfunction of the particle is,

$$\psi(x,y,z)=\sqrt{\frac{8}{abc}}\sin\frac{n_x\pi}{a}x\sin\frac{n_y\pi}{b}y\sin\frac{n_z\pi}{c}z \tag{8}$$

Therefore, the total linear momentum p is $p=p_x^2+p_y^2+p_z^2$

$$=\left[\left(\frac{n_x\hbar\pi}{a}\right)+\left(\frac{n_y\hbar\pi}{b}\right)^2+\left(\frac{n_z\hbar\pi}{c}\right)^2\right]^{1/2}=\pi\hbar\left[\left(\frac{n_x}{a}\right)^2+\left(\frac{n_y}{b}\right)^2+\left(\frac{n_z}{c}\right)^2\right]$$

The eigenvalue of energy of the particle is,

$$E=\frac{p^2}{2m}=\frac{\pi^2\hbar^2}{2m}\left[\left(\frac{n_x}{a}\right)^2+\left(\frac{n_y}{b}\right)^2+\left(\frac{n_z}{c}\right)^2\right]$$

The integers n_x, n_y, n_z are called quantum numbers which are required to describe the stationary state of the particle. Since for any of n_x, n_y, n_z being zero, $\psi(x,y,z)=0$, i.e., the particle is absent in the box. But physically this is not acceptable and hence any of n_x, n_y, n_z cannot be zero.

For the ground state, $n_x = n_y = n_z = 1$ and the ground state wave function of the particle is,

$$\psi_{1,1,1}(x,y,z) = \sqrt{\frac{8}{abc}}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\sin\frac{\pi z}{c}$$

and the energy associated with the ground state called the zero point energy is given by,

$$E_{1,1,1} = \frac{\pi^2\hbar^2}{2m}\left[\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right].$$

The eigenfunctions and eigenvalues of energy for the next higher excited state are as follows:

$$\left.\begin{aligned}\psi_{2,1,1} &= \sqrt{\frac{8}{abc}}\sin\frac{2\pi x}{a}\sin\frac{\pi y}{b}\sin\frac{\pi z}{c}\\ \psi_{1,2,1} &= \sqrt{\frac{8}{abc}}\sin\frac{\pi x}{a}\sin\frac{2\pi y}{b}\sin\frac{\pi z}{c}\\ \psi_{1,1,2} &= \sqrt{\frac{8}{abc}}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\sin\frac{2\pi z}{c}\end{aligned}\right\} \text{ and } \left\{\begin{aligned}E_{2,1,1} &= \frac{\pi^2\hbar^2}{2m}\left[\frac{4}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right]\\ E_{1,2,1} &= \frac{\pi^2\hbar^2}{2m}\left[\frac{1}{a^2}+\frac{4}{b^2}+\frac{1}{c^2}\right]\\ E_{1,1,2} &= \frac{\pi^2\hbar^2}{2m}\left[\frac{1}{a^2}+\frac{1}{b^2}+\frac{4}{c^2}\right]\end{aligned}\right.$$

5.6. Particle in 3-D Cubical box:

If the potential box be cubical instead of rectagular, then $a = b = c$ the eigenfunction corresponding to the state of the particle will be represented by

$$\psi_{n_x n_y n_z}(x,y,z) = \sqrt{\frac{8}{a^3}}\sin\frac{n_x\pi}{a}x\sin\frac{n_y\pi}{a}y\sin\frac{n_z\pi}{a}z$$

The momentum eigen values of the particle will be,

$$p_{n_x,n_y,n_z} = \frac{\pi\hbar}{a}\left[n_x^2+n_y^2+n_z^2\right]^{1/2}$$

and the corresponding energy eigenvalues are, $E_{n_x,n_y,n_z} = \frac{\pi^2\hbar^2}{2ma^2}\left[n_x^2+n_y^2+n_z^2\right]$

Ground state ($n_x = n_y = n_z = 1$):

Wave function: $\psi_{1,1,1} = \sqrt{\frac{8}{a^3}}\sin\frac{\pi x}{a}\sin\frac{\pi y}{a}\sin\frac{\pi z}{a}$

Momentum : $p_{1,1,1} = \frac{\pi\hbar}{a}\sqrt{3}$ **Energy :** $E_{1,1,1} = \frac{3\pi^2\hbar^2}{2ma^2}$

First excited state:

n_x	n_y	n_z
2	1	1
1	2	1
1	1	2

Wave function:
$$\left.\begin{aligned}\psi_{2,1,1} &= \sqrt{\frac{8}{a^3}}\sin\frac{2\pi x}{a}\sin\frac{\pi y}{a}\sin\frac{\pi z}{a}\\ \psi_{1,2,1} &= \sqrt{\frac{8}{a^3}}\sin\frac{\pi x}{a}\sin\frac{2\pi y}{a}\sin\frac{\pi z}{a}\\ \psi_{1,1,2} &= \sqrt{\frac{8}{a^3}}\sin\frac{\pi x}{a}\sin\frac{\pi y}{a}\sin\frac{2\pi z}{a}\end{aligned}\right\}$$

Momentum: $p_{2,1,1} = p_{1,2,1} = p_{1,1,2} = \dfrac{\pi\hbar}{a}\sqrt{6}$

Energy: $E_{2,1,1} = E_{1,2,1} = E_{1,1,2} = \dfrac{\pi^2\hbar^2}{2ma^2}6$

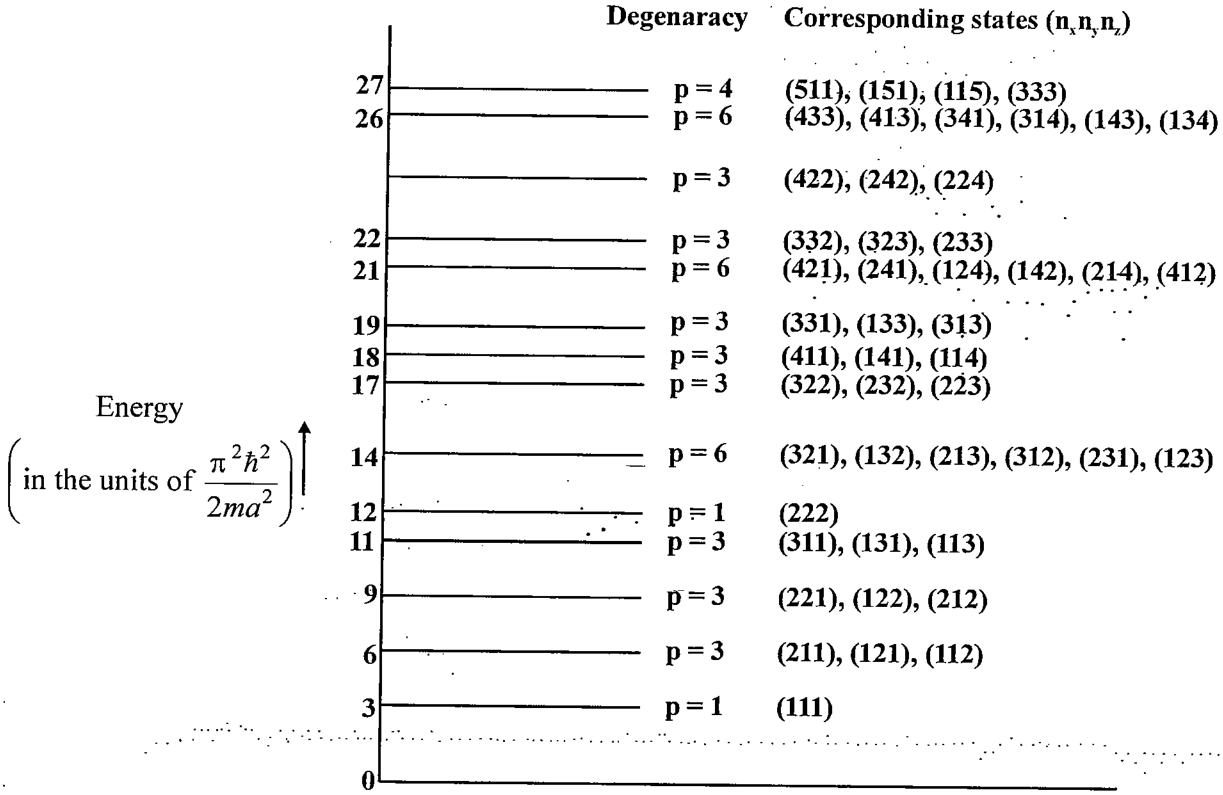

Degenaracy
Corresponding states $(n_x n_y n_z)$
Energy
$\left(\text{in the units of } \dfrac{\pi^2\hbar^2}{2ma^2}\right)$
27
p = 4
(511), (151), (115), (333)
26
p = 6
(433), (413), (341), (314), (143), (134)
p = 3
(422), (242), (224)
22
p = 3
(332), (323), (233)
21
p = 6
(421), (241), (124), (142), (214), (412)
19
p = 3
(331), (133), (313)
18
p = 3
(411), (141), (114)
17
p = 3
(322), (232), (223)
14
p = 6
(321), (132), (213), (312), (231), (123)
12
p = 1
(222)
11
p = 3
(311), (131), (113)
9
p = 3
(221), (122), (212)
6
p = 3
(211), (121), (112)
3
p = 1
(111)
0

Figure : Energy levels of a particle in a cubical potential box

Degeneracy:

The property that, two or more quantum states of a particle, with different sets of quantum numbers and different eigenfunctions having the same value of momentum and energy is called degeneracy.

For example, the three states corresponding to quantum numbers (2, 1, 1), (1, 2, 1) and (1, 1, 2) all have the same momentum $\left(\dfrac{\pi\hbar}{a}\sqrt{6}\right)$ and energy $\left(\dfrac{\pi^2\hbar^2}{2ma^2}6\right)$.

Example 18. An electron is confined in a cubical box of side 1Å. Find the momentum and energy of the electron in the ground state and the first excited state.

oln. Ground state: $n_x = n_y = n_z = 1$

Momentum of the electron is

$$P_{1,1,1} = \frac{\pi\hbar}{\ell}\sqrt{3} = \frac{\pi h}{2\pi\ell}\sqrt{3} = \frac{h}{2\ell}\sqrt{3} = 5.72\times10^{-24}\,kg-m/s$$

Energy of the electron is

$$E_{1,1,1} = \frac{3\pi^2\hbar^2}{2m\ell^2} = 181\times10^{-19}J = 113\,eV$$

First excited state:

n_x	n_y	n_z
2	1	1
1	2	1
1	1	2

Momentum of the electron: $p_{2,1,1} = p_{1,2,1} = p_{1,1,2} = \frac{\pi\hbar}{a}\sqrt{6} = 8.13\times10^{-24} kg - m/s$

Energy of the electron : $E_{2,1,1} = E_{1,2,1} = E_{1,1,2} = \frac{\pi^2\hbar^2}{2ma^2}6 = 362\times10^{-19} J = 226\, eV$

Example 19. An electron is confined in a cubical box of side 1Å. (i) Find its lowest energy (ii) what is the temperature at which the average of the molecule of a perfect gas would be equal to the lowest energy of the electron $\left(k = 1.38\times10^{-23} J/K\right)$?

Soln. (i) Ground state energy of the electron

$$E_{1,1,1} = \frac{3\pi^2\hbar^2}{2ma^2} = \frac{3\pi^2\times\left(1.054\times10^{-34}\right)^2}{2\times9.11\times10^{-31}\times\left(10^{-10}\right)^2} = 18.03\times10^{-18} J$$

(ii) If T K be the required temeprature of the molecules, then

$$\frac{3}{2}kT = \frac{3\pi^2\hbar^2}{2ma^2} \text{ [k is the Boltzmann constant]}$$

$$= 18.03\times10^{-18} J$$

$$\therefore \quad T = \frac{2\times18.03\times10^{-18}}{3\times1.38\times10^{-23}} = 8.71\times10^5 K$$

Example 20. A particle of mass 'm' is in a cubic box of size 'a'. If the particle is in an eigenstate with energy $E = \frac{14\pi^2\hbar^2}{2ma^2}$. What will be its wave function?

Soln. $E = \frac{14\pi^2\hbar^2}{2ma^2} = \frac{\pi^2\hbar^2}{2ma^2}\left(n_x^2 + n_y^2 + n_z^2\right) \Rightarrow n_x^2 + n_y^2 + n_z^2 = 14$

$$\left(n_x, n_y, n_z\right) = (1,2,3)$$

$$\psi_{123}(x,y,z) = \sqrt{\frac{8}{a^3}}\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi y}{a}\right)\sin\left(\frac{3\pi z}{a}\right)$$

$$= \left(\frac{2}{a}\right)^{3/2}\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi y}{a}\right)\sin\left(\frac{3\pi z}{a}\right)$$

Example 21. Find the number of energy states and energy levels in the range $E < [15h^2/(8\text{ma}^2)]$ of a cubical box of side a.

Soln. For a particle in a cubic box of side a, the energy is given by

$$E = \frac{\pi^2\hbar^2}{2ma^2}(n_x^2 + n_y^2 + n_z^2) = \frac{h^2}{8ma^2}(n_x^2 + n_y^2 + n_z^2)$$

Comparing with the given expression, we get $n_x^2 + n_y^2 + n_z^2 < 15$

The number of possible combinations of $(n_x\, n_y\, n_z)$ is

(1 1 1)	1 way
(1 1 2), (1 2 1), (2 1 1)	3 ways
(1 1 3), (1 3 1), (3 1 1)	3 ways
(1 2 2), (2 1 2), (2 2 1)	3 ways
(2 2 2)	1 way
(1 2 3), (1 3 2), (2 1 3), (3 2 1), (3 1 2)	6 ways
Total	17 ways

Hence the No. of possible states = 17. The No. of energy levels = 6.

Example 22. A crystal has some negative ion vacancies, each containing one electron. Treat these electrons as moving freely inside a volume whose dimensions are of the order of lattice constant. Assuming the value of lattice constant, estimate the longest wavelength of electromagnetic radiation absorbed by these electrons.

Soln. The energy levels of an electron in a cubical box of side a is

$$E_{n_x, n_y, n_z} = \frac{\pi^2 \hbar^2}{2ma^2}(n_x^2 + n_y^2 + n_z^2), \quad n_x, n_y, n_z = 1, 2, 3, \ldots$$

Lattice constant a = 1 Å = 10^{-10} m.

The energy of the ground state is given by $E_{1111} = \dfrac{\pi^2 \hbar^2}{2ma^2} \times 3 = 1.795 \times 10^{-17} J$

The longest wavelength corresponds to the transition from energy E_{111} to E_{211}, and hence

$$E_{211} = \frac{\pi^2 \hbar^2 \times 6}{2ma^2} = 3.59 \times 10^{-17}\,\mathrm{J}$$

Longest wavelength $\lambda = \dfrac{c}{v} = \dfrac{ch}{E_{211} - E_{111}} = 11\, nm$

5.7. Particle in 3-D Harmonic Oscillator :

We consider a particle of mass m bound to the origin by a force having components.

$$F_x = -k_x x = -m\omega_x^2 x \;\; ; \;\; F_y = -k_y x = -m\omega_y^2 y \; ; \; F_z = -k_z z = -m\omega_z^2 z$$

along the x, y, z components of the displacement $\vec{r}$ from the origin. Here, k_x, k_y, k_z are the force constants and $\omega_x, \omega_y, \omega_z$ are the angular frequencies along x, y, z-axes respectively. Such a system is called a three-dimensional harmonic oscillator.

Again, $\quad F_x = -\dfrac{\partial V}{\partial x}, \quad F_y = -\dfrac{\partial V}{\partial y}$ and $F_z = -\dfrac{\partial V}{\partial z}$

where the potential energy V of the oscillator is,

$$V(x, y, z) = \frac{1}{2}\left(k_x x^2 + k_y y^2 + k_z z^2\right)$$

The time-independent three-dimensional Schrodinger equation for the system is

$$-\frac{\hbar^2}{2m}\nabla^2\psi(x,y,z)+\frac{1}{2}\left(k_x x^2+k_y y^2+k_z z^2\right)\psi(x,y,z)=E\psi(x,y,z)$$

$$\Rightarrow\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}+\frac{2m}{\hbar^2}\left[E-\frac{1}{2}k_x x^2-\frac{1}{2}k_y y^2-\frac{1}{2}k_z z^2\right]=0$$

$$\Rightarrow\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}+\left[\lambda-\left(\alpha_x^2x^2+\alpha_y^2y^2+\alpha_z^2z^2\right)\right]\psi=0 \tag{9}$$

where, $\lambda=\frac{2mE}{\hbar^2}$, $\alpha_x=\sqrt{\frac{mk_x}{\hbar^2}}$, $\alpha_y=\sqrt{\frac{mk_y}{\hbar^2}}$ and $\alpha_z=\sqrt{\frac{mk_y}{\hbar^2}}$

Equation (9) can be solved by the method of separation of variables. The energy eigenvalues and eigenfunctions of the three-dimensional oscillator are therefore,

$$\left.\begin{aligned}E_x&=\left(n_x+\frac{1}{2}\right)\hbar\omega_x\\E_y&=\left(n_y+\frac{1}{2}\right)\hbar\omega_y\\E_z&=\left(n_z+\frac{1}{2}\right)\hbar\omega_z\end{aligned}\right\}$$

where, n_x, n_y and n_z are positive integers i.e., $n_x, n_y, n_z = 1, 2, 3, ...$ and

$$X(x)=N_{n_x}\,e^{-\alpha_x^2x^2/2}H_{n_x}(\alpha_x x);\;Y(y)=N_{n_y}\,e^{-\alpha_y^2y^2/2}H_{n_y}(\alpha_y y);\;Z(z)=N_{n_z}\,e^{-\alpha_z^2z^2/2}H_{n_z}(\alpha_z z)$$

Therefore, the complete wave function is

$$\psi_{n_xn_yn_z}=N_{n_x}.N_{n_y}N_{n_z}\,e^{-\left(\alpha_x^2x^2+\alpha_y^2y^2+\alpha_z^2z^2\right)/2}H_{n_x}(\alpha_x x)H_{n_y}(\alpha_y y)H_{n_y}(\alpha_z z) \tag{10}$$

where the normalization factor is

$$N_{n_x\,n_y\,n_z}=\left[\frac{\alpha_x\alpha_y\alpha_z}{\pi^{3/2}2^{(n_x+n_y+n_z)}n_x!n_y!n_z!}\right]^{1/2}$$

$$E_{n_xn_yn_z}=\hbar\left[\left(n_x+\frac{1}{2}\right)\omega_x+\left(n_y+\frac{1}{2}\right)\omega_y+\left(n_z+\frac{1}{2}\right)\omega_z\right]$$

For the isotropic oscillator, $\omega_x=\omega_y=\omega_z$ and $\alpha_x=\alpha_y=\alpha_z$

$$\therefore\quad E_n=\left[n_x+n_y+n_z+\frac{3}{2}\right]\hbar\omega=\left(n+\frac{3}{2}\right)\hbar\omega$$

where $n=n_x+n_y+n_z$ is the total quantum number and n = 0, 1, 2, 3, ...

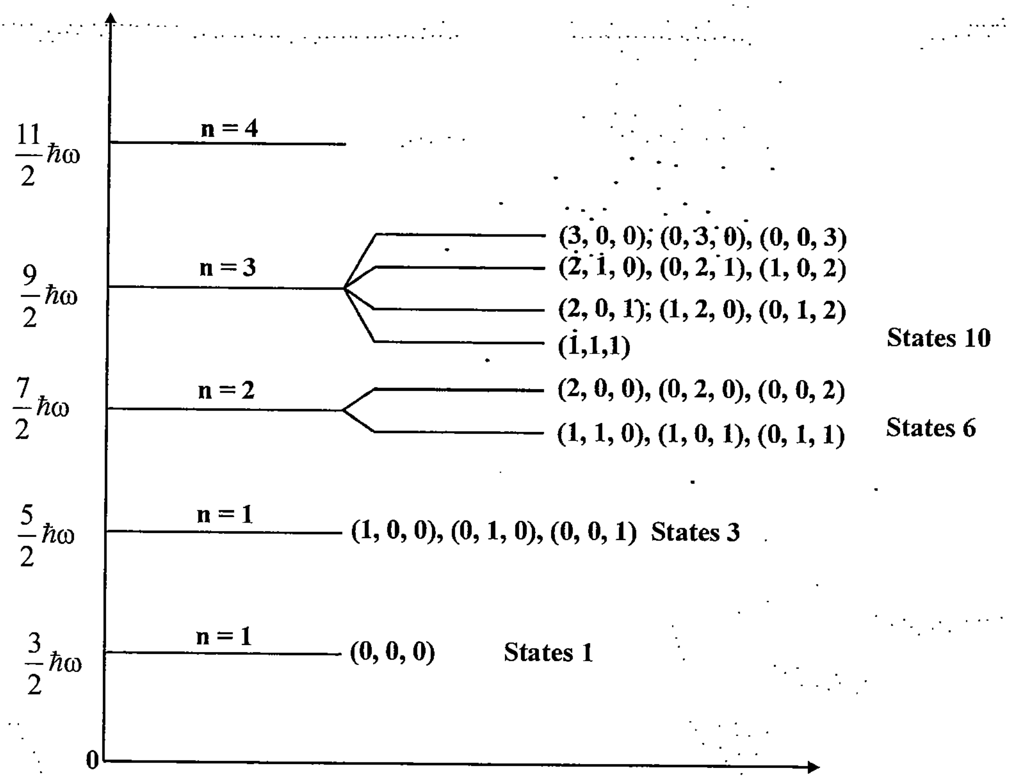
It can be shown that except for the lowest energy level $\left(n_x=n_y=n_z=0, i.e.\, n=0\right)$ all the energy levels are degenerate.

The zero point (or, the ground level) energy E_0 for $n_x=n_y=n_z=0$ of the oscillator is given by

$$E_0=\frac{3}{2}\hbar\omega$$

Thus the zero point energy of a three-dimensional isotropic harmonic oscillator is three times that for the linear harmonic oscillator. the ground level of the oscillator is thus non-degenerate. The state with principal quantum number n = 1, there are three levels each of energy $\frac{5}{2}\hbar\omega$. So, the first excited state is 3-fold degenerate. Similary, the state with n = 2 is 6-fold degenerate. Thus all the energy levels of the isotropic three-dimensional oscillator are degenerate and the degree of degeneracy is $p=\frac{1}{2}(n+1)(n+2)$.

n_x	n_y	n_z	n	Wave function	E_n	Degeneracy (p)
0	0	0	0	$\psi_{0,0,0}$	$\frac{3}{2}\hbar\omega$	$p=1$
1	0	0	1	$\psi_{1,0,0}$	$\frac{3}{2}\hbar\omega$	$p=3$
0	1	0		$\psi_{0,1,0}$		
0	0	1		$\psi_{0,0,1}$		
1	1	0		$\psi_{1,1,0}$	$\frac{3}{2}\hbar\omega$	$p=6$
1	0	1	2	$\psi_{1,0,0}$		
0	1	1		$\psi_{0,1,1}$		
2	0	0		$\psi_{2,0,0}$		
0	2	0	2	$\psi_{0,2,0}$		
0	0	2		$\psi_{0,0,2}$		



$\frac{11}{2}\hbar\omega$
n = 4
$\frac{9}{2}\hbar\omega$
n = 3
(3, 0, 0), (0, 3, 0), (0, 0, 3)
(2, 1, 0), (0, 2, 1), (1, 0, 2)
(2, 0, 1); (1, 2, 0), (0, 1, 2)
(1,1,1)
States 10
$\frac{7}{2}\hbar\omega$
n = 2
(2, 0, 0), (0, 2, 0), (0, 0, 2)
(1, 1, 0), (1, 0, 1), (0, 1, 1)
States 6
$\frac{5}{2}\hbar\omega$
n = 1
(1, 0, 0), (0, 1, 0), (0, 0, 1) States 3
$\frac{3}{2}\hbar\omega$
n = 1
(0, 0, 0)
States 1
0

Figure : Energy levels and degeneracy for the 3-dimensional harmonic oscillator

Example 23. The wave function of a particle confined in a box of length L is

$$\psi(x)=\sqrt{\frac{2}{L}}\sin\frac{\pi x}{L}$$

The probability of finding the particle in the domain $0<x<\frac{L}{2}$ is **[BHU 2010, 2011]**

(a) 1 (b) $\frac{1}{4}$ (c) $\frac{1}{2}$ (d) 0

Soln. Given: $\psi(x)=\sqrt{\frac{2}{L}}\sin\frac{\pi x}{L}$

The probability is $P=\int_0^{L/2}|\psi(x)|^2\,dx=\frac{2}{L}\int_0^{L/2}\sin^2\left(\frac{\pi x}{L}\right)dx$

$$=\frac{2}{2L}\int_0^{L/2}\left\{1-\cos\left(\frac{2\pi x}{L}\right)\right\}dx$$

$$=\frac{1}{L}\left[x-\frac{L}{2\pi}\sin\left(\frac{2\pi x}{L}\right)\right]_0^{L/2}=\frac{1}{L}\left[\frac{L}{2}\right]=\frac{1}{2}.$$

Correct option is (c)

Example 24. The ground state energy of a particle in an one-dimensional quantum well is 4.4 eV. If the width of the well is doubled, then the new ground state energy is **[BHU 2011]**

(a) 1.1 eV (b) 2.2 eV (c) 8.8 eV (d) 17.6 eV

Soln. Ground state energy of a particle in a 1-D quantum well, is $E_1=\frac{\pi^2\hbar^2}{2mL^2}$ (where L is the width of the well).

If the width of the well is doubled, then the ground state energy will decrease 4 times.

Therefore, new ground state energy will be 1.1 eV.

Correct option is (a)

Example 25. If an electron $\left(\mathrm{m}=9.1\times10^{-31}\,\mathrm{kg}\right)$ is confined by an electrical force to move between two rigid walls separated by 1.0×10^{-9} meter. The quantized energy value for the second lowest energy state is

[BHU 2011]

(a) 0.38 eV (b) 0.76 eV (c) 1.5 eV (d) 3.0 eV

Soln. Second lowest energy = First excited state energy =

$$E_2=\frac{4\pi^2\hbar^2}{2mL^2}=\frac{4\times(3.14)^2\times\left(1.05\times10^{-34}\right)^2}{2\times\left(9.1\times10^{-31}\right)\times\left(10^{-9}\right)^2}=2.38\times10^{-19}J=1.49\,eV$$

Correct option is (c)

Example 26. A particle is confined in a one-dimensional box of length l. Its wave function is given by

$$\psi(x)=\begin{cases}\sqrt{\dfrac{2}{l}}\sin\left(\dfrac{\pi x}{l}\right) & \text{for} \quad 0<x<1\\ 0 & \text{for} \quad l<x<0\end{cases}$$ **[BHU 2012]**

The expectation values of the x-component of momentum is

(a) 1 (b) 0.5 (c) $\dfrac{l}{2}$ (d) zero

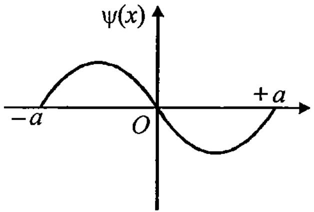
Soln. Given : $\psi(x)=\begin{cases}\sqrt{\dfrac{2}{l}}\sin\left(\dfrac{\pi x}{l}\right) & \text{for} \quad 0<x<1\\ 0 & \text{for} \quad l<x<0\end{cases}$

$$\therefore \quad \langle \hat{p}_x\rangle=\langle\psi\,|\,\hat{p}\,|\,\psi\rangle=\int_0^l\psi^*(x)\left(-i\hbar\frac{d}{dx}\right)\psi(x)\,dx$$

$$=\frac{2}{l}\int_0^l\frac{\pi}{l}\sin\left(\frac{\pi x}{l}\right)(-i\hbar)\cos\left(\frac{\pi x}{l}\right)dx=-\frac{i\hbar\pi}{l^2}\int_0^l\sin\left(\frac{2\pi x}{l}\right)dx=+\frac{i\hbar\pi}{l^2}\left[\frac{l}{2\pi}\cos\left(\frac{2\pi x}{l}\right)\right]_0^l=0.$$

Correct option is (d)

Example 27. In the figure below shows one of the possible energy eigenfunctions $\psi(x)$ for a particle bouncing freely back and forth along the x-axis between impenetrable walls located at $x=-a$ and $x=+a$. The potential energy equals zero for $|x|<a$. If the energy of the particle is 2 electron volts when it is in the quantum state associated with this eigenfunction, what is its energy when it is in the quantum state of lowest possible energy ?



(a) 0 eV (b) $\dfrac{1}{\sqrt{2}}$ eV (c) $\dfrac{1}{2}$ eV (d) 1 eV **[BHU 2012]**

Soln. The energy of the particle is given by $E_n=\dfrac{n^2\pi^2\hbar^2}{2ma^2}$

The given wave function corresponds to $n=2$.

$$\therefore \quad E_2=\frac{4\pi^2\hbar^2}{2ma^2}=2\text{ eV}\Rightarrow\frac{\pi^2\hbar^2}{2ma^2}=\frac{1}{2}\text{ eV}.$$

The lowest possible energy is the corresponds to $n=1$, i.e.,

$$E_1=\frac{\pi^2\hbar^2}{2ma^2}=\frac{1}{2}\text{ eV}.$$

Correct option is (c)

Example 28. A particle is moving in a one-dimensional potential box of infinite height. What is the probability of finding the particle in a small interval Δx at the center of the box when it is in the energy state, next to the least energy state? **[BHU 2012]**

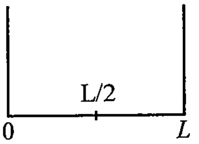
(a) 0 (b) 0.15 (c) 0.35 (d) 0.5

Soln. The wave function of the particle is

$$\psi(x)=\sqrt{\frac{2}{L}}\sin\left(\frac{2\pi x}{L}\right)$$

The required probability is

$$P=\int_{\frac{L}{2}-\frac{\Delta x}{2}}^{\frac{L}{2}+\frac{\Delta x}{2}}|\psi(x)|^2\,dx$$



$$=\frac{2}{L}\int_{\frac{L}{2}-\frac{\Delta x}{2}}^{\frac{L}{2}+\frac{\Delta x}{2}}\sin^2\left(\frac{2\pi x}{L}\right)dx=\frac{2}{2L}\left[x-\frac{L}{4\pi}\sin\left(\frac{4\pi x}{L}\right)\right]_{\frac{L-\Delta x}{2}}^{\frac{L+\Delta x}{2}}$$

$$=\frac{1}{L}\left[(\Delta x)-\frac{L}{4\pi}\left\{\sin\frac{4\pi}{L}\left(\frac{L+\Delta x}{2}\right)-\sin\frac{4\pi}{L}\left(\frac{L-\Delta x}{2}\right)\right\}\right]$$

$$=\left[\Delta x-\frac{2L}{4\pi}\cos 2\pi\sin\frac{2\pi}{L}\Delta x\right]$$

$$=\frac{1}{L}\left[\Delta x-\frac{L}{2\pi}\sin\frac{2\pi}{L}\Delta x\right]$$

For small Δx, $\sin\frac{2\pi}{L}\Delta x\approx\frac{2\pi}{L}\Delta x$

Therefore, $P=\frac{1}{L}\left[\Delta x-\frac{L}{2\pi}\frac{2\pi}{L}\Delta x\right]=0$

Correct option is (a)

Example 29. A particle of mass m is restricted to move in one-dimensional between two points such that $0\le x\le a$. If the potential function is such that **[BHU 2013]**

$$V(x)=\infty \quad x<0 \text{ and } x>a$$
$$=0 \quad 0\le x\le a$$

the particle will have discrete energy spectrum as

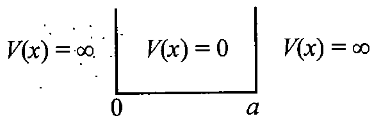
(a) $E_n=\frac{\pi^2\hbar^2}{4ma^2}\cdot n^2,\ \ n=0,1,2,...$ (b) $E_n=\frac{3\pi^2\hbar^2}{2ma^2}(n+1)^2,\ \ n=0,1,2,...$

(c) $E_n=\frac{\pi^2\hbar^2}{2ma^2}(n+1)^2,\ \ n=0,1,2,...$ (d) $E_n=\frac{3}{2}\frac{\pi^2\hbar^2}{ma^2}\cdot n^2,\ \ n=0,1,2,...$

Soln. This is one-dimensional particle in a box problem.

The energy is given by

$$E_n=(n+1)^2\frac{\pi^2\hbar^2}{2ma^2}\ ;\ n=0,1,2,....$$



Correct option is (c)

Example 30. The normalized ground state of a particle of mass m which is constrained to move inside a potential $V(x)=\begin{cases}0 & -L\le x\le +L\\ \infty & \text{elsewhere}\end{cases}$, is given by: **[BHU 2016]**

(a) $\sqrt{\frac{2}{L}}\sin\frac{\pi x}{L}$ (b) $\sqrt{\frac{2}{L}}\cos\frac{\pi x}{L}$ (c) $\sqrt{\frac{1}{L}}\sin\frac{\pi x}{2L}$ (d) $\sqrt{\frac{1}{L}}\cos\frac{\pi x}{2L}$

Soln. The ground state of a particle in a box is

$$\psi(x)=\sqrt{\frac{2}{2L}}\sin\left(\frac{\pi x}{2L}\right)$$

where $V(x)=\begin{cases}0 \;;\; 0\le x\le 2L\\ \infty \;;\; \text{otherwise}\end{cases}$

Shifting $x\to x-L$, we have

$$V(x)=\begin{cases}0 \;;\; -L\le x\le L\\ \infty \;;\; \text{otherwise}\end{cases} \text{ and}$$

$$\psi(x)=\sqrt{\frac{2}{2L}}\sin\left[\frac{\pi}{2L}(x-L)\right]$$

$$=\sqrt{\frac{1}{L}}\sin\left(\frac{\pi x}{2L}-\frac{\pi}{2}\right)$$

$$=\sqrt{\frac{1}{L}}\cos\left(\frac{\pi x}{2L}\right)$$

Correct option is (d)

Example 31. A particle is confined in a one-dimensional box of length l. Its ground state wave function is given by

$$\psi(x)=\begin{cases}\sqrt{\frac{2}{l}}\sin\left(\frac{\pi x}{l}\right) & ;\; \text{for } 0<x<l\\ 0 & ;\; \text{for } \quad x>l\end{cases}$$

The expectation value of the momentum operator with respect to the above wave function is equal to

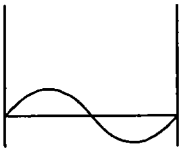
(a) 1 (b) 0.5 (c) $l/2$ (d) 0 **[HCU 2011]**

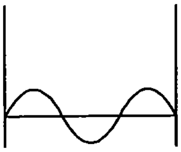
Soln. Given: $\psi(x)=\begin{cases}\sqrt{\frac{2}{l}}\sin\left(\frac{\pi x}{l}\right) & ;\; 0<x<l\\ 0 & ;\; 0>x>l\end{cases}$

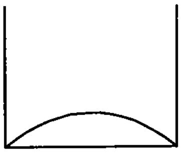
$$\therefore\ \langle\hat{p}_x\rangle=\int_0^l\psi^*(x)\left(-i\hbar\frac{d}{dx}\right)\psi(x)\,dx=(-i\hbar)\frac{2\pi}{l^2}\int_0^l\sin\left(\frac{\pi x}{l}\right)\cos\left(\frac{\pi x}{l}\right)dx=0$$

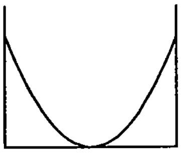
Correct option is (d)

Example 32. Which of the following represents the first excited state wave function of a massive particle confined to move in a one-dimensional box of length L (the x-axis represents the space co-ordinate and the y-axixrepresents the shape of the wave-function, in each figure.). **[HCU 2012]**

(a) 

(b) 

(c) 

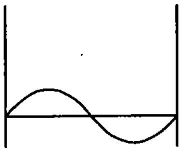
(d) 

Soln. The wave function of the particle inside a box is given by

$$\psi = \sqrt{\frac{2}{L}}\sin\frac{n\pi}{L}x$$

So the wave function of the first excited state

$$\psi = \sqrt{\frac{2}{L}}\sin\frac{2\pi}{L}x$$



Correct option is (a)

Another Method:

The number of nodes of the nth state is $(n-1)$.

So the number of nodes in the 2nd state will be $(2-1)=1$

Correct option is (a)

Example 33. If a particle of mass 1.67×10^{-27} kg is confined to move in a box of length 10^{-14} m and infinite depth, the minimum kinetic energy of the particle would be **[HCU 2012]**

(a) 0.21 MeV (b) 0.33 MeV (c) 2.03 eV (d) 3.3 eV

Soln. The minimum kinetic energy would be

$$E_k = \frac{\pi^2\hbar^2}{2ma^2} = \frac{\pi^2(1.05\times10^{-34})^2}{2\times1.67\times10^{-27}\times(10^{-14})^2} = 3.25\times10^{-13}\text{ J} = 2.03\times10^6\text{ eV} = 2.03\text{ MeV}$$

Correct option is (c)

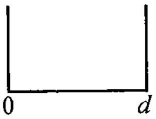
Example 34. In a one dimensional potential well with impenetrable walls occupying the domain $0\le x\le d$, a particle in the lowest energy state has the maximum probability to be found at **[HCU 2013]**

(a) $x=0$ (b) $x=d/2$ (c) $x=d/4$ (d) $x=d$

Soln. The ground state corresponding to the given potential is $\psi=\sqrt{\frac{2}{d}}\sin\frac{\pi}{d}x$

The probability density $p(x)=\psi^*\psi=\frac{2}{d}\sin^2\frac{\pi}{d}x$.

$p(x)$ will be maximum, when $\frac{dp}{dx}=0$



$$\Rightarrow \frac{2}{d}\frac{\pi}{d}2\cdot\sin\frac{\pi}{d}x\cos\frac{\pi}{d}x=0 \Rightarrow \sin\frac{2\pi}{d}x=\pi$$

$$\Rightarrow \frac{2\pi}{d}x=\pi \qquad \Rightarrow x=\frac{d}{2}$$

Correct option is (b)

Example 35. A rigid-walled box that extends from –L to +L is divided into three sections by interior walls at –x to +x where x < L. Each section contains one particle of same mass in its ground state. The total energy of the system in the ground state is **[HCU 2014]**

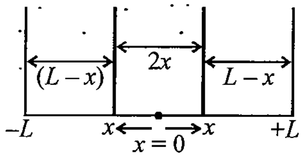
(a) $E_T = \dfrac{\pi^2\hbar^2}{2m}\left[\dfrac{2}{(L-x)^2}+\dfrac{1}{4x^2}\right]$ (b) $E_T = \dfrac{\pi^2\hbar^2}{6mL^2}$

(c) $E_T = \dfrac{\pi^2\hbar^2}{2mL^2}\left[\dfrac{2}{(L+x)^2}+\dfrac{1}{4x^2}\right]$ (d) $E_T = \dfrac{\pi^2\hbar^2}{2m}\left[\dfrac{2}{(L+x)^2}+\dfrac{1}{(L-x)^2}\right]$

Soln. The total energy of system in the ground state is

$$E = \frac{\pi^2\hbar^2}{2m}\left[\frac{1}{(L-x)^2}+\frac{1}{(2x)^2}+\frac{1}{(L-x)^2}\right]$$

$$= \frac{\pi^2\hbar^2}{2m}\left[\frac{2}{(L-x)^2}+\frac{1}{4x^2}\right]$$



Correct option is (a)

Example 36. For a wave function defined as $\psi(x) = \left(\dfrac{2}{L}\right)^{1/2}\sin\left(\dfrac{\pi x}{L}\right)$ in the region $0 < x < L$ and $\psi(x) = 0$ outside this region, the average value of the square of the momentum $\left(p^2\right)$ is, **[DU 2016]**

(a) $\dfrac{\hbar\pi}{L}$ (b) $\dfrac{h^2\pi^2}{L^2}$ (c) $\dfrac{\hbar^2\pi^2}{L^2}$ (d) 0

Soln. Given : $\psi(x) = \sqrt{\dfrac{2}{L}}\sin\left(\dfrac{\pi x}{L}\right)$

$$\therefore\ \langle p^2\rangle = \int_0^L \psi^*(x)\left(-i\hbar\frac{d}{dx}\right)^2\psi(x)\,dx$$

$$= -\frac{2}{L}\int_0^L \sin\left(\frac{\pi x}{L}\right)\left(-i\hbar\frac{\pi}{L}\right)^2\sin\left(\frac{\pi x}{L}\right)dx$$

$$= \frac{2\hbar^2\pi^2}{2L^3}\int_0^L\left(1-\cos\frac{2\pi x}{L}\right)dx$$

$$= \frac{\hbar^2\pi^2}{L^2}$$

Correct option is (c)

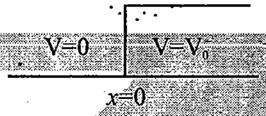
Chapter 6

Potential Step and Potential Barrier

6.1. Single Step Potential:

The potential function $V(x)$ in a one-dimensional single step potential undergoes a discontinuous change as shown in figure. The potential function $V(x)$ of the potential step may be expressed as,

$$V(x) = \begin{cases} 0, & \text{for } x < 0 \\ V_0, & \text{for } x > 0 \end{cases}$$



The potential difference between the dees of a cyclotron or, the potential barrier at the surface of a metal is approximately similar to this step potential.

Let a particle of mass m and kinetic energy E is travelling parallel to the x-axis and is incident on the potential step at $x = 0$. Since $V(x) = 0$ for $x < 0$ and $V(x) = V_0$ for $x > 0$, the energy E of the particle is wholly kinetic for $x < 0$ and partly kinetic and partly potential for $x > 0$.

From classical point of view in region I, the particle can move freely as $V(x) = 0$. But in region II, if $E < V_0$, the particle will remain in region I for ever and sometimes will be reflected back at $x = 0$. But for $E > V_0$ the particle will not be reflected back at $x = 0$ instead enter the region II.

Quantum mechanically, it will be shown that the electrons will be partly reflected and partly transmitted at the discontinuity.

Case A: ($E > V_0$)

The time-independent Schrodinger equation for region I is,

$$-\frac{h^2}{2m}\frac{d^2\psi_1}{dx^2} = E\psi_1 \quad \Rightarrow \quad \frac{d^2\psi_1}{dx^2} + \frac{2mE}{h^2}\psi_1 = 0 \Rightarrow \quad \frac{d^2\psi_1}{dx^2} + \frac{2mE}{h^2}\psi_1 = 0 \qquad \text{... (1)}$$

where, $\alpha = \sqrt{\frac{2mE}{\hbar^2}}$ and ψ_1 is the wave function for the particle moving in region I.

For region II, the time-independent Schrodinger equation is,

$$\frac{d^2\psi_2}{dx^2} + \frac{2m}{\hbar^2}(E - V_0)\psi_2 = 0 \quad \Rightarrow \quad \frac{d^2\psi_2}{dx^2} + \beta^2\psi_2 = 0 \qquad \text{...(2)}$$

where, $\beta = \sqrt{\frac{2m(E - V_0)}{\hbar^2}}$ and ψ_2 is the wave function for the particle in region II.

The general solutions of equations are

$$\psi_1 = Ae^{i\alpha x} + Be^{-i\alpha x}, \quad \psi_2 = Ce^{i\beta x} + De^{-i\beta x} \qquad \text{...(3)}$$

The terms $e^{i\alpha x}$ and $e^{-i\alpha x}$ represent respectively a wave travelling along the positive x-direction and a wave travelling along the negative x-direction, i.e., a reflected wave. Again, the terms $e^{i\beta x}$ and $e^{-i\beta x}$ represent respectively a wave travelling along the positive x-direction and a wave travelling along the negative x-direction (a reflected wave). But due to the presence of only one discontinuity at $x = 0$, there cannot be any reflected waves in region II. Hence the term $De^{-i\beta x}$ is discarded.

$$\therefore \quad \psi_2 = Ce^{i\beta x} \qquad \text{...(4)}$$

The values of the constants A, B and C can be determined by applying the following boundary conditions:

$$\textbf{1. } (\psi_1)_{x=0} = (\psi_2)_{x=0} \text{ and } \quad \textbf{2. } \left(\frac{d\psi_1}{dx}\right)_{x=0} = \left(\frac{d\psi_2}{dx}\right)_{x=0} \qquad \text{....(5)}$$

Using the above boundary condition in equations we get,

$$A + B = C$$

and $\alpha A - \alpha B = \beta C$ or, $\alpha A - \alpha B = \beta(A + B)$

$$\Rightarrow \quad (\alpha - \beta)A = (\alpha + \beta)B \quad \Rightarrow \quad B = \frac{\alpha - \beta}{\alpha + \beta}A \qquad \text{...(6)}$$

Now from equations we get

$$A + \frac{\alpha - \beta}{\alpha + \beta}A = C \quad \Rightarrow \quad C = \left(\frac{2\alpha}{\alpha + \beta}\right)A$$

$$\therefore \quad \psi_1 = A\left[e^{i\alpha x} + \left(\frac{\alpha - \beta}{\alpha + \beta}\right)e^{-i\alpha x}\right] \qquad \text{...(7)}$$

$$\text{and} \quad \psi_2 = \left(\frac{2\alpha}{\alpha + \beta}\right)Ae^{i\alpha_2 x} \qquad \text{...(8)}$$

The probability current density for the incident beam of particles in region I is,

$$J_i = \frac{\hbar\alpha}{m}(Ae^{i\alpha x})(Ae^{i\alpha x})^* = \frac{\hbar\alpha}{m}(AA^* e^{i\alpha x}\, e^{-i\alpha x}) \quad \text{or,} \quad J_i = \frac{\hbar\alpha}{m}|A|^2 \qquad \text{...(9)}$$

The probability current density for the reflected beam is,

$$J_r = \frac{\hbar\alpha}{m}(Be^{-i\alpha x})(Be^{-i\alpha x})^* = \frac{\hbar\alpha}{m}|B|^2 \qquad \text{...(10)}$$

The net probability current density in the direction from left to right is

$$\therefore \quad J = J_i - J_r = \frac{\hbar\alpha}{m}(|A|^2 - |B|^2) \qquad \text{...(11)}$$

But, $|B|^2 = \frac{(\alpha - \beta)^2}{(\alpha + \beta)^2}|A|^2$ [from equation (6)]

$$\therefore \quad J = \frac{\hbar\alpha}{m}|A|^2\left[1 - \frac{(\alpha - \beta)^2}{(\alpha + \beta)^2}\right] = \frac{\hbar\alpha}{m}|A|^2\left[\frac{4\alpha\beta}{(\alpha + \beta)^2}\right] \quad \Rightarrow \quad J = \frac{4\alpha\beta}{(\alpha + \beta)^2}\frac{\hbar\alpha}{m}|A|^2 \qquad \text{....(12)}$$

Again, the probability current density for the transmitted beam of particles in region II is,

$$J_t = \frac{\hbar\beta}{m}\psi_2\psi_2^* = \frac{\hbar\beta}{m}(Ce^{i\beta x})(Ce^{i\beta x})^* = \frac{\hbar\beta}{m}|C|^2 \quad \text{...(13)}$$

But, $C = \dfrac{2\alpha}{\alpha+\beta}A$ ...(14)

$$\therefore \quad J_t = \frac{\hbar\beta}{m}\left(\frac{2\alpha}{\alpha+\beta}\right)^2 |A|^2 = \frac{4\alpha^2}{(\alpha+\beta)^2}\frac{\hbar\beta}{m}|A|^2 \quad \text{...(15)}$$

From equations $J = J_t$, i.e. the net probability density from left to right in region I is equal to the probability density in region II.

$$\text{Again,} \quad J_r + J_t = \frac{\hbar\alpha}{m}|A|^2\frac{(\alpha-\beta)^2}{(\alpha+\beta)^2} + \frac{4\alpha}{(\alpha+\beta)^2}\frac{\hbar\beta}{m}|A|^2 = \left[\frac{(\alpha-\beta)^2}{(\alpha+\beta)^2} + \frac{4\alpha\beta}{(\alpha+\beta)^2}\right]\frac{\hbar\alpha}{m}|A|^2$$

$$= \left[\frac{(\alpha-\beta)^2}{(\alpha+\beta)^2} + \frac{4\alpha\beta}{(\alpha+\beta)^2}\right]J_i \Rightarrow J_r + J_t = J_i \quad \text{...(16)}$$

This shows that the total number of particles is conserved in the process.

Reflection and Transmission Coefficients:

The reflection coefficient is defined as,

$$R = \frac{\text{Reflected flux}\,(J_r)}{\text{Incident flux}\,(J_i)} = \frac{\frac{\hbar\alpha}{m}|B|^2}{\frac{\hbar\alpha}{m}|A|^2} = \frac{|B|^2}{|A|^2} = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2\left[\because B = \frac{\alpha-\beta}{\alpha+\beta}A\right] \quad \text{...(17)}$$

The transmission coefficient is,

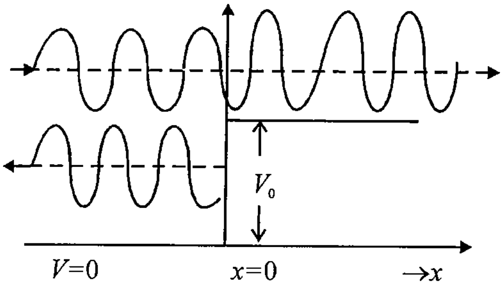
$$T = \frac{\text{Transmitted flux}\,(J_t)}{\text{Incident flux}\,(J_i)} = \frac{\frac{\hbar\beta}{m}|C|^2}{\frac{\hbar\alpha}{m}|A|^2} = \frac{\beta|C|^2}{\alpha|A|^2} \quad \text{...(18)}$$

$$\frac{\beta}{\alpha}\frac{4\alpha^2}{(\alpha+\beta)^2} = \frac{4\alpha\beta}{(\alpha+\beta)^2}\left[\because C = \frac{2\alpha}{\alpha+\beta}A\right]$$

$$\therefore \quad T + R = \frac{4\alpha\beta}{(\alpha+\beta)^2} + \frac{\alpha-\beta}{(\alpha+\beta)^2} = 1 \quad \text{...(19)}$$

Equation indicates that at the boundary between regions I and II, the probability of finding the particle is conserved.

Equation shows, that, $|C| > |A|$ as $\alpha > \beta$. In other words, the amplitude (C) of the transmitted wave is greater than that of the incident wave (i.e., A). The de Broglie wavelength in region I will be shorter than that in region II. This is due to the fact that the kinetic energy of the particle in region I is greater than the kinetic energy is region II. The nature of the wave forms is shown in figure.



The analysis thus shows that, for $E > V_0$, the incident particle has certain probability of being reflected and certain probability of being transmitted.

Case B : ($E < V_0$)

For region I ($x<0$), the Schrodinger wave equation is,

$$\frac{d^2\psi_1}{dx^2}+\alpha^2 x=0\text{, where } \alpha^2=\frac{2mE}{\hbar^2} \qquad ...(20)$$

The general solution of the above equation is as above,

$$\psi_1 = Ae^{i\alpha x}+Be^{-i\alpha x} \qquad ...(21)$$

For region II ($x > 0$), the wave equation is,

$$\frac{d^2\psi_2}{dx^2}-\frac{2m}{\hbar^2}(V_0-E)\psi_2=0\ (\because V_0>E) \quad \text{or,} \quad \frac{d^2\psi_2}{dx^2}-\gamma^2 x=0\left[\gamma^2=\frac{2m}{\hbar^2}(V_0-E)\right] \qquad ...(22)$$

The general solution of the above equation is, $\psi_2 = Ce^{\gamma x}+De^{-\gamma x}$ ...(23)

But the term, $e^{\gamma x}\rightarrow\infty$ as $x\rightarrow 0$ which contradicts the condition of well-behaved wave function.

$$\therefore \quad \psi_2 = De^{-\gamma x} \qquad ...(24)$$

The values of the constants A, B and D can be calculated by applying the following boundary conditions:

$$\textbf{1. } (\psi_1)_{x=0}=(\psi_2)_{x=0} \text{ and } \quad \textbf{2. } \left(\frac{d\psi_1}{dx}\right)_{x=0}=\left(\frac{d\psi_2}{dx}\right)_{x=0} \qquad ...(25)$$

Thus we get, $A + B = D$ and $i\alpha A - i\alpha B = -\gamma D$

Solving we get, $i\alpha(A-B)=-\gamma(A+B) \Rightarrow (i\alpha+\gamma)A=(i\alpha-\gamma)B$

$$\Rightarrow \quad B=\frac{i\alpha+\gamma}{i\alpha-\gamma}A \qquad ...(26)$$

$$\text{and} \quad D = A+\frac{i\alpha+\gamma}{i\alpha-\gamma}A=\left(\frac{i\alpha-\gamma+i\alpha+\gamma}{i\alpha-\gamma}\right)A \Rightarrow D=\frac{2i\alpha}{i\alpha-\gamma}A \qquad ...(27)$$

Now from equation (26), $BB^*=\left(\frac{i\alpha+\gamma}{i\alpha-\gamma}\right)A\left(\frac{i\alpha+\gamma}{i\alpha-\gamma}\right)A^* = \left(\frac{i\alpha+\gamma}{i\alpha-\gamma}\right)\left(\frac{-i\alpha+\gamma}{-i\alpha-\gamma}\right)AA^*$

$$\Rightarrow \quad BB^*=AA^* \Rightarrow |B|^2=|A|^2 \qquad ...(27)$$

Thus the probability current in region I is zero. Equation shows that the intensity of the incident wave is equal to the intensity of the reflected wave. Hence the incident and reflected probability currents cancel each other. This can be proved as follows:

We have the probability current density in region II as,

$$J_t = -\frac{i\hbar}{2m}\left[\psi_2^* \frac{d\psi_2}{dx} - \psi_2 \frac{d\psi_2^*}{dx}\right] = -\frac{i\hbar}{2m}[D^* e^{-\gamma x} D(-\gamma)e^{-\gamma x} - De^{-\gamma x}.D^x(-\gamma)e^{-\gamma x}]$$

$$\Rightarrow \quad J_t = \frac{i\hbar}{2m} DD^* \gamma[e^{-2\gamma x} - e^{-2\gamma x}] = 0 \qquad ...(28)$$

Reflection and Transmission Coefficients:

The reflection coefficient, $R = \frac{J_r}{J_i} = \frac{(Be^{-i\alpha x})(Be^{-i\alpha x})^* \frac{\hbar\alpha}{m}}{(Ae^{i\alpha x})(Ae^{i\alpha x})^* \frac{\hbar\alpha}{m}} = \frac{BB^*}{AA^*} = \frac{|B|^2}{|A|^2}$ ... (29)

Since $|B|^2 = |A|^2$ for $V_0 > E$,

$\therefore \quad R = 1$

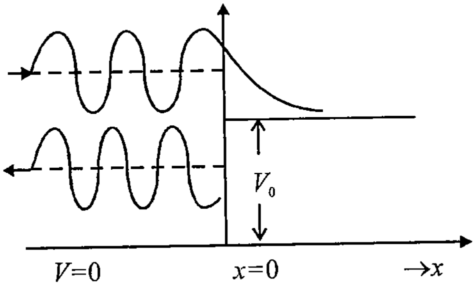
Hence, the incident wave is totally reflected at the potential step at $x = 0$.

The transmission coefficient, $T = \frac{J_t}{J_i}$, but for region II, $J_t = 0$

$\therefore \quad T = 0$

The above analysis leads to the following conclusions:

1. The incident wave is totally reflected at the potential step.
2. Both in region I and II, the probability current is zero.
3. In region II, the wave function $\psi_2 = De^{-\gamma x}$, i.e. ψ_2 is exponentially damped. This leads to a finite probability of finding the particle in region II.



The phenomenon is very much similar to the case of total internal reflection of electromagnetic waves while travelling from denser to a rarer medium.

If for finite energy E, the potential $V_0 \to \infty$, then $\gamma = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}} \to \infty$

The wave function ψ_2 in this case is, $\psi_2 = De^{-\gamma x} \to De^{-\alpha x} \to 0$

Again, $B=\left(\frac{i\alpha+\gamma}{i\alpha-\gamma}\right)A$ and $D=\frac{2i\alpha}{i\alpha-\gamma}A$

$$\therefore \quad \lim_{\gamma\to\infty}\frac{B}{A}=\lim_{\gamma\to 0}\frac{\alpha-i\gamma}{\alpha+i\gamma}=\lim_{\gamma\to 0}\frac{\frac{\alpha}{\gamma}-i}{\frac{\alpha}{\gamma}+i}=\frac{-i}{i}=-1\Rightarrow\frac{B}{A}=-1 \text{ or, } A+B=0$$

Also, $\lim_{\gamma\to\infty}\frac{D}{A}=\frac{2i\alpha}{i\alpha-\gamma}=\frac{2\alpha}{\alpha+i\gamma}=0$

$\therefore \quad D=0$

The wave function in region I is, $\psi_1(x)=Ae^{i\alpha x}+Be^{-i\alpha x}$

When $x=0$, $\psi_1(0)=A+B=0$ and the wave function in region II is, $\psi_2(x)=De^{-\gamma x}$ at $x=0$, $\psi_2(0)=D=0$

The slope of ψ_1 at $x=0$ is, $\left(\frac{d\psi_1}{dx}\right)_{x=0}=A(i\alpha)e^{i\alpha 0}+B(-i\alpha)e^{-i\alpha 0}$

$$=i\alpha(A-B)=i\alpha[A-(-A)]=2i\alpha A[\because A+B=0]$$

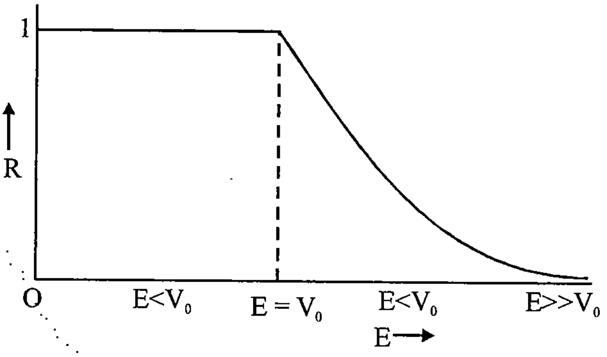
Hence, at $x=0$, where the value of $V(x)$ suddenly increases from 0 to ∞, the wave function $\psi_1(0)$ becomes zero. Also, the slope of ψ_1 at $x=0$, i.e., $\left(\frac{d\psi}{dx}\right)_{x=0}$ suddenly decreases from $2i\alpha A$ to zero.

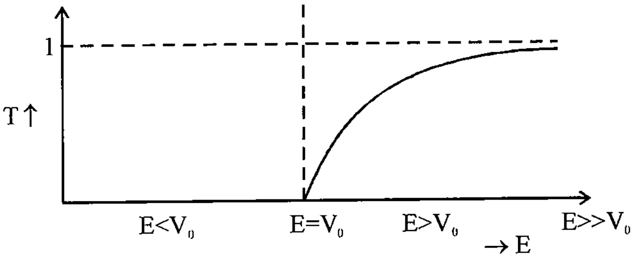
Penetration depth:

It is defined as the distance at which wave function reduces to 1/e of the value of the wave function at the barrier (x = 0) and given by

$$x=\frac{1}{\beta}=\sqrt{\frac{\hbar^2}{2m(V_0-E)}}$$

The fact that particle can penetrate into the classically forbidden region leads to Quantum Mechanical Tunnelling. If the potential barrier is infinitely high i.e. $V_0\to\infty$, then penetration depth becomes very much negligible.





Example 1. Electrons of energy 3.8 eV are subjected to a potential $V = 0$ for $x < 0$ and $V = 7.6$ eV for $x > 0$. (a) Find the wavelength of the sinusoidal variation of the wave function in the classically allowed region (b) the reflected wave differs from the incident wave by a phase factor. Find this phase factor.

Soln. **(a)** The wave number 'k' is $\alpha = \sqrt{\frac{2m_e E}{\hbar^2}} = \sqrt{\frac{3.8\,eV}{38\times 10^{-3} eV\ nm^2}} = 10\ nm^{-1}$

The wavelength is $\lambda = \frac{2\pi}{\alpha} = \frac{6.28}{10\ nm^{-1}} \approx 0.63\ nm$

(b) The wave function in the classically allowed region is

$$\psi = A_1\left[e^{i\alpha x} + \frac{i\alpha+\beta}{i\alpha-\beta}e^{-i\alpha x}\right]$$

Thus the reflected wave differs from the incident wave by a phase factor $f = \frac{ik+\gamma}{ik-\gamma}$

We have $\alpha = 10\ nm^{-1}$ and similarly $\beta = \sqrt{\frac{2m(V_0-E)}{\hbar^2}} = 10\ nm^{-1}$. So, $f = e^{i\frac{3\pi}{2}}$

Example 2. Electrons of energy 10 eV are incident on a potential step of height 13.8 eV. Find the distance in which the probability density of finding the particle decreases to a factor of 0.01 as it penetrates into the classically forbidden region.

Soln. The wave function in the classically forbidden region is of the form

$$\psi = C\,e^{-\beta x}$$

where, $\beta = \sqrt{\frac{2m(V_0-E)}{\hbar^2}} = \sqrt{\frac{(13.8-10)eV}{38\times 10^{-3}\ eV\ nm^2}} = 10\ nm^{-1}$

According to the question,

$$P(x+d) = 0.01P(x) \Rightarrow |C|^2 e^{-2\gamma(x+d)} = 0.01|C|^2 e^{-2\gamma x}$$

$$\Rightarrow e^{2\gamma d} = 100 \Rightarrow 2\gamma d = \ln(100) \Rightarrow d = \frac{\ln 10}{\gamma} = 0.23\ nm$$

Example 3. An electron beam is incident on a potential barrier of height 0.03 eV and of infinite width. Find the reflection coefficient and transmission coefficient at the barrier if the energy of the incident electron beam be (a) 0.04 eV, (b) 0.025 eV.

Soln. **(a)** In this case, $E > V_0$.

Reflection coefficient $R = \left(\frac{\sqrt{E}-\sqrt{(E-V_0}}{\sqrt{E}+\sqrt{E-V_0}}\right)^2 \approx 0.1$

Transmission coefficient $T = 1 - R = 1 - 0.1 = 0.9$

(b) In this case, $E < V_0$. So, $T = 0$ and $R = 1$, i.e. all the electrons that are incident on the barrier are reflected back and no electron will be transmitted through the barrier.

6.2. One-Dimensional Rectangular Potential Barrier (Quantum Mechanical Tunnelling Effect):

The diagrammatic representation of a rectangular potential barrier is shown in figure which extends from $x = 0$ to $x = a$. The potential of the barrier in this region is constant and equal to V_0. On the left and right side of the barrier the potential $V = 0$.

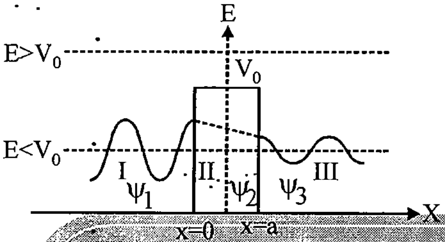


Fig. One-dimensional rectangular potential barrier

Now, we suppose that a stream of particles of energy E be incident from left on the barrier surface at $x = 0$.

The following two cases arise:

1. If $E > V_0$, then according to classical mechanics the particles will be wholly transmitted and no reflection is possible. But quantum mechanically there is always some probability at $x = 0$ and $x = a$.
2. If $E < V_0$, then classically the particles will be wholly reflected and hence penetration through the barrier is impossible. But quantum mechanically there is always some probability of penetration into the barrier and appearance of the particles in region III. This finite probability of transmission through the barrier even $E<V_0$ is called the quantum mechanical tunnelling effect.

Case A: When $E > V_0$

For region I, ($x<0$) where $V(x) = 0$, the time-independent Schrodinger equation is,

$$\frac{d^2\psi_1}{dx^2} + \frac{2m}{\hbar^2}E\psi_1 = 0 \quad \Rightarrow \quad \frac{d^2\psi_1}{dx^2} + \alpha^2\psi_1 = 0 \qquad \text{....(1)}$$

where, $\alpha^2 = \frac{2mE}{\hbar^2}$ and ψ_1 is the wave function in region I.

The general solution of equation is, $\psi_1(x) = Ae^{i\alpha x} + Be^{-i\alpha x}$ ....(2)

For region II, ($0 < x < a$), where $V(x) = V_0$, the Schrodinger equation is,

$$\frac{d^2\psi_2}{dx^2} + \frac{2m}{\hbar^2}(E - V_0)\psi_2 = 0 \quad \Rightarrow \quad \frac{d^2\psi_2}{dx^2} + \beta^2\psi_2 = 0 \qquad \text{....(3)}$$

where, $\beta^2 = \frac{2m(E - V_0)}{\hbar^2}$ and ψ_2 is the wave function in region II.

The general solution of equation is, $\psi_2(x) = Ce^{i\beta x} + De^{-i\beta x}$ ....(4)

Here, $De^{-i\beta x}$ represents the wave which is reflected at the boundary at $x = a$.

For region III, $(x > a)$, where $V(x) = 0$ again, the Schrodinger equation is,

$$\frac{d^2\psi_3}{dx^2} + \alpha^2\psi_3 = 0 \text{ and the general solution is, } \psi_3(x) = Fe^{i\alpha x} \qquad \text{.... (5)}$$

Since in region III, the wave propagates to the right only, there is no question of reflection and as such the $e^{-i\alpha x}$ term is absent from equation.

The five constants A, B, C, D and F can be obtained by applying the boundary conditions at $x = 0$ and $x = a$.

The boundary conditions are:

1. ψ is continuous at $x = 0$ and $x = a$ as the probability density must be continuous, and

2. $\frac{d\psi}{dx}$ at $x = 0$ and $x = a$ is continuous since probability current must be continuous.

Applying the boundary conditions at $x = 0$ and $x = a$ we get,

$$A + B = C + D,\ A - B = \frac{B}{\alpha}(C - D),\ Ce^{i\beta a} + De^{-i\beta a} = Fe^{i\alpha a},\ Ce^{i\beta a} - De^{-i\beta a} = \frac{\alpha}{\beta}e^{i\alpha a}$$

Now solving equations we get, $A = \frac{C}{2}\left(1 + \frac{\beta}{\alpha}\right) + \frac{D}{2}\left(1 - \frac{\beta}{\alpha}\right)$, $B = \frac{C}{2}\left(1 - \frac{\beta}{\alpha}\right) + \frac{D}{2}\left(1 + \frac{\beta}{\alpha}\right)$ ....(5)

Solving equations we get, $C = \frac{F}{2}\left(1 + \frac{\alpha}{\beta}\right)e^{i(\alpha-\beta)a}$, $D = \frac{F}{2}\left(1 - \frac{\alpha}{\beta}\right)e^{i(\alpha+\beta)a}$ ....(6)

Substituting the values of C and D from equation (6) in equation (5) we get,

$$A = \frac{F}{4}e^{i\alpha a}\left[\left(1 + \frac{\beta}{\alpha}\right)\left(1 + \frac{\alpha}{\beta}\right)e^{-i\beta a} + \left(1 - \frac{\beta}{\alpha}\right)\left(1 - \frac{\alpha}{\beta}\right)e^{i\beta a}\right] \qquad \text{....(7)}$$

$$B = \frac{F}{4}e^{i\alpha a}\left[\left(1 - \frac{\beta}{\alpha}\right)\left(1 + \frac{\alpha}{\beta}\right)e^{-i\beta a} + \left(1 + \frac{\beta}{\alpha}\right)\left(1 - \frac{\alpha}{\beta}\right)e^{i\beta a}\right] \qquad \text{....(8)}$$

From equation $\frac{F}{A} = \frac{4e^{-i\alpha a}}{\left(1 + \frac{\beta}{\alpha}\right)\left(1 + \frac{\alpha}{\beta}\right)e^{-i\beta a} + \left(1 - \frac{\beta}{\alpha}\right)\left(1 - \frac{\alpha}{\beta}\right)e^{i\beta a}}$

$$= \frac{2\alpha\beta e^{-i\alpha a}}{(\alpha^2 + \beta^2)\left(\frac{e^{-i\beta a} - e^{i\beta a}}{2}\right) + 2\alpha\beta\left(\frac{e^{-i\beta a} + e^{i\beta a}}{2}\right)}$$

$$= \frac{2\alpha\beta e^{-i\alpha a}}{2\alpha\beta\cos\beta a - i(\alpha^2 + \beta^2)\sin\beta a}$$

Transmission co-efficient $T = \frac{J_t}{J_i} = \frac{\frac{\hbar k}{m}(Fe^{iax})(Fe^{-iax})^*}{\frac{\hbar k}{m}(Ae^{iax})(Ae^{iax})^*} = \frac{FF^*}{AA^*} = \frac{|F|^2}{|A|^2} = \left(\frac{F}{A}\right)\left(\frac{F}{A}\right)^*$

$$= \frac{4\alpha^2\beta^2}{4\alpha^2\beta^2\cos^2\beta a + (\alpha^2+\beta^2)^2\sin^2\beta a} = \frac{1}{1+\frac{(\alpha^2-\beta^2)^2}{4\alpha^2\beta^2}\sin^2\beta a} \qquad \text{....(9)}$$

But, $\alpha^2 - \beta^2 = \frac{2mE}{\hbar^2} - \frac{2m}{\hbar^2}(E-V_0) \Rightarrow (\alpha^2-\beta^2)^2 = \left(\frac{2m}{\hbar^2}\right)^2 V_0^2$

and $\alpha^2\beta^2 = \left(\frac{2mE}{\hbar^2}\right)\left[\frac{2m}{\hbar^2}(E-V_0)\right] = \left(\frac{2m}{\hbar^2}\right)^2 E(E-V_0)$

$$\therefore \quad T = \frac{1}{1+\frac{V_0^2}{4E(E-V_0)}\sin^2\beta a} \qquad \text{....(10)}$$

From equation, we note that T = 1, if $\sin^2\beta a = 0$ or, $\beta a = n\pi$, $n = 1, 2, 3, \ldots\ldots$, then,

$$a = \frac{n\pi}{\beta} = n\frac{\pi\hbar}{\beta\hbar} = n\frac{\frac{h}{2}}{p} = n\frac{\lambda_2}{2}$$

where, $\hbar = \frac{h}{2\pi}$ and $B\hbar = p$, the momentum of the particle in region II, $\lambda_2 = \frac{h}{p}$, the de-Broglie wavelength of the particle.

Hence, if the barrier region contains an integral multiple of half de-Broglie wavelengths, then the barrier transmission is complete. This is Ramsauer-Townsend Effect. It explains the phenomenon of scattering of low energy electrons from the atoms of noble gases.

Equation shows that T oscillates for $\frac{E}{V_0} > 1$ and approaches to unity of E >> V_0

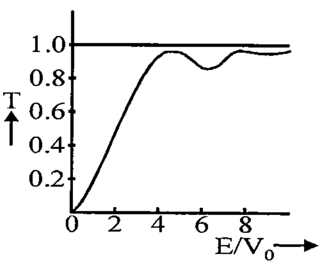


Fig. Transmission through a rectangular potential barrier as a function of energy

Reflection coefficient $R=\left(\frac{B}{A}\right)\left(\frac{B}{A}\right)^*=\frac{\frac{(\alpha^2-\beta^2)^2}{4\alpha^2\beta^2}\sin^2\beta a}{1+\left(\frac{\alpha^2-\beta^2}{4\alpha^2\beta^2}\right)\sin^2\beta a}=\frac{\frac{V_0^2}{4E(E-V_0)}\sin^2\beta a}{1+\frac{V_0^2}{4E(E-V_0)}\sin^2\beta a}$ ...(11)

From equations, $R+T=1$

The natures of the wave functions ψ_1, ψ_2 and ψ_3 in region I, II and III respectively are shown in figure.

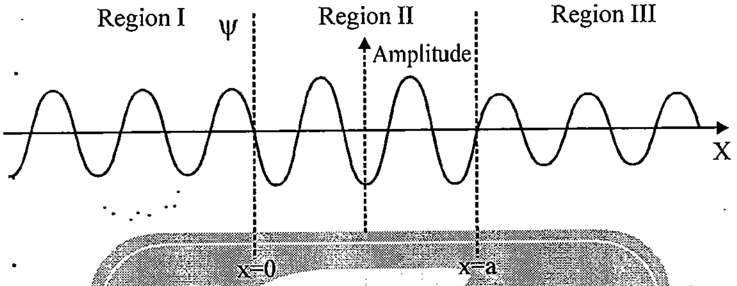


Fig. Wave function of a particle in a rectangular potential barrier

Case B : When $E<V_0$

The time-independent Schrodinger equations for regions I, II and III are given by,

$$\frac{d^2\psi_1}{dx^2}+\alpha^2\psi_1=0,\quad \frac{d^2\psi_2}{dx^2}-\beta^2\psi_2=0 \quad\text{and}\quad \frac{d^2\psi_3}{dx^2}+\alpha^2\psi_3=0 \qquad ....(12)$$

where, $\alpha^2=\frac{2mE}{\hbar^2}$ and $\beta^2=\frac{2m}{\hbar^2}(V_0-E)$

The solutions of the above three equations for regions I, II and III are respectively,

$$\psi_1=Ae^{i\alpha x}+Be^{-i\alpha x},\ \psi_2=Ce^{\beta x}+De^{-\beta x}\ \text{and}\ \psi_3=Fe^{i\alpha x}+Ge^{-i\alpha x} \qquad ....(13)$$

In region III as the wave propagates along the forward direction, there is no reflection and hence $Ge^{-i\alpha x}$ is discarded.

$$\therefore \psi_3=Fe^{i\alpha x} \qquad ....(14)$$

Here A, B, C, D and F are arbitrary constants. To find the values of these constants we apply the following boundary conditions:

1. $(\psi_1)_{x=0}=(\psi_2)_{x=0}$ **2.** $\left(\frac{d\psi_1}{dx}\right)_{x=0}=\left(\frac{d\psi_2}{dx}\right)_{x=0}$ ....(16)

Applying these boundary conditions to equations (13) we get,

$A+B=C+D$ and $i\alpha A-i\alpha B=\beta C-\beta D$

Solving the above two equations we get,

$$A=\left(1-\frac{i\beta}{\alpha}\right)\frac{C}{2}+\left(1+\frac{i\beta}{\alpha}\right)\frac{D}{2} \quad\text{and}\quad B=\left(1+\frac{i\beta}{\alpha}\right)\frac{C}{2}+\left(1-\frac{i\beta}{\alpha}\right)\frac{D}{2} \qquad ....(17)$$

Similarly the boundary conditions at $x = a$ are:

$$\textbf{3.}\ (\psi_2)_{x=a} = (\psi_3)_{x=a} \qquad \textbf{4.}\ \left(\frac{d\psi_2}{dx}\right)_{x=a} = \left(\frac{d\psi_3}{dx}\right)_{x=a} \qquad \text{....(18)}$$

Now from equations (13) we get,

$$Ce^{\beta a} + De^{-\beta a} = Fe^{\alpha a} \quad \text{and} \quad \beta Ce^{\beta a} - D\beta e^{-\beta a} = i\alpha Fe^{i\alpha a} \qquad \text{....(19)}$$

Solving equations we get,

$$C = \left(1 + \frac{i\alpha}{\beta}\right)\frac{F}{2}e^{-\beta\alpha}e^{i\alpha a} \ \text{or,}\ C = \left(1 + \frac{i\alpha}{\beta}\right)\frac{F}{2}e^{(i\alpha-\beta)a} \ \text{and}\ D = \left(1 - \frac{i\alpha}{\beta}\right)\frac{F}{2}e^{(i\alpha+\beta)a} \qquad \text{....(20)}$$

Using these values of C and D in equations (17) we get,

$$A = \frac{F}{4}e^{i\alpha a}\left[\left(1+\frac{i\beta}{\alpha}\right)\left(1+\frac{\alpha}{i\beta}\right)e^{-\beta a} + \left(1-\frac{i\beta}{\alpha}\right)\left(1-\frac{\alpha}{i\beta}\right)e^{\beta a}\right] \qquad \text{....(21)}$$

$$B = \frac{F}{4}e^{i\alpha a}\left[\left(1-\frac{i\beta}{\alpha}\right)\left(1+\frac{\alpha}{i\beta}\right)e^{-\beta a} + \left(1+\frac{i\beta}{\alpha}\right)\left(1-\frac{\alpha}{i\beta}\right)e^{\beta a}\right] \qquad \text{....(22)}$$

Transmission Coefficient and Reflection Coefficient:

The transmission coefficient is given by, $T = \frac{J_t}{J_i} = \frac{\frac{\hbar k}{m}(Fe^{i\alpha x})(Fe^{i\alpha x})^*}{\frac{\hbar k}{m}(Ae^{i\alpha x})(Ae^{i\alpha x})} = \left(\frac{F}{A}\right)\left(\frac{F}{A}\right)^*$ ....(23)

Now from equation we get, $\frac{F}{A} = \frac{4}{e^{i\alpha a}\left[\left(1+\frac{i\beta}{\alpha}\right)\left(1+\frac{\alpha}{i\beta}\right)e^{-\beta a} + \left(1-\frac{i\beta}{\alpha}\right)\left(1-\frac{\alpha}{i\beta}\right)e^{\beta a}\right]}$

$$\therefore T = \left(\frac{F}{A}\right)\left(\frac{F}{A}\right)^* = \frac{1}{\cos h^2(\beta a) + \frac{1}{4}\left(\frac{\beta}{\alpha} - \frac{\alpha}{\beta}\right)^2 \sin h^2(\beta a)} = \frac{1}{1 + \sin h^2(\beta a) + \left(\frac{\beta^2 - \alpha^2}{2\beta\alpha}\right)^2 \sin h^2(\beta a)}$$

$$= \frac{1}{1 + \sin h^2(\beta a)\left[1 + \left(\frac{\beta^2 - \alpha^2}{2\alpha\beta}\right)^2\right]} = \frac{1}{1 + \left(\frac{\beta^2 + \alpha^2}{4\alpha^2\beta^2}\right)\sin h^2 \beta a} \qquad \text{....(24)}$$

Substituting $\alpha^2 = \frac{2mE}{\hbar^2}$ and $\beta^2 = \frac{2m(E - V_0)}{\hbar^2}$ we get, from equations (24)

$$T = \frac{1}{1 + \frac{V_0^2}{4E(V_0 - E)}\sin h^2(\beta a)} \qquad \text{....(25)}$$

Reflection coefficient $R = \dfrac{J_r}{J_i} = \dfrac{\dfrac{\hbar k}{m}(Be^{-i\alpha x})\,(Be^{-i\alpha x})^*}{\dfrac{\hbar k}{m}(Ae^{i\alpha x})\,(Ae^{i\alpha x})^*} = \left(\dfrac{B}{A}\right)\left(\dfrac{B}{A}\right)^* = \dfrac{|B|^2}{|A|^2}$

$$= \frac{\dfrac{(\beta^2+\alpha^2)^2}{4\alpha^2\beta^2}\sinh^2\beta a}{\cosh^2\beta a + \dfrac{1}{4}\dfrac{(\beta^2-\alpha^2)^2}{4\alpha^2\beta^2}} = \frac{\left(\dfrac{\beta^2+\alpha^2}{2\beta\alpha}\right)^2\sinh^2\beta a}{\cosh^2\beta a + \left(\dfrac{\beta^2-\alpha^2}{2\beta\alpha}\right)^2\sinh^2\beta a}$$

$$\Rightarrow \quad R = \frac{\left(\dfrac{\beta^2+\alpha^2}{2\beta\alpha}\right)^2\sinh^2\beta a}{1+\left(\dfrac{\beta^2+\alpha^2}{2\beta\alpha}\right)^2\sinh^2\beta a} \qquad \text{....(26)}$$

Quantum Mechanical Tunnelling:

From equation, we see that the transmission coefficient $T \neq 0$ which means that there is a finite probability of transmission of the particles through the potential barrier of height V_0 and width a even if $E < V_0$. This cannot be accounted for in view of classical theory.

Thus quantum mechanical tunnelling is the phenomenon of transmission of a particle through a potential of finite width and height when its energy is less than the barrier height.

The transmission probability depends on V_0 and a such that T is small if V_0 and a are large.

Again, when $a \to$ large value $\beta a \to \infty$.

$$\therefore \quad \sinh^2(\beta a) = \left(\frac{e^{\beta a} - e^{-\beta a}}{2}\right)^2 = \frac{e^{2\beta a}}{4} \gg 1 \qquad \text{.... (27)}$$

$$\therefore \quad T = \frac{16\alpha^2\beta^2}{(\beta^2+\alpha^2)} e^{-\beta a} = \frac{16(V_0 - E)}{V_0^2} e^{-2\beta a} \qquad \text{....(28)}$$

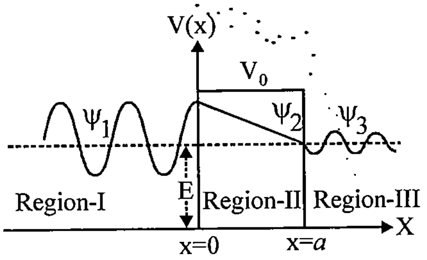


Fig. ψ_1, ψ_2 and ψ_3 for E<V$_0$

The waveform in region I, II and II are shown in figure. The appearance of the incident and transmitted waves are the same as in case $E > V_0$. But for $E < V_0$, there is no wave inside the barrier except an exponentially decaying function as in evident from equation.

From equations it is easily seen that, $R + T = 1$.

From the foregoing discussion the following conclusions are drawn:

Case A: When $E > V_0$

1. When $E \to V_0$, $\beta a = a\sqrt{\dfrac{2m}{\hbar^2}(E-V_0)} \ll 1$

$$\therefore \quad \sin\beta a = a\sqrt{\frac{2m}{\hbar^2}(E-V_0)} \text{ and } \sin^2\beta a = \frac{2m(E-V_0)a^2}{\hbar^2}$$

From equation (25)
$$T = \frac{1}{1+\dfrac{V_0^2}{4E(E-V_0)}\sin^2\beta a} = \frac{1}{1+\dfrac{V_0^2}{4E(E-V_0)}\times\dfrac{2m(E-V_0)a^2}{\hbar^2}}$$

$$= \frac{1}{1+\dfrac{2mV_0^2a^2}{4E\hbar^2}} = \frac{1}{1+\dfrac{mV_0^2a^2}{2E\hbar^2}},$$

2. If $E > V_0$, then $(E - V_0)$ increases, $\beta = \sqrt{\dfrac{2m(E-V_0)}{\hbar^2}}$ also increases. But $(E - V_0)$ increases more rapidly than $\sin^2\beta a$ for given values of a and V_0. Hence T increases with the increase of energy E of the incident particle.

3. In this case as shown before that if $a = n(\lambda/2)$, then there is a perfect transmission of the incident beam of particles.

Case B : When $E < V_0$

1. There is a finite probability of transmission of a particle through a potential barrier of width a and height V_0 even if $E < V_0$.

2. The factor $(V_0 - E)$ decreases with the increase of E.

$$\therefore \quad \beta = \sqrt{\frac{2m(V_0-E)}{\hbar^2}}$$

also decreases. The rate of decrease of $\sin^2\beta a$ for a fixed value of a and V_0 is more than that of $(V_0 - E)$. So T increases steadily with E.

3. When $E \to V_0$, $\beta a = a\sqrt{2m(V_0-E)} \ll 1$

$$\therefore \quad \sinh\beta a = \sqrt{\frac{2m(V_0-E)}{\hbar^2}}a \text{ and } \sinh^2\beta a = \frac{2m(V_0-E)a}{\hbar^2}$$

Then, from equation (25),
$$T = \frac{1}{1+\dfrac{V_0^2}{4E(V_0-E)}\times\dfrac{2m(V_0-E)a}{\hbar^2}}$$

$$\simeq \frac{1}{1+\frac{2mV_0^2a^2}{4E\hbar}} = \frac{1}{1+\frac{mV_0a^2}{2\hbar^2}} \quad (\text{as } E \to V_0)$$

This value T, in case $E > V_0$, is the same as the value of T for $E < V_0$. T is small, if V_0 and a are large.

Example 4: Calculate the percentage transmission of an electron beam of energy 3eV when incident on a potential barrier of height 4eV and width 20Å.

Soln.: Transmission coefficient $T = \frac{16E}{V_0}\left(1-\frac{E}{V_0}\right)e^{-2\beta a}$

Given: $E = 3eV, V_0 = 4\,eV, a = 20A^0$. Here, $2\beta a = 2a \times \sqrt{\frac{2m}{\hbar^2}(V_0 - E)} \approx 20.49$

Therfore, $T = \frac{16E}{V_0}\left(1-\frac{E}{V_0}\right)e^{-2\beta a} = 3.797 \times 10^{-9}$ (approx)

Hence, the % of transmission is = $3.797 \times 10^{-9} \times 100 = 3.797 \times 10^{-7}$.

Example 5. An alpha particle is trapped in a nucleus of radius 1.4×10^{-15}m. What is the probability that it will escape from the nucleus if its energy is 2MeV? The potential barrier at the surface of the nucleus is 4 MeV and the mass of the α-particle = 6.64×10^{-27} kg

Soln. Transmission coefficient $T = \frac{16E}{V_0}\left(1-\frac{E}{V_0}\right)e^{-2\beta a}$

Given: $E = 2MeV, V_0 = 4\,MeV, a = 2 \times 1.4 \times 10^{-15}\,m$

Example 6. Transmission probability of a particle with energy E moving through a one-dimensional rectangular potential barrier of height U_0 ($U_0 < E$) **[BHU 2011]**

(a) is always 1

(b) is never 1

(c) varies from 0 to 1 periodically with increasing $\frac{E}{U_0}$

(d) becomes 1 periodically with increasing $\frac{E}{U_0}$ but never becomes 0

Soln.

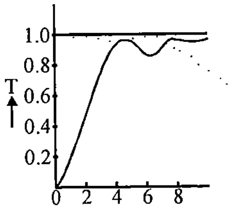


Fig. Transmission through a rectangular potential barrier as a function of energy

Correct option is (d)

Example 7. A free particle with knietic energy E and de Broglie wavelength λ enters a region in which it has potential energy V. What is the particle's new de Broglie wavelength ? **[BHU 2012]**

(a) $\lambda\left(1-\frac{V}{E}\right)$ (b) $\lambda\left(1-\frac{E}{V}\right)^{-1}$ (c) $\lambda\left(1+\frac{V}{E}\right)^{1/2}$ (d) $\lambda\left(1-\frac{V}{E}\right)^{-1/2}$

Soln. The de-Broglie wavelength is $\lambda=\frac{h}{\sqrt{2mE}}$, and the new de-Broglie wavelength λ' is

$$\lambda'=\frac{h}{\sqrt{2m(E-V)}}=\frac{h}{\sqrt{2mE\left(1-\frac{V}{E}\right)}}$$

$$=\frac{h}{\sqrt{2mE}}\left(1-\frac{V}{E}\right)^{-1/2}$$

$$=\lambda\left(1-\frac{V}{E}\right)^{-1/2}$$

Correct option is (d)

Chapter 7

Linear Harmonic Oscillator

If a system vibrates about an equilibrium position such that (a) the magnitude of the force acting on it is proportional to displacement from the equilibrium position and (b) the direction of the force is opposite to that of displacement, it is called a *linear harmonic oscillator* (LHO).

The importance of the LHO in classical as well as quantum physics is that many complex systems can be reduced to a collection of harmonic oscillators of difference frequencies using Fourier analysis. The method is suitable for many-body systems, like the atoms in crystalline solids. In fact in the theory of specific heat of solids, Debye assumed the collective motion of the atoms in the solid as made up of LHO, the frequency of vibration ranging up to a maximum cut-off frequency depending on the nature of the system.

7.1. Energy of the LHO According to Classical Mechanics:

Let at any time t, the displacement of a particle of mass m constraint to move along the X-axis be x from the equilibrium position $x = 0$. The restoring force on the particle is $F = -kx$ where, k is a positive constant. Then the Newton's equation of motion is,

$$m\frac{d^2x}{dt^2} = -kx \quad \text{or,} \quad \frac{d^2x}{dt^2} + \omega^2 x = 0 \;\; \left[\text{where, } \omega^2 = \frac{k}{m}\right] \qquad ....(1)$$

The general solution of equation is, $x = a\sin(\omega t + \phi)$ ....(2)

where a and ψ are constants which may be determined from the initial conditions. Equation represents the simple harmonic motion (SHM) of the particle with frequency,

$$\nu = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad ....(3)$$

The potential energy $V(x)$ at displacement x is,

$$V = \int_0^x \frac{dV}{dx}dx = \int_0^x -F dx = -\int_0^x (-kx)dx = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2 x^2 \left[\because \omega^2 = \frac{k}{m}\right] \qquad ....(4)$$

The kinetic energy T at x is given by,

$$T = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 = \frac{1}{2}ma^2\omega^2\cos^2(\omega t + \phi) = \frac{1}{2}m\omega^2 a^2[1 - \sin^2(\omega t + \phi)] = \frac{1}{2}m\omega^2(a^2 - x^2) \; ....(5)$$

Therefore, the total energy of the particle at x is,

$$E = T + V = \frac{1}{2}m\omega^2(a^2 - x^2) + \frac{1}{2}m\omega^2 x^2, \quad E = \frac{1}{2}m\omega^2 a^2 \qquad ....(6)$$

The total energy E is proportional to a^2. Classically, the particle can oscillate with any amplitude a and as such the total energy increases with the increase of a.

The potential energy, $V = \frac{1}{2}kx^2$ has a parabolic form figure signifying that the particle executes motion in a potential well of parabolic form.

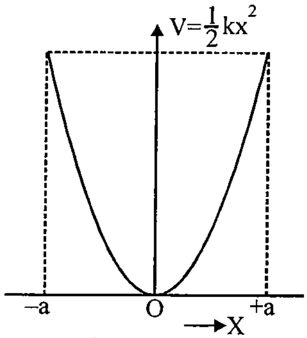


Fig. Potential energy curve of a LHO

Classical Probability of Finding the LHO within a Small Distance dx:

The probability of finding the LHO within x and $x + dx$ from its equilibrium position is

$$= \frac{\text{time taken by the LHO to pass over dx}}{\text{period of oscillation}} \quad \text{or,} \quad P(x)dx = \frac{2dt}{T} = \frac{2dt}{\left(\frac{2\pi}{\omega}\right)} = \frac{\omega dt}{\pi}$$

But, $\because x = a\sin(\omega t + \phi)$, $\therefore dx = a\omega\cos(\omega t + \phi)dt$

$$\Rightarrow \quad dt = \frac{dx}{a\omega\sqrt{1 - x^2/a^2}} = \frac{dx}{\omega\sqrt{a^2 - x^2}} = \frac{dx}{\omega\sqrt{\frac{2E}{m\omega^2} - x^2}}, \left[\because a^2 = \frac{2E}{m\omega^2}\right]$$

$$\text{Therefore,} \quad P(x)dx = \frac{dx}{\pi\sqrt{\frac{2E}{m\omega^2} - x^2}} \qquad \text{....(7)}$$

7.2. Quantum Mechanical Linear Harmonic Oscillator:

The Hamiltonian operator for the one-dimensional linear harmonic oscillator is given by,

$$\hat{H} = \frac{\hat{p}_x^2}{2m} + V(x) = \frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2x^2 \qquad \text{....(8)}$$

The time-independent Schrodinger wave equation for the LHO is,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi \quad \text{or,} \quad \frac{d^2\psi}{dx^2} + \frac{2mE}{\hbar^2}\psi - \frac{2m}{\hbar^2}\left(\frac{1}{2}m\omega^2x^2\right)\psi = 0$$

$$\Rightarrow \quad \frac{d^2\psi}{dx^2} + \left(\frac{2mE}{\hbar^2}\right)\psi - \left(\frac{m^2\omega^2}{\hbar^2}\right)x^2\psi = 0 \qquad \text{....(9)}$$

Equation may be expressed in a simpler form by introducing a dimensional independent variable q such that,

$$q=\sqrt{\frac{m\omega}{\hbar}}x=\sqrt{\alpha}x,\ \left[\text{where, } \alpha=\frac{m\omega}{\hbar}\right] \quad \text{or, } x=\sqrt{\frac{\hbar}{m\omega}}q \qquad ....(10)$$

$$\therefore \quad \frac{d\psi}{dx}=\frac{d\psi}{dq}\frac{dq}{dx}=\frac{d\psi}{dq}\sqrt{\frac{m\omega}{\hbar}} \quad \text{or,} \quad \frac{d^2\psi}{dx^2}=\frac{d^2\psi}{dq^2}\frac{dq}{dx}\sqrt{\frac{m\omega}{\hbar}}=\frac{d^2\psi}{dq^2}\frac{m\omega}{\hbar} \quad \text{or,} \quad \frac{d^2\psi}{dx^2}=\frac{m\omega}{\hbar}\frac{d^2\psi}{dq^2}$$

Now from equations (9) we get,

$$\frac{m\omega}{\hbar}\frac{d^2\psi}{dq^2}+\left(\frac{2mE}{\hbar^2}-\frac{m^2\omega^2}{\hbar^2}\frac{\hbar}{m\omega}q^2\right)\psi=0 \quad \text{or,} \quad \frac{m\omega}{\hbar}\frac{d^2\psi}{dq^2}+\left(\frac{2mE}{\hbar^2}-\frac{m\omega}{\hbar}q^2\right)\psi=0$$

$$\Rightarrow \frac{d^2\psi}{dq^2}+\left(\frac{2E}{\hbar\omega}-q^2\right)\psi=0 \quad \text{or,} \quad \frac{d^2\psi}{dq^2}+(\lambda-q^2)\psi=0 \ \text{ where, } \lambda=\frac{2E}{\hbar\omega} \qquad ....(11)$$

Asymptotic Solution:

For large value of q such that $q^2 \gg \lambda$, we may neglect λ in equation (11), then it reduces to

$$\frac{d^2\psi}{dq^2}-q^2\psi=0 \qquad ....(12)$$

The solution of the above equation is, $\psi(q)=e^{\pm q^2/2}$ which may be verified as follows:

Differentiating equation twice with respect to q we get,

$$\frac{d^2\psi}{dq^2}=q^2e^{\pm q^2/2}\pm e^{\pm q^2/2}=(q^2\pm 1)e^{\pm q^2/2}$$

Since q is very large, $q^2\pm 1 \simeq q^2 \ \therefore \ \frac{d^2\psi}{dq^2}=q^2e^{\pm q^2/2}=q^2\psi$ or, $\frac{d^2\psi}{dq^2}-q^2\psi=0$

which is the same as equation (12).

Now, the probability of finding the LHO along the x-axis is $|\psi(q)|^2$ and for $\psi=e^{q^2/2}$, $|\psi(q)|^2 \to \infty$ as $q \to e^{q^2/2}$. Hence, $\psi(q)=\infty$ is not well behaved and as much this solution is rejected. On the other hand, $|\psi(q)|^2 \to 0$ as $q\to\infty$ in $\psi(q)=e^{-q^2/2}$. So $\psi(q)=e^{-q^2/2}$ is an asymptotic solution of the wave equation.

Let us now try with a possible solution, $\psi(q)=e^{-q^2/2}\phi(q)$ where, $\phi(q)$ is a function of q.

Differentiating equation twice with respect to q we get,

$$\frac{d^2\psi}{dq^2}=e^{-q^2/2}\left[\frac{d^2\phi}{dq^2}-2q\frac{d\phi}{dq}+(q^2-1)\phi\right] \qquad ....(13)$$

Now from equations we get,

$$e^{-q^2/2}\left[\frac{d^2\phi}{dq^2}-2q\frac{d\phi}{dq}+(q^2-1)\phi\right]+e^{-q^2/2}(\lambda-q^2)\psi=0 \text{ or, } \quad e^{-q^2/2}\left[\frac{d^2\phi}{dq^2}-2q\frac{d\phi}{dq}+(\lambda-1)\phi\right]=0$$

But, $e^{-q^2/2}\neq 0$ except for $q\to\pm\infty$

So, $$\frac{d^2\phi}{dq^2}-2q\frac{d\phi}{dq}+(\lambda-1)\phi=0 \qquad ....(14)$$

Power Series Solution of Equation:

The power series solution of equation may be assumed as,

$$\phi(q)=q^s\sum_{r=0}^{\infty}a_r q^r=\sum_r a_r q^{s+r} \qquad ....(15)$$

Differentiating equation with respect to q we get,

$$\frac{d\phi}{dq}=\sum_r a_r(s+r)q^{s+r-1} \text{ and } \frac{d^2\phi}{dq^2}=\sum_r a_r(s+r)(s+r-1)q^{s+r-2}$$

Using these values of $\frac{d\phi}{dq}$ and $\frac{d^2\phi}{dq^2}$ in equation we get,

$$\sum_r a_r(s+r)(s+r-1)q^{s+r-2}-2q\sum_r a_r(s+r)q^{s+r-1}+(\lambda-1)\sum_r a_r q^{s+r}=0$$

$$\Rightarrow \quad \sum_r a_r(s+r)(s+r-1)q^{s+r-2}-\sum_r a_r\{2(s+r)-(\lambda-1)\}q^{s+r}=0$$

The power series expressed by equation will be satisfied only if the coefficient of each power of q vanishes separately.

Equating the coefficient of q^{s-2} to zero, we get, $a_0 s(s-1)=0$, $\because a_0\neq 0, s=0$ or, $(s-1)=0$

Equating the coefficient of q^{s+1} to zero, $a_1(s+1)s=0$

$\therefore$ either, $a_1=0$ or, $s=0$ or, $s=-1$. But, as $s\neq -1$

$\therefore$ either, $a_1=0$ or, $s=0$ or, both.

Similarly, equating the coefficient of q^{s-r} to zero,

$$a_{r+2}(s+r+2)(s+r+1)-a_r[2(s+r)-(\lambda-1)]=0$$

$$\Rightarrow \quad a_{r+2}=\frac{2s+2r+1-\lambda}{(s+r+2)(s+r+1)}a_r \qquad ....(16)$$

Equation is called the recursion formula. From this equation the coefficients of each even power of q, i.e. $q^2, q^4, q^6, \ldots$ can be calculated if q_0 is known. Similarly, the coefficients of $q^3, q^5, q^7, \ldots.$ can be found out if q_1 is known.

Examining the Validity of the Series Solution:

From equation we note that for large value of r, $\lim\limits_{r\to\infty}\dfrac{a_{r+2}}{a_r}=\dfrac{2}{r}$

Again, considering the series expansion,

$$e^{q^2}=1+q^2+\frac{q^4}{2!}+.....+\frac{q^n}{\left(\frac{r}{2}\right)!}+\frac{q^{r+2}}{\left(\frac{r}{2}+1\right)!}+.... = 1+b_2q^2+b_4q^4+....+b_rq^r+b_{r+2}r^{r+2}+....$$

The recursion formula for the expansion of e^{q^2} is,

$$\frac{b_{r+2}}{b_r}=\frac{\left(\frac{r}{2}\right)!}{\left(\frac{r}{2}+1\right)!}=\frac{1}{\frac{r}{2}+1}=\frac{2}{r+2}, \qquad \therefore\ \lim_{r\to\infty}\frac{b_{r+2}}{b_r}=\frac{2}{r}$$

It is, therefore, clear that for large value of n the series for $\phi(q)$ and e^{q^2} behave similarly. Then the corresponding eigenfunction is, $\psi(q)=e^{-q^2/2}\phi(q)=e^{-q^2/2}e^{q^2}=e^{q^2/2}$

Equation shows that for $q\to\infty$, $\psi(q)\to\infty$ and hence the wave function $\psi(q)$ is inadmissible as it does not satisfy the well behaved character of the function.

Eigenvalues of the LHO:

The requirement of the well behaved solution imposes the condition that the series represented by equation must terminate after the rth term. The condition for this is,

$$2s+2r+1-\lambda=0 \text{ or, } \lambda=2s+2r+1$$

$$\Rightarrow \quad \lambda=2(s+r)+1=2n+1 \text{ [where, } n=s+r \text{ and } n=0,1,2,.....]$$

$$\Rightarrow \quad \frac{2E_n}{\hbar\omega}=2n+1\left[\because \lambda=\frac{2E}{\hbar\omega}\right] \Rightarrow E_n=(2n+1)\frac{\hbar\omega}{2}$$

$$\Rightarrow \quad \boxed{E_n=\left(n+\tfrac{1}{2}\right)\hbar\omega} \qquad ....(17)$$

where, $\omega=2\pi\nu$, the angular frequency of the LHO and n = 0, 1, 2, ...

The equation leads to the fact that there is a discrete set of energy eigenvalues which are equispaced in the interval of $\hbar\omega$.

For $n=0$, we get, $E_0=\dfrac{\hbar\omega}{2}=\dfrac{h\nu}{2}$, where E_0 is called the zero point (or, null point) energy of the oscillator.

Wave Functions for the LHO:

Putting $\lambda=2n+1$ in equation we get, $\dfrac{d^2\phi}{dq^2}-2q\dfrac{d\phi}{dq}+2nq=0$ ....(18)

This is known as the Hermite differential equation. We can prove that the solution of equation is of the form,

$$\phi(q)=e^{q^2}\left(\frac{d}{dq}\right)^n e^{-q^2} \qquad ....(19)$$

Multiplying equation by $(-1)^n$ we get Hermite polynomial of order n,

$$H_n(q)=(-1)\phi(q)=(-1)^n e^{q^2}\left(\frac{d}{dq}\right)^n e^{-q^2} \qquad ....(20)$$

The following are a few Hermite polynomials for even and odd values of n:

$n=0,\ H_0(q)=1$ $\quad$ $n=1,\ H_1(q)=2q$

$n=2,\ H_2(q)=4q^2-2$ $\quad$ $n=3,\ H_3(q)=8q^3-12q$

$n=4,\ H_4(q)=16q^4-48q^2+12$ $\quad$ $n=5,\ H_5(q)=32q^5-160q^3+120q$

Now the eigenfunction for the LHO is given by, $\psi_n(q)=N_n e^{-q^2/2}H_n(q)$ ....(21)

Here, N_n is the normalization constant. N_n can be found out from the condition of normalization as follows:

$$\int_{-\infty}^{+\infty}|\psi_n(x)|^2dx=1 \quad \text{or,} \quad \frac{|N_n|^2}{\sqrt{\alpha}}\int_{-\infty}^{+\infty}H_n^2(q)e^{-q^2}dq=1$$

But, $\displaystyle\int_{-\infty}^{+\infty}H_n^2(q)e^{-q^2/2}dq=2^n(n!)\sqrt{\pi}$; So, $\displaystyle\frac{N_n^2}{\sqrt{\alpha}}.2^n(n!)\sqrt{\pi}=1$

$$\Rightarrow \quad N_n=\left[\frac{\sqrt{\alpha}}{2^n n!\sqrt{\pi}}\right]^{1/2} \qquad ....(22)$$

Thus, the normalized wave function for the LHO is,

$$\psi_n(x)=\left[\frac{\sqrt{\alpha}}{2^n n!\sqrt{\pi}}\right]^{1/2}e^{-\alpha x^2/2}H_n\left(\sqrt{\alpha}\,x\right) \qquad ....(23)$$

$$\Rightarrow \quad \psi_n(x)=\left(\frac{\alpha}{\pi}\right)^{1/4}(2^n n!)^{-1/2}e^{-\alpha x^2/2}H_n\left(\sqrt{\alpha}\,x\right) \qquad ....(24)$$

This is the normalized eigenfunction of the oscillator in the nth state. The wave function corresponding to the lowest energy state, i.e., $n=0$ is,

$$\psi_0(x)=\left(\frac{\alpha}{\pi}\right)^{1/4}e^{-\alpha x^2/2} \quad [\because H_0(\sqrt{\alpha}\,x)=1] \qquad ....(25)$$

•The table below shows first few wave functions of the oscillator:

Table : First few wave function of the LHO

n	$\psi_n(x)$
0	$\psi_0(x) = (\alpha/\pi)^{1/4} e^{-\alpha x^2/2}$
1	$\psi_1(x) = (\alpha/\pi)^{1/4} \sqrt{2}\sqrt{\alpha}\, x e^{-\alpha x^2/2}$
2	$\psi_2(x) = (\alpha/\pi)^{1/4} (1/\sqrt{2})(2\alpha x^2 - 1)e^{-\alpha x^2/2}$
3	$\psi_3(x) = (\alpha/\pi)^{1/4} (1/\sqrt{3})[2(\alpha\sqrt{x})^3 - 2\sqrt{\alpha}\, x]e^{-\alpha x^2/2}$
4	$\psi_4(x) = 1/2(\alpha/\pi)^{1/4} (1/\sqrt{6})[4(\sqrt{\alpha}\, x)^4 - 12(\sqrt{\alpha}\, x)^2 + 3]e^{-\alpha x^2/2}$
5	$\psi_5(x) = 1/2(\alpha/\pi)^{1/4} (1/\sqrt{15})[4(\sqrt{\alpha}\, x)^5 - 20(\sqrt{\alpha}\, x)^3 + 15(\sqrt{\alpha} x)]e^{-\alpha x^2/2}$

It is to be noted that $\psi_0(x)$ has no node for finite q. As n increases, the number of nodes progressively increases with n. For example, $\psi_1(x)$ has a single node and $\psi_2(0)$ has two nodes as shown in figure.

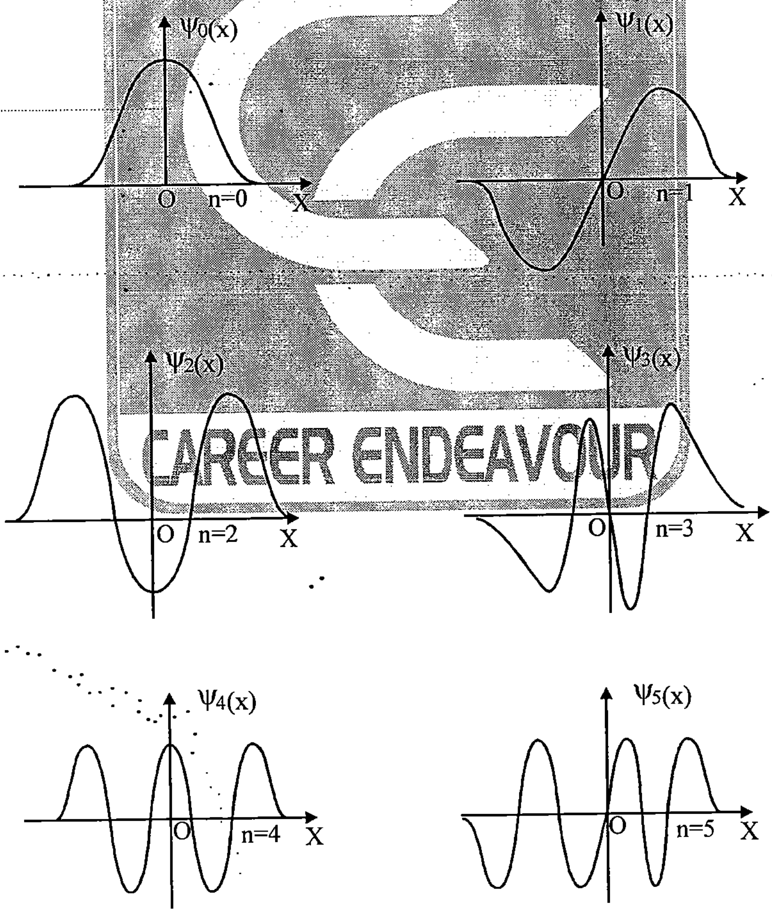


Fig. Harmonic oscillator wave functions for different n

Operator formalism of harmonic Oscillator:

Hamiltonian of the system is $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$

Defining two dimensionless operators

$$\hat{P} = \frac{\hat{p}}{\sqrt{m\hbar\omega}} \text{ and } \hat{X} = \hat{x}\sqrt{\frac{m\omega}{\hbar}}$$

The hamiltonian can be written as, $\hat{H} = \frac{\hbar\omega}{2}\left[\frac{\hat{p}^2}{m\hbar\omega} + \frac{m\omega}{2\hbar}\hat{x}^2\right] = \frac{\hbar\omega}{2}\left[\hat{P}^2 + \hat{X}^2\right]$

Introducing two non-hermitian operators, $\hat{a} = \frac{1}{\sqrt{2}}\left[\hat{X} + i\hat{P}\right]$, $\hat{a}^\dagger = \frac{1}{\sqrt{2}}\left[\hat{X} - i\hat{P}\right]$

where $\hat{a}$ and $\hat{a}^\dagger$ are hermitian conjugate to each other.

Now, $\hat{a}^\dagger\hat{a} = \frac{1}{2}\left[\hat{X}^2 + \hat{P}^2 + i\hat{X}\hat{P} - i\hat{P}\hat{X}\right] = \frac{1}{2}\left(\hat{P}^2 + \hat{X}^2\right) + \frac{i}{2}\left[\hat{X}, \hat{P}\right]$

$$\Rightarrow \hat{a}^\dagger\hat{a} = \frac{\hat{H}}{\hbar\omega} - \frac{1}{2} \Rightarrow \hat{H} = \left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)\hbar\omega \quad \Rightarrow \hat{H} = \left(\hat{N} + \frac{1}{2}\right)\hbar\omega$$

where, $\hat{N} = \hat{a}^\dagger\hat{a}$ = number operator/occupation number operator.

- **Some important commutation relations:**

(i) $\left[\hat{a}, \hat{a}^\dagger\right] = 1$ (ii) $\left[\hat{a}^\dagger\hat{a}, \hat{a}^\dagger\right] = \hat{a}^\dagger$ (iii) $\left[\hat{a}^\dagger\hat{a}, \hat{a}\right] = -\hat{a}$

(iv) $\left[\hat{H}, \hat{a}^\dagger\hat{a}\right] = 0$ (v) $\left[\hat{a}, \hat{H}\right] = \hbar\omega\hat{a}$ (vi) $\left[\hat{a}^\dagger, \hat{H}\right] = -\hbar\omega\hat{a}^\dagger$

- **Lowering operator and Raising operator:**

$\hat{H}\left(\hat{a}|n\rangle\right) = \left(\hat{a}\hat{H} - \hbar\omega\hat{a}\right)|n\rangle = \hat{a}E_n|n\rangle - \hbar\omega\hat{a}|n\rangle = (E_n - \hbar\omega)\left(\hat{a}|n\rangle\right)$

So, that, $\hat{a}|n\rangle$ is the eigenstates of $\hat{H}$ operator corresponding eigenvalue $(E_n - \hbar\omega)$. Action of $\hat{a}$ on $|n\rangle$ generate new energy states that is lower by one unit energy $\hbar\omega$ i.e.

$$\hat{a}|n\rangle = C_n|n-1\rangle \quad \Rightarrow \langle n|\hat{a}^\dagger = \langle n-1|c_n^*$$

Therefore, $\left\langle n\left|\hat{a}^\dagger\hat{a}\right|n\right\rangle = \langle n-1|n-1\rangle|c_n|^2 \Rightarrow n = |c_n|^2 \Rightarrow |c_n| = \sqrt{n}$

So, $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$

Similarly, $\hat{H}\left(\hat{a}^\dagger|n\rangle\right) = \left(\hat{a}^\dagger\hat{H} + \hbar\omega\hat{a}^\dagger\right)|n\rangle = (E_n + \hbar\omega)\left(\hat{a}^\dagger|n\rangle\right)$

So, that, $\hat{a}^\dagger|n\rangle$ are the eigenstates of $\hat{H}$ operator corresponding eigenvalue $(E_n + \hbar\omega)$. Action of $\hat{a}^\dagger$ on $|n\rangle$ generate new energy states that is higher by one unit energy $\hbar\omega$.

$$\hat{a}^\dagger|n\rangle = c_n|n+1\rangle \quad \Rightarrow \langle n|\hat{a} = \langle n+1|c_n^*$$

Therefore, $\left\langle n\left|\hat{a}\hat{a}^{\dagger}\right|n\right\rangle = |c_n|^2 \langle n+1 \mid n+1\rangle \Rightarrow (n+1) = |c_n|^2 \Rightarrow |c_n| = \sqrt{n+1}$

So,
$$\hat{a}^{\dagger}|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

Therefore, the nth state can be written in terms of ground state as

$$|n\rangle = \frac{\left(\hat{a}^{\dagger}\right)^n}{\sqrt{n!}}|0\rangle$$

Calculation of normalized wave function:

Ground state wavefunction:

Since, action of $\hat{a}$ on $|n\rangle$ generate new energy states that is lower by one unit energy $\hbar\omega$, then

$$\hat{a}|0\rangle = 0$$

$$\Rightarrow \left\{\sqrt{\frac{m\omega}{2\hbar}}\hat{x} + i\frac{\hat{p}}{\sqrt{2m\omega\hbar}}\right\}\psi_0 = 0 \Rightarrow \sqrt{\frac{m\omega}{2\hbar}}x\psi_0 + \frac{\hbar}{\sqrt{2m\omega\hbar}}\psi_0 = 0$$

$$\Rightarrow \frac{d\psi_0}{dx} = -\frac{m\omega}{\hbar}x\psi_0 \Rightarrow \frac{d\psi_0}{\psi_0} = -\frac{m\omega}{\hbar}x\,dx \Rightarrow \ell n\psi_0 = -\frac{m\omega}{\hbar}\frac{x^2}{2} + \ell nA$$

$$\Rightarrow \psi_0(x) = A\exp\left(-\frac{m\omega}{\hbar}\frac{x^2}{2}\right) \Rightarrow \psi_0(x) = A\exp\left(-\frac{\alpha^2x^2}{2}\right) \text{ [where } \alpha^2 = \frac{m\omega}{\hbar}\text{]}$$

Normalization Condition: $\int_{-\infty}^{\infty}\psi_0^*(x)\psi_0(x)\,dx = 1$

$$\Rightarrow 2|A|^2\int_0^{\infty}\exp\left(-\alpha^2x^2\right)dx = 1 \Rightarrow A = \left(\frac{\alpha}{\sqrt{\pi}}\right)^{1/2}$$

Therefore, normalized ground state wavefunction will be

$$\psi_0(x) = \left(\frac{\alpha}{\sqrt{\pi}}\right)^{1/2} e^{-\alpha^2x^2/2}$$

First excited state wavefunction:

Since, $\hat{a}^{\dagger}|0\rangle = |1\rangle$

$$\Rightarrow \quad |1\rangle = \left[\sqrt{\frac{m\omega}{2\hbar}}\hat{x} - \frac{i}{\sqrt{2m\omega\hbar}}\hat{p}\right]|0\rangle$$

$$= \left[\sqrt{\frac{m\omega}{2\hbar}}\hat{x} - \frac{i}{\sqrt{2m\omega\hbar}}\left(i\hbar\frac{\partial}{\partial x}\right)\right]|0\rangle$$

$$= \sqrt{\frac{m\omega}{2\hbar}}x\left[\frac{\alpha}{\sqrt{\pi}}\right]^{1/2} e^{\frac{-\alpha^2x^2}{2}} - \frac{\hbar}{\sqrt{2m\omega\hbar}}\left[\frac{\alpha}{\sqrt{\pi}}\right]^{1/2} e^{-\alpha^2x^2/2}\left(-\alpha^2x\right)$$

$$= \left[\frac{\alpha}{\sqrt{4}.\sqrt{\pi}}\right]^{1/2} e^{-\frac{\alpha^2x^2}{2}}(2\alpha x) = \left[\frac{\alpha}{2\sqrt{\pi}}\right]^{1/2} e^{-\frac{\alpha^2x^2}{2}} H_1(\alpha x)$$

Expectation value of a odd operator for a symmetric potential:

The wave function of a particle is moving under a symmetric potential $V(-x)=V(x)$, has a definite parity i.e it will be of either even or odd parity.

Therefore, expectation value of an odd operator $=\langle \hat{A}\rangle=\langle\psi|\hat{A}|\psi\rangle=0$

Similarly, $\langle \hat{A}^m\rangle=\langle\psi|\hat{A}^m|\psi\rangle=0\ (m=\text{odd integer})$

Calculation of expectation value using operator formalism:

As we know, $\hat{a}=\sqrt{\frac{m\omega}{2\hbar}}\hat{x}+\hat{p}\frac{i}{\sqrt{2m\omega\hbar}}$ and $\hat{a}^\dagger=\sqrt{\frac{m\omega}{2\hbar}}\hat{x}-\hat{p}\frac{i}{\sqrt{2m\omega\hbar}}$,

therefore we get,

$$\hat{x}=\sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}+\hat{a}^\dagger\right) \quad \text{and} \quad \hat{p}=\frac{1}{i}\sqrt{\frac{\hbar m\omega}{2}}\left(\hat{a}-\hat{a}^\dagger\right)$$

$$\langle\hat{x}\rangle=\sqrt{\frac{\hbar}{2m\omega}}\left\langle n\left|\left(\hat{a}+\hat{a}^\dagger\right)\right|n\right\rangle=\sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n}\langle n|n-1\rangle+\sqrt{n+1}\langle n|n+1\rangle\right]=0$$

$$\langle\hat{p}\rangle=\frac{1}{i}\sqrt{\frac{\hbar m\omega}{2}}\left\langle n\left|\hat{a}-\hat{a}^\dagger\right|n\right\rangle=\frac{1}{i}\sqrt{\frac{\hbar m\omega}{2}}\left[\sqrt{n}\langle n|n-1\rangle-\sqrt{n+1}\langle n|n+1\rangle\right]=0$$

$$\langle\hat{x}^2\rangle=\frac{\hbar}{2m\omega}\left\langle n\left|\left(\hat{a}+\hat{a}^\dagger\right)\left(\hat{a}^\dagger+\hat{a}\right)\right|n\right\rangle=\frac{\hbar}{2m\omega}\left\langle n\left|\left(\hat{a}^2+\hat{a}^{\dagger 2}+\hat{a}\hat{a}^\dagger+\hat{a}^\dagger\hat{a}\right)\right|n\right\rangle$$

$$=\frac{\hbar}{2m\omega}\left[\sqrt{n}\sqrt{n-1}\langle n|n-2\rangle+\sqrt{n+1}\sqrt{n+2}\langle n|n+1\rangle+\sqrt{n+1}\sqrt{n+1}\langle n|n\rangle+\sqrt{n}\sqrt{n}\langle n|n\rangle\right]$$

$$=\frac{\hbar}{2m\omega}(2n+1)=\frac{\hbar}{m\omega}\left(n+\frac{1}{2}\right)$$

Similarly, $\langle p^2\rangle=\left(n+\frac{1}{2}\right)m\hbar\omega$

Uncertainity in position $\Delta x=\sqrt{\langle x^2\rangle-\langle x\rangle^2}=\sqrt{\frac{\hbar}{m\omega}\left(n+\frac{1}{2}\right)}$

Uncertainity in momentum $\Delta p=\sqrt{\langle p^2\rangle-\langle p\rangle^2}=\sqrt{\left(n+\frac{1}{2}\right)\hbar m\omega}$

Therefore, the uncertainity product becomes $\Delta x\Delta p=\left(n+\frac{1}{2}\right)\hbar$

Example 1: Let $|0\rangle$ and $|1\rangle$ denote the normalized eigenstates corresponding to the ground and first state of a 1-D harmonic oscillator. The uncertainty Δp in the state $\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$ is:

(a) $\frac{\sqrt{m\omega\hbar}}{2}$ (b) $\sqrt{\frac{m\omega\hbar}{2}}$ (c) $\sqrt{m\omega\hbar}$ (d) $\sqrt{2m\omega\hbar}$

Soln. $\langle \hat{p}\rangle = \frac{1}{2}\left[\langle 0|\hat{p}|0\rangle + \langle 0|\hat{p}|1\rangle + \langle 1|\hat{p}|0\rangle + \langle 1|\hat{p}|1\rangle\right] = \frac{1}{2}\left[0 + \frac{1}{i}\sqrt{\frac{m\omega\hbar}{2}} - \frac{1}{i}\sqrt{\frac{m\omega\hbar}{2}} + 0\right] = 0$

$$\langle \hat{p}^2\rangle = \frac{1}{2}\left[\langle 0|\hat{p}^2|0\rangle + \langle 0|\hat{p}^2|1\rangle + \langle 1|\hat{p}^2|0\rangle + \langle 1|\hat{p}^2|1\rangle\right] = \frac{1}{2}\left[m\hbar\omega\frac{1}{2} + 0 + 0 + \frac{3}{2}m\hbar\omega\right] = m\omega\hbar$$

Uncertainity in the momentum in the given state will be

$$\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\hbar m\omega}$$

Example 2: The wave function of a linear harmonic oscillator in the potential $V(x) = \frac{1}{2}m\omega^2x^2$ at a given instant is $\psi(x) = \left(\frac{2}{\pi\beta^2}\right)^{1/4} e^{-x^2/\beta^2}$. For what value of β will the function $a\psi(x)$ be zero?

Soln. $a = \frac{1}{\sqrt{2}}\left(\hat{X} + i\hat{P}_x\right) = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}x + i\frac{p_x}{\sqrt{m\hbar\omega}}\right) = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}x + \sqrt{\frac{\hbar}{m\omega}}\frac{d}{dx}\right)$

Thus, $a\psi(x) = \frac{1}{\sqrt{2}}\left(\frac{2}{\pi\beta^2}\right)^{1/4}\left(\sqrt{\frac{m\omega}{\hbar}}x + \sqrt{\frac{\hbar}{m\omega}}\frac{d}{dx}\right)e^{-x^2/\beta^2}$

$$= \left(\frac{1}{2\pi\beta^2}\right)^{1/4}\left(\sqrt{\frac{m\omega}{\hbar}}x + \sqrt{\frac{\hbar}{m\omega}}\left(-\frac{2x}{\beta^2}\right)\right)e^{-x^2/\beta^2}$$

$$= \left(\frac{1}{2\pi\beta^2}\right)^{1/4}\left(\sqrt{\frac{m\omega}{\hbar}} - \frac{2}{\beta^2}\sqrt{\frac{\hbar}{m\omega}}\right)xe^{-x^2/\beta^2}$$

This will be zero if $\beta = \sqrt{\frac{2\hbar}{m\omega}}$.

Example 3. Consider a particle of mass 'm' and charge 'q' placed in a uniform electric field $E = E_0\hat{i}$. Apart from the force due to the electric field, a restoring force corresponding to the potential $V_1(x) = \frac{1}{2}m\omega^2x^2$ acts on the particle. Find the energy eigenvalues of the particle, taking the electric potential energy to be zero at the origin.

Soln. The electric potential enregy at the position 'x' will be $-qE_0x$. Thus, the net potential is

$$V(x) = \frac{1}{2}m\omega^2x^2 - qE_0x = \frac{1}{2}m\omega^2\left(x^2 - \frac{2qE_0}{m\omega^2}x\right)$$

$$= \frac{1}{2}m\omega^2\left(x - \frac{qE_0}{m\omega^2}\right)^2 - \frac{q^2E_0^2}{2m\omega^2}$$

If you shift the origin of 'x' to $\frac{qE_0}{m\omega^2}$ and define $x'=\left(x-\frac{qE_0}{m\omega^2}\right)$, the potential as a function of x' is

$$V(x')=\frac{1}{2}m\omega^2x'^2-\frac{q^2E_0^2}{2m\omega^2}$$

This is the potential of a linear harmonic oscillator with energy $-\frac{q^2E_0^2}{2m\omega^2}$ at $x'=0$. This is a constant shift in the energy eigenvalues are $\left(n+\frac{1}{2}\right)\hbar\omega-\frac{q^2E_0^2}{2m\omega^2}$.

Example 4. A linear harmonic oscillator is in its ground state. Find the range of 'x' that is allowed classically. Calculate the probability of finding it in the classically forbidden region.

Soln. For classically allowed region, the total energy E should be greater than or equal to the potential energy. For linear harmonic oscillator in ground state, this region is given by

$$\frac{\hbar\omega}{2}\geq\frac{1}{2}m\omega^2x^2\Rightarrow x^2\leq\frac{\hbar}{m\omega}\Rightarrow-\sqrt{\frac{\hbar}{m\omega}}\leq x\sqrt{\frac{\hbar}{m\omega}}$$

The probability of finding the oscillator in this range is

$$\int_{-\sqrt{\hbar/m\omega}}^{\sqrt{\hbar/m\omega}}\left|\psi_0(x)^2\right|^2dx=2\times\int_0^{\sqrt{\hbar/m\omega}}\sqrt{\frac{m\omega}{\pi\hbar}}e^{-\left(\frac{m\omega}{\hbar}\right)x^2}dx$$

Assuming, $t=\sqrt{\frac{m\omega}{\hbar}}x$, the above expression becomes $2\sqrt{\frac{m\omega}{\pi\hbar}}\sqrt{\frac{\hbar}{m\omega}}\int_0^1e^{-t^2}dt\approx0.84$

Thus the probability of finding the oscillator in the classically forbidden region is $1-0.84=0.16$.

Example 5. A particle of mass 'm' moves in the potential $V(x)=\frac{1}{2}m\omega^2x^2$. It has wave function $\psi(x)$ at time t = 0. What will be the wave function at $t=\frac{2\pi}{\omega}$ (one time period)?

Soln. Let the wave function at $t=0$ be $\psi(x)=\sum_n C_n|\phi_n\rangle$

The wave function at time t will be

$$\psi(x,t)=\sum_n C_n|\phi_n\rangle e^{-\frac{i}{\hbar}E_nt}=\sum_n C_n|\phi_n\rangle e^{-\frac{i}{\hbar}\left(n+\frac{1}{2}\right)\hbar\omega t}$$

$$=\sum_n C_n|\phi_n\rangle e^{-i\left(n+\frac{1}{2}\right)2\pi}=\sum_n C_n|\phi_n\rangle e^{-i\pi}=-\sum_n C_n|\phi_n\rangle=-\psi(x)$$

In one time period the wave function becomes negative of its original form. In two time periods, it regains its original shape.

Example 6. A linear harmonic oscillator is in the first excited state. (i) At what point is its probability density maximum? (ii) What is the value of maximum probability density?

oln. The harmonic oscillator wave function in the $n = 1$ state is

$$\psi_1(x) = \left(\frac{\alpha}{2\sqrt{\pi}}\right)^{1/2} 2\alpha x \exp\left(\frac{-\alpha^2 x^2}{2}\right) \quad \text{where } \alpha = \left(\frac{m\omega}{\hbar}\right)^{1/2}$$

(i) Probability density $P(x) = \psi\psi^* = \frac{2\alpha^3}{\sqrt{\pi}} x^2 \exp(-\alpha^2 x^2)$

$P(x)$ is maximum when $dP/dx = 0 \Rightarrow \frac{2\alpha^3}{\sqrt{\pi}}(2x - 2\alpha^2 x^3) = 0 \ \Rightarrow x = \pm\frac{1}{\alpha}$

(ii) Maximum value of $P(x) = \frac{2\alpha}{\sqrt{\pi}}\frac{1}{e} = \frac{2\alpha}{\sqrt{\pi}}\frac{1}{2.718} = 0.415\alpha$

Example 7. The energy of a linear harmonic oscillator in third excited state is 4.1 eV. The frequency of vibration is ($\mu = 6.62\times10^{-34}$ J-sec) **[BHU 2011]**

(a) $3.3\times10^{13}\ Hz$ (b) $2.8\times10^{13}\ Hz$ (c) $3.9\times10^{13}\ Hz$ (d) $4.2\times10^{13}\ Hz$

Soln. Given: Third excited state energy $= E_3 = \frac{7}{2}h\nu = 4.1eV$

Therefore, Frequency of vibration will be $\nu = \frac{2}{7}\frac{E_3}{h} = \frac{2}{7}\times\frac{4.1\times1.6\times10^{-19}}{6.6\times10^{-34}} = 0.28\times10^{15}\ Hz$

Correct option is ( )

Example 8. Let $|n\rangle$ represent the normalized n^{th} energy eigenstate of the one-dimensional harmonic oscillator

$$H|n\rangle = \hbar\omega\left(n+\frac{1}{2}\right)|n\rangle$$

If $|\psi\rangle$ is a normalized ensemble state that can be expanded as a linear combination **[BHU 2012]**

$$|\psi\rangle = \frac{1}{\sqrt{14}}|1\rangle - \frac{2}{\sqrt{14}}|2\rangle + \frac{3}{\sqrt{14}}|3\rangle$$

of the eigenstates, what is the expectation value of the energy operator in this ensemble state ?

(a) $\frac{102}{14}\hbar\omega$ (b) $\frac{43}{14}\hbar\omega$ (c) $\frac{23}{14}\hbar\omega$ (d) $\frac{17}{14}\hbar\omega$

Soln. Given: $H|n\rangle = \hbar\omega\left(n+\frac{1}{2}\right)|n\rangle$ and $|\psi\rangle = \frac{1}{\sqrt{4}}|1\rangle - \frac{2}{\sqrt{14}}|2\rangle + \frac{3}{\sqrt{14}}|3\rangle$

Therefore, $\langle H\rangle = \langle n|H|n\rangle$

$$= \left(\frac{1}{\sqrt{14}}\langle1| - \frac{2}{\sqrt{14}}\langle2| + \frac{3}{\sqrt{14}}\langle3|\right) H \left(\frac{1}{\sqrt{14}}|1\rangle - \frac{2}{\sqrt{14}}|2\rangle + \frac{3}{\sqrt{14}}|3\rangle\right)$$

$$= \left(\frac{1}{\sqrt{14}}\langle1| - \frac{2}{\sqrt{14}}\langle2| + \frac{3}{\sqrt{14}}\langle3|\right)\left(\frac{3}{2}\frac{\hbar\omega}{\sqrt{14}}|1\rangle - \frac{5}{2}\hbar\omega\frac{2}{\sqrt{14}}|2\rangle + \frac{7}{2}\hbar\omega\frac{3}{\sqrt{14}}|3\rangle\right)$$

$$= \left(\frac{3}{2\times14} + \frac{10}{14} + \frac{63}{2\times14}\right)\hbar\omega = \left(\frac{3+20+63}{28}\right)\hbar\omega = \frac{43}{14}\hbar\omega.$$

Correct option is (b)

Example 9. The wave function $\psi(x) = A\exp\left\{-\frac{b^2x^2}{2}\right\}$ where A and b are real constants, is a normalized eigenfunction of the Schrödinger equation for a particle of mass M and energy E in a one-dimensional potential $V(x)$ such that $V(x) = 0$ at $x = 0$. Which of the following is correct? **[BHU 2012]**

(a) $V = \frac{\hbar^2 b^4}{2M}$ (b) $V = \frac{\hbar^2 b^4 x^2}{2M}$ (c) $V = \frac{\hbar^2 b^6 x^4}{2M}$ (d) $E = \hbar^2 b^2(1 - b^2x^2)$

Soln. Given : $\psi(x) = A\exp\left\{-\frac{b^2x^2}{2}\right\}$

The Schrödinger equation is $-\frac{\hbar^2}{2M}\frac{d^2}{dx}\psi(x) + V\,\psi(x) = E\,\psi(x)$

$$\Rightarrow -\frac{\hbar^2}{2M}\frac{d}{dx}\left\{-b^2x\,A\exp\left(-\frac{b^2x^2}{2}\right)\right\} + V\,\psi(x) = E\,\psi(x)$$

$$\Rightarrow \frac{\hbar^2b^2}{2M}\left\{1 - b^2x^2\right\}\psi(x) + V\,\psi(x) = E\,\psi(x)$$

$$\Rightarrow E = \frac{\hbar^2b^2}{2M}\left(1 - b^2x^2\right) + V$$

$$\Rightarrow V = E - \frac{\hbar^2b^2}{2M}\left(1 - b^2x^2\right)$$

$\because V(x) = 0$ at $x = 0$, we have $0 = E - \frac{\hbar^2b^2}{2M}$ or $E = \frac{\hbar^2b^2}{2M}$

$$\therefore V = \frac{\hbar^2b^4x^2}{2M}.$$

Correct option is (b)

Example 10. The ground state eigenfunction for a linear harmonic oscillator, in terms of $\alpha = \sqrt{\frac{mk}{\hbar^2}}$ where k = force constant and m = mass of the linear oscillator, is **[BHU 2013]**

(a) $\psi(x) = \left(\frac{\alpha}{\sqrt{\pi}}\right)^{1/4} e^{\alpha x^2/2}$ (b) $\psi(x) = \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\alpha x^2/2}$

(c) $\psi(x) = \left(\frac{\alpha}{\sqrt{\pi}}\right)^{1/2} e^{-\alpha^2x^2/2}$ (d) $\psi(x) = \left(\frac{\alpha}{\sqrt{\pi}}\right)^{1/4} e^{+\alpha^2x^2/2}$

Soln. The ground state eigen function is

$$\psi(x) = \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\frac{\alpha x^2}{2}}$$

Correct option is (b)

Example 11. The ground state energy of the system described by the Hamiltonian $H = \frac{bx^2}{2m} + ax^2 + bx + c$ is:

[BHU 2014]

(a) $\hbar\sqrt{\frac{2a}{m}} + c - \frac{b^2}{4a}$ (b) $\hbar\sqrt{\frac{2a}{m}} - c + \frac{b^2}{4a}$ (c) $\hbar\sqrt{\frac{a}{2m}} - c + \frac{b^2}{4a}$ (d) $\hbar\sqrt{\frac{a}{2m}} + c - \frac{b^2}{4a}$

Soln. The Hamiltonian is

$$H = ax^2 + bx + c \quad = a\left(x + \frac{b}{2}\right)^2 - \frac{b^2}{4a} + c$$

This is the potential of simple harmonic oscillator with a shift of $-\frac{b^2}{4a} + c$.

On comparing with standard harmonic oscillator potential, $H = \frac{1}{2}kx^2 + H_0$

we have $k = 2a$ and $H_0 = -\frac{b^2}{4a} + c$

Since, the minimum energy is $E = \frac{\hbar}{2}\sqrt{\frac{k}{m}} + H_0 = \hbar\sqrt{\frac{a}{2m}} - \frac{b^2}{4a} + c$

Correct option is (c)

Example 12. Consider one sided simple harmonic oscillator described by the potential $V(x) = \frac{1}{2}mw^2x^2,\ 0 \le x \le \infty$. The value of $\langle x \rangle$ in any state will be **[BHU 2014]**

(a) 0 (b) a positive number (c) always be negative (d) can be positive or negative

Soln. Given : $V(x) = \frac{1}{2}m\omega^2x^2,\ 0 \le x \le \infty$

The wave function for this potential is

$$\psi(x) = \sqrt{\frac{\gamma}{2^n n!\sqrt{\pi}}}\, H_n(\xi)\, e^{-\xi^2/2};\ n = 1, 3, 5, \ldots \text{(where } \xi = \gamma x \text{ and } \gamma = \sqrt{\frac{m\omega}{\hbar}}\text{)}$$

$$\therefore\ \langle x \rangle = \int_0^\infty x|\psi(x)|^2\,dx \qquad = \frac{\gamma}{2^n n!\sqrt{\pi}}\int_0^\infty x\left(H_n(\xi)\,e^{-\xi^2/2}\right)^2 dx$$

Taking $n = 1$, we have $\langle x \rangle = \frac{\gamma}{2\sqrt{\pi}}\int_0^\infty x\,H_1^2(\xi)\,e^{-\xi^2}\,dx$

where $H_1(\xi) = 2\xi$

$$\therefore\ \langle x \rangle = \frac{4\gamma}{2\sqrt{\pi}}\int_0^\infty x\,\xi^2\,e^{-\xi^2}\,dx \qquad = \frac{4\gamma^3}{2\sqrt{\pi}}\int_0^\infty x^3\,e^{-\gamma^2x^2}\,dx$$

Using, $\int_0^\infty x^n\,e^{-ax^2}\,dx = \frac{\left|\overline{\frac{n+1}{2}}\right.}{2(a)^{\frac{n+1}{2}}}$

$$\therefore\ \langle x \rangle = \frac{2\gamma^3}{\sqrt{\pi}}\,\frac{\left|\overline{2}\right.}{2(\gamma^2)^2} = \frac{1}{\gamma\sqrt{\pi}} = \sqrt{\frac{\hbar}{m\omega\pi}} > 0$$

Correct option is (b)

Example 13. The wave function for a 1-d simple harmonic oscillator has odd number of real zeros, Which of the following statements is incorrect? **[BHU 2014]**

(a) The wave function is odd under parity

(b) The wave function passes through origin

(c) The wave function does not have definite parity

(d) The state for this wave function is non-degenerate

Soln. Odd number of zeroes means the wave function has odd parity. Therefore, the wavefunction will pass through the origin and the corresponding state will be non-degenerate.

Correct option is (c)

Example 14. State of a one-dimensional simple harmonic oscillators is $\psi(x,t) = \frac{1}{5}\left[3\phi_0 - 2\sqrt{2}\,\phi_1 + 2\sqrt{2}\,\phi_2\right]$, where ϕ_n are the eigen functions of the Hamiltonian with eigen values $E_n = \left(n+\frac{1}{2}\right)\hbar\omega$. The expectation value of the energy in the state $\psi(x,t)$ is : **[BHU 2014]**

(a) $0.46\,\hbar\omega$ (b) $1.46\,\hbar\omega$ (c) $0.5\,\hbar\omega$ (d) 0

Soln. Given : $\psi(x,t) = \frac{1}{5}\left[3\phi_0 - 2\sqrt{2}\,\phi_1 + 2\sqrt{2}\,\phi_2\right]$

$$\langle E\rangle = \left(\frac{3}{5}\right)^2 E_0 + \left(-\frac{2\sqrt{2}}{5}\right)^2 E_1 + \left(\frac{2\sqrt{2}}{5}\right)^2 E_2$$

$$= \frac{9}{25}\frac{\hbar\omega}{2} + \frac{8}{25}\frac{3}{2}\hbar\omega + \frac{8}{25}\frac{5}{2}\hbar\omega = \frac{73}{50}\hbar\omega = 1.46\,\hbar\omega$$

Correct option is (b)

Example 15. For a linear harmonic oscillator, the potential is $V = \frac{1}{2}m\omega^2 x^2$. Using Schrödinger equation with $\psi(x) = xe^{-m\omega x^2/2\hbar}$ one obtains the energy eigenvalue as **[BHU 2015]**

(a) $E = \frac{5}{2}\hbar\omega$ (b) $E = \frac{3}{2}\hbar\omega$ (c) $E = \frac{1}{2}\hbar\omega$ (d) $E = \hbar\omega$

Soln. The wavefunction for a linear harmonic oscillator is

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(\xi) e^{-\xi^{2/2}}, \text{ where } \xi = \sqrt{\left(\frac{m\omega}{\hbar}\right)}\,x$$

For $n = 1$, $\psi_1(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2}} H_1(\xi) e^{-\xi^{2/2}}$, where $H_1(\xi) = 2\xi$

$$\therefore\ \psi_1(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2}} 2\sqrt{\frac{m\omega}{\hbar}}\, x e^{-\frac{m\omega}{2\hbar}x^2}$$

Comparing with $\psi(x) = xe^{-\frac{m\omega x^2}{2\hbar}}$, we have $n = 1$

$$\therefore\ E_n = \left(n+\frac{1}{2}\right)\hbar\omega = \frac{3}{2}\hbar\omega$$

Correct option is (b)

Example 16. The ground state energy of a one-dimensional harmonic oscillator is 6.2 eV. If the oscillator undergoes a transition from its $n=3$ level to $n=2$ level by emitting a photon, what is the wavelength associated with the photon ? **[HCU 2011]**

(a) 50 nm (b) 62 nm (c) 100 nm (d) 200 nm

Soln. The energy of one-dimensional harmonic oscillator is $E_n = \left(n+\frac{1}{2}\right)h\nu$.

For ground state, $n=0$. $\therefore$ $E_0 = \frac{h\nu}{2} = 6.2$ eV (given).

The energy of the photon is $\frac{hc}{\lambda} = E_3 - E_2 = \frac{7}{2}h\nu - \frac{5}{2}h\nu = h\nu$.

$\Rightarrow \frac{hc}{\lambda} = 12.4 \text{ eV} \Rightarrow \lambda = \frac{6.6\times10^{-34}\times3\times10^{8}}{12.4\times1.6\times10^{-19}} \simeq 100 \text{ nm}.$

Correct option is (c)

Example 17. In a harmonic oscillator, if the transition energy from 3rd state to 2nd state is 4.8 eV, what is the ground state energy? **[HCU 2014]**

(a) 1.2 eV (b) 2.4 eV (c) 4.8 eV (d) 0.6 eV

Soln. The energy of the nth state of a harmonic oscillator is given by $E_n = \left(n+\frac{1}{2}\right)\hbar\omega$.

$\therefore$ $E_0 = \frac{\hbar\omega}{2}$ (ground state); $E_1 = \frac{3\hbar\omega}{2}$ (2nd state); $E_2 = \frac{5\hbar\omega}{2}$ (3rd state)

Given : $E_3 - E_2 = 4.8 \text{ eV} \Rightarrow \frac{5}{2}\hbar\omega - \frac{3\hbar\omega}{2} = 4.8 \text{ eV} \Rightarrow \hbar\omega = 4.8 \text{ eV}$

So the ground state energy $= \frac{1}{2}\hbar\omega = 2.4$ eV.

Correct option is (b)

Example 18. A particle of mass m is in a one-dimensional potential given by $V(x)=\infty$ for $x<0$ and $V(x) = kx^2/2$ for $x>0$. The first excited state has an energy **[DU 2015]**

(a) $(3/2)\hbar\sqrt{(k/m)}$ (b) $(7/2)\hbar\sqrt{(k/m)}$ (c) $(1/2)\hbar\sqrt{(k/m)}$ (d) $(5/2)\hbar\sqrt{(k/m)}$

Soln. Given : $V(x) = \begin{cases} \infty & ; \quad x<0 \\ \frac{kx^2}{2} & ; \quad x>0 \end{cases}$

The energy for this one sided one-dimensional linear harmonic oscillator is $E = \left(n+\frac{1}{2}\right)\hbar\sqrt{\frac{k}{m}}$, where n = 1, 3, 5, ...

The first excited state corresponds to $n=3$ whose energy is given by $E_3 = \frac{7}{2}\hbar\sqrt{\frac{k}{m}}$.

Correct option is (b)

Chapter 8

Nuclear Physics

8.1: Nucleons:

All nuclei are composed of two types of particles, positively charged protons and neutral neutrons, referred collectively as nucleons. Their principal properties are listed in Table 7-1.

	Proton	Neutron
Charge	$+1.6 \times 10^{-19}$ C	0C
Rest Mass	1.67252×10^{-27} kg 938.256 MeV 1.007277 u	1.67482×10^{-27} kg 939.550 MeV 1.008 665 u
Spin	1/2	1/2
Magnetic Moment	$+2.7928\,\beta_n$	$-1.9128\,\beta_n$

Atomic nuclei of the same element have the same number of protons but can have different number of neutrons. The electron structure of the atom was understood before even the composition of its nuclear was known. The reason is that the forces that hold the nucleus together are vastly stronger than the electric forces that hold the electrons to its nucleus, and it is correspondingly harder to break apart a nucleus to find out what is inside. Changes in the electron structure of an atom such as those that occur when a photon is emitted or absorbed or when a chemical bond is formed or broken, involve energies of only few electronvolts. Change in nuclear structure. On the other hand, involve energies in MeV, a million times greater.

All the nuclei contain neutron and proton are jointly called nucleon.

The atomic number of an element is the number of protons in each of its atomic nuclei, which is same as the number of electrons in neutral atom of an element. The varieties of element that differ is number of neutron but same number of protons is the nuclei are called its isotopes.

8.2. Basic characteristics of the atomic nucleus

Atomic number, proton number (Z): Number of protons in the atomic nucleus, hence number of electrons in the neutral atom.

Neutron number (N): Number of neutrons in the atomic nucleus.

Mass number (A): Total number of nucleons in the nucleus $A = Z + N$.

Notation: The atomic number Z is given as left subscript of the atomic symbol X, the mass number A as left superscript, the neutron number N as right subscript: ${}^{A}_{Z}X_{N}$

even-even nuclei: even proton number Z, even neutron number N,

even-odd nuclei: even proton number Z, odd neutron number N,

odd-even nuclei: odd proton number Z, even neutron number N,

odd-odd nuclei: odd proton number Z, odd neutron number N.

Isotopes: Atomic nuclei with the same atomic number Z, but different neutron numbers N.

- ${}^{A}_{Z}X_{N}$ and ${}^{A+1}_{Z}X_{N+1}$ are isotopes. Example, the carbon isotopes ${}^{12}C$, ${}^{13}C$ and ${}^{14}C$.

Basically, isotopes are chemically equivalent. Only processes that depend on mass exhibit a slightly different behavior for different isotopes (differences in the physical chemical equilibria, differences in diffusion velocity, isotopic shifts in atomic spectra, resonance frequencies in molecules, critical temperature of superconductors). These phenomena are called isotope effects.

Isobars: Atomic nuclei with equal mass number A, but different proton numbers Z.

- ${}^{A}_{Z}X_{N}$ and ${}^{A}_{Z+1}Y_{N-1}$ are isobars. Example: ${}^{14}C$ and ${}^{14}N$

Isotones: Atomic nuclei with equal neutron number N, but different atomic numbers Z.

- ${}^{A}_{Z}X_{N}$ and ${}^{A+1}_{Z+1}Y_{N}$ are isotones

Radius of nucleus:

Shape of atomic nuclei is mostly deformed in an axially symmetric manner, spherical near closed nucleon shells. Nuclear radius R, may be estimated by the formula

$$R = r_0 A^{1/3},\ \text{where } r_0 \approx 1.2\ fm = 1.2\times10^{-15}\ m,\ A = \text{atomic mass number}$$

Binding Energy:

Binding energy, B, the energy released when free nucleons are bound together to form a nucleus. SI unit is the joule (J). Usually the binding energy is given in MeV:

$$1\ MeV = 1.6022\times10^{-13}\ J$$

The mass of a stable atomic nucleus is smaller than the sum of the masses of the constituent nucleons.

Mass defect, $\Delta W(A,Z)$ the difference between the sum of masses of all nucleons and the nuclear mass $m_K(A,Z)$:

$$\Delta W(A,Z) = m_p Z + (A-Z)m_n - m_K(A,Z)$$

According to mass-energy equivalence, the mass defect ΔW is related to the binding energy B,

$$B = \Delta W(A,Z)c^2,\quad 1\,\text{MeV}/c^2 = 1.7827\times10^{-30}\ \text{kg}$$

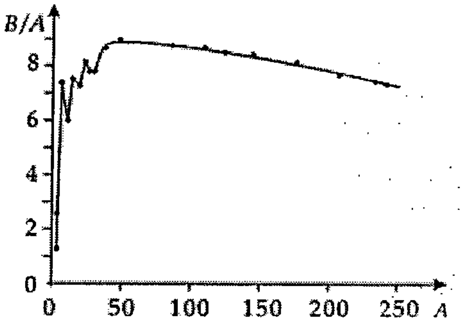
Atomic Mass Unit (u):

It is equal to 1/12 of the mass of a neutral atom of the carbon isotope ^{12}C:

$$u = \frac{1}{12} m\ {}^{12}C = \frac{1\,g}{N_A} = 1.660\times10^{-27}\ kg\quad (N_A\text{: Avogadro's number})$$

Quantity	Symbol	Value	Error /ppm
atomic mass unit	u	$931.494\,32\ \text{MeV}/c^2$	0.30
electron mass	m_e	$0.510\,999\,06\ \text{MeV}/c^2$	0.30
muon mass	m_μ	$105.658\,389\ \text{MeV}/c^2$	0.32
proton mass	m_p	$938.272\,31\ \text{MeV}/c^2$	0.30
neutron mass	m_n	$939.565\,63\ \text{MeV}/c^2$	0.30
Planck's constant	$\hbar$	$6.582\,122\,0\cdot10^{-22}\ \text{MeV}\cdot\text{s}$	0.30

Binding energy per nucleon:
Binding energy per nucleon is not constant. It increases initially, become highest for Fe, then decreases.



Binding energy per nucleon (B/A): A measure for the stability of an atomic nucleus. Mean experimental value: $B/A \approx 8$ MeV.

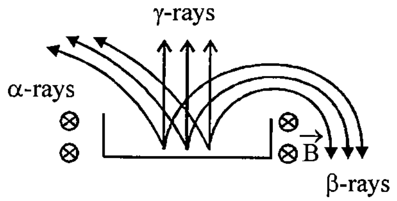
For light nuclei, the binding energy per nucleon increases with the mass number. The most stable atomic nucleus is iron (^{56}Fe) with a binding energy per nucleon of ≈ 8.8 MeV . For A > 56, the binding energy per nucleon decreases with increasing nucleon number.

Therefore, nuclear energy may be released either by fusion of light nuclei or by fission of heavy nuclei.

The local maxima of the binding energy in the range of light nuclei $\left(\text{e.g. for } {}_2^4He\right)$ are caused by closure of neutron and/or proton shells, analogous to the strong binding of the electron shell in inert gas atoms.
Saturation of nuclear forces, the binding energy per nucleon is approximately constant at about 8 MeV.
The magnitude of the binding energy of a nucleus determines its stability against decay.

8.3. Radioactivity:

Radioactivity is the phenomenon of spontaneous distintegration, attended with emission of corpuscular or electromagnetic radiations, of heavy atomic nuclei $\left(\text{for } \frac{N}{Z} > 1.5\right)$ like uranium, radium, etc., at a constant rate unaffected by any physical or chemical changes or influences such as temperature, pressure etc. to which the atom may be subject. It is a nuclear property of the active element and in all radioactive processes a transmutation of the element occure an altogether new nucleus is formed.



Radioactive Decay law:

1. On emission of α or β rays, which is usually but not invariably accompanied by γ-ray emission, the emitting parent nuclide transforms into a new daughter element. The daughter element again is radioactive so that the process of successive disintegration continuous till the original active parent nucleid gets transformed into stable one.

2. The rate of radioactive disintegration, that is, the number of atoms that break up at any instant of time ‘t’ is directly proportional to the number N_t of active nuclides present in the sample under study at that instant.

Decay equation:

Let N_t be the number of active nuclides present in the sample at any time 't' then we have experimentally,

$$-\frac{dN_t}{dt} \propto N_t$$

$$\Rightarrow \quad \frac{dN_t}{dt} = -\lambda N_t$$

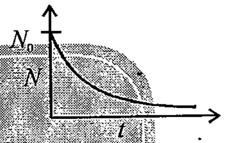
Where, λ, the constant of proportionality, is known as the *decay* constant – a characteristic constant of the element (nuclide). The negative sign hints at the fact that N_t decreases with t.

$$\frac{dN_t}{N_t} = -\lambda dt$$

$$\ell n\, N_t = -\lambda t + A$$

where, A is the constant of integration.

At $t = 0$, $N_t = N_0$, the initial number of nuclides and $A = \ell n\, N_0$



Therefore, $$\ell n\left(\frac{N_t}{N_0}\right) = -\lambda t$$

Or, $$\boxed{N_t = N_0 e^{-\lambda t}}$$

Half-life: The half-life of a radioactive nuclide is defined as the time $T_{½}$ in which the original amount of radioactive atoms is reduced by way of distintegration to half its value.

At $t = T_{½}$, $N_t = N_0/2$

$$\frac{N_0}{2} = N_0 e^{-\lambda T_{½}}$$

$$\therefore \quad T_{½} = \frac{\ell n 2}{\lambda} = \frac{2.303 \log 2}{\lambda} = \frac{0.693}{\lambda}$$

Or, $$\boxed{\lambda T_{½} = 0.693 = \text{constant}}$$

Average life:

The average or mean life time $\overline{T}$ of a radioelement is the average life time of all the atoms in the given sample and is defined as the ratio of the total life time of all the atoms to be total number of atoms.

$$\therefore \quad \overline{T} = \frac{t_1 dN_1 + t_2 dN_2 + \ldots\ldots}{dN_1 + dN_2 + \ldots\ldots} = \frac{\Sigma t dN}{\Sigma dN} = \frac{\int_{t=0}^{\infty} t dN}{\int_{N_0}^{0} dN} = \frac{\int_{t=0}^{\infty} t dN}{-N_0}$$

But, we have,

$$dN = d\left(N_0 e^{-\lambda t}\right) = -\lambda N_0 e^{-\lambda t} dt$$

$$\therefore \quad \overline{T} = \lambda \int_0^{\infty} t e^{-\lambda t} dt = \lambda \left[-\frac{t}{\lambda} e^{-\lambda t} + \frac{1}{\lambda} \int_0^{\infty} e^{-\lambda t} dt \right]$$

$$= \lambda \left[-\frac{t}{\lambda} e^{-\lambda t} - \frac{1}{\lambda^2} e^{-\lambda t} \right]_0^{\infty} = \frac{1}{\lambda}$$

Therefore, $\boxed{\bar{T} = \frac{1}{\lambda}}$

We can write also, $\boxed{T_{½} = \frac{0.693}{\lambda} = 0.693\bar{T}}$

Activity:

The activity or strength A_t of a radioactive sample at any instant t is thus defined as the number of disintegration occurring in the sample in unit time at t, that is,

$$\text{Activity,} \quad \boxed{A_t = \left| \frac{dN_t}{dt} \right| = \lambda N_t = \frac{0.693}{T} N_t}$$

Units of activity:

The customary unit of radioactivity is called the curie (Ci). It is defined as the acitivity of any radioactive substance that disintegrates at the rate of 3.7×10^{10} disintegration per second.

$$1\ \text{Ci} = 1\ \text{curie} = 3.7 \times 10^{10}\ \text{disintegration/sec}$$

$$1\ \text{mCi} = 10^{-3}\ \text{curie}$$

$$1\ \mu\text{Ci} = 10^{-6}\ \text{curie}$$

Radioactive displacement law:

1. The α particles having been identities as helium nuclei. Suppose I have parent nucleus having mass number A and atomic number, after α decay the parent nucleus become a new daughter element whose atomic number $Z - 2$ and mass $A - 4$. It position in periodic table will be shifted towards left by two.

$${}^{A}X_{Z} \xrightarrow{\alpha} {}^{A-4}Y_{Z-2}$$

2. The β-particle defined as ${}_{-1}e^{0}$ due to β emission the parent elements $(Z - A)$ become a daughter element $(Z + 1, A)$. So, it position in periodic table will be shifted towards right by one.

$${}^{A}X_{Z} \xrightarrow{\beta} {}^{A}Y_{Z+1}$$

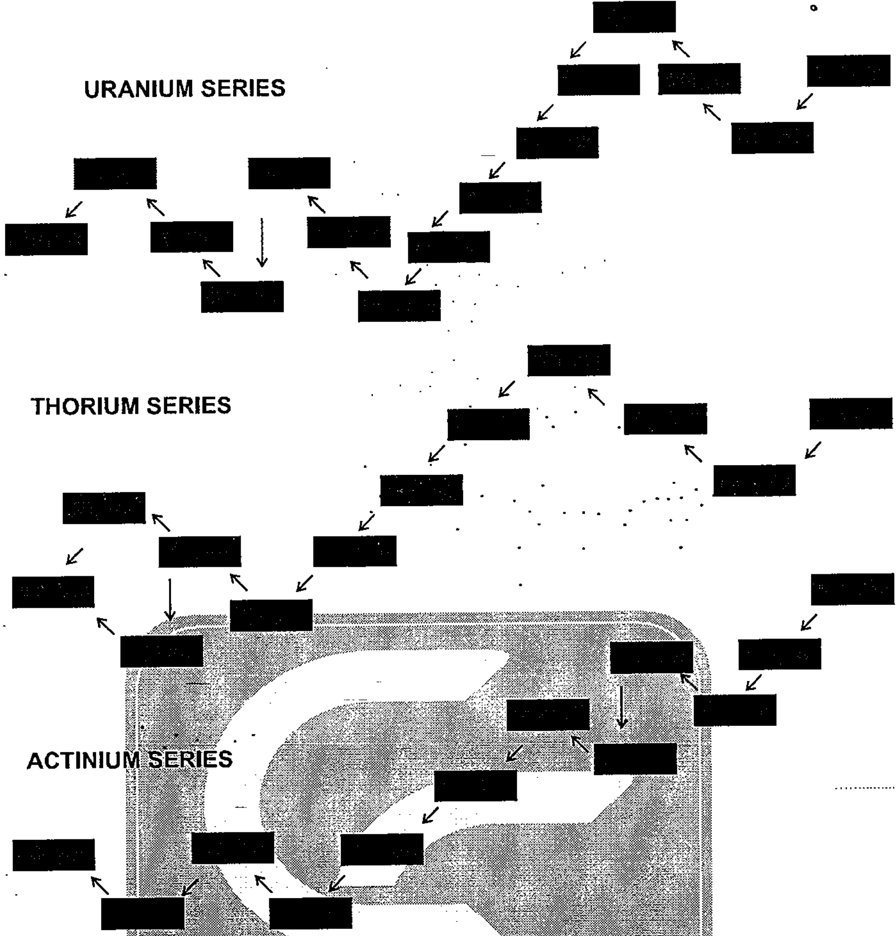
Natural radioactive decay series:

There are three naturally occuring radioactive series. These three series are

(i) Uranium series: $(4n + 2)$

(ii) Actinium series: $(4n + 3)$

(iii) Thorium series: $(4n)$



URANIUM SERIES

THORIUM SERIES

ACTINIUM SERIES

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Note: ↖ $-(\beta\text{-decay})$, ↙ $-(\alpha\text{-decay})$

Successive transformation:
Consider a radioactive nuclide *P, symbolised by script 1, to decay into another radioactive nuclide Q (script 2); the latter again decays into a stable end-product R (script 3). For instance,

$$P(1)\xrightarrow{\lambda_1}Q(2)\xrightarrow{\lambda_2}R(3)\ [stable]$$

Let λ_1, λ_2 be the decay constants of nuclides 1 and 2 respectively, and N_1, N_2, N_3 be the number of atoms of the three kinds at any intant *t*.

Now the second nuclide would be formed at the rate $\lambda_1 N_1$ by the decay of the parent atom and disappear by its own decay at the rate $\lambda_2 N_2$; the atoms of the end-product i.e. R appear at the rate $\lambda_2 N_2$ by the decay of nuclide 2, but being stable do not disappear. So,

$$\frac{dN_1}{dt} = -\lambda_1 N_1 \qquad \text{... (i)}$$

$$\frac{dN_2}{dt} = \lambda_1 N_1 - \lambda_2 N_2 \qquad \text{... (ii)}$$

$$\frac{dN_3}{dt} = \lambda_2 N_2 \qquad \text{... (iii)}$$

Solving equation (i) : $(N_1)_t = N_{10}e^{-\lambda_1 t}$

where N_{10} is the number of atoms of nuclide 1, at time t = 0

From equation (ii), using (iii), we obtain

$$\frac{dN_2}{dt} = \lambda_1 N_{10}\, e^{-\lambda_1 t} - \lambda_2 N_2$$

Or, $$\frac{dN_2}{dt} + \lambda_2 N_2 = \lambda_1 N_{10} e^{-\lambda_1 t} \qquad \text{... (iv)}$$

Multiplying both sides of equation (iv) by $e^{\lambda_2 t}$, the integrating factor,

$$e^{\lambda_2 t}\left(\frac{dN_2}{dt} + \lambda_2 N_2\right) = \lambda_1 N_{10}\, e^{(\lambda_2 - \lambda_1)t}$$

Or, $$\frac{d}{dt}\left(N_2 e^{\lambda_2 t}\right) = \lambda_1 N_{10} e^{(\lambda_2-\lambda_1)t}$$

Integrating,

$$N_2 e^{\lambda_2 t} = \frac{\lambda_1}{\lambda_2 - \lambda_1} N_{10} e^{(\lambda_2-\lambda_1)t} + A \qquad \text{... (v)}$$

Where A is the constant of integration,

Now at $t = 0, (N_2)_t = N_{20} =$ constant. So, from equation (v),

$$A = N_{20} - \frac{\lambda_1}{\lambda_2 - \lambda_1} N_{10}$$

$$\therefore \quad (N_2)_t = \frac{\lambda_1}{\lambda_2 - \lambda_1} N_{10}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right) + N_{20} e^{-\lambda_2 t} \qquad \text{... (vi)}$$

From equation (iii), using (iv), we obtain on integration,

$$(N_3)_t = \left(\frac{\lambda_1}{\lambda_2 - \lambda_1} N_{10} - N_{20}\right) e^{-\lambda_2 t} - \frac{\lambda_2}{\lambda_2 - \lambda_1} N_{10} e^{-\lambda_1 t} + B \qquad \text{... (vii)}$$

At $t = 0, (N_3)_t = N_{30}$. So, from equation (vii), we get

$$\therefore \quad B = N_{30} + N_{20} + N_{10}$$

$$\therefore \quad (N_3)_t = N_{30} + N_{20}\left(1 - e^{-\lambda_2 t}\right) + N_{10}\left(1 + \frac{\lambda_1}{\lambda_2 - \lambda_1} e^{-\lambda_2 t} - \frac{\lambda_2}{\lambda_2 - \lambda_1} e^{-\lambda_1 t}\right) \qquad \text{... (viii)}$$

Equations (i), (vi) and (viii) therefore, constitute the solution of the problem.

In most cases, only the first nuclide is present at $t = 0$. So, $N_{20} = N_{30} = 0$. We thus have from (vi) and (viii).

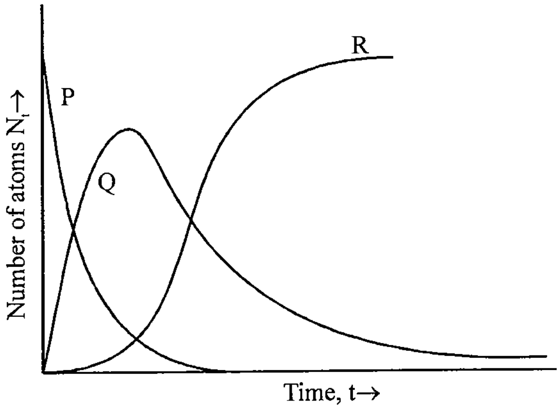
$$(N_2)_t = \frac{\lambda_1}{\lambda_2 - \lambda_1} N_{10}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right)$$

$$(N_3)_t = N_{10}\left(1 + \frac{\lambda_1}{\lambda_2 - \lambda_1} e^{-\lambda_2 t} - \frac{\lambda_2}{\lambda_2 - \lambda_1} e^{-\lambda_1 t}\right)$$

The decay of the first nuclide P, the growth and decay of the second nuclide Q and the growth of the third nuclide R are shown in the figure below .

Time for Q(2) to attain a maximum:

Equation (vi) gives how N_2 varies with t. It shows that $N_2 = 0$ at $t = 0$. It increases with increasing t and attains a maximum at $t = t_m$, say.



Now, we have, $\dfrac{dN_2}{dt} = \dfrac{\lambda_1 N_{10}}{\lambda_2 - \lambda_1}\left(-\lambda_1 e^{-\lambda_1 t} + \lambda_2 e^{-\lambda_2 t}\right)$

$$\therefore \quad \frac{dN_2}{dt} = 0 \Rightarrow \lambda_2 e^{-\lambda_2 t_m} = \lambda_1 e^{-\lambda_1 t_m}$$

$$\therefore \quad e^{(\lambda_1 - \lambda_2)t_m} = \frac{\lambda_1}{\lambda_2}$$

Or, $(\lambda_1 - \lambda_2)t_m = \ell n \dfrac{\lambda_1}{\lambda_2}$

$$\therefore \quad t_m = \frac{1}{\lambda_1 - \lambda_2}\ell n \frac{\lambda_1}{\lambda_2}$$

That is, t_m corresponds to maximum N_2 can be readily verified by obtaining the second time derivative of N_2 which becomes negative at $t = t_m$.

Branching of radioactivity :

When a radio-element shows that α and β activities, there are definite probabilities of α and β decay constants for the branches. Let it contain N atoms at any time t, the total rate of disintegration will be

$$\frac{dN}{dt} = -\left(\lambda_\alpha + \lambda_\beta\right)N = \lambda N, \text{ where } \lambda = \lambda_\alpha + \lambda_\beta \text{ the total decay constant.}$$

If $(T_{1/2})_\alpha$, $(T_{1/2})_\beta$ be the partial half lives of α and β decay.

$$\lambda_\infty = \frac{0.693}{(T_{1/2})_\alpha}, \quad \lambda_\beta = \frac{0.693}{(T_{1/2})_\beta}$$

$$\therefore \quad \lambda = 0.693\left(\frac{1}{(T_{1/2})_\alpha} + \frac{1}{(T_{1/2})_\beta}\right)$$

But if $T_{1/2}$ be the mean half-life of the substance, then

$$\lambda = \frac{0.693}{T_{1/2}}$$

$$\therefore \quad \boxed{\frac{1}{T_{1/2}} = \frac{1}{(T_{1/2})_\alpha} + \frac{1}{(T_{1/2})_\beta}}$$

Determination of constants of radioactivity:

1. For simple decays : We have seen that the activity A of a sample exponentially with time according to the equation,

$$A_t = A_0\, e^{-\lambda t}.$$

$$\therefore \quad \ell n\, A_t = \ell n A_0 - \lambda t$$

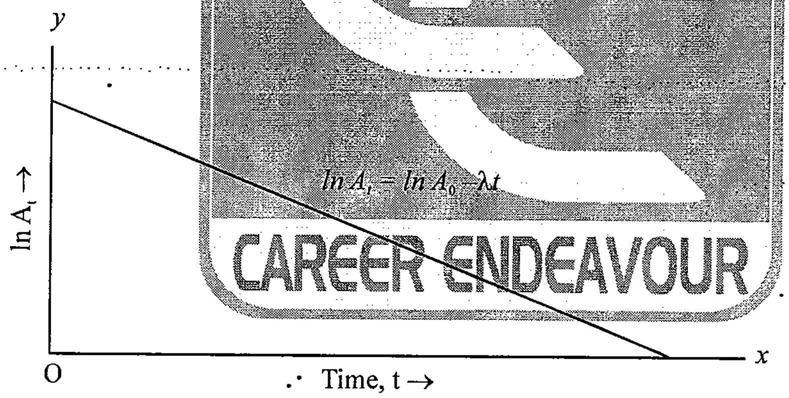
Or, $\quad y = -\lambda t + C \qquad \text{... (i)}$

where, $y = \ell n\, A_t$ and $C = \ell n\, A_0$.

Differentiating (i),

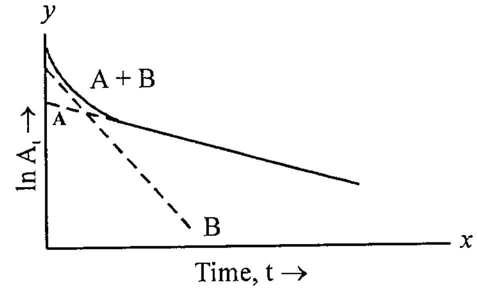
$$\frac{dy}{dt} = -\lambda$$

So, when the logarithm of the activity, $\ln A_t$, is plotted against time t, it will give a straight line with a slope equal to $-\lambda$. This therefore would furnish us a method for evaluating λ, provided we can measure the activity of the sample.



2. Complex decays:

We conclude the section by indicating how one can determine half-lives for complex decays, e.g. two half-lives when a sample contains a mixture of say two activities A and B such that $T_{1/2}(A) >> T_{1/2}(B)$. A graph of the logarithm of the observed activity of the sample is first plotted against time, the observations however are extended over a time large enough compared to $T_{1/2}(B)$. The tail-end of the curve is due to the activity A, since the contribution from B in this region is negligible. So, if the straight part of the graph is extrapolated to $t = 0$, the contribution to the total observed activity of the sample from A at any instant t can be found. Subtracting this activity of A from the observed total activity at t, we get the activity of B at t. Once the contributions from the two activities are known, their half-lives can be evaluated by the usual procedure.



- **α-decay:**

- **Velocity of α-particles:**
  Let 'v' be the velocity of the α-particles, M be its mass, q the charge, r the radius of the track and B the magnetic field, then $v = Bqr/M$.

- **Range of α-particles:**
  The distance through which an α-particle travels in a specified material before stopping to ionise it is called the range of α-particles in that material. The range is thus a sharply defined ionisation path-length.
  The range depends on (i) The initial energy of the α-particles, (ii) the ionisation potential of the gas and (iii) the chances of collision between the α-particles and on the nature and the temperature and pressure of the gas. With increase of pressure, the range decreases; it increases if the temperature of the gas increased.

  The range R in standard air is proportional to v^3, i.e. $R \propto v^3 \Rightarrow R = av^3$
  The relation is known as the Geiger law, which is valid only in a limited velocity range.

  Since $R \propto v^3$ and the energy $E = \frac{1}{2}mv^2$, the range-energy relationship is

  $$R \propto E^{3/2} \Rightarrow R = bE^{3/2}$$

- **Specific ionisation:**
  The number of ion-pairs formed per unit path length at any point in the path of the α-particle is called specific ionisation and is symbolised by I.

  Since, $E \propto R^{2/3} \Rightarrow \dfrac{dE}{dR} \propto R^{-\frac{1}{3}} \propto \dfrac{1}{v}$

- **Geiger-Nuttall law:**
  An important quantitative relation between the range R of the α-particles and the decay constant λ of the emitting nuclei was experimentally discovered by Geiger and Nuttall(1911) and is called the Geiger-Nutall law. The relation runs as:

  $$\ln \lambda = A + B \ln R,$$

  where A and B are constants having values different for different radioactive series.

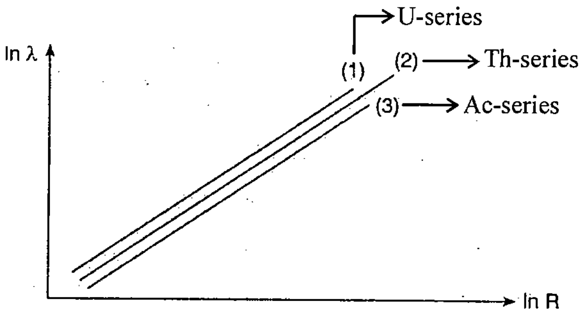


Figure: Variation of in λ with *l*nR: Geiger-Nuttall law

Since $R \propto E^{3/2} \Rightarrow \ln\lambda = C + D\ln E$,
where C, D are two constants.

- α**-disintegration energy:**

  $${}^{A}_{Z}X \rightarrow {}^{A-4}_{Z-2}Y + {}^{4}_{2}He$$

  The Q-value of the decay process is known as the α-disintegration energy which is the total energy released in the disintegration process and is given by $Q_\alpha = (M_X - M_\alpha - M_Y)c^2$
  where M's are the masses of the particles and 'c' the velocity of light in vacuum.

  For heavy nuclei, Q_α is positive, so the decay can occur spontaneously as it does. According to laws of conservation of momentum and energy.

  $$0 = M_\alpha v_\alpha - M_Y v_Y$$

  and $Q_\alpha = \frac{1}{2}M_\alpha v_\alpha^2 + \frac{1}{2}M_Y v_Y^2$

  $$\Rightarrow Q_\alpha = \frac{1}{2}M_\alpha v_\alpha^2\left(1+\frac{M_\alpha}{M_Y}\right) = T_\alpha\left(1+\frac{M_\alpha}{M_Y}\right)$$

  Kinetic energy of ejected α-particles, $T_\alpha = \frac{Q_\alpha}{1+M_\alpha/M_Y}$

  $$\Rightarrow Q_\alpha = T_\alpha\frac{[My + M_\alpha]}{M_y} = T_\alpha\left[\frac{A}{A-4}\right]$$

  $$\Rightarrow \boxed{T_\alpha = Q_\alpha\left[1-\frac{4}{A}\right]} \;\&\; T_d = Q_\alpha - T_\alpha = \frac{4Q}{A}$$

  Where K.E. of α-particle $= T_\alpha$
  As $A \sim 200 \Rightarrow T_\alpha \sim Q_\alpha$

α- decay through tunnelling:

$${}_{Z}X^{A} \longrightarrow {}_{Z-2}Y^{A-4} + 2\text{He}^4$$

$$R = R_0A^{1/3}$$

$$V(R)=\frac{2(Z-2)|e|^2}{4\pi\epsilon_0 R}$$

$$T=e^{-G}, G=A(Z-2)E^{-1/2}-B(Z-2)^{1/2}R^{1/2}$$

$$A=1.587\times10^{-6}\ \mathrm{J}^{1/2}$$

$$B=94\times10^{6}\ m^{-1/2}$$

$$\nu=\frac{v}{2R};\ \lambda=\nu T;\ T_{½}=\frac{\ln 2}{\lambda}$$

Example: ${}_{92}U^{238}\longrightarrow {}_{90}Th^{234}+{}_{2}He^{4}$ and the energy of the emitted α particle is $E=6.72\times10^{-13}$ joule. Determine the half-life time of ${}_{92}\mathrm{U}^{238}$.

Soln. Nuclear radius of parent

$$R=R_0A^{1/3}=(1.4\,\mathrm{fm})(238)^{1/3}=8.7\,\mathrm{fm}=8.7\times10^{-15}\,\mathrm{m}$$

Atomic number of parent $Z=92$

Tunnelling probability,

$$T=e^{-G}=e^{-\left(1.587\times10^{-6}\,\mathrm{J}^{1/2}\right)(92-2)\left(6.72\times10^{-13}\,\mathrm{J}\right)^{-1/2}+\left(94\times10^{6}\,\mathrm{m}^{-1/2}\right)(92-2)^{1/2}\left(8.7\times10^{-15}\,\mathrm{m}\right)^{1/2}}=\frac{6}{10^{39}}$$

Out of 10^{39} strikes α emerges out six times only.

Velocity, $v=\sqrt{\frac{2E}{m}}=\sqrt{\frac{2\left(6.72\times10^{-13}\,\mathrm{J}\right)}{4\times1.67\times10^{-27}\,\mathrm{kg}}}=1.42\times10^{7}\,\mathrm{ms}^{-1}$

Collision frequency, $\nu=\frac{v}{2R}=\frac{1.42\times10^{7}\,\mathrm{ms}^{-1}}{2\left(8.7\times10^{-15}\,\mathrm{m}\right)}=8.2\times10^{20}\mathrm{s}^{-1}$

Decay constant, $\lambda=\nu T=\left(8.2\times10^{20}\mathrm{s}^{-1}\right)\left(6\times10^{-39}\right)=4.92\times10^{-18}\mathrm{s}^{-1}$

Half life $T_{½}=\frac{\ln 2}{\lambda}=\frac{\ln 2}{4.92\times10^{-18}\mathrm{s}^{-1}}=1.4\times10^{17}\mathrm{s}$

$$=\frac{1.4\times10^{17}}{365\times24\times60\times60}\text{years}=4.4\times10^{9}\text{ years}$$

The experimental value of half life is 4.5×10^{9} years. So, we have obtained a rough estimate of half life of ${}_{92}U^{238}$.

- **β –decay:**

Let v be the velocity of a given β-particle, B the magnetic flux density, m the relativistic mass of the β-particle and r the radius of the circular track.

Then $\frac{mv^2}{r} = Bev \Rightarrow r = \frac{mv}{Be} = \frac{m_0}{\sqrt{1-v^2/c^2}}\left(\frac{v}{Be}\right)$

- **β - decay , we write:**

$${}_{Z}^{A}X \rightarrow {}_{Z+1}^{A}Y + {}_{-1}^{0}e$$

The disintegration energy in β^-decay is:

$$Q_{\beta^-} = \left[M_n(A,Z) - M_n(A,Z+1) - m_e\right]c^2$$

$$= \left[M(A,Z) - Zm_e - M(A,Z+1) + (Z+1)m_e - m_e\right]c^2 \text{ (in terms of atomic mass)}$$

$$= \left[M(A,Z) - M(A,Z+1)\right]c^2$$

where M_n is the nuclear mass, M the atomic mass and m_e the mass of electron.

$Q_{\beta^-} > 0$, if $M(A, Z) > M(A, Z+1)$

implying that β^- decay occurs only if the mass of the parent atom is greater than that of the daughter atom.

- **β^+ - decay:**

$${}_{Z}^{A}X \rightarrow {}_{Z-1}^{A}Y + {}_{+1}^{0}e$$

$$\Rightarrow \quad Q_{\beta^+} = \left[M_n(A,Z) - M_n(A,Z-1) - m_e\right]c^2$$

$$= \left[M(A,Z) - Zm_e - M(A,Z-1) + (Z-1)m_e - m_e\right]c^2$$

$$= \left[M(A,Z) - M(A,Z-1) - 2m_e\right]c^2$$

For β^+ - decay to occur: $Q_{\beta^+} > 0$, if $M(A,Z) > M(A,Z-1) + 2m_e$

i.e. the mass of the parent atom is greater than the daughter atom at least twice the electronic mass, i.e. 1.02 MeV.

- **Electron capture:**

$${}_{Z}^{A}Y + {}_{-1}^{0}e \rightarrow {}_{Z-1}^{A}Y$$

Therefore, disintegration energy, $Q_e = \left[M_n(A,Z) + m_e - M_n(A,Z-1)\right]c^2 - B_e$ where B_e is the binding energy of the electron to the orbit.

$$\Rightarrow \quad Q_e = \left[M(A,Z) - Zm_e + m_e - M(A,Z-1) + (Z-1)m_e\right]c^2 - B_e$$

$$= \left[M(A,Z) - M(A,Z-1)\right]c^2 - B_e$$

For electron capture to occur: $Q_e > 0$, if $M(A,Z) > M(A,Z-1) + B_e$

i.e. the mass of the parent atom is greater than that of the daughter atom by at least the binding energy of the electron.

- **Selection rule for β^+ - decay:**

  If ℓ_β is odd, initial and final nuclei must have opposite parities (parity changes in these transitions); for even ℓ_β values the initial and final nuclei must have same parity (no change in parity). Furthermore, as in allowed transitions, the emission of leptons (electron and neutrino) in the singlet state (Fermi-selection rule) requires $\Delta I \le \ell_\beta$, whereas triplet-state (G-T selection rule) emission requires $\Delta I \le \ell_\beta + 1$. Thus selection rules for forbidden transitions are:

  First forbidden – For these transitions $\ell_\beta = 1$ and parity changes.

  Fermi-selection rules: $\Delta I = \pm 1, 0\,(\text{except } 0 \to 0)$

  Gamow Teller rules: $\Delta I = \pm 2, \pm 1, 0\left(\text{except } 0 \to 0, \frac{1}{2} \to \frac{1}{2}, 0 \leftrightarrow 1\right)$

  The examples are:

${}^{39}_{18}Ar \to {}^{39}_{19}K + \beta^-$	$(7/2^- \to 3/2^+)$	*GT.*
${}^{14}_{6}C \to {}^{14}_{7}N + \beta^-$	$0 \to 1$	*F*
${}^{38}_{17}Cl \to {}^{38}_{18}Ar + \beta^-$	$(2^- \to 2^+)$	*F + GT mixed*
${}^{139}_{56}Ba \to {}^{139}_{57}La + \beta^-$	$(7/2^- \to 7/2^+)$	*F + GT mixed*
${}^{143}_{59}Pr \to {}^{143}_{60}Nd + \beta^-$	$(7/2^+ \to 7/2^-)$	*F + GT mixed*
${}^{147}_{61}Pm \to {}^{147}_{62}Sm + \beta^-$	$(7/2^+ \to 7/2^-)$	*F + GT mixed*
${}^{85}_{36}Kr \to {}^{85}_{37}Rb + \beta^-$	$(9/2 \to 5/2)$	*GT.*
${}^{89}_{38}Sr \to {}^{89}_{39}Y + \beta^-$	$(5/2^+ \to 1/2^-)$	*GT.*
${}^{137}_{55}Cs \to {}^{136}_{56}Ba^* + \beta^-$	$(7/2^+ \to 11/2^-)$	*GT.*

Second forbidden. For these transition $\ell_\beta = 2$ and no change in parity.

Fermi-selection rules: $\Delta I = \pm 2, \pm 1\,(\text{except } 0 \leftrightarrow 0)$

Gamow Teller rules: $\Delta I = \pm 3, \pm 2\,(\text{except } 0 \leftrightarrow 2)$

The examples are:

${}^{137}_{55}Cs \to {}^{137}_{56}Ba + \beta^-$	$(7/2^+ \to 3/2^+)$	*F + GT mixed*
${}^{10}_{4}Be \to {}^{10}_{5}B + \beta^-$	$(0^+ \to 3^+)$	*GT.*
${}^{22}_{11}Na \to {}^{22}_{10}Ne + \beta^+$	$(3^+ \to 0^+)$	*GT.*

- **γ-decay:**

  In γ-decay there is no change in Z and A. The γ-ray photons are emitted when the nucleus jumps from an excited state to the lower state (after 10^{-14} sec). It is always not possible that when nucleus jumps from higher energy state to lower γ-photon will be emitted.

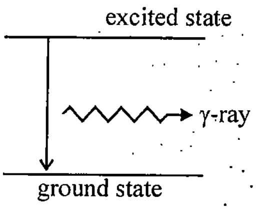
  Sometimes the nucleus in higher energy state directly give energy to the atomic e^- and hence an e^- may come out rather than a γ-ray photon. This process is known as internal conversion and the e^- so emitted is called as conversion e^-.

Energy of emitted γ-ray in α-decay:

A parent nuclei ${}^{A}X_{Z}$, when it emit a particle it goes to the excited state and it comes down to its ground state by emitting γ-ray.

According energy conservation rule we can write,

$$E_\gamma = Q_\alpha - \left(T_\alpha + T_y\right) = Q_\alpha - T_\alpha\left(1 + \frac{M_\alpha}{M_y}\right)$$



$$\therefore \quad \boxed{E_\gamma = Q_\alpha - T_\alpha\left(1 + \frac{M_\alpha}{M_y}\right)}$$

- **Selection rule for γ-decay:**

Type	Symbol	Change in Angular momentum, L	Parity change
Electric dipole	E_1	1	Yes
Magnetic dipole	M_1	1	No
Electric quadrupole	E_2	2	No
Magnetic quadrupole	M_2	2	Yes
Electric octupole	E_3	3	Yes
Magnetic octupole	M_3	3	No
Electric 2^L – pole	E_L	L	[No for even L. Yes for odd L]
Magnetic 2^L – pole	M_L	L	[Yes for even L. No for odd L]

γ - ray interaction with matter :

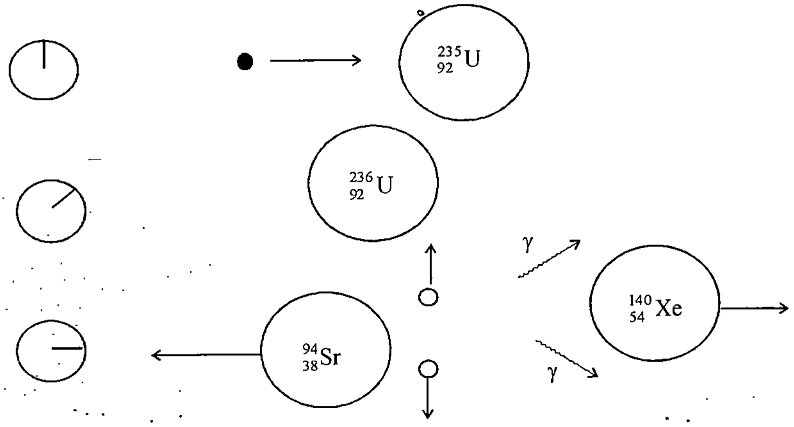
There are three fundamental procedure in which γ-ray interact with matter. These three processes
(i) Compton scattering
(ii) Photoelectric effect
(iii) Pair production

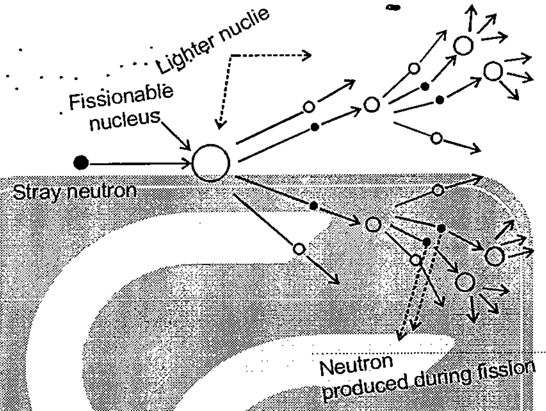
The three factors should therefore be combined into one single (total) absorption coefficient, μ in the absorption law. The intensity of the transmitted γ-ray is given by

$\boxed{I = I_0 e^{-\mu x}}$ where, x is the thickness of medium.

8.4. Nuclear fission:

A nucleus of the uranium isotope ${}^{235}_{92}\text{U}$ undergoes fission when struck by a neutron. It is not the impact of neutron that has this effect. Instead, the ${}^{235}_{92}\text{U}$ nucleus absorbs the neutron to became ${}^{236}_{92}\text{U}$, and the new nucleus is so unstable that almost at once it explodes into two fragments. Later several other heavy nuclides were found to be fissionable by neutrons in similar processes.

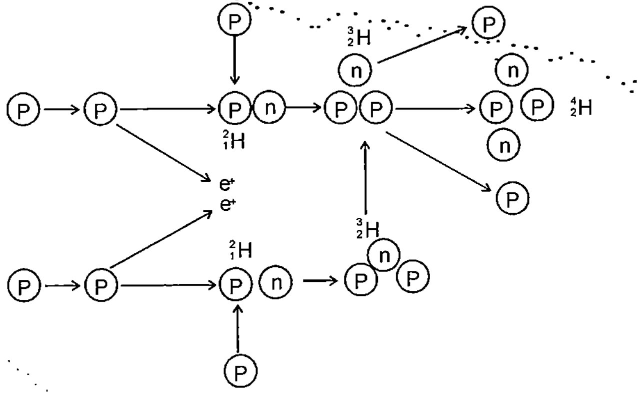

$^{235}_{92}$U
$^{236}_{92}$U
γ
$^{140}_{54}$Xe
$^{94}_{38}$Sr
γ


Lighter nuclie
Fissionable nucleus
Stray neutron
Neutron produced during fission

Reaction is self sustaining if atleast one neutron from each fission event on the average induces another fission event.

8.5. Nuclear fission:

Self-sustaining fusion reactions can occur only under condition of extremely high temperature and density. The high temperature ensures that some nuclei-those in the high velocity tail of the Maxwell-Boltzamann distribution have the energy needed to come close enough together to interact, which they do by tunneling through the electric potential barrier between them. (At 10^7 K temperature, typical of sun is interior, the average proton KE is only about 1KeV, whereas the barrier is about 1MeV). The high density ensures that such collisions are frequent.


P
$^{3}_{2}$H
n
P
n
P
P
P
n
P
P
P
n
$^{4}_{2}$H
$^{2}_{1}$H
n
e+
e+
P
$^{3}_{2}$H
$^{2}_{1}$H
n
P
P
P
n
P
P
P

The combination of four hydrogen nuclie to form a helium nucleus and two positrons.

- **Classification of nuclear reactions:**

(i) **Elastic scattering:** In elastic scattering, the same particles are scattered in different directions and there is no loss of energy. The residual nucleus is the same as the target nucleus and is left in the same state (ground state) as the latter so that it can be represented as X(x, y)X. An example is the scattering of neutrons by graphite: ${}^{12}C(n,n)\,{}^{12}C$. **Example:** ${}_2He^4 + {}_{79}Au^{197} \rightarrow {}_{79}Au^{197} + {}_2He^4$

(ii) **Inelastic scattering:** In inelastic scattering, the same particles are scattered in different directions with different energy, as there is loss of energy due to collision. The residual nucleus which is the same as target nucleus is left in an excited state so that the process can be represented as $X(x,y)X^*$. An example is the collision of fast neutrons with U-238.

(iii) **Radiative capture:** In radiative capture, the incident particle is absorbed or captured by the target nucleus to form the excited compound nucleus which disintegrates to produce one or more γ – photons and goes down to the ground state. The process may be represented as $X(x,\gamma)Y^*$. **Example:** ${}_{92}U^{235} + {}_0n^1 \longrightarrow {}_{92}U^{236}$.

(iv) **Reaction of transformation:** Here the oncoming particle is retained in the nucleus and the compound nucleus emits a different particle so that the product nucleus is different from target nucleus, e.g. ${}^9Be(\alpha,n)^{12}C$

(v) **Photo-disintegration:** In photo-disintegration, a very energetic photon is absorbed by the target nucleus so that it is raised to an excited state and subsequently disintegrates. It can be represented as $X(\gamma,y)Y$. **Example:** ${}_1H^2 + \gamma \longrightarrow {}_1H^1 + {}_0n^1$.

(vi) **Stripping reactions:** In stripping reactions, one or more nucleons from the projectile are captured by the target nucleus, the remaining stripped nucleus is emitted in a different direction.

Example: ${}_{29}Cu^{63} + {}_1d^2 \longrightarrow {}_{29}Cu^{64} + {}_1H^1$

(vii) **Direct reactions:** A collision of an incident particle with the nucleus may immediately pull one of the nucleons out of the target nucleus and is called 'pick up reaction'.

Example: ${}_1H^1 + {}_6C^{13} \longrightarrow {}_1d^2 + {}_6C^{12}$

- **Conservation in Nuclear reactions:**

(i) Conservation of mass number

(ii) Conservation of atomic number

(iii) Conservation of energy (including mass-energy)

(iv) Conservation of linear momentum

(v) Conservation of angular momentum.

(vi) Conservation of parity

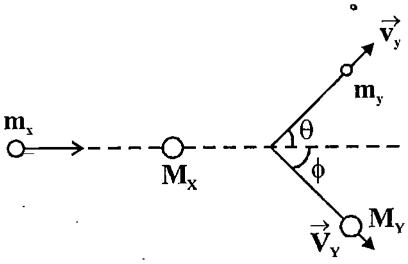
(vii) Conservation of isotopic spin

Kinematics of nuclear reactions:

Let a particle of mass m_x moving along x-axis with a velocity v_x collide elastically with a target nucleus of mass M_X at rest. Due to the nuclear reaction, the product nucleus of mass M_Y is scattered at an angle ϕ with velocity v_Y and the product particle of m_y is emitted with velocity v_y at an angle θ

From the law of conservation of linear momentum along x and y directions, we get

$$\left.\begin{aligned} M_Y u_Y \cos\phi + m_y v_y \cos\theta &= m_x v_x \\ -M_Y v_Y \sin\phi + m_y v_y \sin\theta &= 0 \end{aligned}\right\}$$



Or, $$\left.\begin{aligned} M_Y v_Y \cos\phi &= m_x v_x - m_y v_y \cos\theta \\ M_Y v_Y \sin\phi &= m_y v_y \sin\theta \end{aligned}\right\}$$

Squaring the above two equations and adding, we obtain

$$M_Y^2 v_Y^2 = m_x^2 v_x^2 + m_y^2 v_y^2 - 2m_x m_y v_x v_y \cos\theta$$

Or, $$2M_Y K_Y = 2m_x K_x + 2m_y K_y - 4\sqrt{m_x m_y K_x K_y}\cos\theta$$

since, non-relativistically, K = kinetic energy = $\frac{1}{2}mv^2$

$$\therefore \quad K_Y = \frac{m_x}{M_Y}K_x + \frac{m_y}{M_Y}K_y - \frac{2}{M_Y}\sqrt{m_x m_y K_x K_y}\cos\theta$$

But, $Q = K_Y + K_y - K_x$

$$\therefore \quad Q = K_y\left(1+\frac{m_y}{M_Y}\right) - K_x\left(1-\frac{m_x}{M_Y}\right) - \frac{2}{M_Y}\sqrt{m_x m_y K_x K_y}\cos\theta$$

This gives the Q-value of the reaction in terms of K_y, K_x and θ without involving the kinetic energy of the recoil nucleus, K_Y and the mass M_x of target nucleus and is called the standard form of Q-equation.

If $\theta = 90°$, the Q-expression simplifies to : $Q = K_y\left(1+\frac{m_y}{M_Y}\right) - K_x\left(1-\frac{m_x}{M_Y}\right)$

If Q is negative, it gives the minimum value of threshold kinetic energy to be given to the incident particle for an endoergic reaction to proceed in the forward direction.

- **Q-value and threshold energy of nuclear reaction:**

The law of conservation of energy and momentum imposes certain restrictions on the reactions. These restrictions are called the kinematic restrictions and this mathematical methods is known as kinematics. Consider the nuclear reaction

$$x + X \longrightarrow y + Y$$

Where x, X, y and Y are the bombarding particle, target nucleus, outgoing particle and product nucleus respectively. It is assumed that the target nucleus is in rest. Since total energy is conserved in the nuclear reaction, therefore we get,

$$\left(m_x c^2 + E_x\right) + M_X c^2 = \left(E_y + m_y c^2\right) + \left(E_Y + M_Y c^2\right)$$

E_x, E_y and E_Y are the kinetic energies of respective particles.

Now the quantity $Q = E_y + E_Y - E_x \Rightarrow Q = \left(m_x + M_X - m_y - M_Y\right)c^2$

Where Q is called the Q-value of nuclear reaction.

(i) If Q is positive, the reaction is said to be exoergic (exothermic) and
(ii) If Q is negative, the reaction is called endoergic (endothermic).

The minimum K.E. required for incident particle (x) to start the nuclear reaction is called the threshold energy $(E_x)^{th}$. The relation between Q-values and threshold energy is:

$$\boxed{E_x^{th} = -Q\frac{(m_y + M_Y)}{(M_Y + m_y - m_x)}}$$

If $E_x^{th} = 0$ for exoergic or exothermic reactions i.e. these reaction are spontaneous process.

- **Mechanism of nuclear reactions:**
- **Transmutation by protons:**

(a) (p,α) reactions:

(i) ${}_3Li^7 + {}_1H^1 \rightarrow ({}_4Be^8)^* \rightarrow {}_2He^4 + {}_2He^4$ (ii) ${}_3Li^6 + {}_1H^1 \rightarrow ({}_4Be^7)^* \rightarrow {}_2He^3 + {}_2He^4$

(iii) ${}_4Be^9 + {}_1H^1 \rightarrow ({}_5B^{10})^* \rightarrow {}_3Li^6 + {}_2He^4$ (iv) ${}_5B^{11} + {}_1H^1 \rightarrow ({}_6C^{12})^* \rightarrow {}_4Be^8 + {}_2He^4$

(v) ${}_9F^{19} + {}_1H^1 \rightarrow ({}_{10}Ne^{20})^* \rightarrow {}_8O^{16} + {}_2He^4$

(b) (p–n) Reaction:

(i) ${}_5B^{11} + {}_1H^1 \rightarrow ({}_6C^{12})^* \rightarrow {}_6C^{11} + {}_0n^1$ (ii) ${}_8O^{18} + {}_1H^1 \rightarrow ({}_9F^{19})^* \rightarrow {}_9F^{18} + {}_0n^1$

(iii) ${}_{11}Na^{23} + {}_1H^1 \rightarrow ({}_{12}Mg^{24})^* \rightarrow {}_{12}Mg^{23} + {}_0n^1$ (iv) ${}_{28}Ni^{58} + {}_1H^1 \rightarrow ({}_{29}Cu^{59})^* \rightarrow {}_{29}Cu^{58} + {}_0n^1$

(v) ${}_{29}Cu^{65} + {}_1H^1 \rightarrow ({}_{30}Zn^{66})^* \rightarrow {}_{30}Zn^{65} + {}_0n^1$ (usually endoergic)

(c) (p, d) reaction:

(i) ${}_3Li^7 + {}_1H^1 \rightarrow {}_3Li^6 + {}_1H^2$ (ii) ${}_4Be^9 + {}_1H^1 \rightarrow {}_4Be^8 + {}_1H^2$

(d) Proton capture:

Compound nucleus in excited states come to ground state with γ–ray photon.

(i) ${}_3Li^7 + {}_1H^1 \rightarrow ({}_4Be^8)^* \rightarrow {}_4Be^8 + \gamma$ (ii) ${}_6C^{12} + {}_1H^1 \rightarrow ({}_7N^{13}) \rightarrow {}_7N^{13} + \gamma$

(iii) ${}_7N^{14} + {}_1H^1 \rightarrow ({}_8O^{15})^* \rightarrow {}_8O^{15} + \gamma$ (iv) ${}_9F^{19} + {}_1H^1 \rightarrow ({}_{10}Ne^{20})^* \rightarrow {}_{10}Ne^{20} + \gamma$

(v) ${}_{13}Al^{27} + {}_1H^1 \rightarrow ({}_{14}Si^{28})^* \rightarrow {}_{14}Si^{28} + \gamma$ (vi) ${}_{24}Cr^{50} + {}_1H^1 \rightarrow ({}_{25}Mn^{51})^* \rightarrow {}_{25}Mn^{51} + \gamma$

If incident proton has an energy (> 20 MeV) the compound nucleus has sufficient excitation energy to permit the expulsion of two or more nucleons.

(2) Transmutation by Neutrons:-

Neutrons have no electric charge and can penetrate positively charged nuclei without any experience of repulsive electrostatic force.

(a) (n–α) reaction:

With slow Neutrons

(i) ${}_3Li^6 + {}_0n^1 \rightarrow ({}_3Li^7)^* \rightarrow {}_1H^3 + {}_2He^4$ (ii) ${}_5B^{10} + {}_0n^1 \rightarrow ({}_5B^{11})^* \rightarrow {}_3Li^7 + {}_2He^4$

(iii) ${}_{13}Al^{27} + {}_0n^1 \rightarrow ({}_{13}Al^{28})^* \rightarrow {}_{11}Na^{24} + {}_2He^4$ (iv) ${}_7N^{14} + {}_0n^1 \rightarrow ({}_7N^{15})$

(v) $_5B^{11} + _2He^4$ (vi) $_3Li^7 + _2He^4 + _2He^4$

Capture of fast Neutrons - emission of α - particle are usually radioactive

(i) $_{11}Na^{23} + _0n^1 \rightarrow \left(_{11}Na^{24}\right)^* \rightarrow _9F^{20} + _2He^4$ followed by $_9F^{20} \rightarrow _{10}Ne^{20} + \beta^-$

(ii) $_{13}Al^{27} + _0n^1 \rightarrow \left(_{13}Al^{28}\right)^* \rightarrow _{11}Na^{24} + _2He^4$ followed by $_{11}Na^{24} \rightarrow _{12}Mg^{24} + \beta^-$

(b) (n, p) reaction: Proton in the nucleus is replaced by neutron mass number, does not change but change decreases by one unit.

(i) $_7N^{14} + _0n^1 \rightarrow \left(_7N^{15}\right)^* \rightarrow _6C^{14} + _1H^1 + Q$ followed by $_6C^{14} \rightarrow _7N^{14} + \beta^- + E_{max}$

(ii) $_2He^3 + _0n^1 \rightarrow \left(_2He^4\right)^* \rightarrow _1H^3 + _1H^1 + Q$ followed by $_1H^3 \rightarrow _2He^3 + \beta^- + E_{max}$

(iii) $_{17}Cl^{35} + _0n^1 \rightarrow \left(_{17}Cl^{36}\right)^* \rightarrow _{16}S^{35} + _1H^1 + Q$ followed by $_{16}S^{35} \rightarrow _{17}Cl^{35} + \beta^- + E_{max}$

With fast neutrons:

(i) $_{13}Al^{27} + _0n^1 \rightarrow \left(_{13}Al^{28}\right)^* \rightarrow _{12}Mg^{27} + _1H^1$ **(ii)** $_{16}S^{32} + _0n^1 \rightarrow \left(_{16}S^{33}\right)^* \rightarrow _{15}P^{32} + _1H^1$

(iii) $_{29}Cu^{65} + _0n^1 \rightarrow \left(_{29}Cu^{66}\right)^* \rightarrow _{28}Ni^{65} + _1H^1$ **(iv)** $_{30}Zn^{64} + _0n^1 \rightarrow \left(_{30}Zn^{65}\right)^* \rightarrow _{29}Cu^{64} + _1H^1$

(c) $(n-\gamma)$ **Reaction:**

(i) $_1H^1 + _0n^1 \rightarrow \left(_1H^2\right)^* \rightarrow _1H^2 + \gamma$ **(ii)** $_1H^2 + _0n^1 \rightarrow \left(_1H^3\right)^* \rightarrow _1H^3 + \gamma$

Product Nucleus is Radioactive:

(i) $_{45}Rh^{103} + _0n^1 \rightarrow \left(_{45}Rh^{104}\right)^* \rightarrow _{45}Rh^{104} + \gamma$ **(ii)** $_{49}In^{115} + _0n^1 \rightarrow \left(_{49}In^{116}\right)^* \rightarrow _{49}In^{116} + \gamma$

(iii) $_{92}U^{238} + _0n^1 \rightarrow \left(_{92}U^{239}\right)^* \rightarrow _{92}U^{239} + \gamma$ **(iv)** $_{29}Cu^{65} + _0n^1 \rightarrow \left(_{29}Cu^{66}\right)^* \rightarrow _{29}Cu^{66} + \gamma$

(v) $_{79}Au^{65} + _0n^1 \rightarrow \left(_{79}Au^{198}\right)^* \rightarrow _{79}Au^{198} + \gamma$

(d) (n – d) and (n – t) reaction: Bombardment of 90 MeV neutrons: $_7N^{14} + _0n^1 \rightarrow _6C^{12} + _1H^3$

(e) (n – 2n) reaction: One neutron is captured by nucleus and 2 neutrons are emitted. $Q < 0$ fast neutron are needed . In most cases, residue nucleus is unstable -followed by positron emission.

(i) $_6C^{11} + _0n^1 \rightarrow \left(_6C^{13}\right)^* \rightarrow _6C^{11} + 2\,_0n^1$ (ii) $_{19}K^{39} + _0n^1 \rightarrow \left(_{19}K^{40}\right)^* \rightarrow _{19}K^{38} + 2\,_0n^1$

(iii) $_{51}Sb^{121} + _0n^1 \rightarrow \left(_{51}Sb^{122}\right)^* \rightarrow _{51}Sb^{120} + 2\,_0n^1$

(f) Neutron - Three or more particles: Incident neutron ~ 30 MeV sufficient energy to overcome Coulomb barrier, 3 neutrons or even 2 neutrons and a proton are ejected from compound nucleus. Neutron (~ 100 MeV) Nuclei with moderate mass number undergo spallation and those of high mass number, e.g. Bi and Pb suffer fission probably accompanied by spallation.

- **(3) Transmutation by Deuteron:** High energy Deuteron.

(a) $(d-\alpha)$ reaction

(i) $_3Li^6 + {}_1H^2 \rightarrow \left({}_4Be^8\right)^* \rightarrow {}_2He^4 + {}_2He^4$ (ii) $_8O^{16} + {}_1H^2 \rightarrow \left({}_9F^{18}\right)^* \rightarrow {}_7N^{14} + {}_2He^4$,

(iii) $_{10}Ne^{20} + {}_1H^2 \rightarrow \left({}_{11}Na^{22}\right)^* \rightarrow {}_9F^{18} + {}_2He^4$

(iv) $_{12}Mg^{26} + {}_1H^2 \rightarrow \left({}_{13}Al^{28}\right)^* \rightarrow {}_{11}Na^{24} + {}_2He^4$,

(v) $_{13}Al^{27} + {}_1H^2 \rightarrow \left({}_{14}Si^{29}\right)^* \rightarrow {}_{12}Mg^{25} + {}_2He^4$

(b) (d – p) Reaction: Isotope creation

(i) $_6C^{12} + {}_1H^2 \rightarrow \left({}_7N^{14}\right)^* \rightarrow {}_6C^{13} + {}_1H^1$ (ii) $_3Li^7 + {}_1H^2 \rightarrow \left({}_4Be^9\right)^* \rightarrow {}_3Li^8 + {}_1H^1$

(iii) $_{11}Na^{23} + {}_1H^2 \rightarrow \left({}_{12}Mg^{25}\right)^* \rightarrow {}_{11}Na^{24} + {}_1H^1$ (iv) $_{15}P^{31} + {}_1H^2 \rightarrow \left({}_{16}S^{33}\right)^* \rightarrow {}_{15}P^{32} + {}_1H^1$

(v) $_{48}Cd^{114} + {}_1H^2 \rightarrow \left({}_{49}In^{116}\right)^* \rightarrow {}_{48}Cd^{115} + {}_1H^1$

(vi) $_{83}Bi^{209} + {}_1H^2 \rightarrow \left({}_{84}Po^{211}\right)^* \rightarrow {}_{83}Bi^{210} + {}_1H^1$

(c) (d – n) Reaction:

(i) $_3Li^7 + {}_1H^2 \rightarrow \left({}_4Be^9\right)^* \rightarrow {}_4Be^8 + {}_0n^1$ (ii) $_4Be^9 + {}_1H^2 \rightarrow \left({}_5B^{11}\right)^* \rightarrow {}_5B^{10} + {}_0n^1$

(iii) $_6C^{12} + {}_1H^2 \rightarrow \left({}_7N^{14}\right)^* \rightarrow {}_7N^{13} + {}_0n^1$

When two deuterons interact both the (d, n) and (d, p) reactions have been obtained.

$$_1H^2 + {}_1H^2 \rightarrow {}_2He^4 \rightarrow {}_1H^3 + {}_1H^1 + 4.02\,\text{MeV}$$

$$\rightarrow {}_2He^3 + {}_0n^1 + 3.25\,\text{MeV}$$

- **Cross-section of nuclear reactions (Classical):**

One of the most important parameters in nuclear reactions is the reaction cross-section, symbolised by σ. It is quantitative measure of the probability of occurrence of a nuclear reaction.

Let a parallel beam of N monoenergetic particles be incident per unit time normally on a target foil of surface area A and thickness Δx, having 'n' nuclei per unit volume. Now, the number ΔN of nuclei in the foil undergoing nuclear reaction will be proportional to (i) the intensity of the beam and (ii) the number of target nuclei contained in the the foil.

But the intensity I of the beam $= N/A$ and the number of nuclei in the foil $= nA.\Delta x$

Therefore, number of nuclei transmuted is $\Delta N \propto \frac{N}{A} nA\Delta x$

$\therefore\quad \Delta N = \sigma N n \Delta x = \sigma N n_1$, where σ = constant.

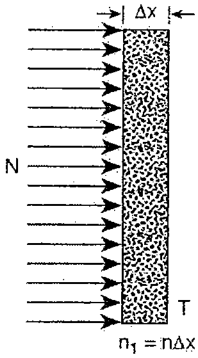


Figure: Nuclear reaction cross-section: bombardment of target foil.

where $n_1 = n\Delta x$, the number of nuclei per unit area of the target foil used.

The constant σ is called the nuclear reaction cross-section.

Therefore, nuclear reaction cross-section, $\boxed{\sigma = \dfrac{\Delta N}{N n_1}}$

- The probability of the incident particle to interact with the target atom will be equal to

$$P = \frac{\text{number of interacting particles}}{\text{number of incident particles}}$$

If *dN* is number of interacting particles and *N* is number of incident particles, then

$$P = \boxed{\frac{dN}{N} = \sigma\, n\, dx} \Rightarrow \int_{N_0}^{N} \frac{dN}{N} = \int_{z=z}^{z=0} \sigma\, n\, dx \Rightarrow \ln\left[\frac{N}{N_0}\right] = -\sigma\, n\,(+x) \Rightarrow \frac{N}{N_0} = e^{-\sigma\, nx}$$

$$\boxed{N = N_0\, e^{-\sigma\, n\, x}}$$

8.6. Nuclear Shell Model:

1. It very well explains the existence of magic numbers and the stability and high binding energy on the basis of closed shells.
2. The shell model provides explanation for the ground state spins and magnetic dipole moment of the nuclei. The neutrons and protons with opposite spins pair off so that the mechanical and magnetic dipole moment cancel and the odd or left out proton or neutron contributes to the spin and magnetic moment of the nuclei as a whole.
3. Nuclear isomerism, i.e., existence of isobaric, isotopic nuclei in different energy states of odd-*A* nuclei between 39-49, 69-81, 111 to 125 has been explained by shell model by the large difference in nuclear spins of isomeric states are their *A*-values are close to magic numbers.

Application of Nuclear Shell Model:

Determination of total angular momentum and parity of the ground state

1. If protons and neutrons both are even, then the total angular momentum become zero and parity is even.
2. If one of the nucleon (proton or neutron) is even and other is odd then the j value and $(-1)^{\ell}$ of the last unpaired nucleon give the total angular momentum and parity.
3. For odd-odd nucleon we use Brennan-Bernstein rules. According to this rules we define

the quantity, $N = (j_P - \ell_P) + (j_N - \ell_N)$

(i) If $N = 0$, then $j = |j_P - j_N|$, parity $= -ve$

(ii) If $N = \pm 1$ and $N = Z$ then $J = |J_P + J_N|$, parity $= \pm 1$

(iii) If $N = \pm 1$ and $N \neq Z$ then $J = (J_P + J_N - 1)$, parity $= +ve$

Example : ${}_8O^{16}$

$$Z = 8 \text{ (even) } \rightarrow 1s_{1/2}^2\ 1p_{3/2}^4\ 1p_{1/2}^2$$

$$N = 8 \text{ (even) } \rightarrow 1s_{1/2}^2\ 1p_{3/2}^4\ 1p_{1/2}^2$$

Since, there have no unpaired nucleon.
Therefore, parity is positive and angular momentum $j = 0$. It is represented by 0^+.

${}_{13}^{27}Al$: $Z = 13(\text{odd}) \rightarrow 1s_{1/2}^2\ 1p_{3/2}^4\ 1p_{1/2}^2\ 1d_{5/2}^5$

$N = 14(\text{even}) \rightarrow$ does not contribute in angular momentum

Therefor, $j = 5/2$; $l = 2$ (for d, $l = z$)

Parity = $(-1)^2 = +1$, even parity. It is represented by $\frac{5^+}{2}$.

${}_{16}^{33}S$: $Z = 16(\text{even}) \rightarrow$ does not contribute in angular momentum

$N = 17(\text{odd}) \rightarrow 1s_{1/2}^2\ 1p_{3/2}^4\ 1p_{1/2}^2\ 1d_{5/2}^5\ 2s_{1/2}^2\ 1d_{3/2}^2$

Therefor, $j = 3/2$; $l = 2$ (for d, $l = 2$)

Parity = $(-1)^2 = +1$, even parity. It is represented by $\frac{3^+}{2}$.

${}^{38}Cl_{17}$:

$Z = 17(\text{odd}) \rightarrow 1s_{1/2}^2\ 1p_{3/2}^4\ 1p_{1/2}^2\ 1d_{5/2}^6\ 2s_{1/2}^2\ 1d_{3/2}^1$ (particle state)

$N = 21(\text{odd}) \rightarrow 1s_{1/2}^2\ 1p_{3/2}^4\ 1p_{1/2}^2\ 1d_{5/2}^6\ 2s_{1/2}^2\ 1d_{3/2}^4\ 1f_{7/2}^1$ (particle state)

$$N = (j_P - \ell_P) + (j_N - \ell_N) = -\frac{1}{2} + \frac{1}{2} = 0$$

$$\therefore \quad j = |j_P - j_N| = \left|\frac{3}{2} - \frac{7}{2}\right| = \frac{4}{2} = 2 \text{ and parity} = -1$$

Therefore, spin parity = 2^-.

${}^{26}Al_{13}$:

$Z = 13(odd) = 1s_{1/2}^2\ 1p_{3/2}^4\ 1p_{1/2}^2\ 1d_{5/2}^5$

$N = 13(odd) = 1s_{1/2}^2\ 1p_{3/2}^4\ 1p_{1/2}^2\ 1d_{5/2}^5$

$$N = (J_P - \ell_P) + (J_N - \ell_N) = \frac{1}{2} + \frac{1}{2} = 1 \quad \text{and } N = Z$$

$$J = |J_P + J_N| = \left|\frac{5}{2} + \frac{5}{2}\right| = 5, \text{ parity} = +\text{ve} \quad \text{and parity is +ve}$$

Therefore, spin parity 5^+.

$^{56}Co_{27}$:

$Z = 27(odd) = 1s_{1/2}^2\ 1p_{3/2}^4\ 1p_{1/2}^2\ 1d_{5/2}^6\ 2s_{1/2}^2\ 1d_{3/2}^4\ 1f_{7/2}^7$

(hole state, because only one particle state is empty)

$N = 29(odd) = 1s_{1/2}^2\ 1p_{3/2}^4\ 1p_{1/2}^2\ 1d_{5/2}^6\ 2s_{1/2}^2\ 1d_{3/2}^4\ 1f_{7/2}^8\ 2p_{3/2}^1$ (particle state)

$$N = (j_P - \ell_P) + (j_N - \ell_N) = \frac{1}{2} + \frac{1}{2} = 1 \text{ and } N \neq Z$$

$$\therefore \quad J = |j_P + j_N - 1| = \frac{7}{2} + \frac{3}{2} - 1 = 4, \text{ parity} = +ve$$

Therefore, spin parity = 4^+.

8.7. Ground State of Deuteron $\left({}_1D^2\right)$

Deuteron is a single two nucleon (one *p* and one *n*) bound system which is found in nature. The important experimental determinal properties about Deuteron are given below:

(i) The *B.E.* of Deuteron/nucleon is very small compare to other nuclei i.e. it is a weakly system.

(ii) The ground state spin of Deuteron $I_d = 1$ (iii) The ground state parity of Deuteron = even (positive)

(iv) The quadrupole momento of Deuteron $(Q_d) \neq 0$

(v) The magnetic moment of Deuteron is slightly different from the sum of intrinsic magnetic moments of neutron + proton i.e. $(\mu_n + \mu_p) - \mu_d \to 0$

These factor represents that the ground state of Deuteron is a mixture of $\underset{(L=0)}{{}^3S}$ and $\underset{(L=2)}{{}^3D}$ states in which ^3S contribution is 96% and ^3D contribution is only 4%.

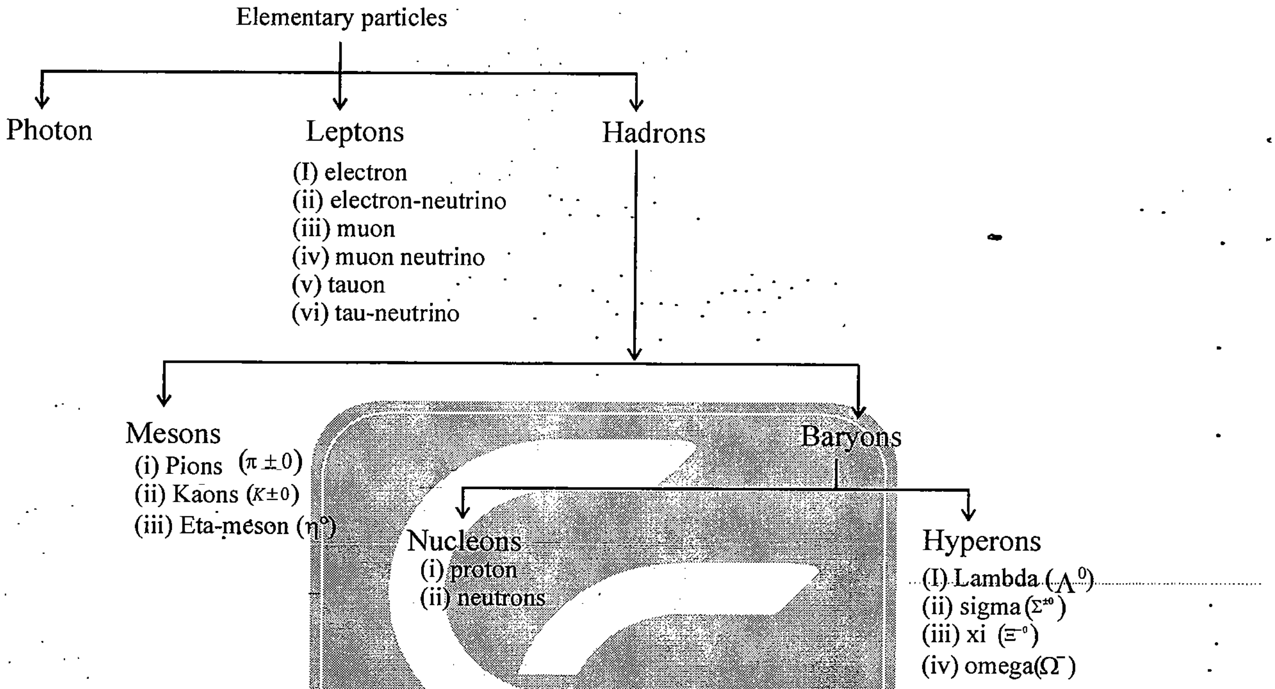
This indicate that nuclear force are non-central as $L \neq$ constant and they are spin dependent. In Deuteron both *p* and *n* have spin parallel to each other.

8.8. Elementary Particles:

It is a dream for physicists to be able to explain the material world, the physical universe entirely. This has driven them continuously to explore the fundamental entities of matter was atom then nucleus, neutron and protons and now the quarks. In this present chapter we will read about all elementary particle their spin charge, isospin, mass etc.

Classification of elementary particles:

The elementary particles may be classified in a number of different ways depending on their masses, interaction, statistics etc. Commonly, however, they are classified into such categories as


Elementary particles
Photon
Leptons
(I) electron
(ii) electron-neutrino
(iii) muon
(iv) muon neutrino
(v) tauon
(vi) tau-neutrino
Hadrons
Mesons
(i) Pions ($\pi \pm 0$)
(ii) Kaons ($K \pm 0$)
(iii) Eta-meson (η^0)
Baryons
Nucleons
(i) proton
(ii) neutrons
Hyperons
(I) Lambda (Λ^0)
(ii) sigma ($\Sigma^{\pm 0}$)
(iii) xi (Ξ^{-0})
(iv) omega (Ω^-)

The four fundamental forces:

In nature, there are four different types of fundamental interactions. They are:

1. **Gravitational force :**
   The gravitational force is the oldest known force. It holds us to the surface of the earth and holds the entire universe together. It is a long-range force varying as $\frac{1}{r^2}$. It has life time, $\tau = 10^{16}$ sec.
2. **Electromagnetic force:**
   The electromagnetic force was the second force known. In fact it was originally two forces the electric force and magnetic force. The electromagnetic force hold the atoms, molecules, solid and liquid together. It is also a long range force varying as $\frac{1}{r^2}$. It has a life time, $\tau = 10^{-20}$ sec.
3. **Weak interaction force:**
   This fundamental interaction involves leptons and hadrons. The β-decay of radioactive nuclei and decays of strong particles are typically of weak interaction. It has a life time, $\tau = 10^{-10}$ sec.
4. **Strong interaction force :**
   The strong nuclear force is responsible for holding the nucleons together. It is the strongest of all the forces but is a very short range force. Its effect occur within a distance of about 10^{-15} m. The strong nuclear force acts only on the hadrons. It has a life time, $\tau = 10^{-23}$ sec.

Some elementary particle and their properties:

Category	Particles Name	Symbols	Charge	Spin	B	L	s	I_3	Y	Quark structure
	Electron	e^-	−1	½	0	1	0		0	
	Electron-neutrino	ν_e	0	½	0	1	0		0	
Leptons	Muon	μ^-	−1	½	0	1	0		0	
	Muon-Neutrino	ν_μ	0	½	0	1	0		0	
	Tauon	τ	−1	½	0	1	0		0	
	Tauon-Neutrino	ν_τ	0	½	0	1	0		0	
		π^+	+1	0	0	0	0	+1	0	$u\bar{d}$
	Pions	π^0	0	0	0	0	0	0	0	$\frac{1}{\sqrt{2}}\left(u\bar{u}-d\bar{d}\right)$
Mesons		π^-	−1	0	0	0	0	−1	0	$d\bar{u}$
		K^+	+1	0	0	0	+1	-½	+1	$u\bar{s}$
	Kaons	K^0	0	0	0	0	+1	-½	+1	$d\bar{s}$
		K^-	−1	0	0	0	−1	½	−1	$s\bar{u}$
	Eta	η^0	0	0	0	0	0	0	0	$\frac{1}{\sqrt{6}}\left(u\bar{u}+d\bar{d}-2s\bar{s}\right)$
	η^1	η'	0	0	0	0	0	0	0	$\left(u\bar{u}+d\bar{d}+s\bar{s}\right)\frac{1}{\sqrt{3}}$
Nucleons	Proton	P	+1	½	1	0	0	½	1	uud
	Neutron	n	0	½	1	0	0	-½	1	udd
	Lambda	Λ^0	0	½	1	0	−1	½	0	uds
		Σ^+	+1	½	1	0	−1	+1	0	uus
	Sigma	Σ^0	0	½	1	0	−1	0	0	uds
Hyperons		Σ^-	-1	½	1	0	−1	0	0	dds
	Xi	Ξ^0	0	½	1	0	-2	½	-1	uss
		Ξ^-	−1	½	1	0	-2	-½	-1	dss
	Omega	Ω^-	-1	½	1	0	−3	0	−2	sss

Conservation laws in elementary particle reactions:

- **Exact conservation laws:**
  (i) Conservation of linear momentum (ii) Conservation of angular momentum
  (iii) Conservation of charge (iv) Conservation of baryon number
  (v) Conservation of lepton number
- **Approximate conservation laws:**
  (i) Isospin or isotopic spin(I):
  It arose however from the idea that pairs of particles like nucleons and triplets like pions hardly differ in their mass and may be considered as isotopes and that their charges, differing from each other by unity, suggest space qunatization similar to electron spin and orbit in a magnetic field.
- **A multiplet number** (M) is defined as the number of their different charge states. For instance, for nucleons i.e. protons or neutrons - the multiplet number $M = 2$. Similarly, for the triplet of pions, $M = 3$; $M = 2$ for two kaons and two antikaons; $M = 1$ for one neutral η^0.

  Now, $M = 2I+1 \quad \Rightarrow I = (M-1)/2$

- Isospin is treated as a vector $\vec{I}$ of magnitude $\sqrt{I(I+1)}$, like angular momentum, but I is dimensionless. Its component along Z-axis, is given by I_3 which have the allowed values.
  $I, (I-1), (I-2), \ldots\ldots\ldots\ldots -I$
- For nucleons $(M=2), I=(M-1)/2=\frac{1}{2}$ and the values of I_3 are $+\frac{1}{2}$ and $-\frac{1}{2}$; $I_3=+\frac{1}{2}$ is assigned to proton, $I_3=-\frac{1}{2}$ to neutron.
- For pions $(M=3), I=(M-1)/2=1$. Hence, $I_3=+1, 0, -1$. $I_3=+1$ is assigned to π^+, $I_3=0$ to π^0 and $I_3=-1$ to π^-.
- Isospin is conserved in strong interactions but is violated in electromagnetic and weak interaction. The Z-component of isospin, I_3- is conserved in strong and electromagnetic interaction and not in weak interaction.

(ii) Hypercharge(Y): It is defined as double of the average charge $\bar{Q}$ of the multiplet.

$$Y=2\bar{Q} \Rightarrow \bar{Q}=Y/2 \quad \Rightarrow \boxed{Q=I_3+Y/2}$$

- For any strong and electromagnetic interaction, the hypercharge is conserved, i.e. remains invariant. But it need not be conserved in weak interaction.

Example: $p+p \to \Lambda^0+K^0+p+\pi^+$

Hypercharge: $1+1=0+1+1+0 \quad \Rightarrow \Delta Y=0$

(iii) Strangeness number (S): It is defined as the difference of the hyperchange Y and the baryon number B.

$$S=Y-B \quad \Rightarrow Y=S+B$$

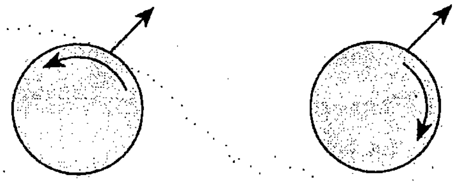
i.e., the hyperchange is the sum of the baryon number and the strangeness number

Therefore, $Q=I_3+\frac{B+S}{2}$

- Strangeness number S is conserved in strong and electromagnetic reactions. For a weak interactions, $\Delta S=0$ or ± 1

(iv) Parity:

When particle like neutrinos are emitted during radioactive decay, they show a preferred spin direction. If a neutrino spins in the direction at which a right-handed screw advances, it is said to posses a helicity +1; if however the spin is in the direction of a left-handed screw, the helicity is -1. As the parity P is related to the spin J, the two quantum numbers are usually combined and is symbolised by J^P. So, $\left(\frac{1}{2}\right)^+$ means the J-value is $\frac{1}{2}$ and $P=+1$; $\left(\frac{1}{2}\right)^-$ indicates $J=\frac{1}{2}, P=-1$.



Neutrino
(a)

Anti-Neutrino
(b)

Figure: Helicity of neutrino and antineutrino

- In strong and electromagnetic interactions, parity is conserved but it is not conserved in weak interaction.

(v) Charge conjugation:

Charge conjugation means reversal of the signs of all types of charge i.e. electric, baryonic and laptonic of the particles (Figure below). If a physical law holding for particles also holds for corresponding antiparticles, the principle of charge conjugation is said to be valid.

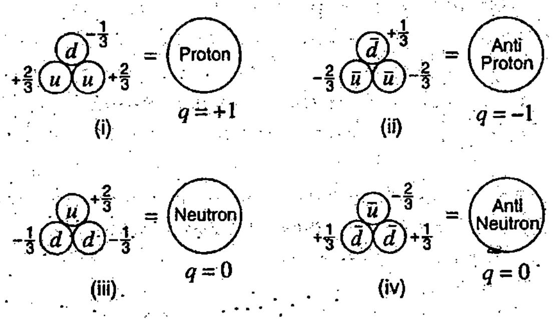

$-\frac{1}{3}$
d
$+\frac{2}{3}$ u u $+\frac{2}{3}$
=
Proton
$q=+1$
(i)
$+\frac{1}{3}$
$\bar{d}$
$-\frac{2}{3}$ $\bar{u}$ $\bar{u}$ $-\frac{2}{3}$
=
Anti Proton
$q=-1$
(ii)
$+\frac{2}{3}$
u
$-\frac{1}{3}$ d d $-\frac{1}{3}$
=
Neutron
$q=0$
(iii)
$-\frac{2}{3}$
$\bar{u}$
$+\frac{1}{3}$ $\bar{d}$ $\bar{d}$ $+\frac{1}{3}$
=
Anti Neutron
$q=0$
(iv)

Figure: Formation of proton, antiproton and neutron, antineutron by charge conjugation

- Strong and electromagnetic interactions are charge conjugate invariant. But the weak interaction like β-decay does not obey charge conjugation.

(vi) Time reversal:

The operation T i.e., time-reversal means replacing the time t by $-t$ in all equations of motion i.e., reflection of time axis at the origin of time coordinate in relativistic space-time continuum. It is thus, like the parity operation, a discrete change.

- T-operation consists of reversing the signs of momenta $\left(\vec{p}=d\vec{r}/dt\right)$ and angular momenta $\left(\vec{L}=\vec{r}\times\vec{p}\right)$.

  T also transforms the wave function to its complex conjugate. If T is conserved i.e., time-reversal invariance occurs, then the reversed equation of motion is also a valid equation of motion of the system concerned. All the known fundamental equations of motion are invariant in time-reversal.
- Strong and electromagnetic interactions are invariant under time-reversal transformation.

***CPT* theorem:**

This is an exact conservation law. It states that all interactions in nature are invariant under joint operations of charge conjugation (C), inversion of space coordinates at origin, i.e. parity (P) and reversal of time (T). The order of operations is immaterial.

The invariance of CPT transformation implies that if any interaction is not invariant under any one of C, P and T operations, its effect gets compensated by the joint effect of the other two.

Conserved quantities in different type of reactions:

Conserved Quantities	Strong	Electromagnetic	Weak
1. Charge	✓	✓	✓
2. Linear Momentum	✓	✓	✓
3. Relativistic Energy	✓	✓	✓
4. Spin	✓	✓	✓
5. L_e	✓	✓	✓
6. L_u	✓	✓	✓
7. L_τ	✓	✓	✓
8. Baryon Number B	✓	✓	✓
9. I_3	✓	✓	×
10. I	✓	May or may not be conserved	×
11. S	✓	✓	May or may not be conserved
12. Y = B + S	✓	✓	May or may not be conserved
13. Parity (P)	✓	✓	×
14. Charge conjugation (C)	✓	✓	×
15. Time reversion (T)	✓	May or may not be conserved	May or may not be conserved
16. CP	✓	✓	May or may not be conserved
17. CPT	✓	✓	✓

Relationship between particles and antiparticles:

(a) Mass → Same
(b) Spin → Same
(c) Charge → Same but opposite in sign.
(d) Mag. moment → Same in magnitude but opposite in sign.
(e) Mean life time of free decay → Same
(f) Annihilation → In pairs
(g) Creation → In pairs
(h) Total isotopic spin → same
(i) Third component of isospin(I_3) → Same in mag. but opposite in sign.
(j) Intrinsic parity → Same for bosons and opposite for Fermions.
(k) Strangeness number (S) → Same in mag. but opposite in sign.
(l) Lepton number (L) → Same in mag. but opposite in sign.
(m) Baryon number (B) → Same in mag. but opposite in sign.
(n) Hypercharge (Y) → Same in mag. but opposite in sign.

The particle and antiparticle have same symbol except a bar over particle for antiparticle.

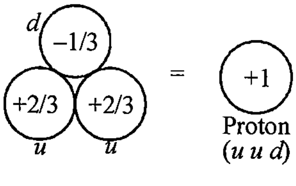
Quark model:

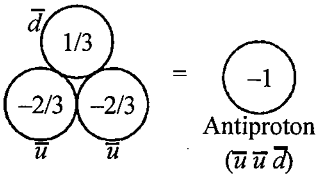
The elementary particles can be conceived (as far as isospin and hyper charge are concerned) as being built out of combination of quarks. The original three quarks were called up (u), down (d) and strange (s).

Each quarks has an anti-quark associated with it $(\bar{u}, \bar{d}, \bar{s})$. Since a quark and its anti-quark have opposite quantum numbers. They can be created from energy. In the reverse process, a quark and its anti-quark annihilate and give energy.

Thus, energy $\rightarrow u + \bar{u}$; $d + \bar{d} \rightarrow$ energy

Quark models for proton and antiproton.





SOLVED PROBLEMS

1. If an electron is confined within a nucleus whose diameter is 10^{-14} m, estimate its minimum kinetic energy.

Soln. The de-Broglie wavelength of a minimum-energy electron confined inside the nucleus would be approximately twice the nuclear diameter (one-half a wavelength would fit into the diameter).

Therefore, the electron's momentum would be of the order of magnitude.

$$p=\frac{h}{\lambda}=\frac{hc}{\lambda c}=\frac{12.4\times10^{3}\text{ eV Å}}{\left(2\times10^{-4}\text{ Å}\right)c}=62\times10^{6}\,\frac{\text{eV}}{c}=62\,\frac{\text{MeV}}{c}$$

corresponding to a kinetic energy of

$$K=\sqrt{(pc)^{2}+E_{0}^{2}}-E_{0}=\sqrt{\left(62\frac{\text{MeV}}{c}\times c\right)^{2}-(0.511\text{ MeV})^{2}}-0.511\text{ MeV}=61\text{ MeV}$$

2. Determine the radii of a ^{16}O and a ^{208}Pb nucleus.

Soln. From $R=r_0A^{1/3}=(1.4\text{ fm})A^{1/3}$,

$$R_0=(1.4\text{ fm})(16)^{1/3}=3.53\text{ fm};\ R_{Pb}=(1.4\text{ fm})(208)^{1/3}=8.29\text{ fm}$$

3. Determine the stable nucleus that has a radius 1/3 that of ^{189}Os.

Soln. Since, $R\propto A^{1/3}$

$$\frac{1}{3}=\frac{R}{R_{Os}}=\left(\frac{A}{A_{Os}}\right)^{1/3}=\left(\frac{A}{189}\right)^{1/3}$$

giving,

$$A=\frac{189}{27}=7 \text{ corresponding to } {}^{7}\text{Li}.$$

4. Calculate the binding energy of ${}^{126}_{52}Te$.

Soln. The binding energy is given by

$$BE=\left(Zm_p\right)c^2+\left(Nm_n\right)c^2-M_{\text{nucleus}}c^2$$

$$=\left(52\times1.007\,825u+74\times1.008\,665\,u-125.903\,322u\right)\times931.5\text{ MeV}/u=1.066\times10^{3}\text{ MeV}$$

5. What is the energy required to remove the least tightly bound neutron from ${}^{40}_{20}Ca$?

Soln. From conservation of mass-energy,

$$M_{^{40}Ca}c^2+E=\left(M_{^{39}Ca}+m_n\right)c^2$$

$$(39.962589u)(931.5\text{ MeV}/u)+E=(38.970691\,u+1.008665\,u)(931.5\text{ MeV}/u)$$

$$\Rightarrow\quad E=15.6\text{ MeV}$$

6. Determine the electrical potential energy of the protons in a nucleus if it is assumed that the charge is uniformly speherically distributed.

Soln. Consider a thin spherical shell of charge, $dq=\rho PV=\rho\left(4\pi r^2dr\right)$ that is added to a sphere that has the same charge density and has total charge

$$q=\rho V=\rho\left(\frac{4}{3}\pi r^3\right)$$

The electrical potential energy dE of the thin shell is then

$$dE=\frac{kq}{r}dq=\frac{k}{r}\left(\frac{4}{3}\rho\pi r^3\right)\left(4\pi\rho r^2dr\right)=3k\left(\frac{4}{3}\rho\pi\right)^2r^4dr$$

The total electrical potential energy of the charged sphere is found by integrating dE from $r = 0$ to $r = R$, the radius of the sphere

$$E = \int_0^R dE = 3k\left(\frac{4}{3}\rho\pi\right)^2 \int_0^R r^4 dr = \frac{3}{5}k\left(\frac{4}{3}\rho\pi\right)^2 R^5 = \frac{3k}{5R}\left(\frac{4}{3}\rho\pi R^3\right)^2$$

Since $\frac{4}{3}\rho\pi R^3 = \rho V = Q = Ze,$ we have

$$E = \frac{3}{5}\frac{kZ^2e^2}{R}$$

The charges forming a nucleus are actually not continuous but must be brought in discrete amounts. For $Z = 1$ the Coulomb energy should be zero, but the above expression gives a finite answer. To correct the above relationship, Z^2 should be changed to $Z(Z-1)$. For large values of Z this is a minor correction, but not for small values of Z. The correct Coulomb energy is:

$$E_C = \frac{3}{5}\frac{kZ(Z-1)e^2}{R}$$

7. What is the activity of one gram of $^{226}_{88}Ra$, whose half-life is 1622 years?

Soln. The number of atoms in 1 g of radius is

$$N(1\ g)\left(\frac{1\text{ g-mole}}{226\text{ g}}\right)\left(6.025\times10^{23}\ \frac{\text{atoms}}{\text{g-mole}}\right) = 2.66\times10^{21}$$

The decay constant is related to the half-life by

$$\lambda = \frac{0.693}{T_{1/2}} = \left(\frac{0.693}{1622\ y}\right)\left(\frac{1\ y}{365\ d}\right)\left(\frac{1\ d}{8.64\times10^4\ s}\right) = 1.355\times10^{-11}s^{-1}$$

The activity is then found from

Activity $= \lambda N = \left(1.355\times10^{-11}s^{-1}\right)\left(2.666\times10^{21}\right) = 3.612\times10^{10}$ disintegrations/s

The definition of the curie is 1 Ci $= 3.700\times10^{10}$ disintegrations/s. This is approximately equal to the value found above.

8. Over what distance in free space will the intensity of a 5 eV neutron beam be reduced by a factor of one-half? ($T_{1/2} = 12.8$ min).

Soln. The speed of the neutron in the beams is found from $\frac{1}{2}mv^2 = K$

$$\frac{1}{2}\left(1.67\times10^{-27}\ kg\right)v^2 = (5\ eV)\left(\frac{1.6\times10^{-19}\ J}{1\ eV}\right)$$

During a time of $T_{1/2} = 12.8$ min, half the neutron will have decayed from the beam. The distance traveled by the undecayed neutrons during this time is

$$d = vt = (31.0\ km/s)(12.8\text{ min})(60\ s/\text{min}) = 23,800\ km$$

or about 2 earth diameters.

9. How much time is required for 5 mg of $^{22}Na\left(T_{1/2} = 2.60\text{ y}\right)$ to reduce to 1 mg?

Soln. Since the mass of a sample will be proportional to the number of atoms in the sample, we may write

$$m = m_0 e^{-\lambda t} = m_0 e^{-\left(0.693/T_{1/2}\right)t}$$

$$\Rightarrow\ 1\text{ mg} = (5\text{ mg})e^{-(0.693/2.60\text{ y})t}$$

$$\Rightarrow\ e^{(0.693/2.60\text{ y})t} = 5$$

Taking logarithms of both sides, we find

$$\frac{0.693\,t}{2.60\,y} = \ln 5 = 1.61 \quad \text{or} \quad t = 6.04\ y$$

10. If 3.6 gm of uranium is to be completely converted to energy, how many Joules of energy will be obtained from it ? **[H.C.U.-2010]**

(a) 100×10^{13} J (b) 3.24×10^{13} J (c) 32.4×10^{13} J (d) 324.0×10^{13} J

Soln. Mass of Uranium $=3.6\ gm=3.6\times10^{-3}kg$

If the mass of Uranium is to be completely converted to energy, then

$$E=mc^2$$

$$=\left(3.6\times10^{-3}\right)\times\left(3\times10^{8}\right)^2 J=32.4\times10^{13}J$$

Correct option is (c)

11. The half life of thorium is 1.4×10^{10} years. What is the time required for 10% of a sample of thorium to disintegrate ? **[H.C.U.-2011]**

(a) 2.13×10^{9} years (b) 2.13×10^{7} years (c) 2.13×10^{6} years (d) 2.13×10^{5} years

Soln. Given : Half life of Thorium $t_{1/2}=1.4\times10^{10}$ years.

Let the initial number of nuclei of Thorium be N_0

$$\therefore\quad N=N_0e^{-\lambda t}$$

Now, $\lambda=\dfrac{\ln 2}{t_{1/2}}$

$$\Rightarrow 0.9N_0=N_0e^{-\frac{\ln 2}{t_{1/2}}t}$$

$$\Rightarrow\quad 0.9=e^{-\ln 2(t/t_{1/2})}\ \Rightarrow\ \ln 0.9=-\ln 2\left(\frac{t}{t_{1/2}}\right)$$

$$\Rightarrow\quad t=-t_{1/2}\frac{\ln 0.9}{\ln 2}=t_{1/2}\frac{\ln(10/9)}{\ln 2}=1.4\times10^{10}\log_2(0.9)$$

$$=1.4\times10^{10}\times0.152=2.13\times10^{9}\text{ years.}$$

Correct option is (a)

12. Let m_{He}, m_p, m_n, m_e represent the masses of ^{4}He nucleus, proton, neutron and electron, respectively. These masses satisfy the following relation

(a) $m_{He}=2\left(m_p+m_n+m_e\right)$ (b) $m_{He}=2\left(m_p+m_n\right)$

(c) $m_{He}>2\left(m_p+m_n\right)$ (d) $m_{He}<2\left(m_p+m_n\right)$ **[H.C.U.-2013]**

Soln. For a given nuclei, its mass is less than total mass of its individual constituents (separated protons and neutrons).

$$\therefore\quad m_{He}<2m_p+2m_n\quad\Rightarrow\quad m_{He}<2\left(m_p+m_n\right)$$

Correct option is (d)

13. An experiment shows that a given radioactive element emits 100 α particles/minute initially and after 10 seconds the rate becomes 10 α particles/minute. The half life period of this α-emitter (upto 2nd decimal place) is

(a) 3.00 sec. (b) 0.33 sec. (c) 30.00 sec. (d) 0.03 sec. **[H.C.U.-2013]**

Soln. Initial activity $\lambda N_i=100$, final activity $\lambda N_f=10$

$$\Rightarrow\ \frac{N_i}{N_f}=10$$

Now, $N_f = N_i e^{-\lambda t}$, $t = 10$ sec (given)

$$\Rightarrow \frac{1}{10} = e^{-\lambda\times 10} \Rightarrow \lambda = \frac{\ln 10}{10}$$

Half life $t_{1/2} = \frac{\ln 2}{\lambda} \Rightarrow t_{1/2} = \frac{\ln 2}{\ln 10}\times 10 \approx 3$ sec

Correct option is (a)

14. The half life of ^{215}At is 100μs. If the sample contains 6 mg of element, the initial activity of the sample is (Assume Avogadro number is 6.03×10^{23} atom/mol)

(a) 0.58×10^{23} Becquerels (b) 1.16×10^{23} Becquerels

(c) 3.32×10^{23} Becquerels (d) 6.64×10^{23} Becquerels **[H.C.U.-2014]**

Soln. Let initial number of nuclei of A be N_0 and that of B be N_0

Half life of $A\left(t_{1/2}\right)_A : 3 = \frac{\ln 2}{\lambda_A} \Rightarrow \lambda_A = \frac{\ln 2}{3}$

Half life of $B\left(t_{1/2}\right)_B : 6 = \frac{\ln 2}{\lambda_B} \Rightarrow \lambda_B = \frac{\ln 2}{6}$

After $t = 18$ hours,

$$N_A = N_0 e^{-\lambda t} = N_0 e^{-\frac{\ln 2}{3}\times 18} = N_0 e^{-6\ln 2}$$

$$N_B = N_0 e^{-\lambda_B t} = N_0 e^{-\frac{\ln 2}{6}\times 18} = N_0 e^{-3\ln 2}$$

$$\frac{N_A}{N_B} = \frac{e^{-\ln 2^6}}{e^{-\ln 2^3}} = \frac{1}{2^6} \div \frac{1}{2^3} = \frac{1}{8}$$

$\therefore\ N_A : N_B = 1:8$

Correct option is (c)

15. Four moles of a radioactive material decays into 0.5 moles in 90 seconds. Then the disintegration constant is

(a) $\frac{0.693}{30}s^{-1}$ (b) $\frac{0.693}{45}s^{-1}$ (c) $\frac{0.693}{15}s^{-1}$ (d) $\frac{0.693}{90}s^{-1}$ **[H.C.U.-2016]**

Soln. Initial number of nuclei = 4 moles $\times N_A$ where N_A is Avogadro number.

Final number of nuclei = 0.5 mole $\times N_A$

We have, $N = N_0 e^{-\lambda t}$

$$\Rightarrow 0.5\,N_A = 4N_A e^{-\lambda\times 90}$$

$$\Rightarrow \frac{40}{5} = e^{90\lambda} \Rightarrow 90\lambda = \ln 8$$

$$\Rightarrow \lambda = \frac{\ln 8}{90} = \frac{3\ln 2}{90} = \frac{0.693}{30}$$

Correct option is (a)

6. Naturally occurring gallium contains two isotopes ${}^{71}_{31}\text{Ga}$ and ${}^{69}_{31}\text{Ga}$ with relative abundance of 40% and 60% respectively. The average number of neutrons in a mixture of gallium nucleus is

(a) 39.2 (b) 37.8 (c) 38.8 (d) 40 **[B.H.U-2010]**

Soln. No. of neutrons in ${}^{71}_{31}Ga \Rightarrow N_1 = 40$

No. of neutrons in ${}^{69}_{31}Ga \Rightarrow N_2 = 38$

Relative abundance of ${}^{71}_{31}Ga$ is 40%, relative abundance of ${}^{69}_{31}Ga$ is 60%

The average number of neutrons in a mixture of Gallium nucleus,

$$= N_1 \times \frac{40}{100} + N_2 \frac{60}{100} = 40 \times 0.4 + 38 \times 0.6 = 38.8$$

Correct option is (c)

17. If λ_1 and λ_2 are the decay constants for parent and daughter nuclei respectively and $N_1(0)$ is the number of parent nuclei at time $t = 0$, then the number of atoms N_2 of daughter nuclei at time t is given by **[B.H.U-2011]**

(a) $N_2 = N_1(0)\frac{\lambda_1}{(\lambda_1 - \lambda_2)}\left[e^{-\lambda_2 t} - e^{-\lambda_1 t}\right]$ (b) $N_2 = N_1(0)\frac{\lambda_1}{(\lambda_1 - \lambda_2)}\left[e^{-\lambda_1 t} - e^{-\lambda_2 t}\right]$

(c) $N_2 = N_1(0)\frac{\lambda_2}{(\lambda_2 - \lambda_1)}\left[e^{-\lambda_2 t} - e^{-\lambda_1 t}\right]$ (d) $N_2 = N_1(0)\frac{\lambda_2}{(\lambda_1 - \lambda_2)}\left[e^{-\lambda_1 t} - e^{-\lambda_2 t}\right]$

Soln. The number of parent nuclei at time$(t = 0) = N_1(0)$

decay constant for parent nuclei is λ_1, decay constant for daughter nuclei is λ_2

$$\frac{dN_1}{dt} = -\lambda_1 N_1 \quad \ldots(i)$$

$$\frac{dN_2}{dt} = +\lambda_1 N_1 - \lambda_2 N_2 \quad \ldots(ii)$$

Solving equation (i)

We have, $N_1 = N_1(0)e^{-\lambda_1 t}$ ...(iii)

Using equation (ii) and (iii),

$$\frac{dN_1}{dt} = \lambda_1 N_1(0)e^{-\lambda_1 t} - \lambda_2 N_2$$

$$\frac{dN_2}{dt} + \lambda_2 N_2 = \lambda_1 N_1(0)e^{-\lambda_1 t}$$

The integrating factor I.F $= e^{\int \lambda_2\, dt} = e^{\lambda_2 t}$

$$\Rightarrow e^{\lambda_2 t}\left(\frac{dN_2}{dt} + \lambda_2 N_2\right) = \lambda_1 N_1(0)e^{(\lambda_2 - \lambda_1)t}$$

$$\Rightarrow \frac{d}{dt}\left(N_2 e^{\lambda_2 t}\right) = \lambda_1 N_1(0)e^{(\lambda_2 - \lambda_1)t}$$

$$\Rightarrow N_2 e^{\lambda_2 t} = \frac{\lambda_1}{\lambda_2-\lambda_1} N_1(0) e^{(\lambda_2-\lambda_1)t} + C$$

Where C is constant.

At $t=0, N_2(t=0)=N_2(0)=0$

$$C = N_2(0) - \frac{\lambda_1}{\lambda_2-\lambda_1} N_1(0) \Rightarrow C = -\frac{\lambda_1}{\lambda_2-\lambda_1} N_1(0)$$

$$\Rightarrow N_2 e^{\lambda_2 t} = \frac{\lambda_1}{\lambda_2-\lambda_1} N_1(0) e^{(\lambda_2-\lambda_1)t} - \frac{\lambda_1}{\lambda_2-\lambda_1} N_1(0)$$

$$\Rightarrow N_2 e^{\lambda_2 t} = \frac{\lambda_1}{\lambda_2-\lambda_1} N_1(0)\left[e^{\lambda_2 t-\lambda_1 t} - 1\right]$$

$$\Rightarrow N_2(t) = \frac{\lambda_1}{\lambda_2-\lambda_1} N_1(0)\left[e^{-\lambda_1 t} - e^{-\lambda_2 t}\right]$$

$$\Rightarrow N_2(t) = \frac{\lambda_1}{\lambda_1-\lambda_2} N_1(0)\left[e^{-\lambda_2 t} - e^{-\lambda_1 t}\right]$$

Correct option is (a)

18. Cobalt 60 is used in hospitals as a radiaoctive source in medicines. It has a half-life of 5.25 years. After a use for certain period its activity has been found to decrease by a factor of 8. How is old the sample?

(a) 15.75 years (b) 10.5 years (c) 21 years (d) 42 years **[B.H.U-2011]**

Soln. Half life of Cobalt-60, $t_{1/2} = 5.25$ years

decay constant $\lambda = \dfrac{\ln 2}{t_{1/2}} = \dfrac{\ln 2}{5.25}$

Now, initial activity $A_1 = \lambda N_1$ ...(i)

Where N_1 – initial no. of nuclei

Final acitvity, $A_2 = \lambda N_2$ ...(ii)

Given, $A_2 = \dfrac{A_1}{8}$ ...(iii)

$$\Rightarrow \lambda N_2 = \frac{\lambda N_1}{8}$$

$$\Rightarrow N_2 = \frac{N_1}{8}$$

We have, $N_2 = N_1 e^{-\lambda t}$

$$\Rightarrow \ln 8 = \lambda t = \frac{t \ln 2}{5.25}$$

$$t = 5.25 \frac{\ln 8}{\ln 2} = 5.25 \log_2 8 = 5.25 \log_2 2^3 = 5.25 \times 3 = 15.75 \text{ years}$$

Correct option is (a)

9. The interaction between neutrons and protons may be represented by the Yukawa potential $u(r) = -\left(\frac{r_0}{r}\right)u_0 e^{-r/a_0}$, then the force $F(r)$ between neutrons and protons is **[B.H.U-2011]**

(a) $-\left(\frac{r_0}{r}\right)u_0 e^{-r/r_0}\left[\frac{1}{r}-\frac{1}{r_0}\right]\hat{r}$
(b) $-\left(\frac{r_0}{r}\right)u_0 e^{-r/r_0}\left[\frac{1}{r}+\frac{1}{r_0}\right]\hat{r}$

(c) $-\left(\frac{r_0}{r}\right)u_0 e^{-r/r_0}\left[\frac{1}{r_0}\right]\hat{r}$
(d) $-\frac{r_0}{r^2}u_0 e^{-r/r_0}\left[\frac{1}{r}-\frac{2}{r_0}\right]$

Soln. Given $U(r) = -\left(\frac{r_0}{r}\right)u_0 e^{-r/r_0}$

$$F(r) = \frac{-\partial U(r)}{\partial r}$$

$$= -\frac{\partial}{\partial r}\left[-\frac{r_0}{r}u_0 e^{-r/r_0}\right]$$

$$= r_0 u_0 \frac{\partial}{\partial r}\left[\frac{e^{-r/r_0}}{r}\right]$$

$$= r_0 u_0\left[-\frac{e^{-r/r_0}}{r_2} - \frac{1}{rr_0}e^{-r/r_0}\right]$$

$$F(r) = -\frac{r_0}{r}u_0 e^{-r/r_0}\left[\frac{1}{r}+\frac{1}{r_0}\right]\hat{r}$$

Correct option is (b)

20. A radioactive nucleus of type 1 decays exponentially with a decay constant λ_1 to stable nucleus of type 2. If at time $t = 0$, the numbers of type 1 and 2 nuclei are respectively $N_1 = N_0$ and $N_2 = 0$, what is the number of type 2 nucleus present at time t? **[B.H.U-2012]**

(a) $N_0 \exp - \lambda_1 t$ (b) $N_0(1-\exp-\lambda_1 t)$ (c) $N_0(1+\exp-\lambda_1 t)$ (d) $1-N_0\exp-\lambda_1 t$

Soln. For nucleus of type 1,

$$\frac{dN_1}{dt} = -\lambda_1 N_1 \qquad \ldots(i)$$

For nucleus of type 2,

$$\frac{dN_2}{dt} = \lambda_1 N_1 \qquad \ldots(ii)$$

Now, from (i),

$$\frac{1}{N_1}dN_1 = -\lambda_1 dt$$

Integrating both sides,

$$\ln N_1 = -\lambda_1 t + c$$

at $t = 0, N_1 = N_0, c = \ln N_0$

$$\Rightarrow \ln N_1 = -\lambda_1 t + \ln N_0$$

$$\Rightarrow N_1 = N_0 e^{-\lambda_1 t}$$

From (ii), $\frac{dN_2}{dt} = \lambda_1 N_1 = \lambda_1 N_0 e^{-\lambda_1 t}$

$$\Rightarrow dN_2 = \lambda_1 N_0 e^{-\lambda_1 t} dt$$

$$\Rightarrow N_2 = \lambda_1 N_0 \frac{e^{-\lambda_1 t}}{-\lambda_1} + D$$

Now, at $t = 0, N_2 = 0 \Rightarrow D = N_0$

$$\Rightarrow N_2 = -N_0 e^{-\lambda_1 t} + N_0$$

$$\Rightarrow N_2 = N_0 \left(1 - e^{-\lambda_1 t}\right)$$

Correct option is (b)

21. Suppose that ${}_Z^A X$ decays by natural radioactivity in two stages to ${}_{Z-1}^{A-4} Y$. The two stages would most likely be which of the following? **[B.H.U-2012]**

	First Stage	Second Stage
(a)	β^- emission with an antineutrino	α emission
(b)	β^- emission	α emission with a neutrino
(c)	β^- emission	γ emission
(d)	Emission of a deuteron	emission of two neutrons

Soln. First state :

$${}_Z^A X \longrightarrow {}_{Z+1}^{A} X + {}_{-1}^{0} e + {}_0^0 \bar{\nu} \qquad \text{(beta decay with an anti-nentrino)}$$

Second state:

$${}_{Z+1}^{A} X \longrightarrow {}_{Z-1}^{A-4} X + {}_2^4 He \qquad \text{(alpha decay)}$$

Correct option is (a)

2. ${}_{92}^{238}\text{U}$ nucleus decay involves 8 α- decays and 6 β- decays. The end product of the series have:
(a) Z = 82, A = 206 (b) Z = 84, A = 224 (c) Z = 88, A = 206 (d) Z = 84, A = 212 **[B.H.U-2014]**

Soln. For uranium $A = 238$, $Z = 92$
In an alpha decay mass number A reduces by 4 and atomic number Z reduced by 2.
In a beta decay A remains unchanged and Z increases by 1.
Therefore, the new mass number $= 238 - 8 \times 4 \Rightarrow A = 206$
The new atomic number $= 92 - 8 \times 2 + 6 \times 1 \Rightarrow Z = 82$
Correct options is (a)

23. What is the number of α decays that occur in a 1 gm sample of thorium in 1 year. The disintegration constant of thorium (232) is 1.58×10^{-18} sec^{-1} and Avogadro's number is 6.023×10^{23}? **[B.H.U-2011]**

(a) 13×10^{12} (b) 13×10^{10} (c) 6×10^{11} (d) 11×10^{15}

Soln. Let the number of α decays be x per second. $1g$ of Thorium–232 contains $=\dfrac{6.023\times10^{23}}{232}$ atoms of Th–232

Decay constant or disintegration constant

$$\lambda=\frac{x\text{ per sec}}{\dfrac{6.023\times10^{23}}{232}}$$

$\therefore$ Alpha decays per sec, $x=\dfrac{\lambda\times6.023\times10^{23}}{232}$

$$x=\frac{1.58\times10^{-18}\times6.023\times10^{23}}{232}$$

$$=0.04\times10^{5}\text{ particles sec}^{-1}$$

For 1 year, the no. of alpha decays
= (1 year) × 0.04 × 10^5 s–1
= (3600 × 24 × 365) × 0.04 × 10^5 ≈ 13 × 10^{10}
Correct option is (b)

24. Consider the fusion reaction $6\,{}_1^2H\rightarrow2\,{}_2^4He+2p+2n+Q(43\ MeV)$. The total energy detained by fusing 1 kg of deuterium to form helium is **[B.H.U-2011]**

(a) $4.3\times10^{27}\ MeV$ (b) $8.6\times10^{27}\ MeV$ (c) $5.2\times10^{27}\ MeV$ (d) $2.15\times10^{27}\ MeV$

Soln. No. of deuterium atoms in 1 kg of deuterium $=\dfrac{N_A}{2}\times1000$

where $N_A=6.023\times10^{23}$

Now, each reaction requires 6 deuterium atoms.

Number of reactions possible $=\dfrac{1}{6}\times\left(\dfrac{6.023\times10^{23}\times10^{3}}{2}\right)$

Q-value for each reaction = 43 MeV
Total energy detained by fusing 1 kg of deutorium = (Number of reactions) × Q-value of one reaction.

$$=\frac{1}{6}\times\frac{6.023\times10^{23}\times10^{3}}{2}\times43\text{ MeV}$$

= 2.15 × 10^{27} MeV
Correct option is (d)

25. What is the amount of energy released when 10 micrograms of uranium (^{235}U) undergoes fission, if energy released per fission is 200 MeV? **[B.H.U-2012]**
(a) 5.126 × 10^{18} MeV (b) 5.126 MeV (c) 5.125 MeV (d) 5.126 × 10^6 MeV

Soln. Number of Uranium–235 atoms in 10 μg of Uranium–235

$$= \frac{6.023\times 10^{23}}{235}\times 10\times 10^{-6}$$

Total energy released = Number of U^{235} atoms × energy per fission

$$= \frac{6.023\times 10^{18}}{235}\times 200\text{ MeV} \approx 5.126\times 10^{18}\text{ MeV}$$

Correct option is (a)

26. A pure Fermi-transition in the nuclear β- decay is the one where:

(a) $\Delta J = \pm 1, O^+ \to O^+$ (b) $\Delta J = 0, O^+ \to O^+$ **[B.H.U-2016]**

(c) $\Delta J = \pm 2, O^+ \to O^+$ (d) $\Delta J > 0, O^+ \to O^+$

Soln. For pure Fermi-transition in the nuclear β-decay, $\Delta J = 0$ and parity does not change.

Only one among the options, i.e., option (b), satisfies the above criteria for pure Fermi-transition in β-decay.

$\Delta J = 0,$ and parity remains same $0^+ \to 0^+$

27. The following nuclear β- decay $\underset{0^+}{\overset{6}{\text{He}}} \to \underset{1^+}{\overset{6}{\text{Li}}} + e^- + \bar{v}, |\Delta J| = 1$ is an explict example ofthe **[B.H.U-2016]**

(a) Forbidden transition   (b) Mixed transition

(c) Pure Fermi transition   (d) Pure Gamow Teller transition

Soln. $\underset{0^+}{\overset{6}{He}} \to \underset{1^+}{\overset{6}{Li}} + e^- + \bar{v}$

Given: $|\Delta J| = 1$

For allowed nuclear β-decay, $\Delta J = 1$ and partiy remains the same $0^+ \to 1^+$. So this is a pure Gamow Teller transition.

28. Geiger's law for the "Range Energy" relationship for the α- particle is well known infact, the range R and velocity v of the α- particle are related by **[B.H.U-2016]**

(a) $R \alpha v^5$ (b) $R \alpha v^4$ (c) $R \alpha v^3$ (d) $R \alpha v^2$

Soln. Geiger's law for the "Range Energy" relationship for the α-particle is

$$R = bE^{3/2} \qquad \ldots(i)$$

Where b is a constant.

Now, $E = \frac{1}{2}mv^2 \Rightarrow E \propto v^2$ ...(ii)

Where v is velocity and m is mass of particle.

Using (ii) in (i),

$$R \propto \left(v^2\right)^{3/2}$$

Therefore, $R \propto v^3$

Correct option is (c)

29. A sample of radioactive nuclie of a certain element can decay only by γ-emission and β-emission. If the half-life for γ-emission is 24 minutes and that for β-emission is 36 minutes, the half-life for the sample is

(a) 30 minutes   (b) 24 minutes   (c) 20.8 minutes   (d) 14.4 minutes **[B.H.U-2012]**

Soln. Given: Half life of γ-decay : $(t_{1/2})_\gamma = 24$ minutes

Half life of β-emission : $(t_{1/2})_\beta = 36$ minutes

Decay constant for γ-decay : $\lambda_\gamma = \dfrac{\ln 2}{(t_{1/2})_\gamma} = \dfrac{\ln 2}{24}$

Decay constant for β-emission : $\lambda_\beta = \dfrac{\ln 2}{(t_{1/2})_\beta} = \dfrac{\ln 2}{36}$

Let the decay constant and half life of sample be λ_s and $(t_{1/2})_s$ respectively.

$$\lambda_s = \lambda_\gamma + \lambda_\beta$$

$$\Rightarrow \frac{\ln 2}{(t_{1/2})_s} = \frac{\ln 2}{(t_{1/2})_\alpha} + \frac{\ln 2}{(t_{1/2})_\beta} \Rightarrow \frac{1}{(t_{1/2})_s} = \frac{1}{24} + \frac{1}{36}$$

$$\Rightarrow (t_{1/2})_s = \frac{24 \times 36}{24 + 36} = 14.4 \text{ minutes}$$

Correct option is (d).

30. An 8-centimetre diameter by 8-centimetre long NaI(Tl) detector detects gamma rays of a specific energy from a point source of radioactivity. When the source is placed just next to the detector at the center of the circular face, 50 percent of all emitted gamma rays at the energy are detected. If the detector is moved to 1 meter away, the fraction of detected gamma rays drops to **[B.H.U-2012]**

(a) 10^{-4} (b) 2×10^{-4} (c) 4×10^{-4} (d) $8\pi \times 10^{-4}$

Soln. For a point source, fraction of gamma rays falling on detector $\alpha = \dfrac{\pi r^2}{4\pi l^2}$, where r is radius of detector, l is the distance between source and detector.

Given: $r = 4$ cm, $l = 100$ cm

$$\alpha = \frac{\pi \times 4^2}{4\pi \times 100^2} = 4 \times 10^{-4}$$

Also, the detector detects only 50% of the gamma rays falling on it.

The fraction of detected gamma rays $= \dfrac{\alpha}{2} = 2 \times 10^{-4}$

Correct option is (b)

31. Geiger-Nuttal law gives the relationship between **[B.H.U-2011]**

(a) energy and range of α particle (b) half-life and decay constant of α particle

(c) half-life and energy of α particle (d) half-life of α emitter and energy of α particle

Soln. Geiger-Nuttal law gives the relationship between the decay constant of a radio isotope with the energy of the alpha particles. In other words, the law relates the half life of α emitter and energy of α particle.

Correct option is (d).

32. The half-life of a radio isotope is 5 years. The fraction of atoms decayed in this isotope in a period of 20 years will be **[B.H.U-2011]**

(a) $\frac{1}{16}$ (b) $\frac{7}{8}$ (c) $\frac{1}{8}$ (d) $\frac{15}{16}$

Soln. Given: $t_{1/2} = 5$ years

$$\therefore \quad \lambda = \frac{\ln 2}{t_{1/2}} = \frac{\ln 2}{5}$$

We know, $N = N_0 e^{-\lambda t}$

$$\Rightarrow \quad N = N_0 e^{-\frac{\ln 2}{5}\times 20} = N_0 e^{-4\ln 2}$$

$$\Rightarrow \quad N = \frac{N_0}{16} \text{ (Fraction of atoms remaining)}$$

$$\therefore \quad \text{Fraction of atoms decayed} = N_0 - \frac{N_0}{16} = \frac{15}{16} N_0$$

Correct option is (d)

33. A reactor is developing energy at the rate of 32×10^6 watts. How many atoms of ^{235}U undergo fission per second ? **[B.H.U-2012]**

(Assume that on the average, an average of 200 MeV is released per fission).

(a) 10^6 (b) 10^{12} (c) 10^{18} (d) 10^{24}

Soln. Energy released per second = 32×10^6 J

Energy released per fission = 200 MeV

$= 200 \times 10^6 \times 1.6 \times 10^{-19}$ J

$= 32 \times 10^{-12}$ J

Number of atoms of ^{235}U undergo fission per sec $= \frac{32\times 10^6}{32\times 10^{-12}} = 10^{18}$ atoms/second

Correct option is (c)

34. The binding energy of the neon isotope $^{20}_{10}$Ne is 160.65 MeV. Thus, its atomic mass, in atomic unit (u), would be **[B.H.U-2015]**

(a) 19.99 u (b) 18.99 u. (c) 17.99 u (d) 16.99 u

Soln. Binding energy $B = \Delta m\, c^2$, where Δm is mass defect and c is speed of light.

$$\therefore \quad \Delta m = \frac{B}{c^2} = \frac{160.65 \text{ MeV}}{\left(3\times 10^8 \text{ m/s}\right)^2}$$

$$= \frac{160.65\times 10^6 \times 1.6\times 10^{-19}}{\left(3\times 10^8\right)^2} = 28.56\times 10^{-29}\, g$$

We know 1 gram = 6.022×10^{23} a.m.u.

$\Delta m = 28.25 \times 10^{-29} \times 6.022 \times 10^{23} = 170.1215 \times 10^{-16} \approx 0.00017$

Atomic mass = $A - \Delta m = 20 - 0.00017 \approx 19.99\, u$

Correct option is (a)

35. If a 2 MeV alpha particle $\left({}_2He^4\right)$ incident on ${}_{47}Ag^{107}$ nucleus is scattered at 90°, then the impact parameter is $\left(e^2/4\pi\in_0 = 1.44 \text{ MeV fm}\right)$

(a) 3.38×10^{-15} m (b) 47×10^{-5} m (c) 6.78×10^{-15} m (d) 0 m **[H.C.U.-2012]**

Soln. K.E. of alpha particle $\left({}_2He^4\right) = 2 \text{ MeV} = 2\times10^6\times1.6\times10^{-19} J = 3.2\times10^{-13} J$

Scattering angle $\theta = 90°$

$Z_1 = 2,\ Z_2 = 47$

Impact parameter $b = \frac{Z_1Z_2ke^2}{2(K.E.)}\cot\frac{\theta}{2}$, where $k = \frac{1}{4\pi\in_0} = 9\times10^9 N$

$$b = \frac{2\times47\times9\times10^9\times\left(1.6\times10^{-19}\right)^2}{2\times3.2\times10^{-13}}$$

$$= \frac{47\times9\times10^9\times10^{-38}\times1.6\times1.6}{2\times1.6\times10^{-13}} = 7.2\times47\times10^{-16}$$

$$= 338.4\times10^{16} = 33.8\times10^{-15}\,m$$

Note: option (a) is misprinted in original question paper.

Correct option is (a)

36. It two deuterons are at a distance of 10^{-12} cm, the nuclear force between them just overcomes the repulsive electrostatic force. The height of the electrostatic potential barrier is **[B.H.U-2010]**

(a) zero (b) 0.14 MeV (c) 0.31 MeV (d) 0.49 MeV

Soln. The electrostatic Potential barrier is

given by $V = \frac{e^2}{4\pi\in_0}\ \frac{Z_a + Z_b}{R_a + R_b}$

For two deuterons, $Z_a = 1, Z_b = 1$

$R_a + R_b = 10^{-12}$ cm $= 10^{-14}$ m

We know, $\frac{e^2}{4\pi\in_0} = 1.44\times MeV\ fm$

$= 1.44\times10^{-15}\ MeV\ m$

$$V = \frac{1.44\times10^{-5}\times1\times1}{10^{-14}} = 0.144\ MeV \approx 0.14\ MeV$$

Correct option is (b)

37. The reaction $\pi^- p \to \Lambda^\circ \pi^\circ$ in strong interaction is not possible because **[B.H.U-2010]**

(a) it does not conserve charge
(b) it does not conserve baryon number
(c) it does not conserve strangeness
(d) it does not conserve isospin

Soln. The reaction is :

$\pi^- + p \to \Lambda^\circ + \pi^\circ$

Charge $Q: -1+1 \to 0+0 \Rightarrow \Delta Q = 0 \quad Q \to$ conserved

Baryon No. $B: 0+1 \rightarrow +1+0 \Rightarrow \Delta B = 0$ $B \rightarrow$ conserved

Strangeness $S: 0+1 \rightarrow -1+0 \Rightarrow \Delta S \neq 0$ $S \rightarrow$ not conserved

Isospin $I: 1+\frac{1}{2} \rightarrow 0+1 \Rightarrow \Delta I \neq 0$ $I \rightarrow$ not conserved

It is a strong reaction where the charge and the Baryon number are conserved. However, since the strangeness number and isospin are not conserved, the reaction is not possible.

Correct options are (c, d)

38. The ground state of a deuteron is **[B.H.U-2016]**

(a) A pure $^3S_1$ state   (b) A pure $^3P_1$ state

(c) A mixture of $^3S_1$ and $^3P_1$ states   (d) A mixture of $^3S_1$ and $^3D_1$ states

Soln. The reason for mixed states of deuteron wave functions arises from the fact that $\mu_d - (\mu_p + \mu_n) \neq 0$, where μ_d is the experimental magnetic dipole, μ_p and μ_n are the magnetic dipole moments of proton and neutron, respectively.

The ground state of the deuteron has total angular momentum $J = 1$.

The corresponding states : $^1P_1, ^3P_1, ^3S_1$ and $^3D_1$. For same parity, one can have two pairs $^1P_1$ and $^3P_1$; $^3S_1$ and $^3D_1$.

Only $^3S_1$ and $^3D_1$ gives proper result with $^3S_1$ state = 96% and $^3D_1$ state = 4%

Correct option is (d)

39. According to the nuclear shell-model, the spin and parity of the ground state of the nucleus $_{19}K^{39}$ would be

(a) $\left(\frac{3}{2}\right)^+$ (b) $\left(\frac{1}{2}\right)^+$ (c) $\left(\frac{3}{2}\right)^-$ (d) $\left(\frac{1}{2}\right)^-$ **[B.H.U-2016]**

Soln. Given : $_{19}K^{39} \Rightarrow Z = 19, A = 39$

No. of neutrons $N = 39 - 19 = 20$

$Z = 19\,(odd) \rightarrow \left(1S_{1/2}^2\right)\left(1P_{3/2}^4\right)\left(1P_{1/2}^2\right)\left(1d_{5/2}^6\right)\left(2s_{1/2}^2\right)\left(1d_{3/2}^3\right)$

For last unpaired nucleon,

$j = \frac{3}{2}$, parity $= (-1)^l = (-1)^2$ $(\because l = 2)$

Therefore, spin parity $= \left(\frac{3}{2}\right)^+$

Correct options are (a)

40. The quark content of a neutron is:

(a) uud   (b) udd   (c) uds   (d) $ud\bar{s}$ **[B.H.U-2016]**

Soln. The quark content of a neutron is 'udd'

$$\begin{array}{ccc} u & d & d \end{array}$$

Since it has charge $Q = \left(\frac{2}{3} \quad -\frac{1}{3} \quad -\frac{1}{3}\right) = 0$

and it has Baryon number $B=\left(\frac{1}{3}+\frac{1}{3}+\frac{1}{3}\right)=1$

It has strangeness number $S=(0+0+0)=0$

$$I_3=\frac{1}{2}-\frac{1}{2}-\frac{1}{2}=-\frac{1}{2}$$

Correct option is (b)

1. Given that ^{113}Cd has density $n=5.58\times10^{27}$ atoms/m^3 and its capture cross-section $\sigma=2\times10^{-24}\,\text{m}^2$ for thermal neutrons, the mean-free path of these neutrons in ^{113}Cd is **[B.H.U-2015]**

(a) 8.93×10^{-7} m (b) 8.93×10^{-5} m (c) 8.93×10^{-9} m (d) 8.93×10^{-10} m

oln. The mean free path

$\lambda=\frac{1}{\Sigma}$ (the reciprocal of the macroscopic cross-section Σ)

or, $\lambda=\frac{1}{n\sigma}$

When n is atomic density and σ is capture cross-section.

$n=5.58\times10^{27}$ atoms/m^3, $\sigma=2\times10^{-24}\,\text{m}^2$

$$\lambda=\frac{1}{5.58\times10^{27}\times2\times10^{-24}}\,\text{m}$$

$$\approx\frac{1}{1.12\times10^{4}}\,\text{m}\approx8.93\times10^{-5}\,\text{m}$$

Correct option is (b)

42 The life time of a free neutron to decay into a proton, an electron and an antineutrino is approximately

(a) 10^{-23} sec (b) 10^{-20} sec (c) 10^{-8} sec (d) 10^{3} sec **[B.H.U-2016]**

Soln. A free neutron decays into a proton, an electron and an anti-neutrino, is an example of weak force decay. The life time is about 15 minutes = 900 seconds ~ 10^3s

Correct option is (d)

PRACTICE SET

1. Find the ratio of the nuclear to the atomic density for hydrogen. (Assume the nuclear radius to be 1 fm). [0.15×10^{15}]

2. A nucleus with $A = 235$ splits into two nuclei whose mass numbers are in the ratio 2:1. Find the radii of the new nuclei. [$r_1 = 5.99$ fm, $r_2 = 7.55$ fm]

3. Calculate the ratio of the nuclear radius of ${}^{208}_{82}Pb$ to the radius of its innermost electrons as calculated from Bohr theory. [1/77.8]

4. What is the mass of a sample of ${}^{14}_{6}C\left(T_{1/2} = 5570 \text{ y}\right)$ that has an activity of 5 Ci. [1.09 g]

5. What is the activity of 5×10^{-7} kg of ${}^{230}_{92}U$, whose half-life is 0.180×10^{7} s? [14.9 Ci]

6. How much time is required for an amount of ${}^{90}_{38}Sr\left(T_{1/2} = 28 \text{ y}\right)$ to be reduced by 75%? [56 y]

Chapter 9

Special Theory of Relativity

9.1: Reference Frame:

A system of coordinates axes which defines the position of particle or an event in two or three dimensional space is called frame of reference.

A reference frame with four coordinates x, y, z, t is referred as space-time frame.

Frame of references (i) Inertial (ii) Non-inertial

1. **Inertial frame of reference:** Newton's law does hold in this frame of reference. This is also called Newtonian, Galilean or inertial frame of reference. All frame of reference, moving with constant velocity with respect to an inertial frame, are also inertial frame.

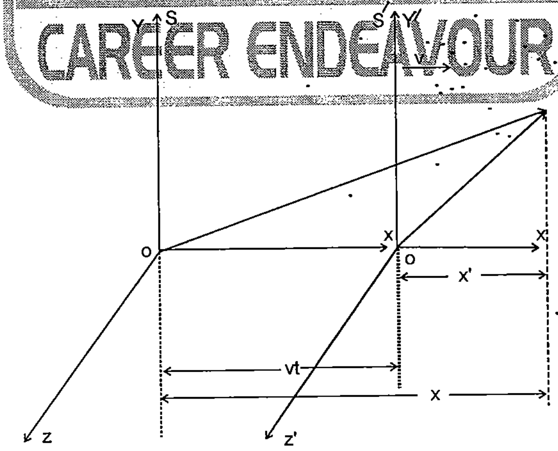
   Inertial frames are non-accelerating frames.

2. **Non-inertial frame:** Accelerated frame of reference in which Newton's law does not hold good.

9.2. Galilean transformation:

An event or a physical phenomenon, observed simultaneously in two separate sets of coordinates corresponding to two frame of references. If two systems are inertial, the transformation relating these two different coordinates is called Galilean transformation.

(a) Transformation of position:



An event or physical phenomena, observed simultaneously in two separate reference frames, will have two separate set of coordinates. Equation relating two coordinates is called transformation equation. And if two system happen to be inertial or Galilean one, the transformation is reffered as Galilean transformation.

Let S and S' be two inertial frame of reference with observer at origin O and O' moving with a relative velocity v along +ve direction of x.

Now if we assume time to be independent of any frame of reference, we have four transformation equation from S to S'.

$$\left.\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned}\right\}$$

(b) Transformation of distance or length: Let two events occurring simultaneously at A_1 and A_2, some distance apart, along axes of x, such that their coordinates in two frame of references are x_1, y, z ; x_2, y, z and x_1, y', z'; x_2', y', z' respectively.

Transformation equation. for event at A_1, (from S to S'): $x_1' = x_1 - vt,\ y' = y,\ z' = z$

Transformation equation for event at A_2 (from S to S'): $x_2' = x_2 - vt,\ y' = y,\ z' = z$

Hence distance between two events

$$x_2' - x_1' = \Delta x' = (x_2 - vt) - (x_1 - vt) = (x_2 - x_1) = \Delta x$$

Now, when a particular property is characterized by the same number in two frame of reference, it is said to be "invariant to transformation."

Hence, distance or length is invariant to Galilean transformation.

(c) Transformation of velocity: Now consider the velocity measurement u and u' as measured by two observer in their frame of reference. Thus, if x_1, t_1 and x_2, t_2 be co-ordinates of initial & final events in frame S and $x_1', t_1'; x_2', t_2'$ be co-ordinates of initial and final events in frame S'.

Velocity relative to frame S, $u = \dfrac{\Delta x}{\Delta t} = \dfrac{x_2 - x_1}{t_2 - t_1}$, since,

$$\begin{aligned} x_1 &= x_1' + vt_1' \\ x_2 &= x_2' + vt_2' \\ t_1 &= t_1' \\ t_2 &= t_2' \end{aligned} \qquad u = \frac{(x_2' + vt_2') - (x_1' + vt_1')}{t_2' - t_1'} = \frac{x_2' - x_1'}{t_2' - t_1'} + \frac{v(t_2' - t_1')}{(t_2' - t_1')}$$

$$\Rightarrow u = u' + v$$

(d) Transformation of acceleration: Let 'a' and 'a'' be the acceleration in S and S' frame of reference.

We have, $a' = \dfrac{du'}{dt} = \dfrac{d(u - v)}{dt} = \dfrac{du}{dt} - 0 = a \quad [\because \text{ v is constant}]$

$\therefore$ Acceleration in invariant under Galilean transformation.

9.3: Ether Hypothesis and Michelson Morley experiment:

Ether hypothesis is an analogy between light and sound propagation. According to this hypothesis, electromagnetic waves are carried by a medium called the ether. The reference frame in which the ether is at rest would constitute an absolute coordinate system.

The value of the speed of light would then hold just in the reference frame in which the ether is at rest.

In particular, the existence of an ether would imply that electromagnetic waves in a moving reference system propagate (analogous to sound propagation) with distinct velocities forward (i.e., direction of motion of the source) and sideways. This hypothesis was tested for the first time in the Michelson-Morley experiment (1887) by means of a Michelson interferometer. Here one observes with an interference setup whether the speed of light changes because of Earth's motion. The moving system in which the experiment was performed is Earth itself on its path around the Sun.

The experiment proves that light propagates with equal velocity c along Earth's orbit and in the perpendicular direction, disproving the ether hypothesis.

9.4: Postulates of special relativity

(1) **Principle of relativity:** The laws of Physics are same in all inertial frames.

(2) **Principle of constancy of speed of light:** The speed of light in empty space (called vacuum) is same in all inertial frames of reference. The velocity of light is 3 x 10^8 m/s in free space.

Invariant Interval: The quantity : $\Delta s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 - c^2\Delta t^2$ is called interval. It is invariant in all frames.

9.5: Lorentz Transformation:

Let S and S' be two reference frames, with their axes parallel and S' moving relative to S with a uniform velocity v along x-axis. Let the origin O and O' of the systems coincide at the instant $t = t' = 0$, when a light originates at O (and O').

Then, by Einstein's postulates, the wavefront of light at time t in S-system would be a sphere given by

$$x^2 + y^2 + z^2 = c^2t^2$$

where c is the velocity of light, and the corresponding equation in S'-system will be

$$x'^2 + y'^2 + z'^2 = c^2t'^2$$

The relative motion of the frames does not change the transverse co-ordinates; $y' = y$ and $z' = z$. So, we may write

$$x^2 - c^2t^2 = x'^2 - c^2t'^2 \qquad \text{...(1)}$$

For a complete equivalence of the two inertial frames $S(x, y, z, t)$ and $S'(x', y', z', t')$, the transformation quations must be symmetrical w.r.t. both and if quantities (x, y, z, t) be finite, (x', y', z', t') must also be so. The above conditions require that the transformation equations should be linear. With these considerations, let

$$x' = \gamma(x - vt) \qquad \text{... (2)}$$

$$x = \gamma'(x' + vt') \qquad \text{... (3)}$$

where γ and γ' are undertermined constants

$$\therefore \quad \frac{x}{\gamma'} = x' + vt', \text{ from (3).}$$

$$\Rightarrow \quad t' = \frac{1}{v}\left(\frac{x}{\gamma'} - x'\right) = \frac{1}{v}\left[\frac{x}{\gamma'} - \gamma(x - vt)\right] \quad \text{using (1)}$$

$$t' = \gamma\left[t - \frac{x}{v}\left(1 - \frac{1}{\gamma\gamma'}\right)\right], \text{ on rearrangement} \qquad \text{... (4)}$$

Substituting for x' from (2) and t' from (4) in (1)

$$x^2 - c^2t^2 = \gamma^2(x - vt)^2 - c^2\gamma^2\left[t - \frac{x}{v}\left(1 - \frac{1}{\gamma\gamma'}\right)\right]^2$$

Equating the coefficients of t^2 and x^2 from both sides.

$$-c^2 = \gamma^2 v^2 - c^2\gamma^2 \qquad \text{... (5)}$$

And $$1 = \gamma^2 - c^2\gamma^2 \cdot \frac{1}{v^2}\left(1 - \frac{1}{\gamma\gamma'}\right)^2 \qquad \text{... (6)}$$

$\therefore$ $\gamma^2\left(c^2 - v^2\right) = c^2$, from (5)

$\Rightarrow$ $$\gamma = \frac{c}{\sqrt{c^2 - v^2}} = \frac{1}{\sqrt{1 - v^2/c^2}} \qquad \text{... (7)}$$

And $\frac{1}{\gamma^2} = 1 - \frac{c^2}{v^2}\left(1 - \frac{1}{\gamma\gamma'}\right)^2$, from (6)

$\Rightarrow$ $1 - \frac{v^2}{c^2} = 1 - \frac{c^2}{v^2}\left(1 - \frac{1}{\gamma\gamma'}\right)^2$, using (7)

$\Rightarrow$ $$1 - \frac{1}{\gamma\gamma'} = \frac{v^2}{c^2} \Rightarrow \frac{1}{\gamma\gamma'} = 1 - \frac{v^2}{c^2} = \frac{1}{\gamma^2}, \text{ using (7)} \qquad \text{... (8)}$$

$\therefore$ $$\gamma' = \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \qquad \text{... (9)}$$

So, the required Lorentz transformation equations are

$$\boxed{\begin{aligned} x' &= \frac{(x - vt)}{\sqrt{1 - v^2/c^2}} \\ y' &= y \\ z' &= z \\ t' &= \frac{\left(t - vx/c^2\right)}{\sqrt{1 - v^2/c^2}} \end{aligned}}$$

The conjugate transformation relations are obtained simply by replacing v and –v in the above set and interchanging the primed and unprimed quantities. So, the transformation relations from S' to S-sytem are :

$$\boxed{\begin{aligned} x &= \frac{x' + vt'}{\sqrt{1 - v^2/c^2}} \\ y &= y' \\ z &= z' \\ t &= \frac{t' + vx'/c^2}{\sqrt{1 - v^2/c^2}} \end{aligned}}$$

9.6: Length contraction:

The length of a rod, as measured in a frame of reference, at rest with respect to observer, is called its proper length and obviously be same in all stationary frame of reference. It is only in moving frame that it will be different. Let a rod laid along the axis of x is frame of reference S, at rest with respect to the observer and let x_1 and x_2 be the co-ordinates of its two ends.

∴ Proper length of rod, $L_0 = (x_2 - x_1)$

If x–coordinates of the ends of the rod in reference frame S', moving with uniform velocity v with respect to frame S be x_1' and x_2', as noted simultaneously at same instant t', we have length of rod in moving reference frame S' given by $L = (x_2' - x_1')$.

From inverse Lorentz transformation:

$$x_1 = \frac{x_1' + vt'}{\sqrt{1-v^2/c^2}}; \quad x_2 = \frac{x_2' + vt'}{\sqrt{1-v^2/c^2}} \quad \therefore \quad L_0 = x_2 - x_1 = \frac{1}{\sqrt{1-v^2/c^2}}\left[x_2' - x_1'\right]$$

$$\Rightarrow \quad L_0 = \frac{L}{\sqrt{1-v^2/c^2}}$$

Therefore, in moving frame length is shorter than in rest frame.

9.7: Time dilation:

Time interval too is affected by relative motion. To an observer, a clock appears to run slower when in motion, than when at rest with respect to him.

Let we have two frame of reference, S and S', the former at rest and latter moving with uniform velocity v, along +ve x-direction.

If two events occur at any given point x' in frame S', at time t_1' and t_2', as noted an the clock carried by it and at times t_1 and t_2, as noted on the clock carried by frame S, we clearly have time interval between two events, as noted in frame S'.

$$\Delta t' = (t_2' - t_1')$$

And, time interval between two events in frame S, $\Delta t = (t_2 - t_1)$

In accordance with inverse Lorentz-transformation

$$t_1 = \frac{t_1' + v_x'/c^2}{\sqrt{1-\frac{v^2}{c^2}}}; \quad t_2 = \frac{t_2' + v_x'/c^2}{\sqrt{1-v^2/c^2}} \quad \therefore \quad \Delta t = (t_2 - t_1) = \frac{(t_2' - t_1')}{\sqrt{1-v^2/c^2}}$$

$$\Rightarrow \quad \Delta t = \gamma \Delta t'$$

Since factor, $\gamma > 1$, we have $\Delta t > \Delta t'$

Relativity of simultaneity : The difference in time measurements in different inertial frames raises the question of re-examination of simultaneity of two different events in different frames.

Let two events occur at x_1 and x_2 in S-frame at times t_1 and t_2 and let them be simultaneous so that $t_1 = t_2$, but x_1 and x_2 should be different for otherwise the events would not appear as two different events.

If the S' frame the events occur at times t'_1 and t'_2 respectively, then

$$t'_1 = \gamma\left(t_1 - vx_1/c^2\right); \; t'_2 = \gamma\left(t_2 - vx_2/c^2\right)$$

$\therefore \quad t'_1 \neq t'_2$ since $x_1 \neq x_2$

Thus, the events are not simultaneous in S' frame. So, called simultaneity is also relative and the idea of a universal time is ruled out.

9.8: Velocity addition theorem:

Suppose a particles has a velocity 'u' in a frame of reference S at rest, such that its component along three coordinates are $u_x = \frac{dx}{dt},\ u_y = \frac{dy}{dt},\ u_z = \frac{dz}{dt}$, and its velocity in a frame of reference S', moving with uniform velocity v relative to S along +ve direction of x-axis, is u', with its component

$u'_x = \frac{dx'}{dt'},\ u'_y = \frac{dy'}{dt'},\ u'_z = \frac{dz'}{dt'}$ along three coordinates axes, as measured in S'.

Now, in accordance with Lorentz transformation:

$$x' = \gamma(x - vt);\ y' = y;\ z' = z;\ t' = \gamma\left(t - \frac{vx}{c^2}\right); \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

$$\Rightarrow \quad u'_x = \frac{dx'}{dt} = \frac{\gamma(dx - vdt)}{\gamma\left(dt - v\frac{dx}{c^2}\right)} = \frac{dx - vdt}{dt - v\frac{dx}{c^2}} \Rightarrow u'_x = \frac{dx/dt - v}{1 - \frac{vdx}{c^2dt}} = \frac{u_x - v}{1 - \frac{u_x v}{c^2}}$$

$$\Rightarrow \quad u'_y = \frac{dy'}{dt'} = \frac{dy}{\gamma\left(dt - v\frac{dx}{c^2}\right)} = \frac{dy/dt}{\gamma\left(1 - \frac{v}{c^2}\frac{dx}{dt}\right)} \Rightarrow u'_y = \frac{u_y}{\gamma\left(1 - \frac{vu_x}{c^2}\right)} \Rightarrow u'_z = \frac{u_z}{\gamma\left(1 - \frac{vu_z}{c^2}\right)}$$

Therefore, the Lorentz velocity transformation is given by

$$\boxed{u'_x = \frac{u_x - v}{1 - u_x v/c^2};\ u'_y = \frac{u_y\sqrt{1 - v^2/c^2}}{1 - u_x v/c^2};\ u'_z = \frac{u_z\sqrt{1 - v^2/c^2}}{1 - u_x v/c^2}}$$

Conjugate equation

$$\boxed{u_x = \frac{u'_x - v}{1 + u'_x v/c^2};\ u_y = \frac{u'_y\sqrt{1 - v^2/c^2}}{1 + u'_x v/c^2};\ u_z = \frac{u'_z\sqrt{1 - v^2/c^2}}{1 + u'_x v/c^2}}$$

Derivation of mass-variation relation:

We shall now derive the mass-variation relation following the procedure of R.C. Tolman.

Consider two frames S and S' in uniform relative translation, parallel to x (or x')-axis, with velocity v. Let two balls of equal mass m' in S' frame collide elastically, moving along the x-axis with velocity μ' and $-\mu'$ in this frame and, on collision, coalesce together. At the moment of collision, their velocity is zero in S' frame, but they still have a velocity v in S frame.

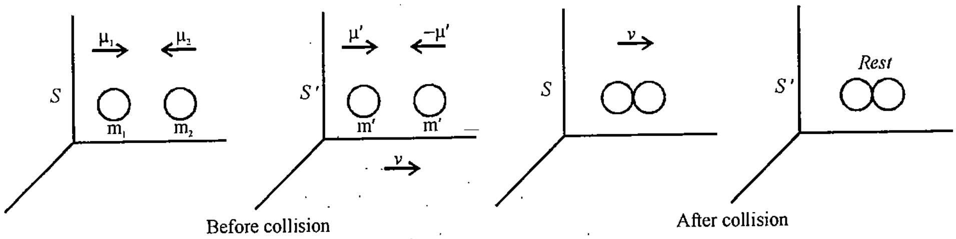


Figure: Collisions between two bodies as observed from two different frames

We shall describe the collision in S and S' frames. The conservation of momentum gives

$$m'\mu' - m'\mu' = 0 \qquad \text{in } S' \text{ frame} \qquad \ldots(1)$$

And $$m_1\mu_1 + m_2\mu_2 = (m_1+m_2)v \qquad \text{in } S \text{ frame} \qquad \ldots(2)$$

By velocity addition theorem, μ_1 and μ_2, the velocities of masses in S frame transform as

$$\mu_1 = \frac{\mu'+v}{1+\mu' v/c^2};\quad \mu_2 = \frac{-\mu'+v}{1-\mu' v/c^2} \qquad \ldots(3)$$

Substituting these values of μ_1 and μ_2 in (2) and on simplification,

$$\frac{m_1}{m_2} = \frac{1+\mu' v/c^2}{1-\mu' v/c^2} = \frac{\sqrt{1-\mu_2^2/c^2}}{1-\mu_1^2/c^2} \qquad \ldots(4)$$

Simplification from equation (3), we may write

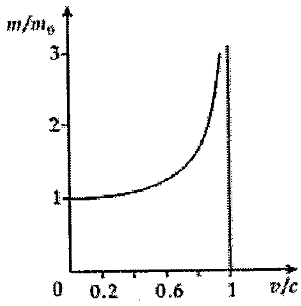
$$1-\frac{\mu_1^2}{c^2} = 1-\frac{(\mu'/c+v/c)^2}{(1+\mu' v/c^2)^2} = \frac{(1+\mu' v/c^2)-(\mu'/c+v/c)^2}{(1+\mu' v/c^2)^2}$$

$$= \frac{(1-v^2/c^2)(1-\mu'^2/c^2)}{(1+\mu' v/c^2)^2} = \frac{(1-v^2/c^2)(1-\mu'^2/c^2)}{(1+\mu' v/c^2)^2}$$

Similarly, $1-\dfrac{\mu_2^2}{c^2} = \dfrac{(1-v^2/c^2)(1-\mu'^2/c^2)}{(1-\mu' v/c^2)^2} \quad \therefore \dfrac{\sqrt{1-\mu_2^2/c^2}}{\sqrt{1-\mu_1^2/c^2}} = \dfrac{1+\mu' v/c^2}{1-\mu' v/c^2}$

The two balls are assumed identical when at rest in the same frame. Let m_0 be the mass under this condition. By (4), the masses are inversely proportional to the quantity $\sqrt{1-\mu^2/c^2}$, when they move with velocity μ.

$$\therefore \quad m_1\sqrt{1-\mu_1^2/c^2} = m_2\sqrt{1-\mu_2^2/c^2} = m_0$$



where, m_0 is the rest mass of the body.
The mass in motion must, therefore, be

$$\boxed{m = \frac{m_0}{\sqrt{1-v^2/c^2}}}$$

Relativistic energy: Mass-energy equivalence:

We define the kinetic energy T of a particle in relativistic mechanics as the work done by an external force in increasing the velocity of the particle from zero to its present value v.

Considering, for simplicity, the one-dimensional motion of a particle in x-direction.

$$T = \int_0^x F dx$$

Where F is the force-component along the displacement dx and x is the distance over which the force F acts.

But $F = \frac{dp}{dt} = \frac{d}{dt}(mv)$

where, m is the mass of the particle moving with velocity v.

$$\therefore \quad T = \int_0^x \frac{d}{dt}(mv)\, dx = \int_0^{mv} v d(mv) = \int_0^v v d\left(\frac{m_0 v}{\sqrt{1-v^2/c^2}}\right)$$

Integrating by parts, we get

$$T = \frac{m_0 v^2}{\sqrt{1-v^2/c^2}} - m_0 \int_0^v \frac{v dv}{\sqrt{1-v^2/c^2}}$$

$$= \frac{m_0 v^2}{\sqrt{1-v^2/c^2}} - m_0 \left| -c^2\sqrt{1-v^2/c^2} \right|_0^v$$

$$= \frac{m_0 c^2}{\sqrt{1-v^2/c^2}} - m_0 c^2 = mc^2 - m_0 c^2$$

$$\therefore \quad \boxed{T = mc^2 - m_0 c^2}$$

Total energy = kinetic energy + rest mass energy

$$\therefore \quad E = T + m_0 c^2 = (m - m_0)c^2 + m_0 c^2 = mc^2$$

$$\boxed{E = mc^2}$$

Total relativistic energy in terms of momentum:

Sometimes, it is convenient and useful to express the total relativistic energy E in terms of the momentum p. We have,

$$m^2c^4 - m_0^2c^4 = m_0^2c^4\left(\frac{1}{1-v^2/c^2} - 1\right)$$

$$= m_0^2c^4 \frac{v^2/c^2}{1-v^2/c^2} = c^2 \frac{m_0^2 v^2}{1-v^2/c^2} = c^2 p^2$$

$$\therefore \quad m^2c^4 = p^2c^2 + m_0^2c^4 \Rightarrow E^2 = p^2c^2 + m_0^2c^4$$

$$\therefore \quad \boxed{E = \sqrt{p^2c^2 + m_0^2c^4}}$$

An important theorem:

If u and u' be the velocities of a particle in S and S' frames respectively, then

$$\sqrt{1-\frac{u'^2}{c^2}}=\frac{\sqrt{\left(1+v^2/c^2\right)\left(1-u^2/c^2\right)}}{1-u_xv/c^2}$$

Proof: From the relativistic velocity addition theorem,

$$u'_x=\frac{v_x-v}{1-\frac{u_xv}{c^2}};\quad u'_y=\frac{u_y\sqrt{1-v^2/c^2}}{1-\frac{u_xv}{c^2}};\quad u'_z=\frac{u_z\sqrt{1-v^2/c^2}}{1-\frac{u_xv}{c^2}} \qquad \ldots(1)$$

where, u_x's and u'_x's are the component velocities.

$$\therefore\quad u'^2=u'^2_x+u'^2_y+u'^2_z=\frac{\left(u_x-v\right)^2+\left(u_y^2+u_z^2\right)\left(1-v^2/c^2\right)}{\left(1-u_xv/c^2\right)^2}\text{, using (1)}$$

$$\therefore\quad \frac{u'^2}{c^2}=\frac{\left(\frac{u_x}{c}-\frac{v}{c}\right)^2+\left(\frac{u_x}{c}-\frac{u_x^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)}{\left(1-u_xv/c^2\right)^2}\qquad\left[\because\ u_y^2+u_z^2=u^2-u_x^2\right]$$

$$\Rightarrow\quad 1-\frac{u'^2}{c^2}=\frac{\left(1-\frac{u_xv}{c^2}\right)-\left(\frac{u_x}{c}-\frac{v}{c}\right)^2-\left(\frac{u^2}{c^2}-\frac{u_x^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)}{\left(1-u_xv/c^2\right)^2}$$

$$=\frac{\left(1-\frac{v^2}{c^2}\right)-\frac{u^2}{c^2}\left(1-\frac{v^2}{c^2}\right)}{\left(1-u_xv/c^2\right)^2}=\frac{\left(1-v^2/c^2\right)\left(1-u^2/c^2\right)}{\left(1-u_xv/c^2\right)^2}\qquad\ldots(2)$$

$$\boxed{1-\frac{u'^2}{c^2}=\frac{\left(1-v^2/c^2\right)\left(1-u^2/c^2\right)}{\left(1-u_xv/c^2\right)^2}}$$

Transformation formulae for momentum, energy and force:

Momentum in S frame, the components of momentum are

$$p_x=mu_x=\frac{m_0u_x}{\sqrt{1-u^2/c^2}};\quad p_y=\frac{m_0v_y}{\sqrt{1-u^2/c^2}};\quad p_z=\frac{m_0u_z}{\sqrt{1-u^2/c^2}}$$

where, u_x's are the component velocities.

If p'_x, p'_y and p'_z be the momentum components in S' frame

$$p'_x = \frac{m_0 u'_x}{\sqrt{1-u'^2/c^2}} = \frac{m_0\left(1-u_x v/c^2\right)}{\sqrt{\left(1-v^2/c^2\right)\left(1-u^2/c^2\right)}} \times \frac{u_x - v}{1-u_x v/c^2}$$

$$= \frac{m_0 u_x - m_0 v}{\sqrt{1-v^2/c^2}\sqrt{1-u^2/c^2}} - \frac{m_0 c^2\left(v/c^2\right)}{\sqrt{\left(1-v^2/c^2\right)\left(1-u^2/c^2\right)}}$$

$$= \frac{p_x - \left(v/c^2\right)E}{\sqrt{1-v^2/c^2}} \quad \left(\because\ E = mc^2 = m_0 c^2/\sqrt{1-u^2/c^2}\right)$$

Also, $$p'_y = \frac{m_0 u'_y}{\sqrt{1-u'^2/c^2}} = \frac{m_0}{\sqrt{1-u'^2/c^2}} \times \frac{u_y\sqrt{1-v^2/c^2}}{\left(1-u_x v/c^2\right)}$$

$$= \frac{m_0\left(1-v_x v/c^2\right)}{\sqrt{\left(1-u^2/c^2\right)\left(1-v^2/c^2\right)}} \times \frac{u_y\sqrt{1-v^2/c^2}}{1-u_x v/c^2}$$

$$= \frac{m_0 u_y}{\sqrt{1-u^2/c^2}} = p_y$$, similarly, we shall have $p'_z = p_z$

So, the transformation formulae for the momentum are

$$\boxed{p'_x = \frac{p_x - vE/c^2}{\sqrt{1-v^2/c^2}};\ p'_y = p_y;\ p'_z = p_z}$$

Energy : The energy E' of a particle of mass m' is S'-frame is

$$E' = m'c^2 = \frac{m_0 c^2}{\sqrt{1-u'^2/c^2}} = \frac{m_0 c^2\left(1-u_x v/c^2\right)}{\sqrt{\left(1-v^2/c^2\right)\left(1-u^2/c^2\right)}}$$

$$= \frac{mc^2}{\sqrt{1-v^2/c^2}} - \frac{mu_x}{\sqrt{1-v^2/c^2}}v = \frac{E - vp_x}{\sqrt{1-v^2/c^2}}$$

$$\therefore \quad \boxed{E' = \frac{E - vp_x}{\sqrt{1-v^2/c^2}}}$$

The conjugate transformation relation is

$$\boxed{E=\frac{E'+vp'_x}{\sqrt{1-v^2/c^2}}}$$

Force: Let a particle of mass m, be under the action of force $\vec{F}$ and have a momentum $\vec{p}$ in an interial frame S. Then, $\vec{F}=\frac{d\vec{p}}{dt}$

$$\therefore \qquad F_x=\frac{dp_x}{dt};\ F_y=\frac{dp_y}{dt};\ F_z=\frac{dp_z}{dt}$$

where F_x's and p_x's are the components of force and momentum respectively.

The corresponding values in S' frame moving relative to S frame with a uniform velocity v along x (or x')-axis are

$$F'_x=\frac{dp'_x}{dt'};\ F'_y=\frac{dp'_y}{dt'};\ F'_z=\frac{dp'_z}{dt'}$$

But the transformation equations for momentum give

$$p'_x=\gamma\left(p_x-vE/c^2\right);\ p'_y=p_y;\ p'_z=p_z;\ \text{also, we have } t'=\gamma\left(t-vx/c^2\right)$$

where, $\gamma=1/\sqrt{1-v^2/c^2}$

$$\therefore \qquad F'_x=\frac{dp'_x}{dt'}=\frac{dp'_x}{dt}\cdot\frac{dt}{dt'}=\frac{\frac{dp_x}{dt}-\frac{v}{c^2}\frac{dE}{dt}}{\sqrt{1-v^2/c^2}}\cdot\frac{\sqrt{1-v^2/c^2}}{1-\frac{v}{c^2}\frac{dx}{dt}}$$

$$=\frac{F_x-\frac{v}{c^2}\frac{dE}{dt}}{1-\frac{v}{c^2}u_x}\cdot \qquad (\because\ dx/dt=u_x)$$

But, from Dirac equation,

$$E^2=p^2c^2+m_0^2c^4=\left(\vec{p}\cdot\vec{p}\right)c^2+m_0^2c^4$$

$$\Rightarrow \qquad 2E\left(dE/dt\right)=2\vec{p}\cdot\left(d\vec{p}/dt\right)c^2=2mc^2\left(\vec{u}\cdot\vec{F}\right)=2E\left(\vec{u}\cdot\vec{F}\right)$$

$$\left(\because\ \vec{p}=m\vec{u};\ \vec{F}=d\vec{p}/dt\right)$$

$$\therefore \qquad \frac{dE}{dt}=\vec{u}\cdot\vec{F}=u_xF_x+u_yF_y+u_zF_z$$

$$\therefore \quad F'_x = \frac{F_x - \left(u_x F_x + u_y F_y + u_z F_z\right) v / c^2}{1 - u_x v / c^2}$$

$$= \frac{F_x\left(1 - u_x v / c^2\right) - v / c^2 \left(v_y F_y + u_z F_z\right)}{1 - u_x v / c^2}$$

$$\boxed{F'_x = F_x - \frac{v / c^2}{1 - u_x v / c^2}\left(u_y F_y + u_z F_z\right)}$$

Again, $\boxed{F'_y = \frac{dp'_y}{dt'} = \frac{dp'_y}{dt}\frac{dt}{dt'} = \frac{dp_y}{dt}\frac{\sqrt{1 - v^2 / c^2}}{1 - u_x v / c^2} = \frac{F_y\sqrt{1 - v^2 / c^2}}{1 - u_x v / c^2}}$

Similarly, $\boxed{F'_z = \frac{F_z\sqrt{1 - v^2 / c^2}}{1 - u_x v / c^2}}$

Time-like and space-like intervals:

Let two events occur in S-frame at (x, y, z, t) and at $(x + \Delta x, y, , z, t + \Delta t)$ respectively. If the space-separation Δx between the two events is less than the distance that light travels in vacuum during the time-separation Δt, i.e., if $\Delta x - c\Delta t < 0$, it is called time-like interval, and if $\Delta x - c\Delta t > 0$, it is called space-like interval.

SOLVED PROBLEMS

1. The period of a pendulum is measured to be 3.0 s in the reference frame of the pendulum. What is the period when measured by an observer moving at a speed of 0.95c relative to the pendulum?

Soln: Instead of the observer moving at 0.95c, we can take the equivalent point of view that the observer is at rest and the pendulum is moving at 0.95c past the stationary observer. Hence, the pendulum is an example of a moving clock.

The proper time is $\Delta t_p = 3.0\ s$. Because a moving clock runs more slowly than a stationary clock by a factor of

$$\Delta t = \gamma \Delta t_p = \frac{1}{\sqrt{1-\frac{(0.95\,c)^2}{c^2}}}\Delta t_p = \frac{1}{\sqrt{1-0.902}}\Delta t_p = (3.2)(3.0\ s) = 9.6\ s$$

That is, a moving pendulum takes longer to complete a period than a pendulum at rest does.

2. Suppose you are driving your car on a business trip and are traveling at 30 m/s. Your boss, who is waiting at your destination, expects the trip to take 5.0 h. When you arrive late, your excuse is that your car clock registered the passage of 5.0 h but that you were driving fast and so your clock ran more slowly than your boss's clock. If your car clock actually did indicate a 5.0-h trip, how much time passed on your boss's clock, which was at rest on the Earth?:

Soln: Here, $\gamma = \frac{1}{\sqrt{1-v^2/c^2}} = \frac{1}{\sqrt{1-\frac{(3\times10^1\ m/s)^2}{(3\times10^8\ m/s)^2}}} = \frac{1}{\sqrt{1-10^{-14}}}$

Perform binomial expansion, we get

$$\gamma = \left(1-10^{-14}\right)^{-1/2} \approx 1+\frac{1}{9}\left(10^{-14}\right) = 1+5.0\times10^{-15}$$

In daily life, γ is not much different from 1

$$\Delta t = \gamma\,\Delta t_p = \left(1+5.0\times10^{-15}\right)(5.0\,h) = 5.0\,h + 2.5\times10^{-14}\,h = \boxed{5.0h + 0.09\ ns}$$

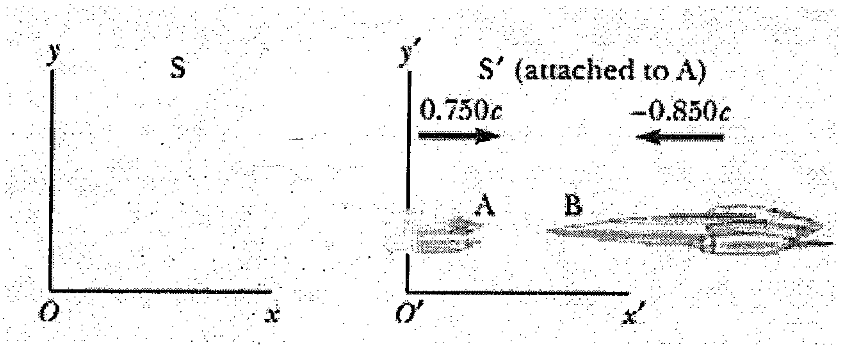
Your boss's clock would be only 0.09 ns ahead of your car clock. You might want to try another excuse!

3. A spaceship is measured to be 120.0 m long and 20.0 m in diameter while at rest relative to an observer. If this spaceship now flies by the observer with a speed of 0.99c, what length and diameter does the observer measure?

Soln: $L = L_p\sqrt{1-v^2/c^2} = (120.0\ m)\sqrt{1-\frac{(0.99\,c^2)}{c^2}} = \boxed{17\ m}$

The diameter measured by the observer is still 20.0 m because the diameter is a dimension perpendicular to the motion and length contraction occurs only along the direction of motion.

4. Two spaceships A and B are moving in opposite directions. An observer on the Earth measures the speed of ship A to be 0.750c and the speed of ship B to be 0.850c. Find the velocity of ship B as observed by the crew on ship A?



Soln: We can solve this problem by taking the S' frame as being attached to ship A, so that relative to the Earth (the S frame). We can consider ship B as moving with a velocity relative to the Earth. Hence, we can obtain the velocity of ship B relative to ship A by using velocity addition formula

$$u_x' = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} = \frac{-0.850\,c - 0.75\,c}{1 - \frac{(-0.850\,c)(0.750\,c)}{c^2}} = \boxed{-0.977\,c}$$

The negative sign indicates that ship B is moving in the negative x direction as observed by the crew on ship A.

5. Imagine a motorcycle moving with a speed 0.80c past a stationary observer. If the rider tosses a ball in the forward direction with a speed of 0.70c relative to himself, what is the speed of the ball relative to the stationary observer?

Soln: The speed of the motorcycle relative to the stationary observer is $v = 0.8c$. The speed of the ball in the frame of reference of the motorcyclist is $u_x' = 0.7c$. Therefore, the speed u_x of the ball relative to the stationary observer is

$$u_x = \frac{u_x' + v}{1 + \frac{u_x' v}{c^2}} = \frac{0.70\,c + 0.80\,c}{1 + \frac{(0.70\,c)(0.80\,c)}{c^2}} = \boxed{0.96\,c}$$

6. An electron, which has a mass of 9.11×10^{-31} kg, moves with a speed of 0.750c. Find its relativistic momentum.

Soln: $$p = \frac{m_e u}{\sqrt{1 - \frac{u^2}{c^2}}} = \frac{(9.11\times10^{-31}\,kg)(0.750\times3.00\times10^{8}\,m/s)}{\sqrt{1 - \frac{(0.750\,c)^2}{c^2}}} = \boxed{3.10\times10^{-22}\,kg\,m/s}$$

7. An electron in a television picture tube typically moves with a speed Find its total energy and kinetic energy in electron volts.

Soln: Rest energy of electron is 0.511 MeV

$$E = \frac{m_e c^2}{\sqrt{1 - \frac{u^2}{c^2}}} = \frac{0.511\,MeV}{\sqrt{1 - \frac{(0.250\,c)^2}{c^2}}} = 1.03(0.511\,MeV) = 0.528\,MeV$$

This is 3% greater than the rest energy.
We obtain the kinetic energy by subtracting the rest energy from the total energy:

$$K = E - m_e c^2 = 0.528\,MeV - 0.511\,MeV = \boxed{0.017\,MeV}$$

3. (a) Find the rest energy of a proton in electron volts.
(b) If the total energy of a proton is three times its rest energy, with what speed is the proton moving?
(c) Determine the kinetic energy of the proton in electron volts.
(d) What is the proton's momentum?

Soln: (a) $E_R = m_p c^2 = \left(1.67\times10^{-27}\ kg\right)\left(3.00\times10^{8}\ m/s\right)^2$

$$= \left(1.50\times10^{-10}\ J\right)\left(1.00\,eV/1.60\times10^{-19}\ J\right) = \boxed{938\ MeV}$$

(b) $E = 3m_p c^2 = \dfrac{m_p c^2}{\sqrt{1-\dfrac{u^2}{c^2}}}$ $\qquad \Rightarrow 3 = \dfrac{1}{\sqrt{1-\dfrac{u^2}{c^2}}}$

Solving for 'u' gives

$$1-\frac{u^2}{c^2} = \frac{1}{9} \quad \Rightarrow \quad \frac{u^2}{c^2} = \frac{8}{9} \quad \Rightarrow u = \frac{\sqrt{3}}{8}c = \boxed{2.83\times10^{8}\ m/s}$$

(c) $K = E - m_p c^2 = 3m_p c^2 - m_p c^2 = 2m_p c^2$

Because, $m_p c^2 = 938\ MeV$, $K = \boxed{1.880\ MeV}$

(d) $E^2 = p^2c^2 + \left(m_p c^2\right)^2 = \left(3\,m_p c^2\right)^2 \Rightarrow p^2c^2 = 9\left(m_p c^2\right)^2 - \left(m_p c^2\right)^2 = 8\left(m_p c^2\right)^2$

$$\Rightarrow p = \sqrt{8}\frac{m_p c^2}{c} = \sqrt{8}\frac{\left(938\ MeV\right)}{c} = \boxed{2\,650\ MeV/c}$$

9. A deuteron, which is the nucleus of a deuterium atom, contains one proton and one neutron and has a mass of 2.013 553 u. This total deuteron mass is not equal to the sum of the masses of the proton and neutron. Calculate the mass difference and determine its energy equivalence, which is called the binding energy of the nucleus.

Soln: Using atomic mass units (u), we have

m_p = mass of proton = 1.007 276 u

m_n = mass of neutron = 1.008 665 u

$$m_p + m_n = 2.015\,941\,u$$

The mass difference Δm is therefore 0.002 388 u.

By definition; 1 u = 1.66×10^{-27} kg

$$\Delta m = 0.002\,388\,u = \boxed{3.96\times10^{-30}\ kg}$$

Using $E = \Delta mc^2$, we find that the binding energy is:

$$E = \Delta mc^2 = \left(3.96\times10^{-30}\ kg\right)\left(3.00\times10^{8}\ m/s\right)^2 = 3.56\times10^{-13}\ J = \boxed{2.23\ MeV}$$

Therefore, the minimum energy required to separate the proton from the neutron of the deuterium nucleus (the binding energy) is 2.23 MeV.

10. If (x, y, z, t) be an event in *S*-system and (x', y', z', t') be the same event in S'-system that moves relative to *S*-system with a uniform velocity *v* along *x*-direction, show that $ds^2 = dx^2 + dy^2 + dz^2 - c^2dt^2$ is invariant under Lorentz transformation.

Soln. Using Lorentz transformation in differential form, $dx = \gamma\left(dx' + vdt'\right)$; $dy = dy'$; $dz = dz'$;

$dt = \gamma\left[dt' + \left(v/c^2\right)dx'\right]$ where *v* is constant and γ =Lorentz factor.

$$\therefore \qquad ds^2 = dx^2 + dy^2 + dz^2 - c^2dt^2$$

$$= \gamma^2 \left(dx'+vdt'\right)^2 + d'y^2 + dz'^2 - c^2\gamma^2 \left(dt'+\left(v/c^2\right)dx'\right)^2$$

$$= \gamma^2 \left(dx'^2 + v^2dt'^2 + 2vdx'dt'\right) - \gamma^2c^2\left(dt'^2 + 2\frac{v}{c^2}dx'dt'\right) + dy'^2 + dx'^2$$

$$= \gamma^2 dx'^2\left(1-\frac{v^2}{c^2}\right) + dy'^2 + dz'^2 - \gamma^2c^2dt'^2\left(1-\frac{v^2}{c^2}\right)$$

$$= dx'^2 + dy'^2 + dz'^2 - c^2dt'^2 \qquad \left(\because \ \gamma = 1/\sqrt{1-v^2/c^2}\right)$$

$$= ds'^2$$

Therefore, ds^2 is invariant under Lorentz transformation.

11. A rigid bar of length 1.5m is at rest relative to frame S'. If it makes an angle $\alpha' = 45^\circ$ with $x'-axis$, find the length of the bar and its orientation relative to frame S, when $v = 0.95c$?

Soln. Resolving the length of the bar into components parallel to x' and y' axes, the corresponding length in S' are:

$$L'_x = L'\cos\alpha' = 1.5\times 1/\sqrt{2} = 1.06m;\ L'_y = L'\sin\alpha' = 1.5\times 1/\sqrt{2} = 1.06\,m$$

The vertical components would show no contraction, when viewed from S. Horizontal component, being parallel to v, would

$$\therefore \qquad L_y = L'_y = 1.06\,m$$

$$L_x = L'_x\sqrt{1-v^2/c^2} = 1.06\sqrt{1-0.95^2} = 0.331\,m$$

Therefore, length of the relative S is:

$$L_1 = \sqrt{L_x^2 + L_y^2} = \sqrt{(1.06)^2 + (0.331)^2} = 1.11\,m$$

Orientation of the bar relative to S is

$$\tan\alpha = \frac{L_y}{L_x} = \frac{1.06}{0.331} = 3.20 \qquad \therefore\ \alpha = 72^\circ 40'$$

12. Show that if l_0^3 is the rest volume of a cube, then $l_0^3\left(1-\beta^2\right)^{1/2}$ is the volume as viewed from a reference frame among moving with uniform velocity v parallel to an edge of the cube, $\beta = v/c$

Soln. The rest length of the cube was l_0 at each side so that the rest volume was l_0^3. Now, along the direction of motion, the length l_0 contracts to $\ell_0\left(1-\beta^2\right)^{1/2}$. In the other two directions, there is no Lorentz contraction and the length remains l_0.

Therefore, moving volume of the cube $\ell_0\left(1-\beta^2\right)^{1/2}\times\ell_0\times\ell_0 = \ell_0^3\left(1-\beta^2\right)^{1/2}$

3. An experimental intends to study a beam of π-mesons of velocity $0.9c$. How far can he place his apparatus from the target where π-measons are produced and still expect to get-sufficient number of π-mesons.

Soln. With relativistic time dilation,

$$\Delta t = \Delta t' / \sqrt{1-0.9^2} = 2.3\Delta t'$$

Since $\Delta t'$ is the time-interval measured by an observer moving with π-mesons, it is the real life-time, so that $\Delta t' = 2\times10^{-8}s$. So, in the lab-frame, the mean life is

$$\Delta t = 2.3\times\Delta t' = 2.3\times2\times10^{-8}s = 4.6\times10^{-8}s$$

Therefore, average distance travelled by π-mesons $= v.\Delta t = 0.9\times3\times10^{8}\times4.6\times10^{-8}m = 12.42\ m$

14. The half-life of a particular particle as measured in the lab is $4.0\times10^{-8}s$ when its speed is 0.80c and $3\times10^{-8}s$ when its speed is 0.60c. Find its actual life time.

Soln. Due to time-dilation, the half-life of a moving particle appears lengthened. Let Δt_0 be the actual half-life, i.e., in the frame of reference of the particle, and Δt that measured by a stationary observer.

$$\therefore \quad \Delta t = \Delta t_0 / \sqrt{1-v^2/c^2}$$

Therefore, by the problem,

$$4.0\times10^{-8} = \frac{\Delta t}{\sqrt{1-0.8^2}};\ 3.0\times10^{-8} = \frac{\Delta t_0}{\sqrt{1-0.6^2}}$$

Either relation gives, $\Delta t_0 = 2.4\times10^{-8}s$, the actual life-time.

15. What will be the period of the 'seconds' pendulum measured by an observer moving with a speed of 0.8c?

Soln. If $T_0\ (=2s)$ be the period of a seconds pendulum as measured by an observer at rest, the same would be, say T to the moving observer.

Therefore, from the time-dilation relation,

$$T = \frac{T_0}{\sqrt{1-v^2/c^2}} = \frac{2}{\sqrt{1-0.8^2}} = 3.33s$$

16. The average life-time of neutron is 15 min. It disintegrates spontaneously into an electron, a proton and a neutrino. If the distance of the sun from the earth is $11\times10^{10}m$, find the average minimum velocity with which a neutron must leave the sun so as to reach the earth before breaking up.

Soln. Let v be the required velocity of the neutron. The time t_1 for neutron to reach the earth from the sun before decay is

$$t_1 = \frac{11\times10^{10}m}{v}$$

If t_0 be the life-time of neutron at rest, then the average life-time t of the 'moving' neutro, as measured by an observer on earth, is

$$t = \frac{t_0}{\sqrt{1-v^2/c^2}} = \frac{15\times60s}{\sqrt{1-v^2/c^2}}$$

Plainly, this t and previous t_1 must be equal,

$$\therefore \quad \frac{11\times10^{10}}{v} = \frac{15\times60}{\sqrt{1-v^2/\left(3\times10^8\right)^2}}$$

Simplifying, we obtain,

$$v = 1.132\times10^{8}\ ms^{-1}$$

17. A rocket is travelling towards the moon with a velocity 0.6c. When half-way to the moon, it fires a message rocket back towards the earth with a velocity 0.8c relative to the primary rocket. What a velocity of the message rocket as seen by a terrestrial observer? Neglect the recoil.

Soln. The velocity of the message rocket relative to a terrestrial observer is, by velocity addition theorem,

$$v=\frac{(0.6-0.8)c}{1-0.8\times0.6}=\frac{-0.2c}{1-0.48}=-\frac{0.2c}{0.52}=-0.385c$$

The negative sign indicates that the rocket is moving towards the earth.

18. The speed of an electron in an uniform electric field changes from 0.95c to 0.98c. Find the change in mass and the workd one on the electron to change the velocity.

Soln. Let m_1, m_2 be the masses of the electron corresponding to velocities 0.95c and 0.98c.

$$\therefore\quad m_1=\frac{m_0}{\sqrt{1-0.95^2}}=3.2\,m_0;\; m_2=\frac{m_0}{\sqrt{1-0.98^2}}=5.0\,m_0$$

where m_0 = rest mass of electron = $9.11\times10^{-31}\,kg$

Therefore, change in mass,

$$\Delta m=m_2-m_1=(5.0-3.2)m_0$$

$$=1.8\times9.11\times10^{-31}\,kg=16.4\times10^{-31}\,kg$$

Work done on electron = change in its kinetic energy.

$$\Delta T=T_2-T_1=(\Delta m)c^2=1.8m_0c^2=1.8\times0.51\,MeV=0.92\,MeV$$

19. An electron of rest mass $9.1\times10^{-31}\,kg$ is moving with a speed 0.99c. What is its total energy? Find theratio of Newtonian kinetic energy to the relativistic kinetic energy.

Soln. The mass of electron corresponding to velocity v = 099c is

$$m=\frac{m_0}{\sqrt{1-0.99^2}}=\frac{9.1\times10^{-31}}{0.141}=64.54\times10^{-31}\,kg$$

Therefore, total energy of electron,

$$E=mc^2=(64.54\times10^{-31})\times(3\times10^8)^2=5.81\times10^{-13}\,J$$

$$\frac{(K.E.)_{Newton}}{(K.E.)_{relativistic}}=\frac{\frac{1}{2}m_0v^2}{mc^2-m_0c^2}=\frac{1}{2}\frac{m_0}{(m-m_0)}\left(\frac{v}{c}\right)^2=\frac{9.1}{2(64.54-9.10)}(0.99)^2=0.08$$

0. The rest mass of electron is $9.11\times10^{-31}\,kg$. What is its momentum (relativistic) when it is moving down the axis of a linear accelerator tube of a speed of 0.4c relative to the accelertor tube?

oln. The required relativistc momentum of the electron is

$$p=\frac{m_0v}{\sqrt{1-v^2/c^2}}=\frac{9.11\times10^{-31}\times0.4c}{\sqrt{1-0.4^2}}=1.19\times10^{-22}\,kg.m/s$$

1. A certain quantity of ice at 0°C melts into water at 0°C and thereby gains 1 kg mass. Find its initial mass.

oln. $1\,kg$ mass $=1\times(3\times10^8)^2\,J$ energy. Let m be the initial mass of ice in kg.

Gain in energy per kg. of ice at 0°C melting into water at 0°C is :

$$80\times1000\,calorie=80\times1000\times4.181J$$

Therefore, total energy gain $=80\times1000\times4.18\times m\ joule$

Therefore, by the problem, $80\times 1000\times 4.18\times m=\left(3\times 10^{8}\right)^{2}$

Simplifying, $m=2.69\times 10^{11}\ kg$

22. A star of radius r contracts to a radius r' while its temperature remains unchanged. Find an expression for the energy radiated by the star, assuming uniform density in each case.

Soln. The density of the star of mass m is $\rho=m/\frac{4}{3}\pi r^{3}$. The new mass m', when the star contacts to a radius r', is

$$m'=\frac{4}{3}\pi r'^{3}\rho=\frac{4}{3}\pi r'^{3}\cdot\frac{m}{\frac{4}{3}\pi r^{3}}=m\left(r'/r\right)^{3}$$

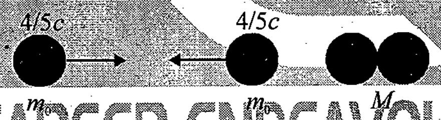
Therefore, mass loss due to contraction, $\Delta m=m-m'=m\left[1-\left(r'/r\right)^{3}\right]$

Therefore, energy radiated by the star $=\Delta m\cdot c^{2}=mc^{2}\left[1-\left(r'/r\right)^{3}\right]$

23. Two lumps of clay, each of first mass m_0, collide head on at $\frac{4}{5}c$ and stick together. Calculate the mass of the composite lump. Explain the result obtained.

Soln. Let M be the mass of composite lump. The conservation of momentum is already satisfied. : zero before collision, zero after collision.

The energy of each lump before collision is $m_0c^{2}/\sqrt{1-\left(4/5\right)^{2}}=\frac{5}{3}m_0c^{2}$. The energy of composite lump after collision is Mc^{2}. So, from conservation of energy principle,



$$Mc^{2}=2\times\frac{5}{3}m_0c^{2}\ \Rightarrow M=\frac{10}{3}m_0$$

The result shows that the mass of the composite lump is more than the sum of initial masses. Thus, in the collision mass is not conserved; the mass-increase is due to the kinetic energy being converted into rest energy.

24. A particle leaving a cyclotron has relativistic energy 10 GeV and a relativistic momentum 8 GeV/C. What is the rest mass of particle? **[H.C.U.-2014]**

(a) 0.25 GeV/C^2 (b) 1.20 GeV/C^2 (c) 2.00 GeV/C^2 (d) 6.00 GeV/C^2

Soln. $E=10\ \text{GeV}$; $p=8\ \text{GeV/c}$

Relationship between relativistic energy, momentum and mass is given by,

$$E=\sqrt{p^{2}c^{2}+m_0^{2}c^{4}}$$

$$\Rightarrow\ 10=\sqrt{\frac{8^{2}}{c^{2}}\cdot c^{2}+m_0^{2}c^{4}}\ \Rightarrow\ 10^{2}=8^{2}+m_0^{2}c^{4}$$

$$\Rightarrow\ m_0^{2}c^{4}=10^{2}-8^{2}=36\ \Rightarrow\ m_0c^{2}=6\ \text{GeV}$$

$$\Rightarrow\ m_0=6\ \text{GeV}/c^{2}$$

Correct option is (d)

25. If the total energy of a particle is thrice its rest mass energy, then the velocity of particle in terms of velocity of light c is

(a) $\frac{c}{3}$ (b) $\frac{2c}{3}$ (c) $\frac{2\sqrt{2}}{3}c$ (d) $\frac{\sqrt{2}}{3}c$ **[BHU 2010]**

Soln. The total energy E is $E = mc^2$

$$\Rightarrow 3m_0c^2 = \frac{m_0 c^2}{\sqrt{1-v^2/c^2}} \Rightarrow 1-\frac{v^2}{c^2}=\frac{1}{9} \Rightarrow v=\frac{2\sqrt{2}}{3}c.$$

Correct option is (c)

26. The kinetic energy of an electron moving with speed of 0.6 c and rest mass energy 0.511 MeV, is **[BHU 2010]**

(a) 0.224 MeV (b) 0.128 MeV (c) 0.64 MeV (d) 0.540 MeV

Soln. The kinetic energy is

$$\text{K.E.} = (m-m_0)c^2 = \left(\frac{m_0}{\sqrt{1-v^2/c^2}} - m_0\right)c^2$$

$$= m_0c^2\left(\frac{1}{\sqrt{1-v^2/c^2}} - 1\right)$$

$$= 0.511\left(\frac{1}{\sqrt{1-0.36}} - 1\right)$$

$$= \frac{0.511}{4} \simeq 0.128 \text{ MeV}.$$

Correct option is (b)

27. Two successive Lorentz transformation with velocities v_1 and v_2 in same direction are equivalent to a single Lorentz transformation with a velocity **[BHU 2010]**

(a) v_1+v_2 (b) $\frac{v_1+v_2}{1+\frac{v_1v_2}{c^2}}$ (c) $\frac{v_1+v_2}{1-\frac{v_1v_2}{c^2}}$ (d) $\frac{v_1-v_2}{1-\frac{v_1v_2}{c^2}}$

Soln. The single Lorentz transformation velocity or the relative velocity is

$$v = \frac{v_1-v_2}{1-\frac{v_1v_2}{c^2}}$$

Correct option is (d)

28. For which value of $\frac{v}{c}$ for a particle will the relativistic mass of the particle exceed its rest mass by a given fraction f? **[BHU 2011]**

(a) $\frac{\sqrt{f(1+f)}}{(2+f)}$ (b) $\frac{\sqrt{f(2+f)}}{(1+f)}$ (c) $\frac{\sqrt{2f(1+f)}}{(2+f)}$ (d) $\frac{\sqrt{f(2+f)}}{2(1+f)}$

Soln. GIven: $\frac{m-m_0}{m_0}=f \Rightarrow \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}-1=f \Rightarrow 1-\frac{v^2}{c^2}=\frac{1}{(1+f)^2}$

$$\Rightarrow \frac{v^2}{c^2} = 1 - \frac{1}{(1+f)^2} = \frac{f^2+2f}{(1+f^2)} \quad \Rightarrow \frac{v}{c} = \frac{\sqrt{f(f+2)}}{1+f}$$

Correct option is (b)

29. The velocity of an object whose length appears to be contracted to half of its proper length is **[BHU 2011]**

(a) $\frac{1}{\sqrt{2}}c$ (b) $\frac{2}{\sqrt{3}}c$ (c) $\frac{1}{2}c$ (d) $\frac{\sqrt{3}}{2}c$

Soln. Given: $l = l_0\sqrt{1-\frac{v^2}{c^2}} = \frac{l_0}{2} \Rightarrow \sqrt{1-\frac{v^2}{c^2}} = \frac{1}{2} \Rightarrow v = \frac{\sqrt{3}}{2}c$

Correct option is (d)

30. If the total energy of the particle is thrice its rest mass energy, then the velocity of the particle is **[BHU 2011]**

(a) $\frac{c}{3}$ (b) $\frac{2c}{3}$ (c) $\frac{\sqrt{2}}{3}c$ (d) $\frac{2\sqrt{2}}{3}c$

Soln. The total energy E is $E = mc^2$

$$\Rightarrow 3m_0c^2 = \frac{m_0c^2}{\sqrt{1-v^2/c^2}} \Rightarrow 1-\frac{v^2}{c^2} = \frac{1}{9} \Rightarrow v = \frac{2\sqrt{2}}{3}c.$$

Correct option is (d)

31. If the proper mean lifetime τ of π^+ meson is 2.5×10^{-8} sec, then the distance travelled by a burst of π^+ mesons travelling with v = 0.73c is **[BHU 2011]**

(a) 800 cm (b) 600 cm (c) 500 cm (d) 300 cm

Soln. Lifetime of meson measured in Lab frame

$$t = \frac{t_0}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{2.5\times10^{-8}}{\sqrt{1-(0.73)^2}}s = 3.68\times10^{-8}\ s$$

Distance travelled by the mesons will be

$$d = \left(3.68\times10^{-8}\right)\times\left(0.73\times3\times10^{8}\right)m = 8.05\ m$$

Correct option is (a)

32. If the total energy of a particle of mass m is equal to twice its rest energy, then the magnitude of the particle relativistic mechanism is **[BHU 2012]**

(a) $\frac{mc}{2}$ (b) $\frac{mc}{\sqrt{2}}$ (c) mc (d) $\sqrt{3}\,mc$

Soln. The total energy is $E^2 = m^2c^4 + p^2c^2$

$\Rightarrow (2mc^2) = m^2c^4 + p^2c^2 \Rightarrow p^2c^2 = 3m^2c^4 \Rightarrow p = \sqrt{3}mc$

Correct option is (d)

33. If a charged pion that decays in 10^{-8} second in its own rest frame is to travel 30 meters in the laboratory before decaying, the pion's speed must be most nearly **[BHU 2012]**

(a) 0.43×10^8 m/s (b) 2.84×10^8 m/s (c) 2.90×10^8 m/s (d) 2.98×10^8 m/s

Soln. Given : $\tau_0 = 10^{-8}$ sec

So the time in the laboratory frame is

$$\tau = \frac{\tau_0}{\sqrt{1-v^2/c^2}} = \frac{10^{-8}}{\sqrt{1-v^2/c^2}}$$

$\therefore \quad L = \tau v$

$$\Rightarrow 30 = \frac{10^{-8}}{\sqrt{1-v^2/c^2}}v \qquad \Rightarrow 900\left(1-\frac{v^2}{c^2}\right) = 10^{-16}v^2$$

$$\Rightarrow 900 = \left(\frac{900}{c^2}+10^{-16}\right)v^2 \qquad \Rightarrow 900 = \left(10^{-14}+10^{-16}\right)v^2$$

$$\Rightarrow v^2 = \frac{900}{1.01\times10^{-14}} \qquad \Rightarrow v = 29.8\times10^7 \text{ m/s} = 2.98\times10^8 \text{ m/s.}$$

Correct option is (d)

34. A particle leaving a cyclotron has a total relativistic energy of 10 GeV and a relativistic momentum of 9 GeV/c. What is the rest mass of this particle ? **[BHU 2012]**

(a) 0.25 GeV/c^2 (b) 1.20 GeV/c^2 (c) 2.00 GeV/c^2 (d) 6.00 GeV/c^2

Soln. The energy momentum equation is $E^2 = p^2c^2 + m_0^2c^4$

$\Rightarrow (10)^2 = (9)^2 + m_0^2c^4$

$\Rightarrow m_0^2c^4 = 19$

$\Rightarrow m_0 = 4.35$ GeV/c^2

Correct option is ( )

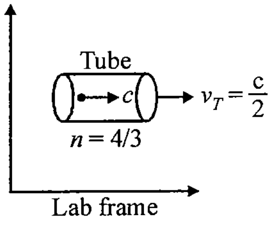
35. A tube of water is travelling at $\frac{1}{2}c$ relative to the lab frame when a beam of light travelling in the same direction as the tube enters it. What is the speed of light in the water relative to the lab frame ? (The index of refraction of water is 4/3). **[BHU 2012]**

(a) $\frac{1}{2c}$ (b) $\frac{2}{3c}$ (c) $\frac{5}{6c}$ (d) $\frac{10c}{11}$

Soln. The speed of light in water relative to the lab frame is

Given : $u_x = c$; $v_T = \frac{c}{2}$; $n = \frac{4}{3}$

The speed of light in the water relative to the lab frame is



$$u'_x = \frac{u_x/n + v_T}{1+u_x v_T/nc^2}$$

$$\Rightarrow \quad u'_x = \frac{\frac{3c}{4}+\frac{c}{2}}{1+\frac{3c^2}{24c^2}} = \frac{5c/4}{11/8} = \frac{10}{11}c.$$

Correct option is (d)

36. If the boost is along x-axis with a uniform velocity $\vec{v}$, the Lorentz invariant quantity is **[BHU 2013]**

(a) $y^2 - c^2t^2$ (b) $z^2 - c^2t^2$ (c) $y^2 + z^2 - c^2t^2$ (d) $x^2 - c^2t^2$

Soln. The Lorentz invariant quantity is $x^2 - c^2t^2$.

Correct option is (d)

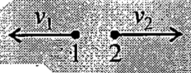
37. Two particles are travelling in the opposite directions with speed 0.9c relative to the laboratory frame. Their relative speed would be **[BHU 2013]**

(a) 0.995c (b) 1.8c (c) 0.895c (d) 0.905c

Soln. Given : $v_1 = v_2 = 0.9c$

The relative speed is

$$v = \frac{v_1+v_2}{1+\frac{v_1 v_2}{c^2}} = \frac{1.8c}{1+0.81} = \frac{1.8}{1.81}c \simeq 0.99c.$$



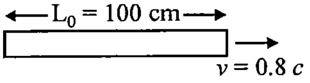
Correct option is (a)

38. The length of a rod moving with velocity equal to 0.8c, would be modified if the proper length is equal to 100 cm. The modified length of this moving rod would be equal to **[BHU 2013]**

(a) 50 cm (b) 60 cm (c) 70 cm (d) 65 cm

Soln. The modified length L is

$$L = L_0\sqrt{1-v^2/c^2}$$
$$= 100\sqrt{1-(0.8)^2}$$
$$= 60 \text{ cm}$$



Correct option is (b)

39. If the boost is along z-axis with uniform velocity, the Lorentz invariant quantity would be **[BHU 2013]**

(a) $z^2 + c^2t^2$ (b) $z^2 + x^2$ (c) $z^2 + y^2$ (d) $x^2 + y^2$

Soln. The distance perpendicular to z-axis, i.e., $x^2 + y^2$ would be invariant.

Correct option is (d)

40. If $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, $\beta^2 = \frac{v^2}{c^2}$ it is clear that $\gamma^2 - \beta^2\gamma^2 = 1$. The choices of γ and $\beta\gamma$ can be made in the language of trigonometrical functions as **[BHU 2013]**

(a) $\gamma = \operatorname{cosec}\theta,\ \beta\gamma = \cot v$ (b) $\gamma = \sec\theta,\ \beta\gamma = \tan\theta$

(c) $\gamma = \cosh\theta,\ \beta\gamma = \sinh\theta$ (d) None of the above

Soln. We choose $\frac{v}{c} = \tanh\theta$. It means, $\beta = \frac{v}{c} = \tanh\theta$.

It gives $\gamma^2(1-\beta^2) = 1$ as

$\Rightarrow \gamma = \cosh\theta$

$\therefore \beta\gamma = \sinh\theta$

Correct option is (c)

41. Which of the following remains invariant under Lorentz transformation ? **[BHU 2014]**

(a) charge density (b) current (c) charge (d) current density

Soln. The charge remains invariant under Lorentz transformation.

Correct option is (c)

42. The speed at which the kinetic energy of an electron is equal to twice its rest energy is : **[BHU 2014]**

(a) $\frac{2\sqrt{2}}{3}C$ (b) $\frac{\sqrt{2}}{3}C$ (c) $\frac{\sqrt{3}}{2}C$ (d) $\frac{C}{2}$

Soln. The kinetic energy is given by K.E. $= (m-m_0)c^2$, where m and m_0 are the relativistic and the rest masses respectively.

As given, K.E. $= 2m_0c^2$

$$\therefore \quad 2m_0c^2 = (m-m_0)c^2 \quad \Rightarrow 3m_0c^2 = \frac{m_0}{\sqrt{1-v^2/c^2}}$$

$$\Rightarrow \quad 1-\frac{v^2}{c^2} = \frac{1}{9} \quad \Rightarrow v = \frac{2\sqrt{2}}{3}c$$

Correct option is (a)

43. A meter scale which is moving with a speed v along its length appears to be centimeter scale to a stationary observer. Which of the following is correct ? **[BHU 2014]**

(a) $v^2 = 0.99\,c^2$ (b) $v^2 = 0.9999\,c^2$ (c) $v = 0.99\,c$ (d) $v^2 = 0.9\,c^2$

Soln. Given : $L_0 = 1\text{ m} = 100\text{ cm}$ and $L = 1\text{ cm}$

$$\text{Using, } L = L_0\sqrt{1-v^2/c^2} \quad \Rightarrow 1 = 100\sqrt{1-v^2/c^2}$$

$$\Rightarrow \frac{1}{10000} = 1-\frac{v^2}{c^2} \quad \Rightarrow \frac{v^2}{c^2} = \frac{9999}{10000} = 0.9999 \quad \Rightarrow v^2 = 0.9999c^2$$

Correct option is (b)

44. Kinetic energy of a free relativistic particle is given as (m_0 is rest mass, p is the momentum of the particle):

(a) $\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}m_0c^2$ (b) pc (c) $\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}-1\right)m_0c^2$ (d) mc^2 **[BHU 2014]**

Soln. Kinetic energy is

$$\text{K.E.} = (m - m_0)c^2$$

$$= \left[\frac{m_0}{\left(1 - v^2/c^2\right)^{1/2}} - m_0\right]c^2 \qquad = \left[\frac{1}{\sqrt{1 - v^2/c^2}} - 1\right]m_0 c^2$$

Correct option is (c)

45. Which of the following is incorrect ? **[BHU 2014]**
(a) Laws of mechanics are covariant under Galilean transformation
(b) Maxwell's equations are covariant under Galilean transformation
(c) Laws of mechanics are covariant under Lorentz transformation
(d) Maxwell's equations are covariant under Lorentz transformation

Soln. Maxwell's equations change their form under Lorentz transformation and hence are not covariant.

Correct option is (d)

46. Which of the following does not remain invariant in special theory of relativity ? **[BHU 2014]**

(a) $\frac{E^2}{c^2} - \vec{p}^2$ (b) $d^3\vec{p}$ (c) $\frac{d^3\vec{p}}{E}$ (d)

Soln. $d^3\vec{p} = dx\,dy\,dz$ is not invariant in special theory of relativity.

Correct option is (b)

47. If a clock moves with a very high velocity the time interval in that clock will appear to a stationary observer as:
(a) shorter (b) longer **[BHU 2014]**
(c) unchanged (d) shorter or longer on the direction of the velocity

Soln. The moving clock appears to be slower to a stationary observer.

Correct option is (a)

48. Which of the following statements is incorrect ? **[BHU 2014]**
(a) No signal can travel with velocity greater than C
(b) Simultanity is frame in dependent
(c) Proper time is same in all inertial frame
(d) Total energy of a particle does not depend on the choice of the inertial frame

Soln. Proper time depends on the velocity of the moving frame and hence depends on the inertial frame.

Correct option is (c)

49. Two photons A and B are moving in the opposite direction. The relative velocity of A w.r.t. B would be

(a) $2c$ (b) 0 (c) c (d) $\frac{2}{3}c$ **[BHU 2014]**

Soln. Relative velocity of A w.r.t. B will be

$$v = \frac{v_1 - v_2}{1 - \frac{v_1 v_2}{c^2}} = \frac{c - (-c)}{1 - \frac{(c)(-c)}{c^2}} = c$$

Correct option is (c)

50. The inertial reference frame is used in the description of special theory of relativity. An example of an inertial reference frame is **[BHU 2015]**

(a) rotating frame with a uniform angular velocity w.r.t. a stationary frame
(b) two frames stationary with respect to each other
(c) A freely failing frame under uniform gravitation
(d) two frames having uniform acceleration with respect to each other

Soln. An inertial reference frame is either at rest or moving with constant velocity with respect to another frame.

Correct option is (b)

51. An electron is moving with a speed of $0.85c$ in a direction opposite to that of a photon. The relative velocity of the electron w.r.t. the photon is **[BHU 2015]**

(a) $0.85c$ (b) $0.65c$ (c) $0.75c$ (d) c

Soln. $v_{ep} = \dfrac{v_e + v_p}{1+\dfrac{v_e v_p}{c^2}} = \dfrac{0.85c + c}{1+\dfrac{0.85c^2}{c^2}} = \dfrac{1.85}{1.85}c = c.$

Correct option is (d)

52. The energy equivalent of the rest mass of an electron is **[BHU 2015]**

(a) 1.12 MeV (b) 0.51 MeV (c) 0.91 MeV (d) 0.85 MeV

Soln. $E = mc^2 = (9.1\times10^{-31})\,(3\times10^{8})^2\ \text{J}$

$$= \frac{(9.1\times10^{-31})\,(3\times10^{8})^2}{1.6\times10^{-19}}\ \text{eV}$$

$= 0.51$ MeV.

Correct option is (b)

53. The Lorentz invariant length between two space-time points (x, y, z, ct) and (x', y', z', ct') is as follows:

(a) $(x'-x)^2+(y'-y)^2+(z'-z)^2-c^2(t'-t)^2$ **[BHU 2015]**
(b) $(x'-x)^2+(y'-y)^2-(z'-z)^2+c^2(t'-t)^2$
(c) $(x'-x)^2-(y'-y)^2+(z'-z)^2+c^2(t'-t)^2$
(d) $(x'-x)^2+(y'-y)^2+(z'-z)^2+c^2(t'-t)^2$

Soln. The Lorentz invariant length is $ds^2 = (x'-x)^2+(y'-y)^2+(z'-z)^2-c^2(t'-t)^2$.

Correct option is (a)

54. Total energy of a relativistic particle, (which is the sum of kinetic energy and rest mass energy) is :

(a) m_0c^2 (b) $\dfrac{m_0c^2}{\sqrt{1-\dfrac{v^2}{c^2}}}$ (c) $m_0c^2\sqrt{1-\dfrac{v^2}{c^2}}$ (d) $\dfrac{m_0v^2}{\sqrt{1-\dfrac{v^2}{c^2}}}$ **[BHU 2015]**

where m_0 is the rest mass if the particle moving with velocity v and c is the speed of light.

Soln. The total energy of a relativistic particle is

$$E = mc^2 = \frac{m_0}{\sqrt{1-v^2/c^2}}c^2$$

Correct option is (b)

55. In the laboratory frame of reference, a particle moving with speed 2.8×10^{10} cm/sec, has the life-time 2.5×10^{-7} sec. The proper life-time of the particle is **[BHU 2015]**

(a) $\approx 5\times10^{-10}$ sec (b) $\approx 6\times10^{-11}$ sec (c) $\approx 7\times10^{-8}$ sec (d) $\approx 7\times10^{12}$ sec

Soln. The proper life-time is

$$\tau_0 = \tau\sqrt{1-v^2/c^2} = 2.5\times10^{-7}\sqrt{1-\left(\frac{2.8\times10^8}{3\times10^8}\right)^2} \doteq 8.9\times10^{-8}\text{ sec}.$$

Correct option is (b)

56. The proper length of a rod is 100 cm. If the rod starts moving with speed 0.8*c*, its length would be

(a) 110 cm (b) 90 cm (c) 60 cm (d) 120 cm **[BHU 2015]**

Soln. If the proper length of a rod is L_0, then its length would be

$$L = L_0\sqrt{1-v^2/c^2} = 100\sqrt{1-(0.8)^2} = 60\text{ cm}.$$

Correct option is (c)

57. An example of a non-inertial reference frame is when **[BHU 2016]**

(a) a frame of reference is stationary w.r.t. the other frame

(b) a frame of reference moves with a uniform velocity towards the other reference frame

(c) a reference frame moves away from the other reference frame with a uniform velocity

(d) a reference frame rotates with a uniform angular velocity w.r.t. the other frame

Soln. An inertial reference frame is either at rest or moving with constant velocity with respect to another frame.

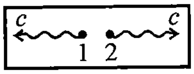
Therefore, reference frame mentioned option (d) is not an inertial frame.

Correct option is (d)

58. Two electrons move in the opposite directions each with speed 0.9 C. The relative velocity of one with respect to the other is **[BHU 2016]**

(a) 1.80 C (b) 0.99 C (c) 0.88 C (d) 0.77 C

Soln. Since, photons travel with the speed of light, so $v_1 = v_2 = c$.



Their relative speed is $v = \dfrac{v_1+v_2}{1+\dfrac{v_1 v_2}{c^2}} = \dfrac{c+c}{1+\dfrac{c^2}{c^2}} = c$

Correct option is (b)

59. A clock moves away from an observer with a uniform velocity. This clock would seem to lose 1 minute per day if it moves with the velocity: **[BHU 2016]**

(a) 1.12×10^7 m/sec (b) 2.24×10^7 m/sec (c) 12.24×10^7 m/sec (d) 1.12×10^6 m/sec

Soln. If there are 1 day ≡ 1440 minutes on earth, there are 1339 minutes on the moving clock.

Using $t = \dfrac{t_0}{\sqrt{1-v^2/c^2}}$, we have

$$\Rightarrow \quad 1440 = \frac{1339}{\sqrt{1 - v^2/c^2}}$$

$$\text{or} \quad 1 - \frac{v^2}{c^2} = \left(\frac{1339}{1440}\right)^2$$

$$\text{or} \quad v = c\sqrt{1 - \left(\frac{1339}{1440}\right)^2} \simeq 1.103 \times 10^8 \text{ m/s}$$

$$= 11.03 \times 10^7 \text{ m/s}$$

Correct option is (c)

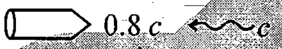
60. An astronaut moves in a spaceship travelling at a speed of 0.8 *c* (*c* is the velocity of light), and observes a photon approaching him from space. The speed of photon with respect to the spaceship is

(a) 1.8 *c* (b) 0.2 *c* (c) 0.9 *c* (d) *c* **[HCU 2010]**

Soln. Given: $v_1 = 0.8c,\ v_2 = c$

The speed of the photon with respect to the spaceship is

$$v = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}} = \frac{0.8c + c}{1 + \frac{0.8c^2}{c^2}} = \frac{1.8}{1.8}c = c.$$



Correct option is (d)

61. Special relativity predicts that an outside inertial observer sees a moving object as bearing one of the following relationships with respect to an observer moving with the object

(a) longer (b) shorter (c) brighter (d) unchanged **[HCU 2010]**

Soln. The outside inertial observer sees the object to be shorter than the moving observer with the object.

Correct option is (b)

62. The proper length of a space vehicle is l_0. According to an observer on earth, the length of the spaceship is 25% of its proper length. The speed of the spaceship according to the observer on earth is given by

(a) $\frac{\sqrt{3}}{2}c$ (b) $\sqrt{\frac{3}{2}}c$ (c) c (d) $0.968\,c$ **[HCU 2010]**

Soln. Given: $L_0 = l_0$ and $L = \frac{l_0}{4}$

Using $L = L_0\sqrt{1 - v^2/c^2} \Rightarrow \frac{1}{4} = \frac{1 - v^2}{c^2} \Rightarrow v = \frac{\sqrt{3}}{2}c.$

Correct option is (a)

63. Two electrons move towards each other, the speed of each being 0.9 *c*. What is their relative velocity according to special theory of relativity?

(a) 0.995 *c* (b) 1.09 *c* (c) 0.89 *c* (d) 0.91 *c* **[HCU 2010]**

Soln. The relative velocity is

$$v = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}} = \frac{1.8c}{1.81} \simeq 0.995c. \qquad \underset{\bullet\!\longrightarrow}{v_1 = 0.9\,c} \quad \underset{\longleftarrow\!\bullet}{v_2 = 0.9\,c}$$

Correct option is (a)

64. Two events occur at an interval Δt at the same point in a rest frame. In a frame which is moving with a uniform velocity v, the time interval between the events will be given by

(a) $\dfrac{\Delta t}{\sqrt{1-v^2}}$ (b) $(\Delta t)\sqrt{\dfrac{1-v^2}{c^2}}$ (c) $\dfrac{\Delta t}{\sqrt{1-v^2/c^2}}$ (d) $\dfrac{\Delta t}{\sqrt{1+v^2/c^2}}$ **[HCU 2010]**

Soln. The two time instants are $t_1' = \dfrac{1}{\sqrt{1-v^2/c^2}}\left(t_1 - \dfrac{vx}{c^2}\right)$ and $t_2' = \dfrac{1}{\sqrt{1-v^2/c^2}}\left(t_2 - \dfrac{vx}{c^2}\right)$.

$$\therefore\ \Delta t' = \frac{1}{\sqrt{1-v^2/c^2}}(t_1 - t_2) = \frac{\Delta t}{\sqrt{1-v^2/c^2}}.$$

Correct option is (c)

65. A measuring rule of rest length l moves relative to an observer with the velocity v. The observer measures the length of the rule as $\frac{2}{3}l$. What is the velocity v ?

(a) c (b) $0.1\ c$ (c) $0.745\ c$ (d) $0.345\ c$ **[HCU 2010]**

Soln. Given : $L_0 = l$ and $L = \dfrac{2}{3}l$

Using, $L = L_0\sqrt{1-v^2/c^2} \Rightarrow \dfrac{4}{9} = 1 - \dfrac{v^2}{c^2} \Rightarrow v = 0.745c.$

Correct option is (c)

66. An astronaut moves in a spaceship travelling at a speed of 0.8 c (c is the speed of light). If the astronaut observes a photon approaching the ship from space, then the speed of this photon with respect to the observer is

(a) 1.8 c (b) c (c) 0.2 c (d) 0.9 c **[HCU 2012]**

Soln. The speed of the photon with respect to the observer is

$$v = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}} = \frac{0.8c + c}{1 + \frac{0.8c^2}{c^2}} = c.$$

Correct option is (b)

67. Two observes having identical instruments carry out identical experiments to study the motion of a massive particle. First observer concludes that the particle is moving in a straight line with constant velocity while the second observer concludes that the particle is moving with acceleration. Which of the following four statements made about these two observers are correct :

(i) both observers are at rest

(ii) second observer is moving with constant velocity with respect to the first observer

(iii) second observer is accelerating with respect to the first observer

(a) (i) (b) (ii) (c) (i) and (iii) (d) (iii) **[HCU 2014]**

Soln. If the particle is moving with a constant velocity w.r.t. an inertial frame, then it will also move with a constant velocity w.r.t. an another inertial frame.

Therefore, from the information given in the question we can say that both observers are not inertial i.e. second observer is accelerating w.r.t the first observer.

Correct option is (d).

68. A particle is accelerated such that its energy is three times the its rest mass energy. Then velocity of the particle would be

(a) $\sqrt{2}\frac{c}{3}$ (b) $\frac{3c}{2\sqrt{2}}$ (c) $2\sqrt{2}\frac{c}{3}$ (d) $\frac{c}{\sqrt{3}}$ **[HCU 2014]**

Soln. The energy is given by $E = mc^2$

$$\Rightarrow 3m_0c^2 = \frac{m_0}{\sqrt{1-v^2/c^2}}c^2 \Rightarrow 1-\frac{v^2}{c^2} = \frac{1}{9} \text{ or } v = \frac{2\sqrt{2}}{3}c.$$

Correct option is (c)

69. A particle of mass 1.40×10^5 kg has a relativistic momentum of magnitude 3.15×10^{13} kg m/s. How fast is the particle traveling? **[HCU 2015]**

(a) $2.68\times10^8\ m/s$ (b) $1.8\times10^8\ m/s$ (c) $3.3\times10^{16}\ m/s$ (d) $3\times10^8\ m/s$

Soln. The relativistic moment of a particle is given by $p = \frac{m_0}{\sqrt{1-v^2/c^2}}v$.

Given: $p = 3.15\times10^{13}$ kg-m/sec, $m_0 = 1.40\times10^5$ kg

$$\therefore\ 2.25\times10^8 = \frac{v}{\sqrt{1-v^2/c^2}} \Rightarrow (2.25\times10^8)^2 = \frac{v^2}{1-v^2/c^2}$$

$$\Rightarrow v = \sqrt{\frac{(2.25\times10^8)^2}{1+\frac{(2.25\times10^8)}{3\times10^8}}} = 1.8\times10^8 \text{ m/sec.}$$

Correct option is (b)

70. A proton of rest mass $1.68\times10^{-27}\ kg$ is traveling at $2.5\times10^8\ m/s$. What is kinetic energy of the proton? **[HCU 2015]**

(a) $1.18\times10^{-10}\ J$ (b) $0.18\times10^{-10}\ J$ (c) $2.18\times10^{-27}\ J$ (d) $1.32\times10^{-27}\ J$

Soln. The kinetic energy of the proton is given by

$$K = mc^2 - m_0c^2 = \frac{m_0c^2}{\sqrt{1-v^2/c^2}} - m_0c^2.$$

Given: $m_0 = 1.68\times10^{-27}$ kg, $v = 2.5\times10^8$ m/sec.

$$\therefore\ K = \frac{1.68\times10^{-27}\times9\times10^6}{\sqrt{1-\left(\frac{2.5}{3}\right)^2}} - 1.68\times10^{-27}\times9\times10^{16}$$

$$= \left(\frac{15.12\times10^{-11}}{0.552} - 15.12\times10^{-11}\right)\text{J} \quad = (27.35\times10^{-11} - 15.12\times10^{-11})\text{J} \quad = 1.22\times10^{-10}\text{ J}$$

Correct option is (a)

71. A spaceship passes you at a speed of 0.85 c. You meausre its length to be 48.2 m. What is its length at rest?

(a) 80 m (b) 100 m (c) 91.5 m (d) 81 m **[HCU 2015]**

Soln. According to Lorentz transform, $L_0' = L_0\sqrt{1-v^2/c^2}$

$$\Rightarrow 48.2 = L_0\sqrt{1-(0.85)^2} \Rightarrow L_0 = 91.49\text{ m} \simeq 91.5\text{ m}.$$

Correct option is (c)

72. A particle at rest has a lifetime of 3 μsec. An observer measures the speed of this particle to be $\frac{3c}{5}$. The distance travelled by this particle, measured by the observer, is **[HCU 2016]**

(a) $\frac{9c}{4}\times 10^{-6}\ m$ (b) $\frac{15c}{4}\times 10^{-6}\ m$ (c) $\frac{18c}{5}\times 10^{-6}\ m$ (d) $\frac{9c}{16}\times 10^{-6}\ m$

Soln. The life time of the particle as measured by the observer in its own frame is

$$\tau = \frac{\tau_0}{\sqrt{1-v^2/c^2}} = \frac{3\mu\sec}{\sqrt{1-\left(\frac{3c}{5c}\right)^2}} = \frac{15}{4}\mu\sec.$$

Therefore, distance travelled $= v\tau$

$$= \frac{3c}{5}\times\frac{15}{4}\mu\sec = \frac{9c}{4}\times 10^{-6}\text{ m}.$$

Correct option is (a)

74. Two electrons are ejected in opposite directions from a sample of radioactive atoms. Each electron has a speed of 0.67c as measured by a laboratory observer. Their relative velocity is given by **[HCU 2013]**

(a) 1.34 c (b) 1.19 c (c) 0.92 c (d) 0.87 c

Soln. The velocity of the pairs of electron w.r.t. lab frame are $v_1 = 0.67c$ and $v_2 = 0.67c$.

The velocity of electron w.r.t. other electron is

$$v = \frac{v_1+v_2}{1+\frac{v_1 v_2}{c^2}}$$

$$= \frac{0.67c+0.67c}{1+\frac{0.67c\times 0.67c}{c^2}} = 0.92c.$$

Correct option is (c)

75. A muon, whose life-time at rest is 2 × 10^{-6} sec, is travelling through the laboratory at three-fifth of the speed of light. It will last in **[DU 2014]**

(a) 2 × 10^{-6} sec. (b) 2.5 × 10^{-6} sec (c) 3.5 × 10^{-6} sec (d) 3.0 × 10^{-5} sec

Soln. Life-time of the muon w.r.t Laboratoty will be

$$t = \frac{t_0}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{2\times 10^{-6}}{\sqrt{1-(0.6)^2}}s = 2.5\times 10^{-6}s$$

Correct option is (b)

76. Relative to a stationary observer, a rod of length 1.0 metre is moving at 0.8 times the speed of light in vacuum. It would appear to the observer that the rod's length is: **[DU 2015]**

(a) 0.8 m (b) 0.6 m (c) 1.0 m (d) 1.25 m

Soln. Length of the rod w.r.t. the observer will be

$$l = l_0\sqrt{1-\frac{v^2}{c^2}} = 1\times\sqrt{1-(0.8)^2}\,m = 0.6\,m$$

Correct option is (b)

77. Two events take place at the same place in a lab frame but occur with a time difference of 3 seconds. The same events occur with a time difference of 5 seconds in a rocket frame. The relative speed of rocket and the laboratory is **[DU 2016]**

(a) 0.5c (b) 0.8c

(c) 0.6c (d) cannot be determined from this data

Soln. The relation between $\Delta t'$ (the time in the rocket frame) and Δt (the time in the lab frame) if the two events take place at the same location in a lab frame is $\Delta t' = \frac{\Delta t}{\sqrt{1-v^2/c^2}}$.

$$\Rightarrow 1-\frac{v^2}{c^2} = \left(\frac{\Delta t}{\Delta t'}\right)^2 = \left(\frac{3}{5}\right)^2 \Rightarrow v = \frac{4}{5}c = 0.8c.$$

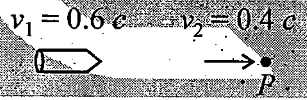
Correct option is (b)

78. A spaceship approaches a planet with a speed 0.6c. At some point it fires a projectile with a speed 0.4c towards the planet. The speed of the projectile, as seen by an observer on the planet would be approximately **[DU 2016]**

(a) 0.76c (b) 0.80c (c) 0.99c (d) 0.40c

Soln. The speed of the projectile as seen by an observer on the planet is

$$v = \frac{v_1+v_2}{1+\frac{v_1 v_2}{c^2}} = \frac{0.6c+0.4c}{1+\frac{0.24c^2}{c^2}} = \frac{1}{1.24}c = 0.80c.$$



Correct option is (b)

79. A particle is moving at a speed of $2.6\times10^8\,ms^{-1}$ relative to the laboratory. Its lifetime as measured by an observer in the laboratory is 4.7×10^{-6} s. The lifetime of the particle in its own rest frame is about **[DU 2016]**

(a) $2.3\times10^{-6}s$ (b) $9.4\times10^{-6}s$ (c) $4.7\times10^{-6}s$ (d) $14.4\times10^{2}s$

Soln. Given : $\tau = 4.7\times10^{-6}$ sec, $v = 2.6\times10^{-8}$ m/s.

The lifetime of the particle in its own frame is

$$\tau_0 = \tau\sqrt{1-v^2/c^2}$$

$$= 4.7\times10^{-6}\sqrt{1-\left(\frac{2.6\times10^8}{3\times10^8}\right)^2}$$

$$= 2.3\times10^{-6}\text{ sec.}$$

Correct option is (a)

Chapter 10

Atomic Physics

10.1. Bohr's Atom Model :

Bohr's postulates :

The postulates of Bohr's theory for hydrogen atom are enumerated.

(1) An electron in atom moves in a circular orbit about the nucleus under the influence of coulomb attraction between the electron and the nucleus, and obeys the laws of classical mechanics.

(2) But instead of an infinite number of orbits which would be possible in classical mechanics, there are only certain discrete orbits (quantization) for which the angular momentum of the electron, L, is an integral multiple of

$h/2\pi\left(=\hbar\right)$, where h is Planck's constant.

i.e. $L = nh/2\pi = n\hbar$; $n = 1, 2, ...$etc.

(3) An electron moving in such a quantised orbit (also called stationary energy state) does not radiate electromagnetic energy, as predicted by electromagnetic theory. Thus, its total energy remains constant.

(4) Electromagnetic radiation is either emitted or absorbed only when transition of an electron between two stationary states takes place. The radiation is emitted or absorbed as a single quantum whose energy E is equal to the difference in energy of the two states, and is given by E = hν, according to Planck's law where 'ν' is the frequency of the radiation.

$\therefore \quad E = E_m - E_n = h\nu_{nm}$ (Bohr's frequency condition)

Note : If the electron is initially at the lower state E_1 then transition to the higher state E_2 is possible only if it absorbs an amount of energy $E = E_2 - E_1 = h\nu$ incident on it.

Expression for orbital radius:

Consider a hydrogen-like atom of atomic number Z consisting of a nucleus of charge +Ze (Z = 1 for hydrogen) and a single electron of mass m and charge $-e$ revolving in a circular orbit of radius r_n. For the mechanical stability of the electron,

$$\frac{Ze^2}{4\pi\varepsilon_0 r_n^2} = \frac{mv_n^2}{r_n} \qquad \text{... (i)}$$

where, v_n is the velocity of the electron in its orbit of radius r_n. The left side of (i) is the Coulomb pull that provides the required centripetal force $\frac{mv^2}{r}$ to keep the electron in orbit.

From Bohr's quantum condition, we obtain

$$L = I\omega = mr_n^2\left(\frac{v_n}{r_n}\right) = mv_n r_n = \frac{nh}{2\pi} \qquad \text{... (ii)}$$

From (i) and (ii), eliminating r_n, we get

$$Ze^2 = 4\pi \epsilon_0 v_n (nh/2\pi)$$

$$\therefore \qquad v_n = \frac{Ze^2}{2\epsilon_0 nh} \qquad \text{... (iii)}$$

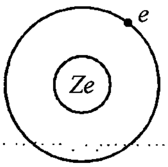
From (ii) and (iii),

$$r_n = \frac{n^2h^2\epsilon_0}{\pi mZe^2} \qquad \text{... (iv)}$$

The radii r_n and the electron's orbit velocity v_n both depend on the integer n, and the radii is proportional to the square of the n-value, n being known as a quantum number, specifically the principal quantum number.

Expression for total energy:

The total energy E_n of the electron in its n^{th} orbit is the sum of its kinetic and potential energy.



Kinetic energy, $E_k = \frac{1}{2}mv_n^2 = Ze^2/8\pi\epsilon_0 r_n$, using the relation (i)

Potential energy, $V = \int_{\infty}^{r_n} \frac{Ze^2}{4\pi\epsilon_0 r_n^2}dr_n = -\frac{Ze^2}{4\pi\epsilon_0 r_n}$

Therefore, total energy, $E_n = E_k + V = \frac{Ze^2}{8\pi\epsilon_0 r_n} - \frac{Ze^2}{4\pi\epsilon_0 r_n} = -\frac{Ze^2}{8\pi\epsilon_0 r_n}$ ... (v)

Therefore, energy in the nth orbit, $E_n = -\frac{Ze^2}{8\pi\epsilon_0}\left(\frac{\pi mZe^2}{n^2h^2\epsilon_0}\right)$, using the relation (iv)

$$\text{Or,} \qquad E_n = -\left(\frac{1}{n^2}\right)\frac{mZ^2e^4}{8\epsilon_0^2 h^2} \qquad \text{... (vi)}$$

The energy of electron within an atom is also quantised, that is, it can take only certain discrete values and comes out to be negative. So, as n increases, i.e., as electron goes to orbits of larger radii, E_n increases and converges to the limiting value $E_n = 0$ as $n \to \infty$. E_n is lowest for the smallest orbit for which $n = 1$.

Frequency of emitted radiation:

Let the electron make a transition from the quantum state n_i (corresponding to total energy E_i) to the quantum state n_f (corresponding to total energy E_f). If $n_i > n_f, E_i > E_f$, then there will be emission of radiation of frequency v, given by Bohr's frequency condition,

$$v=\frac{E_i-E_f}{h}=\frac{mZ^2e^4}{8\in_0^2h^3}\left(\frac{1}{n_f^2}-\frac{1}{n_i^2}\right) \qquad \text{... (vii)}$$

In terms of wave number, $\bar{v}$ (defined as reciprocal of wavelength, that is, $\bar{v}=\frac{1}{\lambda}=\frac{v}{c}$)

$$\bar{v}=\frac{me^4}{8\in_0^2ch^3}Z^2\left(\frac{1}{n_f^2}-\frac{1}{n_i^2}\right)=R_\infty Z^2\left(\frac{1}{n_f^2}-\frac{1}{n_i^2}\right) \qquad \text{... (viii)}$$

where, $R_\infty=\frac{me^4}{8\in_0^2ch^3}$, a constant whose value is $1.09737\times10^7\ m^{-1}$

Origin of spectral series:

The orgin of spectral series can easily be understood now from Bohr's theory.

By putting $n_f=2$ and $n_i=3,4,5,......$etc in the equation (viii), we get a series of spectral lines of wave number.

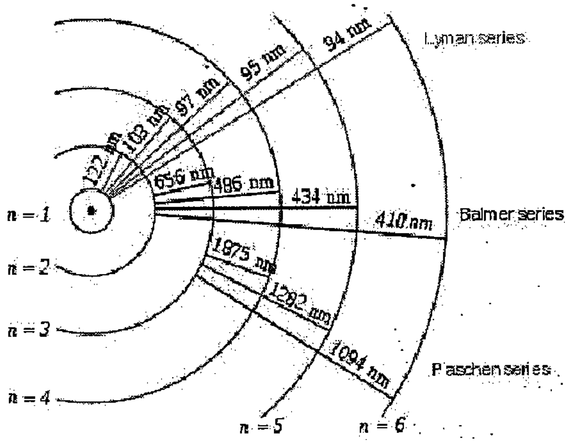
$$\bar{v}=R_\infty Z^2\left(\frac{1}{2^2}-\frac{1}{n_i^2}\right),\ n_i=3,4,.....etc.$$

$$=R_\infty\left(\frac{1}{2^2}-\frac{1}{n_i^2}\right) \text{ for hydrogen (Z = 1)}$$

The series correponds to the Balmer series of hydrogen as given by the empirical formula.

$$\bar{v}=R_H\left(\frac{1}{2^2}-\frac{1}{n_i^2}\right) \qquad \text{where, } R_H=R_\infty=\frac{me^4}{8\in_0^2ch^3}$$

The constant R_H is known as the Rydberg constant.


94 nm
95 nm
97 nm
103 nm
122 nm
656 nm
486 nm
434 nm
410 nm
1875 nm
1282 nm
1094 nm
Lyman series
Balmer series
Paschen series
n = 1
n = 2
n = 3
n = 4
n = 5
n = 6

The equation (viii) suggests that series other than Balmer should also exist, corresponding to $n_f=1, n_f=3, n_f=5$ etc. in regions of the spectrum away from the visible. Thus, the other series lines emitted by hydrogen are :

Lyman series, $\bar{v} = R_\infty\left(\frac{1}{1^2}-\frac{1}{n^2}\right)$, $n = 2,3,4,.........$

Balmer series, $\bar{v} = R_\infty\left(\frac{1}{2^2}-\frac{1}{n^2}\right)$, $n = 3,4,5,.........$

Paschen series, $\bar{v} = R_\infty\left(\frac{1}{3^2}-\frac{1}{n^2}\right)$, $n = 4,5,6,.........$

Brackett series, $\bar{v} = R_\infty\left(\frac{1}{4^2}-\frac{1}{n^2}\right)$, $n = 5,6,7,.........$

Pfund series, $\bar{v} = R_\infty\left(\frac{1}{5^2}-\frac{1}{n^2}\right)$, $n = 6,7,8,.........$

Correction in Bohr's theory for finite nuclear mass

In the general Bohr's theory, the nucleus is considered stationary, that is, it is assumed to be infinitely massive compared to the mass of the electron. However, a nucleus of finite mass cannot be at rest. To modify the theory on this account is rather simple.

Consider the electron-nucleus system as shown in figure. Both the electron of mass m and the nucleus of mass M must move such that the centre-of-mass O of the composite system remains fixed in space.

Let r_1 and r_2 be the radii of the orbits (assumed circular) of the electron and the nucleus respectively. Then,

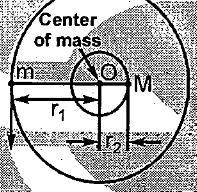


Fig. Correction for finite nucleus mass

$$mr_1 = Mr_2 \Rightarrow r_2 = \frac{m}{M}r_1 \qquad ...(i)$$

Therefore, the distance the nucleus and the electron

$$r = r_1 + r_2 = r_1\left(1+\frac{m}{M}\right), \text{ using (i)} \qquad ...(ii)$$

The electron and the nucleus must rotate about the centre-of-mass with a common angular velocity ω. So the total angular momentum L of the system is given by

$$L = mr_1^2\omega + Mr_2^2\omega$$

$$= mr_1^2\omega\left(1+\frac{m}{M}\right), \text{ using (i)}$$

$$= \frac{mr^2\omega}{1+\frac{m}{M}}, \text{ using (ii)}$$

$$= \frac{mM}{m+M}rv \quad (\because v = r\omega) \qquad ...(iii)$$

For an infinitely heavy nucleus, the electron rotates about the stationary nucleus with

$$L = mrv \qquad ...(iv)$$

Thus the net effect is to replace the mass m of the electron by the so called reduced mass

$$\frac{1}{\mu}=\frac{1}{M}+\frac{1}{m} \qquad \text{...(v)}$$

The Rydberg constant would now become

$$R_H=\frac{\mu e^4}{8\varepsilon_0^2 ch^3}=\frac{m}{\left(1+\dfrac{m}{M}\right)}\left(\frac{e^4}{8\varepsilon_0^2 ch^3}\right)=\frac{R_\infty}{1+\dfrac{m}{M}}$$

where R_∞ is the Rydberg constant for an infinitely heavy mass.

It is clear from (v) that the reduced mass $\mu < m$ and increases with increasing nucleus mass M. When $M \to \infty$, $\mu \to m$, the electronic mass. Therefore, R_H increases as M increases and as $M \to \infty$, $R_M \to R_\infty$. It is observed that the spectral lines of Pickering series of He$^+$ have slightly higher wave numbers of shorter wavelengths than the corresponding lines of Balmer series of hydrogen. The explanation lies in the fact that $R_{\text{He}} > R_H$.

Bohr's correspondence principle

Neither in Bohr's theory nor in Planck's quantum which Bohr utilised, any attempt was made in developing any new theory of atomic mechanics. Hence they were *ad hoc* in nature. Bohr however noted from the energy expression that as the quantum number n becomes very large, the successive values of the allowed energy differ very little from each other and appear almost continuous. Also, as the spectral lines tend to the series limit they crowd so much that the spectrum appears continuous. In this context, Bohr in 1923 enunciated an important principle called the correspondence principle. It states that the behaviour of an atomic system as predicted by quantum theory tends asymptotically (i.e. corresponds) to that expected in classical theory in transitions involving states of large quantum numbers. Bohr's correspondence principle forms an important guideline in formulating the laws of atomic mechanics.

We give here an elementary deduction (illustration) of the principle.

The frequency of revolution f of the electron in an orbit of radius r is

$$f=\frac{\text{electron speed}}{\text{orbit circumference}}=\frac{v}{2\pi r}$$

Substituting the values of v and r

$$v_n=\frac{Ze^2}{2\epsilon_0 nh}\ ; \quad r=\frac{n^2h^2\epsilon_0}{\pi m Ze^2}$$

$$f=\frac{mZ^2e^4}{8\varepsilon_0^2h^3}\left(\frac{2}{n^3}\right) \qquad \text{...(i)}$$

A radiated frequency in quantum theory however is, from (vii) of Art 1.2,

$$\nu=\frac{mZ^2e^4}{8\varepsilon_0^2h^3}\left(\frac{1}{n_f^2}-\frac{1}{n_i^2}\right)=\frac{mZ^2e^4}{8\varepsilon_0^2h^3}\,\frac{(n_i-n_f)(n_i+n_f)}{n_i^2\,n_f^2}$$

$$=\frac{mZ^2e^4}{8\varepsilon_0^2h^3}\,\frac{(2np-p^2)}{n^2(n-p)^2} \qquad \text{...(ii)}$$

writing n for n_i and $n-p$ $(p=1,2,3,...)$ for n_f.

For very large n_i and n_f, $n \gg p$ and we obtain

$$\frac{2np-p^2}{n^2(n-p)^2}\simeq\frac{2np}{n^4}$$

Therefore, from (ii) above, $v = \frac{mZ^2e^4}{8\varepsilon_0^2 h^3}\left(\frac{2p}{n^3}\right)$.

When $p = 1$, $v = f$, the frequency of rotation of electron, from (i). Harmonies of this frequency are radiated when $p = 2, 3, 4, \ldots$. Thus for very large quantum numbers, the classical orbit frequency equals the quantum theory frequency.

Example 1. Calculate the velocity, radius and energy of the first Bohr orbit and also the Rydberg constant for *H*-atom.

Soln. **Orbital speed :** We have from the relation, $v_n = \frac{e^2}{2h\varepsilon_0}\left(\frac{Z}{n}\right)$

Therefore, $v_1 = \frac{e^2}{2h\varepsilon_0}\ (\because\ Z = 1, n = 1) = \frac{(1.6\times10^{-19})^2}{2\times6.62\times10^{-34}\times8.85\times10^{-12}} = 2.18\times10^6$ m/s.

Orbital radius : $r_n = \frac{n^2h^2\varepsilon_0}{\pi me^2}\ (\because\ Z = 1)$

Therefore, $r_1 = \frac{h^2\varepsilon_0}{\pi me^2}\ (\because\ n = 1) = \frac{(6.62\times10^{-34})^2\times8.85\times10^{-12}}{3.14\times9.1\times10^{-31}\times(1.6\times10^{-19})^2} = 0.53\times10^{-10}\,\text{m} = 0.53\ \text{Å}$

Orbital energy : $E_1 = \frac{me^4}{8\varepsilon_0^2h^2}$ (save the sign)

Therefore, $E_1 = \frac{9.1\times10^{-31}\times(1.6\times10^{-19})^4}{8\times(8.85\times10^{-12})^2\times(6.62\times10^{-34})^2} = 2.17\times10^{-18}\ \text{J} = 13.6\ \text{eV}$

Rydberg constant : $R_H = \frac{me^4}{8\varepsilon_0^2ch^3}$

Therefore, $R_H = \frac{9.1\times10^{-31}\times(1.6\times10^{-19})^4}{8\times(8.85\times10^{-12})^2\times3\times10^8\times(6.62\times10^{-34})^3} = 1.093\times10^7\ \text{m}^{-1}$

Note: The radius of the first Bohr orbit of hydrogen is called Bohr radius.

Example 2. Show that the energy required to raise *H*-atom from the ground state ($n = 1$) to the first excited state ($n = 2$) is about 10 eV.

Soln. Let E_1, E_2 be the energies of the electron in states $n = 1$ and $n = 2$ respectively.

Therefore, $E_1 = \frac{me^4}{8\varepsilon_0^2h^2}$; $E_2 = \frac{me^4}{2^2\times8\varepsilon_0^2h^2}$

Therefore, required energy $E = E_1 - E_2 = \frac{3me^4}{4\times8\varepsilon_0^2h^2}$

Therefore, $E = \frac{3\times9.1\times10^{-31}\times(1.60\times10^{-19})^4}{32\times(8.85\times10^{-12})^2\times(6.62\times10^{-34})^2}\ \text{J} \simeq 10\,\text{eV}$

Example 3. Given the Rydberg constant as $1.097 \times 10^7\ \text{m}^{-1}$, calculate the wavelength of the first line of Balmer series in Angstrom unit.

Soln. For the first line i.e. H_α-line of Balmer series we have $n_i = 3, n_f = 2$.

Therefore, Wave number of H_α-line : $\bar{v} = R_H\left(\frac{1}{2^2} - \frac{1}{3^2}\right) = \frac{5}{36}R_H$

Therefore, Wavelength of H_α-line : $\lambda = \frac{1}{\bar{v}} = \frac{36}{5R_H} = \frac{36}{5 \times 1.097 \times 10^7}\ \text{m} = 6563\ \text{A}$

Example 4. What is the velocity of electron in the ground state of Bohr's H-atom in terms of the speed of light? What is this called?

Soln. From the relation $v_n = \frac{e^2 Z}{2\varepsilon_0 hn}$, we have $v_1 = \frac{e^2}{2\varepsilon_0 h}$ $(\because\ Z = 1, n = 1)$. In terms of the speed of light, the electron velocity is $\frac{e^2}{2\varepsilon_0 ch} = \frac{(1.60 \times 10^{-19})^2}{2 \times 8.85 \times 10^{-12} \times 3 \times 10^8 \times 6.62 \times 10^{-34}} \simeq \frac{1}{137}$

This known as the fine structure constant.

Example 5. Calculate what will be the appropriate quantum number n for an electron in an orbit of radius 0.1 mm.

Soln. The radius of Bohr's orbit, in terms of Bohr's radius r_1, is given by $r_n = 0.53n^2\ \text{Å}$.

For $r_n = 0.1\ \text{mm} = 10^6\ \text{Å}$, we therefore have from above $10^6 = 0.53n^2$

$\therefore\ n^2 = \frac{10^6}{0.53} = 18.87 \times 10^5$

$\therefore\ n \simeq 1374$

Example 6. The average of life time of an electron in an excited state of H-atom is about 10^{-8} s. How many revolutions does an electron in the $n = 2$ state make before its transition to $n = 1$ state. The Rydberg constant for H-atom is $1.097 \times 10^7\ \text{m}^{-1}$.

Soln. The frequency f of revolution of the electron in an orbit is $f = \frac{\text{electron speed}}{\text{orbital circumference}} = \frac{v_n}{2\pi r_n}$.

But $v_n = \frac{Ze^2}{2\varepsilon_0 nh}$ and $r_n = \frac{n^2 h^2 \varepsilon_0}{\pi m Z e^2}$. $\therefore\ f = \frac{mZ^2e^4}{8\varepsilon_0^2 h^3}\left(\frac{2}{n^3}\right) = \frac{2R_{H^c}}{n^3}$.

For $n = 2,\ f = \frac{2R_{H^c}}{8} = \frac{2 \times 1.097 \times 10^7 \times 3 \times 10^8}{8} = 8.2 \times 10^{14}\ \text{s}^{-1}$

Therefore, number of revolution in 10^{-8} s is $N = f \times 10^{-8} = 8.2 \times 10^{14} \times 10^{-8} = 8.2 \times 10^6$.

Example 7. Calculate the series limit of the Balmer series given $R_H = 1.097 \times 10^7\ \text{m}^{-1}$

Soln. The wavelength of spectral lines of the Balmer series are given by

$$\frac{1}{\lambda} = R_H\left(\frac{1}{2^2} - \frac{1}{n^2}\right);\ n = 3, 4, 5, \ldots.$$

For the series limit, $n = \infty$. We thus obtain

$$\frac{1}{\lambda}=R_H\left(\frac{1}{2^2}-\frac{1}{\infty^2}\right)=\frac{R_H}{4}$$

$$\therefore\ \lambda_\infty=\frac{4}{R_H}=\frac{4}{1.097\times10^7}=3.646\times10^{-7}\,\text{m}=3646\,\text{Å}$$

Example 8. The first line of the Balmer series of hydrogen has a wavelength 6563 A. Calculate the wavelength of the second line.

Soln. The wavelengths of the spectral lines of Balmer series are given by $\frac{1}{\lambda}=R_H\left(\frac{1}{2^2}-\frac{1}{n^2}\right)$; $n=3,4,5,\ldots.$

For the first line, $n=3$ so that the wavelength λ_1 is given by $\frac{1}{\lambda_1}=R_H\left(\frac{1}{2^2}-\frac{1}{3^2}\right)=\frac{5R_H}{36}$.

For the second line $n=4$. Therefore, the wavelength λ_2 is given by $\frac{1}{\lambda_2}=R_H\left(\frac{1}{2^2}-\frac{1}{4^2}\right)=\frac{3R_H}{16}$

$$\therefore\ \frac{\lambda_1}{\lambda_2}=\frac{3R_H}{16}\times\frac{36}{5R_H}=\frac{27}{20}$$

$$\therefore\ \lambda_2=\lambda_1\times\frac{20}{27}=6363\times\frac{20}{27}=4861\,\text{Å}.$$

Example 9. Calculate the ionisation potential of (i) H-atom and (ii) He-atom in the ground state.

Soln. We have : $E_n=-\frac{me^4Z^2}{8\varepsilon_0^2n^2h^2}$

(i) For H-atom in ground state $Z=1, n=1$.

$$\therefore\ E_1=-\frac{9.1\times10^{-31}\times(1.60\times10^{-19})^4}{8\times(8.85\times10^{-12})^2\times(6.62\times10^{-34})^2}\,\text{J}=-13.6\,\text{eV}.$$

∴ Energy of the first Bohr orbit of H-atom $=-13.6$ eV

∴ Ionisation potential of H-atom in ground state $=13.6$ eV

(ii) For the He-atom $Z=2$.

$$\therefore\ (E_1)_{He}=-\frac{me^4}{8\varepsilon_0^2h^2}\times4=+4(E_1)_H$$

$$\therefore\ (E_1)_{He}=+4\times(-13.6)=-54.4\,\text{eV}$$

∴ Ionisation potential of He-atom $=54.4$ eV

Example 10. Find the wavelength separation of the first member of the Balmer series due to ^{1}H and ^{2}H(= ^{2}D).

Soln. We have $\bar{\nu}_H=R_H\left(\frac{1}{2^2}-\frac{1}{n^2}\right)$.... for ^{1}H and $\bar{\nu}_D=R_D\left(\frac{1}{2^2}-\frac{1}{n^2}\right)$.... for ^{2}H(= ^{2}D).

$$\therefore\ \frac{\bar{\nu}_H}{\bar{\nu}_D}=\frac{R_H}{R_D}=\frac{1+m/M_D}{1+m/M_H}=\frac{1+m/2M_H}{1+m/M_H}\ \text{or}\ \frac{\bar{\nu}_H\sim\bar{\nu}_D}{\bar{\nu}_D}=\frac{1}{2(1+M_H/m)}=\frac{1}{2\times1840}=\frac{1}{3680}$$

$$\therefore \quad \frac{\Delta\bar{\nu}}{\bar{\nu}} = \frac{\Delta\lambda}{\lambda} = \frac{1}{3680}$$

At $\lambda = 656.3$ nm, $\Delta\lambda = \lambda/3680 = 656.3/3680 = 0.178$ nm.

Example 11. A positronium atom is a system that consists of a positron and an electron. Calculate the reduced mass, the Rydberg constant and the wavelength of the first Balmer line for positronium. Given $m = 9.1\times10^{-31}\,kg$, $R_H = 1.09737\times10^{-3}\,\text{Å}^{-1}$ and $A_\alpha = 6563\,\text{Å}$.

Soln. The positron has the same mass m as the electron and has equal but positive charge

$\therefore$ Reduced mass of positronium is

$$\mu = \frac{(m)(m)}{m+m} = \frac{1}{2}m = \frac{1}{2}\times 9.1\times10^{-31}\,\text{kg}$$

$$= 4.55\times10^{-31}\,\text{kg}$$

The Rydberg constant $\left(= \frac{\mu e^4}{8\varepsilon_0^2 ch^3}\right)$ for positronium is therefore half that for hydrogen.

$$\therefore \quad R_P = \frac{1}{2}R_H = \frac{1}{2}\times 1.09737\times10^{-3}\,\text{Å}^{-1}$$

$$= 0.54868\times10^{-3}\,\text{Å}^{-1}$$

The wavelength of first Balmer line, H_α, in H-atom is given by $\frac{1}{\lambda_H} = R_H\left(\frac{1}{2^2} - \frac{1}{3^2}\right)$, while that in positronium is given by $\frac{1}{\lambda_P} = R_P\left(\frac{1}{2^2} - \frac{1}{3^2}\right)$

$$\therefore \quad \frac{\lambda_P}{\lambda_H} = \frac{R_H}{R_P} = 2$$

$$\therefore \quad \lambda_P = 2\lambda_H = 2\times 6563\,\text{Å} = 13126\,\text{Å}$$

Example 12. An electron circles a nucleus of charge Ze. Of the two orbits 1 and 2 of radii r_1 and r_2 respectively, its total energy is greater while in orbit 1. Prove that $r_1 > r_2$, also show that the velocity and acceleration in orbit 2 are greater than those in orbit 1.

Soln. Energy of electron, $E = -\frac{Ze^2}{8\pi\varepsilon_0 r} = -\frac{1}{2}mv^2$ ...(i)

Since $E_1 > E_2$, we have $E_1 - E_2 > 0$ or $-\frac{Ze^2}{8\pi\varepsilon_0 r_1} + \frac{Ze^2}{8\pi\varepsilon_0 r_2} > 0$, using (i)

or $\frac{Ze^2}{8\pi\varepsilon_0}\left(\frac{1}{r_2} - \frac{1}{r_1}\right) > 0$ ...(ii)

$\therefore \quad r_1 > r_2$

From equation (i), again, $\frac{1}{2}m\left(v_2^2 - v_1^2\right) = \frac{Ze^2}{8\pi\varepsilon_0}\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$ or $v_2^2 - v_1^2 > 0$, using (ii) above

$\therefore\ v_2 > v_1$

The acceleration of electron in circular orbit is the normal acceleration, $f = v^2/r$.

We have : $\dfrac{mv^2}{r} = \dfrac{Ze^2}{4\pi\varepsilon_0 r^2}$

$$\therefore\ f_n = \frac{Ze^2}{4\pi m\varepsilon_0 r_n^2}$$

$$\therefore\ f_2 - f_1 = \frac{Ze^2}{4\pi m\varepsilon_0}\left(\frac{1}{r_2^2} - \frac{1}{r_1^2}\right) \text{ or } f_2 - f_1 > 0\ (\because\ r_1 > r_2)$$

$\therefore\ f_2 > f_1$

Example 13. Show that in a Bohr atom if the electron is considered as a wave travelling along the circular path, then the orbit will contain n complete de-Broglie waves.

Soln. The radius of the nth Bohr orbit is given by $r_n = \dfrac{n^2h^2\varepsilon_0}{\pi me^2}$

Therefore, circumference of the orbit is $2\pi r_n = \dfrac{2n^2h^2\varepsilon_0}{me^2}$. But the wavelength of the de-Broglie wave is given by $\lambda = \dfrac{h}{mv} = \dfrac{h}{m}\left(\dfrac{2nh\varepsilon_0}{e^2}\right)\left(\because\ v_n = \dfrac{e^2}{2nh\varepsilon_0}\right)$.

Therefore, number of complete de-Broglie waves in the nth orbit is $\dfrac{2n^2h^2\varepsilon_0}{me^2} \div \dfrac{h}{m}\left(\dfrac{2nh\varepsilon_0}{e^2}\right) = n$.

10.2. Stern-Gerlach Experiment:

Stern and Gerlach, in 1921, perfomed an experiment which demonstrated directly that an atom placed in a magnetic field can take only certain discrete orientations with respect to the field (space quantisation). It also demonstrated the existence of electron spin; thus providing an experimental verification of the vector model of atom.

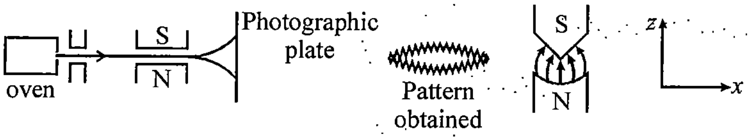
The plan of the experiment is shown in figure (below). A beam of neutral silver atoms was formed by heating silver in an oven. It was collimated by a few fine slits and then passed through a non-homogeneous magnetic field. The beam leaving the mgnetic field was received on a photographic plate P.

Physical Significance:

(i) Experimental evidence of electron spin.

(ii) Direct demonstration of space quantisation.

Experiment : The experiment of Stern and Gerlach was intended to study the deflection of atomic beams in non-uniform magnetic fields.



A beam of neutral silver atoms produced by heating silver in an oven. It was collimated by a few fine slits and then passed through a non-homogeneous magnetic field in which the field increases in z-direction. The beam leaving the magnetic field was received on a photographic plate P.

Observation : Instead of a trace of direct beam, two were obtained symmetrically situated w.r.t. the direct beam as two discrete components, one in +*z*-direction and other in –*z*-direction.

(i) The intensity of both the trace is the same.

(ii) The results obtained were independent of the choice of the *z*-direction.

Inference :

A magnetic dipole experiences a net deflecting force in a non-homogeneous magnetic field which depends on the orientation of the magnet in the field. Since atoms are tiny magnetic dipoles, they experience deflecting forces when passing through the field. If an atom could have any orientation in the magnetic field then for millions of atoms present in the beam all possible orientations would be obtained and the beam would be deflected into a continuous band. In the experiment however, there was no band but discrete traces on the photographic plate. This showed that the atoms passing through the magnetic field were oriented in space in discrete directions so that beam deflected in certain discrete directions only and gave discrete traces on the plate.

Thus the experimental results are qualitatively direct proof of space quantisation of atoms.

10.3. Spectroscopic Terms: L-S and J-J Couplings

1. Calculation of spectroscopic terms: Spectroscopic term is one, where electron in an atom will more probably live.

The spectroscopic terms are determined as

$$\boxed{n^{2s+1}L_J^P}$$

where, n is principal quantum number, s is spin quantum number, L is orbital quantum number, J is total angular quantum number and P stands for parity.

If the spectroscopic term is obtained for a system of odd electronic configurtion, we write 'O' and for even electronic configuration we don't write anything (The odd and even configuration is determined by the total l values).

If L = 0, S state ; L = 1, P state; L = 2, D state etc.

2. Various types of coupling:

There are two types of coupling, arises due to three kind of interactions viz. spin-spin correlation, orbital-orbital interaction (residual electrostatic interaction) and spin-orbit interaction. There are

(i) L-S coupling (in lighter atoms)

(ii) j-j coupling (in heavieur atoms)

(i) L-S coupling: This is also known as Russel-Saunder's coupling. This type of coupling happens in lighter atoms. In this coupling, spin-spin correlation and orbital-orbital interaction is stronger in comparision of spin-orbit interactions. It is divided into three steps:

(a) Spin-spin correlation: Due to this stronger interaction, the spin angular momentum of individual optical electrons combines to give a resultant spin angular momentum $(\vec{S})$, given as

$$\vec{S} = \vec{S}_1 + \vec{S}_2 + \vec{S}_3 + \ldots\ldots\ldots\ldots$$

The magnitude of resultant spin, angular momentum $\vec{S}$ is given by $|\vec{S}| = \hbar\sqrt{s(s+1)}$

Where the total spin quantum number (s) can take the values.

$$S = |s_1 + s_2 + \ldots.|_{\min},\ |s_1 + s_2 + \ldots.|_{\min} + 1, \ldots..(s_1 + s_2 + \ldots..)$$

Example: For one electron, $S = s_1 = \frac{1}{2}$ (doublet as 2s+1=2)

For two electrons, $s_1 = \frac{1}{2}, s_2 = \frac{1}{2}$

$\Rightarrow \quad S = |s_1 - s_2| ....(s_1 + s_2)$

$= 0, 1$ (singlet, triplets)

For three electrons; $s_1 = \frac{1}{2},\ s_2 = \frac{1}{2},\ s_3 = \frac{1}{2}$

$S' = 0, 1$

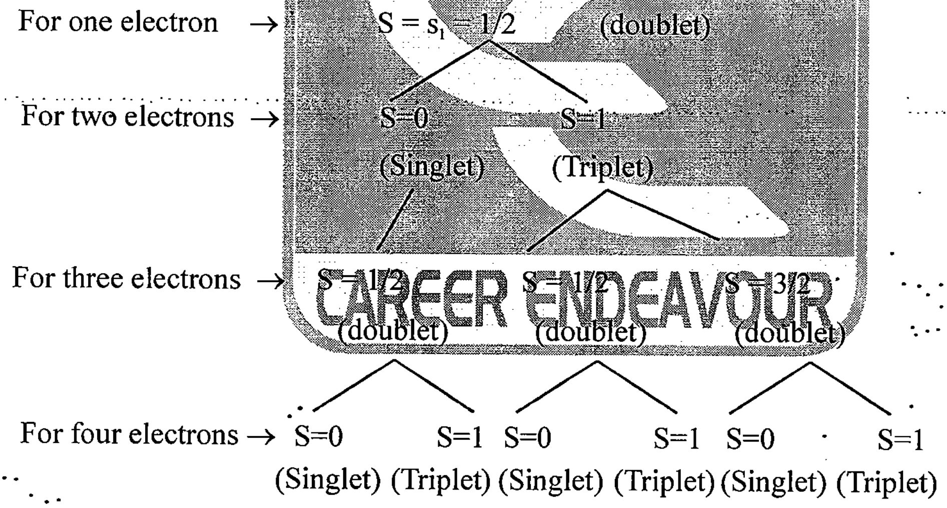
$S = \frac{1}{2}, \frac{1}{2}, \frac{3}{2}$ (Two doublets and one quartets)

In general,

1. For N even electrons → $S = 0, 1, 2 ....N/2$

2. For N odd electrons → $S = \frac{1}{2}, \frac{3}{2}, \frac{5}{2} ....\frac{N}{2}$

OR


For one electron → S = s₁ = 1/2 (doublet)
For two electrons → S=0 (Singlet) S=1 (Triplet)
For three electrons → S=1/2 (doublet) S=1/2 (doublet) S=3/2 (doublet)
For four electrons → S=0 S=1 S=0 S=1 S=0 S=1
(Singlet) (Triplet) (Singlet) (Triplet) (Singlet) (Triplet)

(b) Residual electrostatic interaction: As a result of this stronger interaction, the orbital angular moment of individual electrons coupled to give the resultant orbital angular momentum $\vec{L}$, given as:

$\vec{L} = \vec{\ell}_1 + \vec{\ell}_2 + \vec{\ell}_3 + ...$

Magnitude given by, $|\vec{L}| = \hbar\sqrt{L(L+1)}$

Where L is known as total orbital angular momentum quantum number and it is given as

$L = |\vec{\ell}_1 + \vec{\ell}_2 + ...|_{\min},\ |\vec{\ell}_1 + \vec{\ell}_2 + ...|_{\min} + 1,\ .....(\ell_1 + \ell_2 + ...)$

Example: For electron in 2p → $\ell_1 = 1 = L$ (P state)

For electrons in 3p 3d → $\ell_1 = 1;\ \ell_2 = 2$

$$\Rightarrow \quad L=(2-1).......(2+1)$$

$$=1,2,3 \qquad (P, D, F \text{ state})$$

For electrons in 2p 3p 3d → $\underbrace{l_1=1 \qquad l_2=1}_{L'=0,1,2} \qquad l_3=2$

$\left.\begin{matrix}L'=0\\ l_3=2\end{matrix}\right]$ L=2 (D state)

$\left.\begin{matrix}L'=1\\ l_3=2\end{matrix}\right]$ L=1,2,3 (P,D,F states)

$\left.\begin{matrix}L'=2\\ l_3=2\end{matrix}\right]$ L=(0,1,2,3,4) (S,P,D,F,G states)

(c) Spin-orbit interaction: As a result of this weak interaction, the resultant $\vec{S}$ & $\vec{L}$, couples less strongly to give total resultamt angular momentum $(\vec{J})$ given by $\vec{J}=\vec{L}+\vec{S}$ of magnitude $|\vec{J}|=\hbar\sqrt{J(J+1)}$, where total angular momentum quantum number J is given as

$$J=|L-S|_{\min}, (|L-S|_{\min}+1), ...(L+S)$$

(ii) j-j coupling: This coupling is just opposite to that of L–S coupling. It happens in heavier atoms. In it spin-orbit interaction is relatively stronger than spin-spin correlation and redisual electrostatic interaction. It happens in the following steps:

(a) Spin-orbit interaction: As a result of this relatively stronger interaction the spin angular momentum of individual electrons combine with the orbital angular momentum to give the resultant angular momentum vectors $\vec{j}_1, \vec{j}_2$.........

$\vec{j}_1=\vec{\ell}_1+\vec{s}_1, \vec{j}_2=\vec{\ell}_2+\vec{s}_2$,............ where $|\vec{j}_1|=\hbar\sqrt{j_1(j_1+1)}$ & $j_1=\ell_1 \pm s_1=\ell_1 \pm \frac{1}{2}$

(b) Spin-spin correlation and residual electrostatic interaction:

As a result of these weaker interactions, $\vec{j}_1, \vec{j}_2$..... combines to give the total resultant angular momentum $\vec{J}$ given by:

$\vec{J}=\vec{j}_1+\vec{j}_2+$.........

of magnitude $|\vec{J}|=\hbar\sqrt{J(J+1)}$

And $J=|\vec{j}_1+\vec{j}_2+..........|_{\min}, |\vec{j}_1+\vec{j}_2+......|_{\min}+1, ...(j_1+j_2+........)$

J is total angular momentum quantum number.

Example: For 4p 4d electrons: $\ell_1=1, S_1=\frac{1}{2} \Rightarrow j_1=\frac{1}{2},\frac{3}{2}$ for p electron

$\ell_2=2, S_2=\frac{1}{2} \Rightarrow j_2=\frac{3}{2},\frac{5}{2}$ for D electron

Now (j_1, j_2) gives J

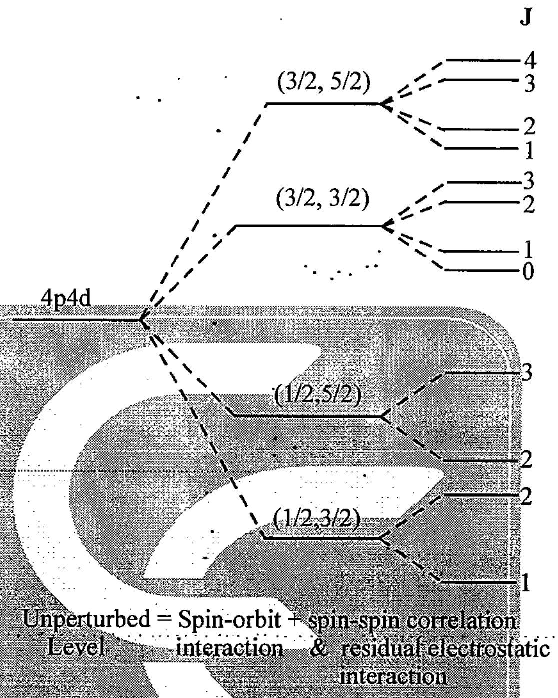
Values of j_1 & j_2 has 4 possible (j_1, j_2) combinations

$\Rightarrow \left(\frac{1}{2},\frac{3}{2}\right)$ gives J=1,2

$\left(\frac{1}{2},\frac{5}{2}\right)$ gives J=2,3

$\left(\frac{3}{2},\frac{3}{2}\right)$ gives J=0,1,2,3

$\left(\frac{3}{2},\frac{5}{2}\right)$ gives J=1,2,3,4


J
(3/2, 5/2)
4
3
2
1
(3/2, 3/2)
3
2
1
0
4p4d
(1/2,5/2)
3
2
(1/2,3/2)
2
1
Unperturbed Level
= Spin-orbit interaction
+ spin-spin correlation & residual electrostatic interaction

Determination of spectral terms under L-S coupling:

Since the core electrons do not contribute to the angular momentum of the atom. We must consider only the (optical) electron outside the completed subshell for completed subshell for computing the term. We can divide the various atoms into three categories:

(i) Atom with one optical electrons:

For a hydrogen like atom ground state configuration is 1s for this we have

$\Rightarrow S_1 = \frac{1}{2},\ \ell_1 = 0$

So, that $S = s_1 = \frac{1}{2}$

Multiplicity $\left(2\times\frac{1}{2}+1\right) = 2$ (doublet)

$L = \ell_1 = 0$ (S-state) and $j = |L-S|,.......|L-S|+1.......(L+S) = \frac{1}{2}$

Thus the ground state term of H-atom is $^2S_{1/2}$. The ground state configuration of an alkali atoms Li, Na, K etc which have optical electron and electronic configuration $(n-1)s^2p^6ns^1$. The ground state term is $^2S_{1/2}$.

The excited state configuration and corresponding term for a H like atom would be

$2s, 3s, 4s.......... {}^2S_{1/2}$

$2p, 3p, 4p........ {}^2p^0_{1/2}, {}^2p^0_{3/2}$ or ${}^2p^0_{1/2, 3/2}$

$3d, 4d.............. {}^2D_{3/2}, {}^2D_{5/2}$

(ii) Atoms with two or more non-equivalent optical electrons:

Let us consider an atom having optical electrons with outer configuration (2s 2p), (3s 4s) (3s, 4p) (In this, n & ℓ are not same, one of them is different or both are different).

Let us calculate terms for $\boxed{2p\,3d}$

For this we have $S_1 = \frac{1}{2}, S_2 = \frac{1}{2}, \ell_1 = 1, \ell_2 = 2$ the possible value of S & L are

$S = |S_1 - S_2|, |S_1 - S_2| + 1.......(S_1 + S_2)$

$S = 0, 1$

multiplicity $= (2s+1) = 1, 3$ (singlet, triplet)

and $L = |\ell_1 - \ell_2|, |\ell_1 - \ell_2| + 1..........|\ell_1 + \ell_2|$

$L = 1, 2, 3$ $(P, D, F$ states$)$

All the terms are odd because the configuration 4p 4d is odd $(\Sigma\ell = 1 + 2 = 3)$

Now for singlets terms we have $\begin{cases} S = 0, & J = 1 \\ L = 1 \end{cases}$

$\Rightarrow$ term ${}^1P_1^0$

$\begin{cases} S = 0, & j = 2 \\ L = 2 \end{cases} \Rightarrow$ term ${}^1D_2^0$

and $\begin{cases} S = 0, & j = 3 \\ L = 3 \end{cases} \Rightarrow$ term ${}^1F_3^0$

For triplet terms, we have

$\begin{cases} S = 1, & j = 0, 1, 2 \\ L = 1 \end{cases} \Rightarrow$ term ${}^3p_0^0, 3p_1^0, 3p_2^0$

$\begin{cases} S = 1, & j = 1, 2, 3 \\ L = 2 \end{cases} \Rightarrow$ term $3D_1^0, 3D_2^0, 3D_3^0$

$\begin{cases} S = 1, & j = 2, 3, 4 \\ L = 3 \end{cases} \Rightarrow$ term $3F_2^0\ 3F_3^0\ 3F_4^0$

For practice, certain outer electrons configuration and their term are given in table.

1. SS → ${}^1S_0,\ {}^3S_1$
2. SP → ${}^1P_1^0,\ {}^3P_{0,1,2}^0$
3. SD → ${}^1D_2, {}^3D_{1,2,3}$

4. PP $\rightarrow$ $^1S_0, {}^1P_1, {}^1D_2, {}^3S_1, {}^3P_{0,1,2}, {}^3D_{1,2,3}$

$^3D_{1,2,3}, {}^3F_{2,3,4}, {}^3G_{3,4,5}$

(i) For SS $\rightarrow$ $S_1 = \frac{1}{2}, S_2 = \frac{1}{2}, \ell_1 = 0, \ell_2 = 0$

$S = 0,1$ multiplicity $(2s+1) = 1,3$ (singlet triplet)

For singlet $\rightarrow$ $s = 0,\quad J = 0$

$L = 0$

For triplet $S = 1,\quad J = 1$

$L = 0$ term $^3s_1$

(ii) For SP: $s_1 = \frac{1}{2}, s_2 = \frac{1}{2}, \ell_1 = 0, \ell_2 = 1$

$S = 0,1, j = 0,1,2$

term $^1p_0^1$

For triplet $S = 1\quad J = 0,1,2$

$L = 1$ term $^3p_{0,1,2}^0$

(iii) For SD: $s_1 = \frac{1}{2}, s_2 = \frac{1}{2}, \ell_1 = 0, \ell_2 = 2$

$S = 0,1$

$L = 2$

For singlet $S = 0,\quad J = 2$

$L = 2$ term $^1D_2$

For triplet $S = 1\quad J = 1,2,3$

$L = 2$ term $^3D_1, {}^3D_2, {}^3D_3$

OR

terms are $^1D_2, {}^3D_{1,2,3}$

(iv) For PP: $s_1 = \frac{1}{2}, s_2 = \frac{1}{2}, \ell_1 = 1, \ell_2 = 1$

$S = 0,1$

$L = 0,1,2$

For singlet $\begin{cases} S = 0 \\ L = 0 \end{cases}$, $J = 0$

term $^1S_0$

$\begin{cases} S = 0 \\ L = 1 \end{cases}$, $J = 1$ term $^1p_1$

$\begin{cases} S = 0 \\ L = 2 \end{cases}$, $J = 2$ term $^1D_2$

For triplet $\begin{cases} S=1 \\ L=0 \end{cases}$, $J=1$

term ${}^3S_1$

$\begin{cases} S=1 \\ L=1 \end{cases}$, $J=0,1,2$ term ${}^3p_0, {}^3p_1, {}^3p_2$

$\begin{cases} S=1 \\ L=2 \end{cases}$, $J=1,2,3$

term ${}^3D_1, {}^3D_2, {}^3D_3$

Similarly, we can determine the spectroscopic terms for three or more non-equivalent electrons.

(iii) Atoms with two or more equivalent electrons:

For two equivalent electrons (same n and l values for both or more electrons) the values atleast one of the remaining quantum numbers $(m_\ell \text{ or } m_s)$ must differ to satisfy Pauli exclusion principle. Hence certain terms which were possible for two or more non equivalent electrons are now not possible here. Before the determination of terms, we must discuss two important factors:

(i) A closed subshell such as s^2, p^6, d^{10}, f^{14} always form a ${}^1S_0$ term only since for them:

$$\Sigma m_\ell = 0$$

$$\Sigma m_s = 0$$

This means that $M_L = 0,\ M_s = 0$

and so, $L=0, S=0 \Rightarrow J=0 \rightarrow {}^1S_0$

(s-state) (singlet)

Therefore, J = 0, i.e. the only possible term is ${}^1S_0$. Hence, we conclude that when a subshell is completely filled the only allowed state is one in which the total spin angular momentum, Total orbital angular momentum and total angular momentum all are zero and hence no magnetic moment.

(ii) The term of configuration $(n\ell)^q$ are the same as the terms of the configuration $(n\ell)^{r-q}$ where r is the max. numbers of electrons i.e. $2(2\ell+1)$.

Example:

Terms for p^5 the same as those of $p^{6-5} = p^1$ term

Terms for p^4 the same as those of $p^{6-4} = p^2$ the term

Terms for d^8 the same as those of d^2 and so on.

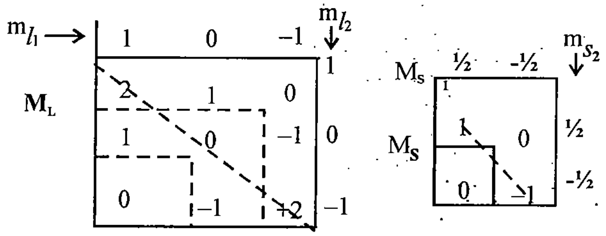
Calculation of spectroscopic term for $\mathbf{p^2}$:

$$\text{Here} \quad \left.\begin{array}{l} \ell_1 = 1;\ s_1 = \dfrac{1}{2} \\ \ell_2 = 1;\ s_2 = \dfrac{1}{2} \end{array}\right] \Rightarrow L = 0,1,2 \ \&\ S = 0,1$$

For the calculation of spectroscopic terms in case of equivalent electrons, we have to consider the Pauli's exclusion principle i.e the four quantum number $(n, \ell, m_l \text{ and } m_s)$ should not be same for any two electrons in an atom. For the calculation of spectroscopic terms in case equivalent electrons we use the Breit-Scheme which is follows as:

Breit scheme for possible M_L-values is:

(i) For S =1



We take, only L = 1

We have to leave the diagonal values for the validity. Because of Pauli's exclusion principle

$\therefore \quad J = 0,1,2$

$\therefore \quad$ terms are ${}^3p_{0,1,2}$

(ii) For S = 0

We take the remaining L values i.e.

L = 0, 2

For $\quad$ L = 0, $\;$ S = 0 $\quad$ J = 0, term is ${}^1S_0$ $\qquad$ Parity is even, as $\Sigma\ell$ for p^2 = even

For $\quad$ L = 2 $\;$ S = 0 $\quad$ J = 2 $\quad$ term is ${}^1D_2$

The terms for various electrons configurtion are given as:

Electonic configuration	Terms
s^1	${}^2s_{1/2}$
s^2, p^6, d^{10}, f^{14}	${}^1s_0$
p^1, p^5	${}^2p^0_{1/2, 3/2}$
p^2, p^4	${}^1s_0, {}^1D_2, {}^3p_{0,1,2}$
p^3	${}^2p^0_{1/2, 3/2}, {}^2D_{3/2, 5/2}, {}^4s^0_{3/2}$
d^1, d^9	${}^2D_{3/2, 5/2}$
d^2, d^8	${}^1S_0, {}^1D_2, {}^1G_4, {}^3P_{0,1,2}, {}^3F_{2,3,4}$

Order of terms and fine structure levels:

The relative energies of the various terms and levels which arise from a given (equivalent) electron configuration may be deduced from a set of rules given by Hund. These rules are:

(i) Of the terms arising from equivalent electron, those with largest multiplicity (2s+1) lie lowest.

(ii) Of the terms with given multiplicity and arising from equivalent electron, that with largest L, value lies lowest.

(iii) (a) In the multiplets formed from equivalent electron in a less than half-filled subshell, the level with lowest J, lies lowest (normal order).

(b) In the multiplets formed from equivalent electron in a more than half filled subshell, the level with highest J lies lowest (inverted order)

(c) Terms arising from half filled subshell show only very slight fine structure splitting.

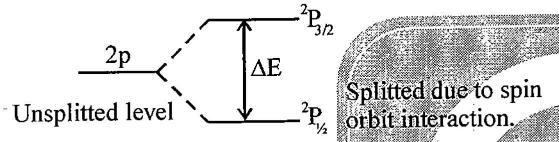
(d) The lowest terms arising from half filled subshell are the s-terms and are specially stable. These terms are ${}^2S_{1/2}$ for a half filled s-subshell, ${}^4S_{3/2}$ for a half-filled p-subshell, ${}^6S_{5/2}$ for a half filled d-subshell and ${}^8S_{7/2}$ for a half-filled f-subshell.

Hund was quite successful in predicting the lowest terms and levels of many atoms. However exceptions do occur:

Example:

Atom	Electronic configuration	Terms
${}_5B$	$1s^2, 2s^2, 2p^1$	${}^2P^0_{1/2}$ (ground), ${}^2P^0_{3/2}$
${}_6C$	$1s^2, 2s^2, 2p^2$	${}^1S_0, {}^1D_2, {}^3P_0$ (ground), ${}^3P_1, {}^3P_2$
${}_7N$	$1s^2, 2s^2, 2p^3$	${}^2p^0_{1/2}, {}^2p^0_{3/2}, {}^2D_{3/2}, {}^2D^0_{5/2}, {}^4s^0_{3/2}$ (ground)
${}_8O$	$1s^2, 2s^2 2p^4$	${}^1S_0, {}^1D_2, {}^3P_0, {}^3P_1, {}^3P_2$ (ground)
${}_9F$	$1s^2, 2s^2; 2p^5$	${}^2P^0_{1/2}, {}^2P^0_{3/2}$ (ground)
${}_{17}Cl$	$1s^2, 2s^2, 2p^6, 3s^2, 3p^5$	${}^2P^0_{1/2}, {}^2P^0_{3/2}$ (ground)

Spin-Orbit Interaction



The term shift in cm^{-1},

$$\Delta T = -\frac{\Delta E}{hc} = 5.84\frac{Z^4}{n^3\ell(\ell+1)}cm^{-1}$$

Or, $\Delta T \propto Z^4$.

10.4. Spectra of Alkali Elements

Broad feature of Alkali Spectra:

After H-atom, the simplest atoms to study with regard to their spectra are the 'Alkali atoms' Li, Na, K, Rb, Cs & Fr. The experimental study of their spectra was taken up by living, Dewar, Rydberg and Bergmann spectral lines in the emission spectrum of an alkali atom formed four series a principal series of bright and most presistant lines, a sharp series of fine lines a diffuse series a comparatively broader lines and fundamental (or Bergmann) series in the infrared region.

Taking clue from of the Balmer's formula for H.

$$\boxed{v_m^B = v_\infty^B - \frac{R}{n^2}}$$

Rydberg pointed out that the alkali series could be represented by similar formula such as

(i) Principal : $v_m^P = v_\infty^P - \frac{R}{(m+p)^2}$; $m = 2, 3, 4.......\infty$

(ii) Sharp : $v_m^s = v_\infty^s - \frac{R}{(m+s)^2}$; $m = 2, 3, 4..........\infty$

(iii) Diffuse : $v_m^D = v_\infty^D - \frac{R}{(m+d)^2}$; $m = 3, 4, 5.........\infty$

(iv) Fundamental : $v_m^f = v_m^f - \frac{R}{(m+f)^2}$; $m = 3, 4, 5......\infty$

where v_∞^s are the wave number of convergence limits of the corresponding series and are called 'fixed terms'. In the other so called running terms. The p, s, d and f represent Rydberg corrections for the corresponding series.

Rydberg noticed the following relations among different series of the same atom. These are

(i) The sharp and diffuse series have a common convergence limit $\left[v_\infty^s = v_\infty^d\right]$

(ii) The common convergence limit is equal to the first running term (with m = 2) of the principal series

$$\left[v_\infty^s = v_\infty^d = \frac{R}{(2+p)^2}\right]$$

(iii) The convergence limit of the principal series is equal to the running term (with m = 1) of the sharp series.

$$\left[v_\infty^p = \frac{R}{(1+s)^2}\right]$$

(iv) The convergence limit of the fundamental series is equal to the first running term (with m = 3) of the diffuse series).

$$\left[v_\infty^f = \frac{R}{(3+d)^2}\right]$$

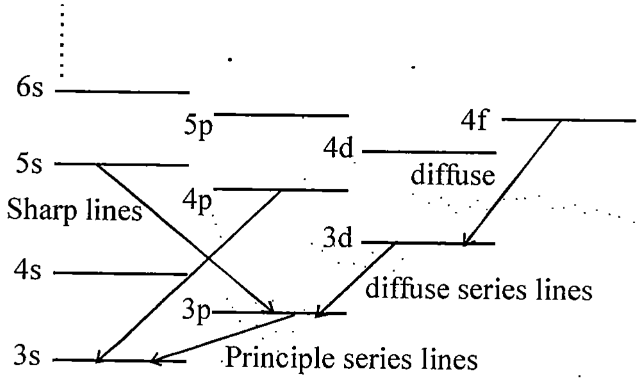
In view of these observations the above formula may be written in the following form:

(i) Principal : $v_m^p = \frac{R}{(1+s)^2} - \frac{R}{(m+p)^2}$; $m = 2, 3, 4....\infty$

(ii) Sharp : $v_m^s = \frac{R}{(2+p)^2} - \frac{R}{(m+s)^2}$; $m = 2, 3, 4....\infty$

(iii) Diffuse : $v_m^d = \frac{R}{(2+p)^2} - \frac{R}{(m+d)^2}$; $m = 4, 5, ...\infty$

(iv) Fundamental : $v_m^f = \frac{R}{(3+d)^2} - \frac{1}{(m+1)^2}$; $m = 4, 5, ...\infty$



[Emission spectra of sodium]

10.5. Zeeman Effect

The splitting of spectral lines by a magnetic field is called Zeeman effect. It was discovered by Zeeman in 1896. It is a configuration of space quantisation. When an atom is placed in a magnetic field, its spectral lines further split into several polarised components. The splitting is proportional to the strength of the applied field.

1. Normal Zeeman effect :

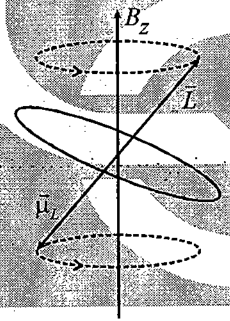
The normal Zeeman effect which is shown by all lines due to transitions between the singlet ($s = 0$) states of an atom, can be explained from the classical theory and also from the quantum theory without taking note of electronic spin.

In the normal Zeeman effect a single line of wavelength λ_0 in the presence of a magnetic field is split into three components with wavelengths $\lambda_0 - \Delta\lambda$, λ_0 and $\lambda_0 + \Delta\lambda$ when viewed at right angles to the magnetic field $\vec{B}$.

The lines of wavelength $\lambda_0 + \Delta\lambda_0$ are plane-polarised with electric vector perpendicular to $\vec{B}$, while the central line has the wavelength of the source and is plane polarized with electric vector parallel to the field $\vec{B}$. This is called a "normal triplet".

Explanation : In terms of quantum theory, an atom possesses an angular polyelectronic momentum $\vec{L}$ and an orbital magnetic moment $\vec{\mu}_L$ with following relation

$$\frac{|\vec{\mu}_L|}{|\vec{L}|} = \frac{e}{2m} \quad (\vec{\mu}_L \text{ is directed opposite to } \vec{L})$$



The magnetic potential energy of an atom in a magnetic field is given by

$$E = -\vec{\mu}_L \cdot \vec{B} = -\mu_{L_z} B_z \quad (\because\ B \text{ is in } z\text{-direction})$$

$$\therefore\ E = -\left(-\frac{e}{2m} m_L \hbar\right) B_z = \left(\frac{eB\hbar}{2m}\right) m_L = m_L \left(\frac{e\hbar}{2m}\right) B \quad (\because\ B_z = B)$$

$\boxed{E = m_L\, \mu_B \cdot B}$, where $\mu_B = \dfrac{e\hbar}{2m}$ (Bohr magneton)

Because m_L can have $2L + 1$ values, a state of given orbital quantum number L is split into $2L + 1$ substates that differ in energy by $\mu_B \cdot B$ when the atom is in magnetic field. But the selection rules permit only.

$\boxed{\Delta m_L = 0, \pm 1}$

So only three components

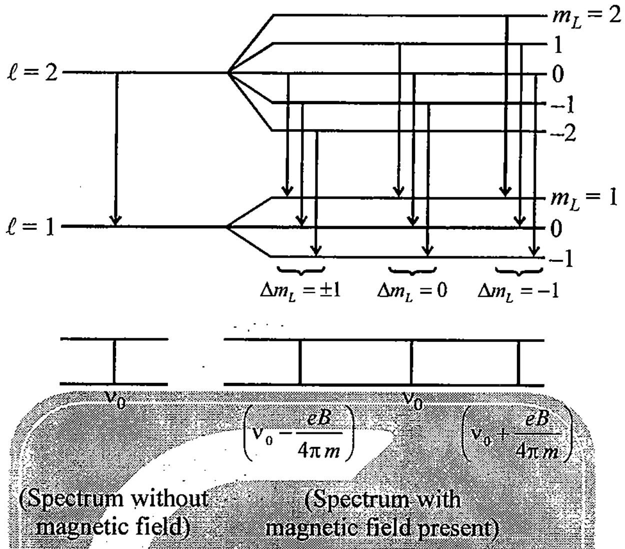
$$\boxed{\nu_i = \nu_0 - \frac{\Delta E}{h}} \quad (\Delta E = \Delta m_L\, \mu_B \cdot B)$$

Note: $\boxed{\Delta = \text{final} - \text{initial}}$

$$\nu_1 = \nu_0 - \frac{\mu_B \cdot B}{\hbar} \qquad (\text{for } \Delta m_L = \pm 1)$$

$$\nu_2 = \nu_0 \qquad (\text{for } \Delta m_L = 0)$$

$$\nu_3 = \nu_0 + \frac{\mu_B \cdot B}{h} \qquad (\text{for } \Delta m_L = -1)$$


$\ell = 2$
$m_L = 2$
1
0
−1
−2
$\ell = 1$
$m_L = 1$
0
−1
$\Delta m_L = \pm 1$
$\Delta m_L = 0$
$\Delta m_L = -1$
ν_0
ν_0
$\left(\nu_0 - \frac{eB}{4\pi m}\right)$
$\left(\nu_0 + \frac{eB}{4\pi m}\right)$
(Spectrum without magnetic field)
(Spectrum with magnetic field present)

Quantum mechanics leads not only to selection rules but also to the polarisation rules. For observation normal to the field, the transition $\Delta m_L = 0$ results in spectral line polarised with the electric vector parallel to the field (π-component) while the transition $\Delta m_L = \pm 1$ results in lines polarized with the electric vector perpendicular to the field (σ-component).

The separation of the Zeeman components is

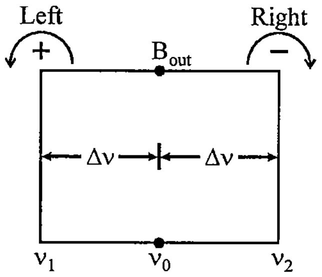
$$\boxed{\Delta\nu = \frac{eB}{4\pi m}} \quad \text{or} \quad \boxed{\Delta\lambda = \frac{\lambda^2 \Delta\nu}{c} = \frac{eB\lambda^2}{4\pi mc}}$$

We can find e/m from $\boxed{\frac{e}{m} = \frac{4\pi\,\Delta\nu}{B}}$

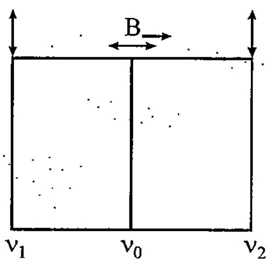
Normal transverse Zeeman effect and longitudinal Zeeman effect.

(a) Transverse Zeeman effect: when the source is viewed at right angles to magnetic field direction, it splits into three components one π and two σ. This is called transverse Zeeman effect.

(b) Longitudinal Zeeman effect: when source is viewed in the direction of the magnetic field, along the z-axis, we obtain two lines which are circularly polarized in opposite sense. It is called longitudinal Zeeman effect.


Left
+
Right
−
B_{out}
Δν
Δν
ν_1
ν_0
ν_2

(Viewed in direction of the field $\vec{B}$)
(Longitudinal Zeeman effect)


B
ν_1
ν_0
ν_2

(Viewed perpendicular to field)
(Transverse Zeeman effect)

2. Anomalous Zeeman Effect :

When an atom is placed in an external magnetic field and then excited, the spectral line it emits in the de-excitation process are split into several components. This is called *Anomalous Zeeman Effect.*

Under the effect of external magnetic field, the magnetic moment associated with the total angular momentum $\vec{J}$ where $(\vec{J} = \vec{L} + \vec{S})$; interacts with the external magnetic field and the degeneracy in the energy due to the quantum number J is broken.

The energy level corresponding to a particular J, splits up into $(2J+1)$ levels, and the magnitude of the splitting is given by $\Delta E = -\vec{\mu}_J \cdot \vec{B}$.

Before the advent of quantum theory, these splittings were divided into two categories: Normal and anomalous.

3. Normal Zeeman effect:

It is a special case of anomalous Zeeman effect where the transition takes place between the singlet states only. In this case, since $s = 0$ therefore, $J = L$ and $\vec{\mu}_J$ and $\vec{J}$ are anti-parallel. This results in equispaced energy splitting and we get less number of lines compared to anomalous Zeeman effect.

The precession of component of $\vec{\mu}$ perpendicular to $\vec{J}$ averages to zero. Only the component antiparallel to $\vec{J}$ remains a constant of magnitude $|\vec{\mu}_J|$ which contributes to the magnetic moment of the atom. Thus,

$$\vec{\mu}_J = \frac{(\vec{\mu}\cdot\vec{J})}{|\vec{J}|}\frac{\vec{J}}{|\vec{J}|}$$

$$\vec{\mu}_J = -\frac{\mu_B}{\hbar}\frac{\{(\vec{L}+2\vec{S})\cdot(\vec{L}+\vec{S})\}\vec{J}}{|\vec{J}|^2} \qquad \text{[from equation (iii)]}$$

$$\text{and } \Delta E = -\vec{\mu}_J \cdot \vec{B} = +\frac{\mu_B}{\hbar}\left\{\frac{(\vec{L}+2\vec{S})\cdot(\vec{L}+\vec{S})}{|\vec{J}|^2}\right\}(\vec{J}\cdot\vec{B})$$

$$\Delta E = \frac{\mu_B}{\hbar}\left\{\frac{L^2+2S^2+3\vec{L}\cdot\vec{S}}{|\vec{J}|^2}\right\}J_zB \qquad \text{... (iv)} \qquad \text{(since direction of } B \text{ is in } z\text{-direction)}$$

$$\text{Since, } \vec{J} = \vec{L}+\vec{S} \text{ or } J^2 = L^2+S^2+2\vec{L}\cdot\vec{S}$$

$$\text{or } \vec{L}\cdot\vec{S} = \frac{J^2-L^2-S^2}{2} \qquad \text{... (v)}$$

Putting this in equation (iv), we have

$$\Delta E = \frac{\mu_B}{\hbar}\left\{L^2+2S^2+\frac{3(J^2-L^2-S^2)}{2}\right\}\frac{J_zB}{|\vec{J}|^2}$$

$$\text{So, } \Delta E = \frac{\mu_B}{\hbar}\left\{\frac{3J^2+S^2-L^2}{2J^2}\right\}J_zB = \frac{\mu_B}{\hbar}\left\{1+\frac{J^2+S^2-L^2}{2J^2}\right\}J_zB$$

Since, $J = \sqrt{j(j+1)}\,\hbar$, $L = \sqrt{\ell(\ell+1)}\,\hbar$, $S = \sqrt{s(s+1)}\,\hbar$ and $J_z = m_j\hbar$

Therefore, $\Delta E = \mu_B \left\{1 + \frac{j(j+1)+s(s+1)-\ell(\ell+1)}{2j(j+1)}\right\} B\,m_j$

Therefore, $\Delta E = \mu_B B\,g\,m_j$ ... (vi)

where $g = 1 + \frac{j(j+1)+s(s+1)-\ell(\ell+1)}{2j(j+1)}$ ... (vii)

This quantity g is called "Lande's g-factor". We can see from the above equation (vii), that when $s = 0$, $j = \ell$ and $g = 1$ and ΔE from equation (vi) becomes $\Delta E = \mu_B B\,m_j$ and hence we get equal splitting proportional to m_j. While for the case $s \neq 0$, the g-factor will vary and we get unequal splittings.

4. Applications of Zeeman Effect:

(i) By analyzing the Zeeman splitting of the spectral lines of an atom, the spectroscopist determines the Zeeman splittings of the energy levels of the atom. These can conclusively confirm the assignment of quantum number j to each level.

(ii) The g values of the energy levels can be deduced from the splitting for a known applied field, giving information about the L and S values of the electron undergoing transitions. Overall, then the term symbols for various atomic states can be deduced by Zeeman experiments.

Much of what is known about the stars has come from measurements and analysis of optical lines spectra. The pattern of lines observed in emission spectra is used to identify the composition of stars; the intensity of lines observed in absorption spectra is used to measure the temperatures of stellar surfaces; the Doppler shift of the spectral lines is used to measure the velocity of stars and the Zeeman effect is used to measure the magnetic fields produced by stars. Generally polarization of spectra (σ and π) is used to separate out components.

Example: Zeeman splitting of D_1 and D_2 lines of sodium.

$^2P_{3/2} \rightarrow {}^2S_{1/2}$ D_2 line

$^2P_{1/2} \rightarrow {}^2S_{1/2}$ D_1 line

The selection rule for transitions is $\Delta m_j = 0, \pm 1$ (but not $m_j = 0$ to $m_j = 0$ if $\Delta j = 0$)

Terms	No. of Zeeman levels	g	$m_j(+J,\ldots,-J)$	$\Delta E = g\,m_j\,\mu_B B\,(g\,m_j)$
$^2S_{1/2}$	2	2	$\pm\frac{1}{2}$	± 1
$^2P_{1/2}$	2	$\frac{2}{3}$	$\pm\frac{1}{2}$	$\pm\frac{1}{3}$
$^2P_{3/2}$	4	$\frac{4}{3}$	$\pm\frac{3}{2}, \pm\frac{1}{2}$	$\pm 2, \pm\frac{2}{3}$

For $\Delta m_j = 0$; components are polarised with electric vector parallel to the magnetic field (π-components)

For $\Delta m_j = \pm 1$; components are polarised with electric vector parallel to the field (σ-components).

5. Calculation of shift in energy:

Since $\Delta E = \mu_B B\,m_j g$

(a) For D_1 transition (i.e., $^2P_{1/2} \rightarrow {}^2S_{1/2}$)

(i) Transition line $m_j = -\frac{1}{2}$ to $m_j = \frac{1}{2}$ line (a)

$$E_2 = E_i - E_f = E_0 + \frac{2}{3} \times \left(-\frac{1}{2}\right)\mu_B B - 2 \times \frac{1}{2}\mu_B B$$

$$E_1 = E_0 - \frac{4}{3}\mu_B B$$

(ii) Transition line $m_j = -\frac{1}{2}$ to $m_j = -\frac{1}{2}$ line (c)

$$E_2 = E_0 + \frac{2}{3} \times \left(-\frac{1}{2}\right)\mu_B B - 2 \times \left(-\frac{1}{2}\right)\mu_B B = E_0 + \frac{2}{3}\mu_B B$$

(iii) Transition line $m_j = \frac{1}{2}$ to $m_j = \frac{1}{2}$ line (b)

$$E_3 = E_0 + \frac{2}{3} \times \frac{1}{2}\mu_B B - 2 \times \left(-\frac{1}{2}\right)\mu_B B$$

$$E_3 = E_0 - \frac{2}{3}\mu_B B$$

(iv) Transition line $m_j = \frac{1}{2}$ to $m_j = -\frac{1}{2}$ line (d)

$$E_4 = E_0 + \frac{2}{3} \times \frac{1}{2}\mu_B B - 2 \times \left(-\frac{1}{2}\right)\mu_B B$$

$$E_4 = E_0 + \frac{4}{3}\mu_B B$$

So, energies are in the order $E_4 > E_2 > E_3 > E_1$

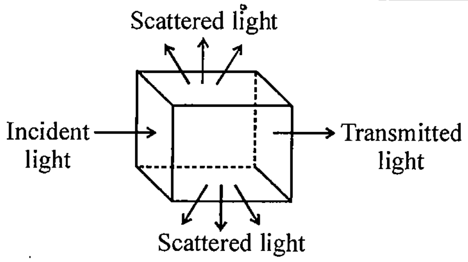
Zeeman observed the effect of external magnetic field ($\left(\sim \frac{1}{10}\text{ Tesla}\right)$ on spectrum lines and he observed that the spectrum line is splitted into a number of components. The electric vector of some of the lines is parallel to the applied magnetic field (π-component) while the electric field vector of some of the lines is transverse to the external magnetic field and are called σ-components.

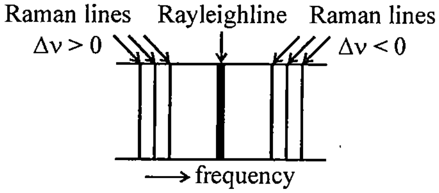
10.6. Raman Spectra

Discovered by C.V. Raman in 1928 and won nobel prize in 1930.

1. Raman Effect :

If a beam of monochromatic light is passed through the substance, nearly all of it is transmitted and a very small fraction of it is scattered in all directions. If the scattered light in a direction perpendicular to the incident beam is spectrographed, it is found not only to contain a line corresponding to the frequency of incident light, but also a pattern of relatively, weak lines on the low frequency side of the incident light and a similar pattern of still more weak lines on the high frequency side.





The line at the incident frequency is known as "Rayleigh line". The other weak lines are called the Raman lines. The difference between the frequencies of the Rayleigh line and a Raman line is known as the "Raman shift", and may be expressed as $\Delta\bar{\nu} = |\bar{\nu}_0 - \bar{\nu}_R|$, where the subscripts 0 and R stands for incident and Raman lines, respectively.

For a given substance, the Raman shift is found to be independent of the frequency of the incident radiation, i.e., it has the same value for the different frequency of the incident light. The difference is a characteristic of the substance and is found to be equal to its vibrational or rotational transition.

2. Nature of Raman Spectra: Molecular spectra

Wave number of Rayleigh (excitting line or incident monochromatic light) = $\boxed{\nu_0 = \frac{1}{\lambda_0}}\ cm^{-1}$

Wave number of stokes Raman line = $(\nu_0 - \Delta\nu)\,cm^{-1}$ and wave number of anti-stokes Raman line $= (\nu_0 + \Delta\nu)\,cm^{-1}$, $\Delta\nu$ is fixed for a susbtance.

Intensity of Rayleigh line > Intensity of Stokes line > Intensity of anti-Stokes Raman line

The change in the energy of the molecule is either of vibrational energy or rotational energy. Therefore, Raman spectra is of two kind.

(i) Vibrational Raman spectrum

(ii) Pure rotational Raman spectrum.

3. Quantum theory of Raman effect :

The occurence of Raman scattering may be most easily by considering a radiation of frequency ν as consisting of a stream of particles called photons having energy $h\nu$. Photons can be imagined to undergo collisions with molecules and, if the collision is perfectly elastic, they will be deflected without any change in energy. A detector placed to collect energy at right angles to an incident beam will thus receive photons of frequency ν. This gives rise to "Rayleigh scattering".

On the other hand, if the collision is inelastic, there can occur an exchange of energy between molecule and photon. This exchange of energy can be of two types.

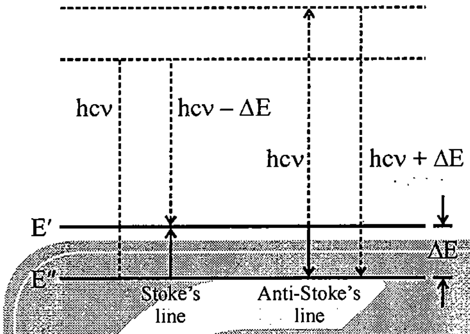
- The molecule before the collision occupies a lower vibrational or rotational energy level and after the emission of the photon, it may be present in any one of the higher energy levels as permitted by selection rule. In this case, the molecule gains energy equivalent to the energy difference between the two involved levels. This energy is provided by the photon and hence its energy is decreased and thus a Raman line on the low frequency side is observed. Such a Raman line is called a "Stokes" line.
- The molecule before the collision occupies one of the higher vibrational or rotational energy level and after the emission of the photon may be present in any one of the lower energy levels as permitted by the selection rule. In this case, the molecule loses an energy equivalent to the energy difference between the two involved levels. This, energy is gained by photon and hence its energy is increased and thus a Raman line on high frequency side is observed. Such a Raman line is called an "anti-Stokes" line.

Stokes line is generally more intense than anti-stokes line.

The Stokes lines frequency is $\dfrac{hc\nu - \Delta E}{hc}$.

The anti-stokes lines frequency is $\dfrac{hc\nu + \Delta E}{hc}$.

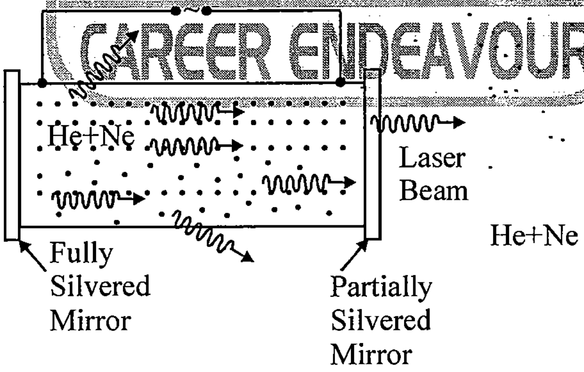
The wave number shift in each case being $\dfrac{\Delta E}{hc}$.



10.7. The Helium-Neon Laser (Four level Laser):

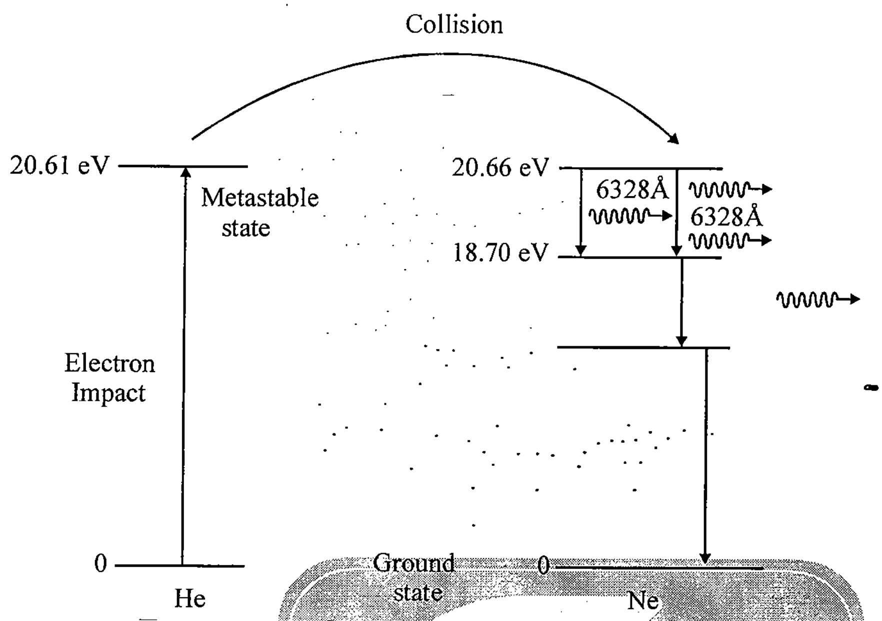
The Helium-Neon laser was the first gas laser to be operated successfully. It was fabricated by Javan in 1961. It is a four-level laser in which population inversion is achieved by electric discharge.

The Helium-Neion laser consists of a long and narrow discharge tube filled with a mixture of He and Ne in a ratio of about 10:1 at a pressure of about 1 mm of mercury. The gas mixtrue (He+Ne) forms the laser medium.



It is placed between a pair of optically plane and parallel mirrors which form a resonant cavity. One of the mirrors is fully silvered and the other is partially silvered. The spacing of mirrors is equal to an integral number of half-wavelengths of the laser light. An electric discharge may be produced in the gas-mixture by electrodes connected to a high-frequency electric source.

Working: When a discharge through the gas mixture, electrons are accelerated down the tube. These accelerated electrons collide with, and "pump" (excite) the He and Ne atoms to metastable states 20.61 eV and 20.66 eV respectively above their ground states (Figure below). Some of the excited He atoms transfer their energy to ground-state


Collision
20.61 eV
Metastable state
Electron Impact
0
He
20.66 eV
6328Å
6328Å
18.70 eV
0
Ground state
Ne

A four-level laser

Ne atoms by collision, with the 0.05 eV of addition energy being provided by the kinetic energy of atoms. Thus, He atoms help in achieving a population inversion in the Ne atoms.

When an excited Ne atom passes spontaneously from the metastable state at 20.66 eV to be the state at 18.70 eV, it emits a 6328 Å photon. This photon travels through the gas-mixture, and if it is moving parallel to the axis of the tube, is reflected back and forth by the mirror-ends until it stimulates an excited Ne atom and causes it to emit a fresh 6328 Å photon in phase with the stimulating photon. This process is continued and a beam of coherent radiation builds up in the tube. When this red-light becomes sufficiently intense, a portion of it escapes through the partially-silvered end.

SOLVED PROBLEMS

1. When the Zeeman lines are observed perpendicular to the magnetic field, they will be
(a) plane polarized (b) circularly polarized
(c) elliptically polarized (d) unpolarized **[B.H.U-2010]**

Soln. When the Zeeman lines are observed perpendicular to the magnetic field, they will be plane polarised.

For $\Delta m_J = \pm 1(\sigma \text{ lines})$ lines are plane polarised perpendicular to $\vec{B}$ direction.

And for $\Delta m_J = 0(\pi \text{ lines})$ lines are plane polarised parallel to $\vec{B}$.

Correct option is (a)

2. Stern-Gerlach apparatus breaks up the atomic beam of silver atoms into following number of components
(a) One (b) Two (c) Three (d) Four **[B.H.U-2010]**

Soln. In Stern-Gerlach experiment, a beam of silver atoms is passed through an inhomogeneous magnetic field. In an inhomogeneous magnetic field, the force on the magnetic moment μ will be $\mu_z x$ (Gradient of the magnetic field in the z-direction) where $\mu_z = \pm\frac{eh}{4m_e}$, where e is the charge of the electron, m_e is its mass, h is Planck's constant, and z is the field direction.

Thus, depending on the orientaton of the magnetic moment relative to the magnetic field there will be either an attractive or repulsive force and the beam will split in two components, exhibiting spatial quantization.

Correct option is (b)

3. Which of the following sets of quantum numbers is not possible ? **[B.H.U-2014]**
(a) $n = 2, l = 3, m_1 = 2$ (b) $n = 3, l = 2, m_1 = -2$
(c) $n = 6, l = 2, m_1 = 0$ (d) $n = 7, l = 3, m_1 = -3$

Soln. **Option (a) :** $n = 2, \ell = 3, m_\ell = 2$

The values of l lies between 0 and $n-1$, including 0 and $n-1$.

l cannot be greator than n.

$\therefore$ option (a) is incorrect

Option (b) : $n = 3, \ell = 2, m_\ell = -2$

For $n = 3$

Possible values of $\ell = 0, 1, 2$

Possible values of m_ℓ(corresponding to $\ell = 2) = -2, -1, 0, 1, 2$

$\therefore$ option (b) is correct

Option (c) : $n = 6, \ell = 2, m_\ell = 0$

Possible values of $\ell = 0, 1, 2, 3, 4, 5$

Possible value of m_ℓ (corresponding to $\ell = 0) = 0$

$\therefore$ option (c) is correct.

Option (d) : $n = 7, \ell = 3, m_\ell = -3$

Possible values of $\ell = 0, 1, 2, 3, 4, 5, 6$

Possible values of m_ℓ (corresponding to $\ell = 3) = -3, -2, -1, 0, 1, 2, 3$

$\therefore$ option (d) is correct.

Correct option is (a)

4. An atomic orbital with principle quantum number n can accommodate N number of electrons, which of the following statements is false ? **[B.H.U-2014]**

(a) $n^2 = N$ (b) $2n^2 = N$ (c) $N = \sum_{l=0}^{n-1} 2(2l+1)$ (d) $N = 2[1+3+5+\cdots 2n-1]$

Soln. An atomic orbital with principle quantum number n can accomodate $2n^2$ number of electrons.

$\therefore$ option (b) is correct.

option (c) and (d) are same. Total number of electrons in an atomic orbital is given by

$$N = \sum_{\ell=0}^{n-1} 2(2\ell+1) = 2\left[1+3+5+\cdots+(2n-1)\right]$$

$\therefore$ Options (c) and (d) are correct.

Option (a) is false as $N \neq n^2$.

Correct option is (a)

5. If the outer orbit of an atom contains two electrons, the possible multiplicity would be

(a) 1, 3 (b) 1 (c) 3 (d) 0 **[B.H.U-2016]**

Soln. For an atom with its outer orbit having two electrons can have two configuration :

(i) Singlet state : opposite spins, therefore total spin S = 0
multiplicity = 2S + 1 = 2 × 0 + 1 = 1

(ii) Triplet state : same spin, total spin = 1
multiplicity = 2S + 1 = 2 × 1 + 1 = 3

Correct option is (a)

5. For an electron with the quantum number $l = 2$, the possible values of j are **[B.H.U-2016]**

(a) $\frac{3}{2}, \frac{3}{2}$ (b) $\frac{5}{2}, \frac{3}{2}$ (c) $\frac{5}{2}, \frac{3}{2}, \frac{1}{2}$ (d) $\frac{3}{2}, \frac{1}{2}$

Soln. Given: for an electron with quantum number $\ell = 2$,

$$j = |\ell - s| \text{ to } |\ell + s|$$

$$= \left|2 - \frac{1}{2}\right| \text{ to } \left|2 + \frac{1}{2}\right| = \frac{3}{2}, \frac{5}{2}$$

Correct option is (b)

7. The ground state of nitrogen atom would be

(a) ${}^4S_{3/2}$ (b) ${}^2D_{3/2}$ (c) ${}^2D_{5/2}$ (d) ${}^2P_{3/2}$ **[B.H.U-2010]**

Soln. Atomic number of nitrogen atom : $Z = 7$

Electronic configuration of Nitrogen : $1s^2 2s^2 2p^3$

$2p^3$ is half-filled, so these electrons will contribute to the total angular momentum.

m_l	1	0	–1
	↑	↑	↑
m_s	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

$$M_L = \sum m_\ell = 0 \Rightarrow L = 0$$

$$M_S = \sum m_s = \frac{3}{2} \Rightarrow S = \frac{3}{2}$$

$$J = L + S = \frac{3}{2}$$

Term symbol: ${}^{2S+1}L_J \Rightarrow {}^4S_{3/2}$

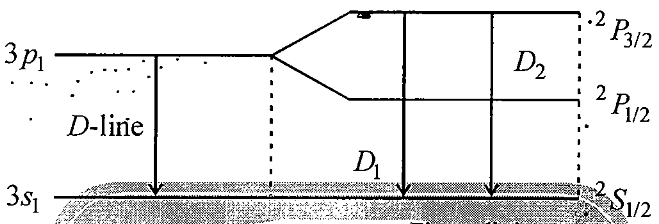
Correct option is (a)

8. The two sodium D-lines have wavelengths 5890 and 5896 Å. These arise due to transition from ${}^2S_{1/2}$ to
(a) ${}^2P_{3/2}$ and ${}^2P_{1/2}$ (b) ${}^2P_{1/2}$ and ${}^2P_{3/2}$ (c) ${}^2D_{5/2}$ and ${}^2D_{3/2}$ (d) ${}^2D_{3/2}$ and ${}^2D_{5/2}$ **[B.H.U-2013]**

Soln. Transitions in Zeeman splitting for Na D-line

$Na = 1s^2 2s^2 2p^6 3s^1$ – Ground state

$Na^* = 1s^2 2s^2 2p^6 3p^1$ – Excited state



Fine splitting

$\Delta L = \pm 1$

$\Delta S = 0, S = \frac{1}{2}$

$\Delta J = 0, \pm 1$

For P electron, $S = \frac{1}{2}, J = \frac{1}{2}, \frac{3}{2} \Rightarrow {}^2P_{1/2}, {}^2P_{3/2}$

For S electron, $L = 0, S = \frac{1}{2}, J = \frac{1}{2} \Rightarrow {}^2S_{1/2}$

$D_1 = 5896\ \text{Å} : {}^2P_{1/2} \rightarrow {}^2S_{1/2}$

$D_2 = 5896\ \text{Å} : {}^2P_{3/2} \rightarrow {}^2S_{1/2}$

Correct option is (a)

9. In a He-Ne laser, the key factor in the process of population inversion is **[B.H.U-2016]**
(a) The energy transfer from He to Ne (b) The energy transfer from Ne to He
(c) The material state of Ne (d) The collision of electron with the wall

Soln. In He-Ne laser, the function of He is to provide the assistance in the population inversion. The energy transfer from He to Ne occurs by collision and is thus the key factor in the process of population inversion.

Correct option is (a)

10. Principal series arises due to transitions **[B.H.U-2015]**
(a) $nf \rightarrow 3d, n > 3$ (b) $nd \rightarrow 3p, n > 2$ (c) $ns \rightarrow 3p, n > 3$ (d) $np \rightarrow 3s, n > 2$

Soln. The principal series arises due to transitions

$np \rightarrow 3s$, where $n > 2$.

The sharp series arises due to transitions

$ns \rightarrow 3p, n > 3$

The diffuse series arises due to transitions

$nd \to 3p, n > 2$

The foundamental series arises due to transitions

$nf \to 3d, n > 3$

Correct option is (d)

11. If the existing line in the Raman effect experiment is 5460 Å and the Stokes' line is at 5520 Å, then the wavelength of the corresponding anti-Stokes line is **[B.H.U-2011]**

(a) 5400 Å (b) 5490 Å (c) 5580 Å (d) 5550 Å

Soln. The wave number of anti-Stokes line is given by

$$\bar{v}_{\text{anti-Stokes}} - \bar{v}_{\text{Raman}} = \bar{v}_{\text{Raman}} - \bar{v}_{\text{Stokes}}$$

$$\Rightarrow \bar{v}_{\text{antiStokes}} = 2\bar{v}_{\text{Raman}} - \bar{v}_{\text{Raman}} - \bar{v}_{\text{Stokes}}$$

$$= 2 \times \frac{1}{5460} - \frac{1}{5520}$$

$$= \frac{2 \times 5520 - 5400}{5460 \times 5520} = 0.0001851\,\text{Å}^{-1}$$

$\therefore$ Wavelength of anti-Stokes line $= \dfrac{1}{\bar{v}_{\text{anti-Stokes}}} = \dfrac{1}{0.0001851}\,\text{Å} \approx 5400\,\text{Å}$

Correct option is (a)

12. Which of the following transitions is allowed ? **[B.H.U-2010]**

(a) ${}^2S_{1/2} \to {}^2D_{3/2}$ (b) ${}^2P_{1/2} \to {}^2D_{3/2}$ (c) ${}^2S_{1/2} \to {}^2S_{1/2}$ (d) ${}^2D_{5/2} \to {}^2P_{1/2}$

Soln. Transition selection rule

$\Delta S = 0$

$\Delta L = 0, \pm 1$

$\Delta J = 0, \pm 1$ but $J = 0 \not\leftrightarrow J = 0$

$${}^2S_{1/2} \longrightarrow {}^2D_{3/2}$$

$S = \frac{1}{2}$ $S = \frac{1}{2} \Rightarrow \Delta S = 0$

$J = \frac{1}{2}$ $J = \frac{1}{2} \Rightarrow \Delta J = 0$

$L = 0$ $L = 2 \Rightarrow \Delta L = 2$ (not allowed)

$\therefore$ Option (a) is incorrect

$${}^2P_{1/2} \longrightarrow {}^2D_{3/2}$$

$S=\frac{1}{2} \qquad S=\frac{1}{2}\Rightarrow \Delta S=0$

$J=\frac{1}{2} \qquad J=\frac{3}{2}\Rightarrow \Delta J=1$

$L=1 \qquad L=2\Rightarrow \Delta L=1$

$\therefore$ Option (b) is correct

$^2S_{1/2}\longrightarrow {}^2S_{1/2}$

Some state transition not allowed

$\therefore$ Option (c) is incorrect

$^2D_{5/2}\longrightarrow {}^2P_{1/2}$

$S=\frac{1}{2} \qquad S=\frac{1}{2}\Rightarrow \Delta S=0$

$J=\frac{5}{2} \qquad J=\frac{1}{2}\Rightarrow \Delta J=2$ (not allowed)

$L=2 \qquad L=1\Rightarrow \Delta L=1$

Correct option is (b)

13. An atom of magnetic quantum number m_l and mass m is placed in a magnetic field B, then magnetic energy is **[B.H.U-2011]**

(a) $m_\ell\left(\frac{e\hbar}{2m}\right)B^2$ (b) $m\left(\frac{e\hbar}{2m_\ell}\right)B$ (c) $m_\ell\left(\frac{e\hbar}{2m}\right)B$ (d) $m\left(\frac{e\hbar}{2m_\ell}\right)B^2$

Soln. The magnetic potential energy of an atom in a magnetic field is given by

$$E=-\vec{\mu}_L\cdot\vec{B}=-\mu_{L_z}B_z \qquad \text{(Assuming } B \text{ in } z\text{-direction)}$$

Where $\mu_L=-\frac{e}{2m}m_l\hbar$ (Orbital magnetic moment)

$$E=-\left(-\frac{e}{2m}m_l\hbar\right)B_z=m_l\left(\frac{e\hbar}{2m}\right)B$$

Correct option is (c)

14. A sample of certain element is placed in a 0.300 tesla magnetic field and suitably excited. How far apart are the Zeeman components of the 450 nm spectral line of this element? **[B.H.U-2011]**

(a) 5.66×10^{-12} metre (b) 2.83×10^{-12} metre (c) 0.0014 nm (d) 0.0042 nm

Soln. Given : $|\vec{B}|=0.3T,\ \lambda=450\ nm$

$$\Delta E=\left(\frac{e\hbar}{2m}\right)B=h\,\Delta f=h\Delta\left(\frac{c}{\lambda}\right)=-hc\frac{\Delta\lambda}{\lambda^2}$$

$$\Rightarrow |\Delta\lambda|=\left(\frac{e\hbar}{2m}\right)B\frac{\lambda^2}{hc}=\frac{eB\lambda^2}{4\pi mc}$$

$$|\Delta\lambda| = \frac{(1.6\times10^{-19})\times(0.3)\times(450\times10^{-9})^2}{4\pi\times(9.11\times10^{-3})\times(3\times10^8)} = 0.00203 \text{ nm} = 2.83\times10^{-12}\text{m}$$

Correct option is (a)

15. The total number of terms arising due to one $4p$ and one $4d$ electrons i.e., outer configuration of an atom as $4p$ $4d$ is

(a) 6 (b) 12 (c) 9 (d) 3 **[B.H.U-2015]**

Soln. For $4p\,4d$:

For this, $s_1 = \frac{1}{2},\ s_2 = \frac{1}{2},\ \ell_1 = 1,\ \ell_2 = 2$

Therefore, $s = |s_1 - s_2|, |s_1 - s_2| + 1 = 0, 1$

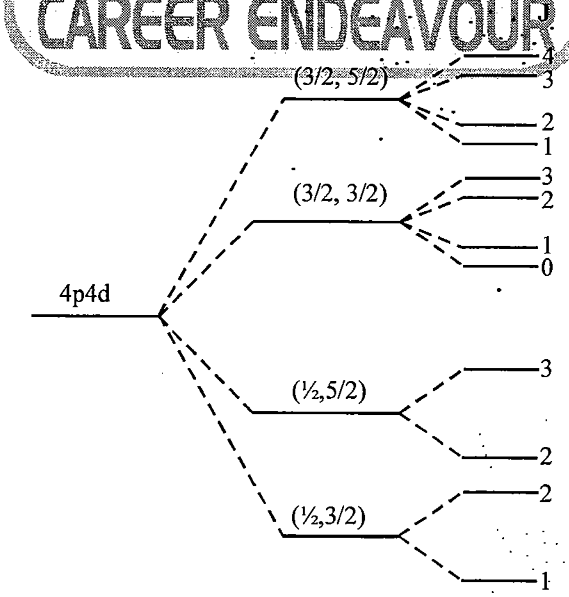
Multiplicity $(2s+1) = 1, 3$ (single, triplet)

$L = |\ell_1 - \ell_2|, |\ell_1 - \ell_2| + 1, \ldots = 1, 2, 3$ (P, D, F states)

Since, $J = |L - S|, \ldots, |t - s|$

Singlet terms ($s = 0$)	Triplet terms ($s = 1$)
$\left.\begin{matrix} S=0 \\ L=1 \end{matrix}\right\}$ $J = 1,\ {}^1P_1$	$\left.\begin{matrix} S=1 \\ L=1 \end{matrix}\right\}$ $J = 0,1,2;\ {}^3P_{0,1,2}$
$\left.\begin{matrix} S=0 \\ L=2 \end{matrix}\right\}$ $J = 2,\ {}^1D_2$	$\left.\begin{matrix} S=1 \\ L=2 \end{matrix}\right\}$ $J = 1,2,3;\ {}^3D_{1,2,3}$
$\left.\begin{matrix} S=0 \\ L=3 \end{matrix}\right\}$ $J = 3,\ {}^1F_3$	$\left.\begin{matrix} S=1 \\ L=3 \end{matrix}\right\}$ $J = 2,3,4;\ {}^3F_{2,3,4}$

Thus a single degenerate level of configuration $4p\,4d$ is splitted into 12 levels as shown below.



Unperturbed Level = Spin-orbit interaction + spin-spin correlation & residual electrostatic interaction

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&
Electronics
for
M.Sc. Entrance

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Chapter 1

CRYSTAL STRUCTURE

Solids: The solids are the forms of matter in which atoms and molecules are strongly bound and maintain definite volume and shape unless changed by applying external forces.

Solid state physics: The solid state physics is the study of the physical properties of the atomic/molecular matter in the solid state. These physical properties include crystal structure, bonding, thermal, electrical and magnetic properties.

Classification of solids:

Solids are classified into two categories:
(i) crystalline solids
(ii) non-crystalline solids (amorphous)

Crystalline solids:

This crystalline solids are defined as an ordered periodic arrangement of atoms, molecules or ions.

Properties of crystalline solids:

(1) The internal arrangement of atoms, ions or molecules of crystalline solid is in a definite order.
(2) The crystalline solids have a sharp melting point.
(3) Below the temperature of crystallization, the crystalline materials are in a stable state. As the stable state is the state of minimum energy, materials acquire a state of minimum energy on crystallization.
(4) The orderly arrangement of atoms, ions and molecules in a crystal is extended over a large volume of the crystal, indicating a long range order by the atoms, ions and molecules.
(5) Crystalline solids are anisotropic. This implies that physical properties such as refractive index, conductivity, thermal properties such as expansion etc. are different in different directions. This is due to orderly arrangement of the constituents.
(6) Crystalline solids have uniform chemical composition throughout.

Examples: Diamond, Fe, Ag, Au, Salt, Sugar and Mica etc.

Non-crystalline solids:

Solids in which constituents (atoms, ions or molecules) are arranged in irregular or random manner are called amorphous solids or non-crystalline solids.

Properties of non-crystalline solids:

(1) The constituents are arranged in irregular manner.
(2) They do not have sharp melting point because the bonds between the atoms are not equally strong.
(3) They are isotropic in nature.
(4) Amorphous solids are regarded as supercooled liquids or pseudo solids.

Single crystalline solids:

When the periodicity of the pattern extends throughout a certain piece of solid material, it is called single crystalline solids. Example: Diamond. In a single crystal, same symmetry exists everywhere throughout the crystal.

Polycrystalline solids:
When a large number of small crystal sections (called grains) of various shapes and sizes are packed to one another separated by the interfaces (called grain boundaries) so that the periodicity is interrupted at the grain boundaries, though the structure may be more stable, the solid is called polycrystalline.

Crystallography: The science which deals with the structure of crystalline solids is called crystallography.
Crystal structure: The atomic arrangement in a crystal is called crystal structure.

In nature, most of the solids exit in crystalline form because crystalline form is more stable as compared to amorphous form of solids. The energy released during the formation of crystalline solids is more than the energy released during the formation of amorphous solids. As more released during the formation, the crystal will have less energy, thus the crystal will be more stable.

In order to describe crystal structure in detail, first we will discuss the following terms:

Crystal translation vectors, symmetry operations, crystal lattices and basis.

Crystal translation vectors:

A crystal is regular arrangement of atoms. We can imagine points in space about which these atoms are located. These points are called as lattice points. Thus an ideal crystal is a body composed of atoms arranged on a lattice defined by three fundamental translation vectors $\vec{a}, \vec{b}, \vec{c}$ such that the atomic arrangement looks the same in every respect when viewed from any point $\vec{r}$ as when viewed from the point $\vec{r}'$.

Thus, $\vec{r}' = \vec{r} + \vec{T} \Rightarrow \vec{r}' = \vec{r} + n_1\vec{a} + n_2\vec{b} + n_3\vec{c}$ ...(i)

where n_1, n_2 and n_3 are arbitrary integers and $\vec{T}$ is called a lattice translation operation or crystal translation vector.

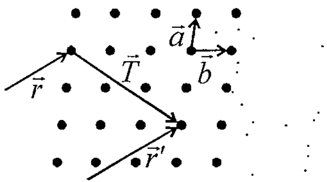
The crystal translation operation is defined as the displacement of a crystal parallel to itself by a crystal translation vector

$$\vec{T} = n_1\vec{a} + n_2\vec{b} + n_3\vec{c} \quad \text{... (ii)}$$

In two dimensional lattice, the vector $\vec{T}$ connects any two points, as shown in figure and given by the following relation

$$\vec{T} = n_1\vec{a} + n_2\vec{b} \quad \text{... (iii)}$$

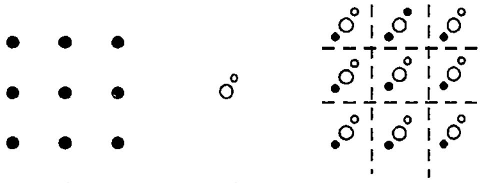
Lattice: A lattice is defined as parallel net like arrangement of points provided the environment about any point is identical with environment about any other point.
The figure shown below is an array of points in two dimensions, is called the plane lattice. If we extend this array of points in three dimensions then the array of points is called space lattice.



Basis: In order to get a crystal structure, an atom or group of atoms must be placed on each lattice point in regular manner. Such an atom or group of atoms is called the basis and acts as a building-block or unit for the crystal structure.

Mathematically, crystal structure can be expressed as

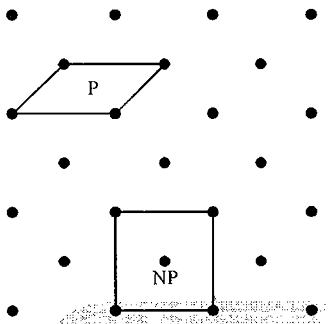
$\boxed{\text{Space lattice + Basis = Crystal structure}}$

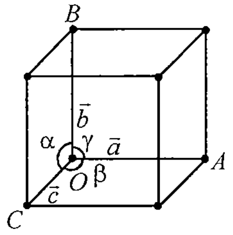

Lattice Basis Crystal structure

Unit cell: The parallelograms formed by the translation vectors are regarded as builiding blocks for constructing the complete lattice and are called as unit cells of the lattice. In three dimensions, this unit cell will be a parallelopiped. Therefore, a unit cell is defined as the smallest unit of lattice points which on repetition produces the complete lattice. There are two types of unit cells:

(*i*) **Primitive unit cell (P):** The lattice points (atoms) belonging to primitive unit cell lie at its corners. The effective number of lattice points in a primitive unit cell is one.

(*ii*) **Non-primitive unit cell (NP):** The lattice points (atoms) belonging to non-primitive cell lie at corners as well as other locations. The effective number of lattice points in a non-primitive unit cell is greater than one.





Wigner-Seitz cell:

A method for the construction of unit cell from the lattice points was given by Wigner-Seitz. The procedure of the construction is as follows:

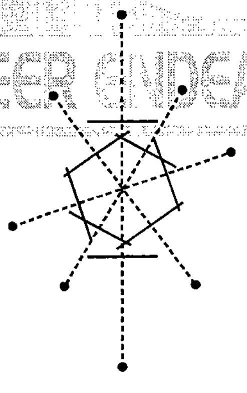
(1) connect a given lattice point to all the nearby lattice points

(2) draw normals at the midpoints of lines connecting the lattice points.

The smallest volume enclosed by the normals is the required primitive cell.

The cell constructed by this procedure is known as Wigner-Seitz cell.

In the figure, a two dimensional oblique lattice is constructed by this procedure and a hexagon is the result. Similarly, for a three dimensional BCC lattice, it is a truncated octahedron and for a FCC lattice, it is a rhombic dodecahedron, respectively.



Construction of Wigner-Seitz primitive cell

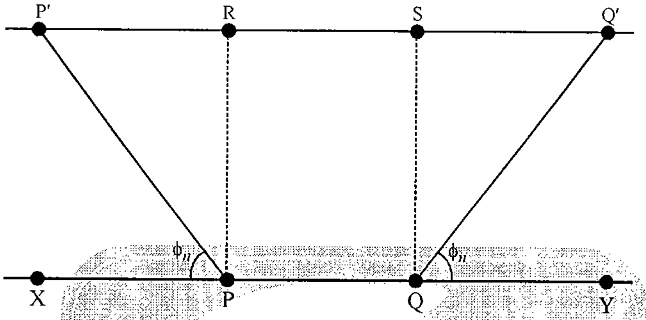
Crystal symmetry:

The operations that leaves the crystal and its environment invariant are called symmetry operations. There are four symmetry operations:

(i) Translational symmetry: A crystal translation is already defined in equations (ii) and (iii). Translation operations repeat the lattice infinite number of times along a given direction and leaves it unchanged.

(ii) Rotational symmetry: If after rotation of a lattice about an axis (or a point in two dimensional lattice) by some angle ϕ_n, the body appears as it did prior to rotation, the lattice is said to possess rotational symmetry. Axis of symmetry may thus be defined as a line such that the crystal assumes a congruent position (self-coincidence), for every rotation of $\left[\dfrac{360°}{n}\right]$. The value of n decide the fold of the axis.

Only one, two, three, four and six-fold rotation axes of symmetry alone are possible in single crystal. The fact that 5-fold rotation axis is not compatible with translation symmetry operation is proved as follows: Consider a lattice row XPQY.



Let the lattice translation be a and the lattice have n-fold rotation axis of symmetry.

Let us rotate the vectors PX and QY through an angle $\phi_n = \left(\dfrac{360°}{n}\right)$ in clockwise and anticlockwise directions respectively. The tips of the vectors P' and Q' in the new position must be lattice points if the lattice possess n-fold rotation axis of symmetry.

As per definition, rotation operation must be the lattice invariant. Clearly $P'Q'$ must be parallel to PQ and must be equal to an integral multiple of the lattice translation a,

$P'Q' = ma$ where m is an integer.

Thus, $\cos\phi_n = \dfrac{P'R}{PP'}$ and $P'R = PP'\cos\phi_n = a\cos\phi_n$ and $SQ' = a\cos\phi_n$ $\Rightarrow$ $P'Q' = P'R + RS + SQ'$

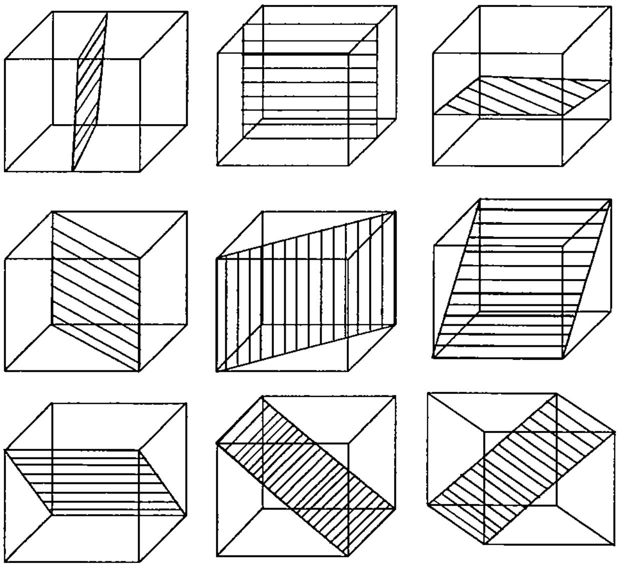
$$\Rightarrow P'Q' = 2a\cos\phi_n + a = ma \Rightarrow 1 + 2\cos\phi_n = m \Rightarrow 2\cos\phi_n = m - 1 = N \Rightarrow \cos\phi_n = \left(\frac{m-1}{2}\right) = \frac{N}{2}$$

N	$N/2$	$\cos\phi_n$	ϕ_n	N
–2	–1	–1	180°	2
–1	–1/2	–1/2	120°	3
0	0	0	90°	4
1	1/2	1/2	60°	6
2	1	1	360°	1

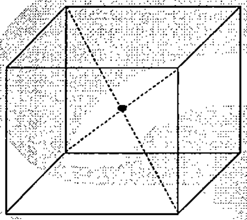
Hence, the permissible values of n are 2, 3, 4, 6, 1. Thus we conclude that 5-fold rotation is not permissible because it is not compatible with lattice translation symmetry.

Similarly, we can say that other rotations such as 7-fold rotation are also not permissible.

(iii) Reflection symmetry : A lattice possess reflection symmetry if there exist a plane (or a line into two dimensions) in the lattice which divides it into two identical halves which are mirror image of each other. Examples of this symmetry are shown in following figures.



(iv) Inversion symmetry : This is a point operation which is applicable to three-dimensional lattices only. This symmetry element implies that each point located at $\vec{r}$ relative to a lattice point has an identical point located at $-\vec{r}$ relative to the same lattice point.



Point group : Symmetry operations performed about a point or a line are called point group. There are total 10 point groups in two dimensional lattices and 32 point groups in three-dimensional lattices.

Space group : Symmetry operations performed by translation as well are called space group. There are total 17 space groups in two dimensional lattices and 230 space groups in three-dimensional lattices.

Types of lattices :

(i) Two-dimensional lattices : The four crystal systems of two dimensional lattices are given in following table.

Sr. No.	Crystal system	Bravais lattice	Conventional unit cell / Unit cell parameters
1.	Oblique	Oblique	Parallelogram $a \neq b, \gamma \neq 90°$
2.	Rectangular	(a) Rectangular primitive (b) Rectangular centered	Rectangle $a \neq b, \gamma = 90°$
3.	Square	Square	Square $a = b, \gamma = 90°$
4.	Hexagonal	Hexagonal	60° Rhombus $a = b, \gamma = 120°$

(ii) Three-dimensional lattices : There are 32 crystal structures based on geometrical considerations (i.e, symmetry and internal structure). But it is common practice to divide all crystal systems into seven groups or basic systems.

These seven basic crystal systems are shown in table with decreasing order of symmetry.

The crystallographic axes $\vec{a}, \vec{b}$ and $\vec{c}$ determines the size and shape of the unit cell. The angles α, β and γ

represent the angles between the vectors $\vec{b}$ and $\vec{c}$, $\vec{c}$ and $\vec{a}$ and $\vec{a}$ and $\vec{b}$ respectively. These α, β, γ and a, b, c are collectively known as lattice parameters.

Sr. No.	Crystal system	Lattice parameters	Bravais lattice	Lattice symbol	Examples
1.	Cubic	$a = b = c$ $\alpha = \beta = \gamma = 90°$	Simple Body-centered Face-centered	P I F	Cu, Ag, Fe, Na, NaCl, CsCl
2.	Tetragonal	$a = b \neq c$ $\alpha = \beta = \gamma = 90°$	Simple Body-centered	P I	β-Sn, TiO_2
3.	Orthorhombic	$a \neq b \neq c$ $\alpha = \beta = \gamma = 90°$	Simple Body-centered End-centered Face-centered	P I C F	Ga, Fe_3C (cementite)
4.	Rhombohedral or Trigonal	$a = b = c$ $\alpha = \beta = \gamma \neq 90°$	Simple	P	As, Sb, Bi
5.	Hexagonal	$a = b \neq c$ $\alpha = \beta = 90°$, $\gamma = 120°$	Simple	P	Mg, Zn, Cd, NiAs
6.	Monoclinic	$a \neq b \neq c$ $\alpha = \gamma = 90° \neq \beta$	Simple End-centered	P C	$CaSO_4.2H_2O$ (gypsum)
7.	Triclinic	$a \neq b \neq c$ $\alpha \neq \beta \neq \gamma \neq 90°$	Simple	P	$K_2Cr_2O_7$

Miller indices:

Crystal lattice may be regarded as made up of an aggregate of a set of parallel equidistant planes passing through the lattice points. These equidistant planes are called as lattice planes. The scheme to represent the orientation of lattice planes was given by Miller (a British crystallographer). He has given some indices to represent a set of planes. The indices of planes are known as Miller indices. For a general set of planes, Miller indices are written as $(h\,k\,l)$.

The steps involved in calculation of Miller indices is given below.

(i) Determine the coordinates of the intercepts made by the plane along the three crystallographic axes (x, y, z axes).

$$\begin{array}{lll} x & y & z \\ pa & qb & rc \\ 2a & 3b & c \ \text{(suppose)} \end{array}$$

$\Rightarrow p = 2, q = 3$ and $r = 1$

(ii) Express the intercepts as multiples of the unit cell dimensions or lattice parameter along the axes i.e.,

$$\begin{array}{ccc} \frac{2a}{a} & \frac{3b}{b} & \frac{c}{c} \\ 2 & 3 & 1 \end{array}$$

(iii) Determine the reciprocals of these numbers $\frac{1}{2}\quad\frac{1}{3}\quad\frac{1}{1}$.

(iv) Reduce these reciprocals to the smallest set of integral numbers and enclose them in bracket with comma.

$$6\times\frac{1}{2}\qquad 6\times\frac{1}{3}\qquad 6\times 1$$

$$(3\qquad 2\qquad 6)$$

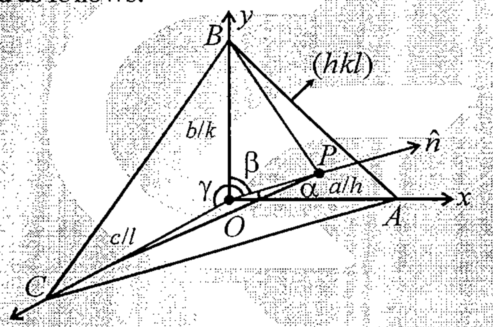
Thus the Miller indices of the planes is (3 2 6).

Thus, Miller indices may be defined as the reciprocals of the intercepts made by plane on the cyrstallographic axes when reduces to smallest numbers.

$$h:k:l=\frac{a}{\text{OA}}:\frac{b}{\text{OB}}:\frac{c}{\text{OC}}$$

where, a, b and c are lattice parameters along x-, y- and z-axes respectively. OA, OB and OC are intercepts on a-, b- and c-axis respectively.

Relation between interplanar spacing and lattice parameters : If the plane have ($h\,k\,l$) miller indices then its intercepts can be calculated as follows:



$$h:k:l=\frac{a}{OA}:\frac{b}{OB}:\frac{c}{OC}$$

Therefore, intercepts $OA=\frac{a}{h}$, $OB=\frac{b}{k}$ and $OC=\frac{c}{l}$.

Now, if we draw a perpendicular (normal) from origin (O) to the plane ($h\,k\,l$) at P. Then OP will be the interplanar spacing (d) between two planes which are parallel to ($h\,k\,l$) and also passes through O.

$$\therefore\quad d=OP=\frac{a}{h}\cos\alpha=\frac{b}{k}\cos\beta=\frac{c}{l}\cos\gamma$$

where α, β and γ represent the angles between the normal and the axes $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively and $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ represent the direction cosines of the normal to the plane ($h\,k\,l$).

Cosine rule: $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$

$$\Rightarrow\quad\left(\frac{hd}{a}\right)^2+\left(\frac{kd}{b}\right)^2+\left(\frac{ld}{c}\right)^2=1$$

$$\Rightarrow\quad d^2\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}+\frac{l^2}{c^2}\right)=1$$

$$\Rightarrow \quad d=\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}+\frac{l^2}{c^2}\right)^{-1/2} \quad \text{or } d=\frac{1}{\sqrt{\frac{h^2}{a^2}+\frac{k^2}{b^2}+\frac{l^2}{c^2}}}$$

There is another way to show this relation. Let $\hat{n}$ be the unit vector in the direction $\overline{OP}$.

$$\therefore \quad d=\hat{n}\cdot\frac{\vec{a}}{h}=\hat{n}\cdot\frac{\vec{b}}{k}=\hat{n}\cdot\frac{\vec{c}}{l}$$

The equation of plane, $f(x, y, z)=\frac{hx}{a}+\frac{ky}{b}+\frac{lz}{c}$

For a surface $f(x, y, z)=\text{constant}$, ∇f represents the vector normal to it.

$$\therefore \quad \hat{n}=\frac{\nabla f}{|\nabla f|}=\frac{\left(\frac{h}{a}\right)\hat{i}+\left(\frac{k}{b}\right)\hat{j}+\left(\frac{l}{c}\right)\hat{k}}{\sqrt{\frac{h^2}{a^2}+\frac{k^2}{b^2}+\frac{l^2}{c^2}}}$$

$$\therefore \quad d=\hat{n}\bullet\frac{\vec{a}}{h}=\left\{\frac{\left(\frac{h}{a}\right)\hat{i}+\left(\frac{k}{b}\right)\hat{j}+\left(\frac{l}{c}\right)\hat{k}}{\sqrt{\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}+\frac{l^2}{c^2}\right)}}\right\}\bullet\frac{\vec{a}}{h}$$

$$\Rightarrow \quad d=\frac{1}{\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}+\frac{l^2}{c^2}\right)^{1/2}}$$

(where $a \neq b \neq c$, therefore this relation corresponds to orthorhombic structure)

For cubic structure, $a=b=c$, $d=\frac{a}{\sqrt{h^2+k^2+l^2}}$

For tetragonal structure, $a=b\neq c$, $d=\frac{1}{\sqrt{\left(\frac{h^2+k^2}{a^2}\right)+\frac{l^2}{c^2}}}$.

Different crystal structures:

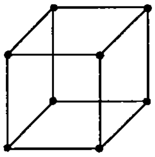
Cubic crystal structures:

(i) Simple cubic crystal structure (SC):

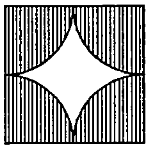
- In a simple cubic structure there is one lattice point (atom) at each of the eight corners (shares 1/8 part of the atom) of the cubic unit cell. This is primitive unit cell structure of cubic crystal system. Properties of simple cubic strucutre are given below.

(a) Number of nearest neighbours:

If we take a lattice point at one corner as the origin, it is observed that this atom is surrounded by six equidistant nearest neighbours at coordinates (+a, 0, 0), (-a, 0 ,0), (0, +a, 0), (0, -a, 0), (0, 0, a) and (0, 0, -a). The coordination number is defined as the number of nearest neighbours to a given atom in the crystal lattice. Thus, for simple cubic structure, the coordination number is six. The distance of these nearest neighbours is a.



Unit cell of SC structure

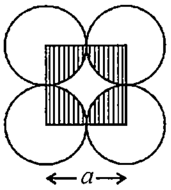


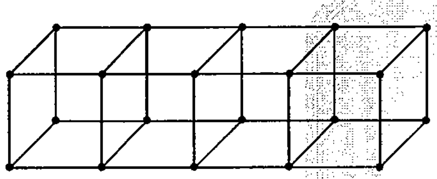
Plane view of SC structure

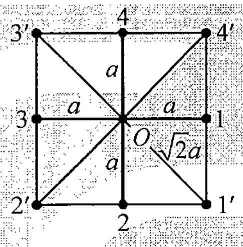
In this structure, each corner atom is shared by eight unit cell.

(b) Number of second nearest neighbours: There are 12 second nearest neighbours at distance $\sqrt{2}a$.

(c) Atomic radius: From this figure, $a = 2r \Rightarrow r = \frac{a}{2}$.







(d) Number of atoms per unit cell: The number of effective lattice points (atoms) per unit cell is given by

$$n_{eff} = n_i + \frac{n_f}{2} + \frac{n_c}{8}$$

where, n_i is the number of lattice points (atoms) completely inside the cell and n_f and n_c represent the lattice points occupying face center and corner positions of the cell, respectively.

For SC structure, $n_i = 0, n_f = 0$ and $n_c = 8$, $n_{eff} = 1 \times 0 + \frac{1}{2} \times 0 + \frac{1}{8} \times 8 = 1$.

(e) Atomic packing fraction: The packing of atoms in a unit cell of the crystal structure of a material is represented by atomic packing fraction (APF). The APF is defined as the ratio of volume occupied by the atoms in a unit cell to that of the volume of the unit cell.

$$\text{APF} = \left(\frac{\text{volume of the atoms per unit cell}}{\text{volume of the unit cell}}\right) = \left(\frac{n_{eff} \times \text{volume of each atom}}{\text{volume of unit cell}}\right)$$

For simple cubic structure, $n_{eff} = 1$

$$\therefore \quad \text{APF} = \frac{1 \times \frac{4}{3}\pi r^3}{a^3} = \frac{1 \times \frac{4}{3}\pi \left(\frac{a}{2}\right)^3}{a^3} = \frac{4}{3}\pi \times \frac{1}{8} = \frac{\pi}{6} = 0.52.$$

$$\therefore \quad \text{APF}\% = 52\%.$$

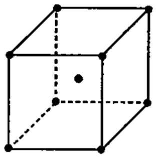
(f) Type of packing: Simple cubic structure is a loose-packed structure.

(g) Example : Only Polonium exhibits this type of structure at room temperature.

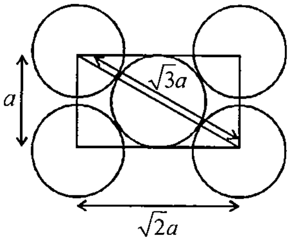
(ii) Body centered cubic structure (BCC):

In the BCC structure, there is one lattice point (atom) at each corners (shares 1/8th part of the atom) of the cubic unit cell and one lattice point (atom) exit at the body center. This is non-primitive unit cell structure of cubic crystal system.

The corner atoms do not touch each other, but each corner atom touches the body centered atom along the body diagonal.



Conventional unit cell of BCC structure



Plane view of BCC structure

(a) Number of nearest neighbours: If we take, the body centered lattice point (atom) of cell as origin then all the corner atoms at coordinates $\left(+\frac{a}{2},+\frac{a}{2},+\frac{a}{2}\right)$, $\left(+\frac{a}{2},+\frac{a}{2},-\frac{a}{2}\right)$, $\left(+\frac{a}{2},-\frac{a}{2},+\frac{a}{2}\right)$, $\left(+\frac{a}{2},-\frac{a}{2},-\frac{a}{2}\right)$, $\left(-\frac{a}{2},+\frac{a}{2},+\frac{a}{2}\right)$, $\left(-\frac{a}{2},+\frac{a}{2},-\frac{a}{2}\right)$, $\left(-\frac{a}{2},-\frac{a}{2},+\frac{a}{2}\right)$ and $\left(-\frac{a}{2},-\frac{a}{2},-\frac{a}{2}\right)$ will be nearest neighbours to the body centered atom. Since, there are 8 corners therefore, number of nearest neighbours (i.e. coordination number) will be 8. The distance of these nearest neighbours from reference lattice point is $\frac{\sqrt{3}a}{2}$.

(b) Number of second nearest neighbours: There are 6 second nearest neighbours at distance a in BCC structures.

(c) Atomic radius: From above figure, $4r=\sqrt{3}a \Rightarrow r=\frac{\sqrt{3}a}{4}$.

(d) Number of atoms per unit cell: The number of effective lattice points (atoms) per unit cell is given by

$$n_{eff}=n_i+\frac{n_f}{2}+\frac{n_c}{8}$$

where, n_i is the number of lattice points (atoms) completely inside the cell and n_f and n_c represent the lattice points occupying face center and corner positions of the cell, respectively.

For BCC structure, $n_i=1, n_f=0$, and $n_c=8$, $n_{eff}=1\times 1+\frac{1}{2}\times 0+\frac{1}{8}\times 8=2$.

(e) Atomic packing fraction: $\text{APF}=\left[\frac{n_{eff}\times \text{volume of each atom}}{\text{volume of unit cell}}\right]$

For BCC structure, $n_{eff}=2$

$$\therefore \quad \text{APF}=\frac{2\times\frac{4}{3}\pi r^3}{a^3}=\frac{2\times\frac{4}{3}\pi\left(\frac{a\sqrt{3}}{4}\right)^3}{a^3}=\frac{\sqrt{3}\pi}{8}=0.68$$

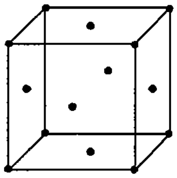
$\therefore \quad \text{APF\%} = 68\%$

(f) Type of packing: Body centered cubic structure is a loose-packed structure.

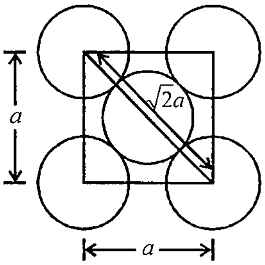
(g) Examples: Li, Na, K, Mo etc.

(iii) Face-centered cubic structure (FCC):

In FCC structure, one atom lies at each corner of the cube in addition to one atom at the center of each face. This is also a non-primitive unit cell structure of cubic crystal system.



Conventional unit cell of FCC structure



Plane view of FCC structure

(a) Number of nearest neighbours : If we take, one of the face centered lattice point (atom) of unit cell as origin then all the 12 corner atoms at coordinates $\left(+\frac{a}{2},+\frac{a}{2},0\right),\left(+\frac{a}{2},-\frac{a}{2},0\right),\left(-\frac{a}{2},-\frac{a}{2},0\right),\left(-\frac{a}{2},+\frac{a}{2},0\right)$, $\left(+\frac{a}{2},0,+\frac{a}{2}\right),\left(+\frac{a}{2},0,-\frac{a}{2}\right),\left(-\frac{a}{2},0,-\frac{a}{2}\right),\left(-\frac{a}{2},0,+\frac{a}{2}\right),\left(0,+\frac{a}{2},+\frac{a}{2}\right),\left(0,+\frac{a}{2},-\frac{a}{2}\right),\left(0,-\frac{a}{2},-\frac{a}{2}\right)$ and $\left(0,-\frac{a}{2},+\frac{a}{2}\right)$ will be nearest neighbours to this face centered atom. These atoms will lie at corners of the three faces in x, y and z-axes passing through the reference atom. Hence, total number of nearest neighbours (i.e. coordination number) will be 12. The distance of these nearest neighbours from reference lattice point is $\frac{\sqrt{2}a}{2}$ or $\frac{a}{\sqrt{2}}$.

(b) Number of second nearest neighbours : There are 6 second nearest neighbours at distance a in FCC structures.

(c) Atomic radius: From above figure, $4r = \sqrt{2}a \Rightarrow r = \frac{\sqrt{2}a}{4} = \frac{a}{2\sqrt{2}}$.

(d) Number of atoms per unit cell: The number of effective lattice points (atoms) per unit cell is given by

$$\therefore \quad n_{eff} = n_i + \frac{n_f}{2} + \frac{n_c}{8}$$

where, n_i is the number of lattice points (atoms) completely inside the cell and n_f and n_c represent the lattice points occupying face center and corner positions of the cell, respectively.

For FCC structure, $n_i = 0$, $n_f = 6$ and $n_c = 8$, $n_{eff} = 1 \times 0 + \frac{1}{2} \times 6 + \frac{1}{8} \times 8 = 3 + 1 = 4$.

(e) Atomic packing fraction : $\text{APF} = \left[\frac{n_{eff} \times \text{volume of each atom}}{\text{volume of unit cell}}\right]$

For FCC structure, $n_{eff} = 4$

$$\therefore\quad \text{APF} = \frac{4\times\frac{4}{3}\pi r^3}{a^3} = \frac{4\times\frac{4}{3}\pi\left(\frac{\sqrt{2}a}{4}\right)^3}{a^3} = \frac{\sqrt{2}\pi}{6} = 0.74$$

$\therefore\quad \text{APF}\% = 74\%$

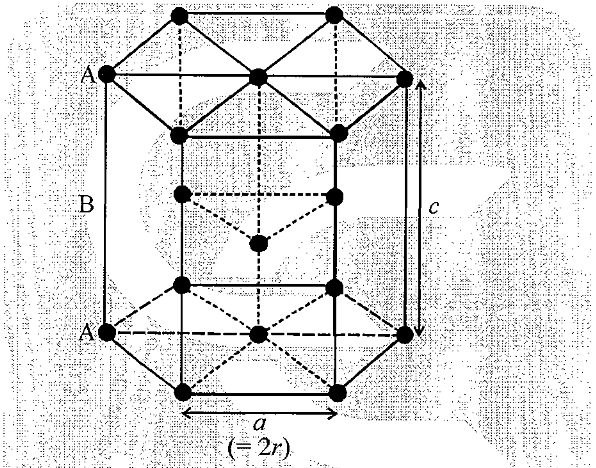
(f) Type of packing: Face-centered cubic strucutre is a close-packed structure.

(g) Examples: Cu, Au, Ag, Al etc.

Hexagonal close packed structure (HCP):

The specific hexagonal structure formed by Magnesium is shown in figure. The unit cell contains one atom at each corner, one atom at the centre of each hexagonal faces and three more atoms within the body of the cell.

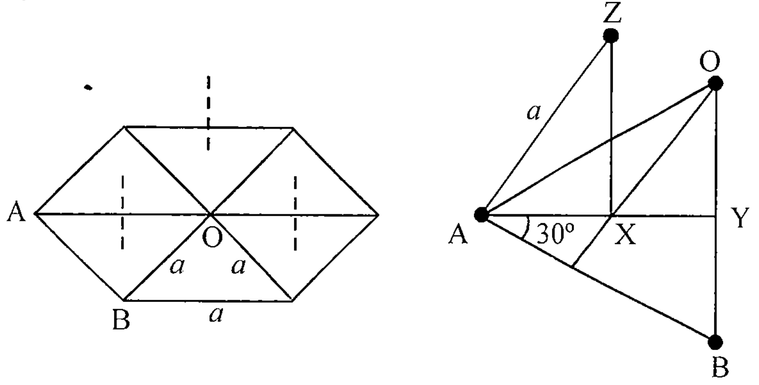
(a) Number of nearest neighbours: Each atom touches six atoms in its plane, three atoms in the layer below its plane, and three atoms in the layer above. Hence the coordination number of this structure is 12. Further the atoms touch each other along the edge of the hexagon. Thus, $a=2r$. The top layer contains seven atoms. Each corner atom is shared by 6 surrounding hexagon cells and the centre atom is shared by two adjacent cells. The three atoms within the body of the cell are fully contributing to the cell.



Conventional unit cell of HCP structure

(b) Number of atoms per unit cell: In this structure, there are 12 atoms at 12 corners of the hexagonal unit cell, two atoms at the center of top and bottom hexagons and three atoms inside the unit cell. Thus, the effective number of atoms in a unit cell is $\frac{1}{6}\times 12 + \frac{1}{2}\times 2 + 1\times 3 = 6$.

(c) Calculation of (c/a) ratio for an ideal HCP structure : Let c be the height of the unit cell and a equilateral triangles at the top or base of the hexagonal cell. These three atoms just rest on the three atoms at the corners of the triangles.



Plane view of HCP structure

In the triangle ABY, $\cos 30^\circ = \dfrac{AY}{AB}$.

$$AY = a\cos 30^\circ = \frac{a\sqrt{3}}{2} \text{ and } (AZ)^2 = (AX)^2 + (ZX)^2 \qquad \ldots \text{(i)}$$

In triangle AXZ, $AX = \dfrac{2}{3}AY = \dfrac{2}{3}\left(\dfrac{a\sqrt{3}}{2}\right) = \dfrac{a}{\sqrt{3}}$

Substituting AZ, AX and ZX in equation (i), we get

$$a^2 = \frac{a^2}{3} + \frac{c^2}{4} \Rightarrow \frac{c^2}{4} = a^2 - \frac{a^2}{3} = \frac{2a^2}{3} \Rightarrow \frac{c^2}{a^2} = \frac{8}{3} \Rightarrow \frac{c}{a} = \sqrt{\left(\frac{8}{3}\right)}$$

(d) Volume of the unit cell :
Area of the base = six times the area of the triangle AOB

$$\text{Area of the triangle, } AOB = \frac{1}{2} \times BO \times AY = \frac{1}{2} \times a \times \frac{\sqrt{3}a}{2} = \frac{\sqrt{3}a^2}{4}$$

$$\text{Thus, area of the base} = 6 \times \frac{\sqrt{3}a^2}{4} = \frac{3}{2}\sqrt{3}a^2$$

$$\text{Volume of the cell } = \frac{3}{2}\sqrt{3}a^2c.$$

(e) Calculation of APF for HCP structure :
Coordination number, $N = 12$; Lattice constant, $a = 2r$
Nearest neighbour distance, $2r = a$; Number of atoms per unit cell, $n = 6$

$$\text{Volume of one atom in a unit cell} = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\left(\frac{a}{2}\right)^3 = \frac{\pi a^3}{6}$$

$$\text{Volume of unit cell } = \frac{3}{2}\sqrt{3}a^2c \text{ with } c/a = \left(\frac{8}{3}\right)^{1/2}$$

$$\text{Atomic packing factor, APF} = \left[\frac{n_{eff} \times \text{volume of each atom}}{\text{volume of unit cell}}\right]$$

$$\therefore \quad APF = \frac{6 \times (\pi a^3/6)}{\frac{3}{2}\sqrt{3}\, a^2 c} = \frac{2\pi}{3\sqrt{3}}\left(\frac{a}{c}\right) = \frac{2\pi}{3\sqrt{3}}\left(\frac{3}{8}\right)^{1/2} = \frac{\sqrt{2}\pi}{6} = \frac{\pi}{3\sqrt{2}} = 0.74$$

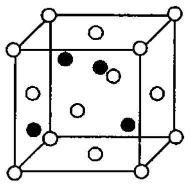
$\therefore \quad$ APF% = 74%

(f) Type of packing: The packing fraction of FCC and HCP are 74% and hence they are known as close-packed structures.

(g) Examples: Magnesium, zinc, cadmium etc.

Diamond Cubic Structure:

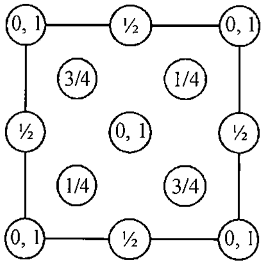
The diamond lattice can be considered to be formed by two interpenetrating FCC lattices along the body diagonal by one-fourth of cube edge. One sublattice has its origin at the point (0, 0, 0) and the other at a point quarter of the way along the body diagonal [i.e. at the point (a/4, a/4, a/4)]. The basic diamond lattice and the atomic positions in the cubic cell of diamond lattice are shown in figure. The fractions denote height about the base in units of cube edge. The point at 0 and 1/2 are on the FCC lattice, those at 1/4 and 3/4 are on a similar lattice displaced along the body diagonal by 1/4th of the cube edge.



○ : Carbon atom occupying DC structure

● : Carbon atom occupying tetrahedral sites

Unit cell of DC structure



Plane view of atomic positions in DC unit cell. Numbers in the circles indicate fractional heights of the carbon atoms.

This is not a Bravais lattice.

(a) Number of nearest neighbours (coordination numbers): Each carbon atom has four nearest neighbours at distance $\sqrt{3}a/4$. Therefore, the co-ordination number is 4.

(b) Number of second nearest neighbours: There are 12 second nearest neighbours at distance $a/\sqrt{2}$.

(c) Atomic radius: From the plane view of DC structure it is clear that $8r=\sqrt{3}a \Rightarrow r=\frac{\sqrt{3}a}{8}$.

(d) Number of atoms per unit cell: $n_{eff}=\frac{1}{8}\times 8+\frac{1}{2}\times 6+4=8$

(e) Atomic packing fraction : $\text{APF}=\left[\frac{n_{eff}\times \text{volume of each atom}}{\text{volume of unit cell}}\right]$

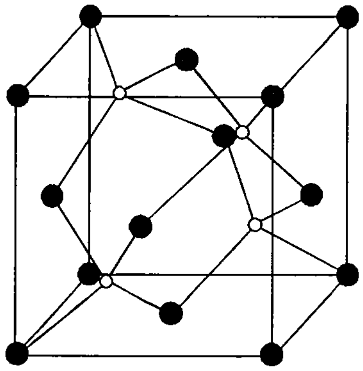
$$\therefore \quad \text{APF}=\frac{8\times\frac{4}{3}\pi r^3}{a^3}=\frac{8\times\frac{4}{3}\pi\left(\frac{\sqrt{3}a}{8}\right)^3}{a^3}=8\times\frac{4}{3}\pi\times\frac{3\sqrt{3}}{8\times8\times8}=\frac{\sqrt{3}\pi}{16}=0.34$$

$$\therefore \quad \text{APF}\%=34\%.$$

(f) Type of packing: Diamond cubic is a loose-packed structure.

(g) Example : Carbon, silicon, germanium etc.

Zinc Blende structure (ZnS) : Zinc blende structure is almost identical to the diamond structure except that the two interpenetrating FCC sublattices are of different atoms and displaced from each other by one-quarter of the body diagonal.



○ Zn-atom occupying tetrahedral site

● S-atoms occupying FCC position

The unit cell of Zinc blende (ZnS) structure

The cubic zinc sulphide structure results when Zn-atoms are placed on one FCC lattice and S-atoms on the other FCC lattice as shown in figure. The conventional cell of this structure is a cube. There are four molecules per conventional cell. For each atom, there are four equally distant atoms of opposite kind arranged at a regular tetrahedron.

The ZnS is not an atomic structure. It is a molecular structure. For this structure, nearest neighbour distance $= r_{\text{Zn}} + r_{\text{S}}$. In a unit cell of ZnS, there will be 4 Zn-atoms and 4 S-atoms.

(a) Atomic radius: $a = \frac{4}{\sqrt{3}}\left(r_{\text{Zn}} + r_{\text{S}}\right)$

(b) Atomic packing fraction: $\text{APF} = \left[\frac{n_{eff} \times \text{volume of each atom}}{\text{volume of unit cell}}\right]$

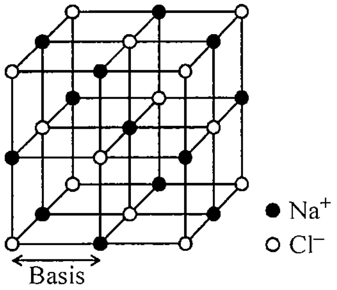
$$\therefore\quad \text{APF} = \frac{4\times\left(\frac{4}{3}\pi r_{\text{Zn}}^3 + \frac{4}{3}\pi r_{\text{S}}^3\right)}{a^3} = \frac{4\times\frac{4}{3}\pi r_{\text{Zn}}^3 + 4\times\frac{4}{3}\pi r_{\text{S}}^3}{\left(\frac{4}{\sqrt{3}}\right)^3\left(r_{\text{Zn}} + r_{\text{S}}\right)^3} = \left\{\frac{r_{\text{S}}^3\left(1 + \frac{r_{\text{Zn}}^3}{r_{\text{S}}^3}\right)}{r_{\text{S}}^3\left(1 + \frac{r_{\text{Zn}}}{r_{\text{S}}}\right)^3}\right\}\times\left(\frac{\sqrt{3}\pi}{4}\right)$$

$$= \frac{\sqrt{3}}{4}\pi\left\{\frac{\left[1 + \left(\frac{r_{\text{Zn}}}{r_{\text{S}}}\right)^3\right]}{\left(1 + \frac{r_{\text{Zn}}}{r_{\text{S}}}\right)^3}\right\}$$

(c) Examples: The compounds which have cubic zinc sulphide structure are CuCl, ZnS, InSb and CdS.

Sodium chloride structure:

In sodium chloride, sodium loses its outer electron and acquires an excess of positive charge while the chlorine atom accepts one electron from sodium and becomes a negative ion. Now, due to the electrostatic forces between the excess charges, the two ions attract each other. Further, due to the strong forces of repulsion as their outer electron shells come into close proximity, the two ions cannot approach each other to within less than certain distance. When attraction and repulsion balance, equilibrium is obtained. The Bravais lattice is a face centred cube, with a basis of one Na^+ ion and one Cl^- ion separated by one-half the body diagonal of the unit cell. There are four molecules of NaCl in a unit cell, with ions in the positions.



Unit cell of sodium chloride structure

$$\text{Na}^+ : \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \left(0, 0, \frac{1}{2}\right), \left(0, \frac{1}{2}, 0\right), \left(\frac{1}{2}, 0, 0\right)$$

$$\text{Cl}^- : (0, 0, 0), \left(\frac{1}{2}, \frac{1}{2}, 0\right), \left(\frac{1}{2}, 0, \frac{1}{2}\right), \left(0, \frac{1}{2}, \frac{1}{2}\right)$$

(a) Number of nearest neighbours (coordination number): Nearest neighbours of Na^+ ions are 6 Cl^- ions. Similarly nearest neighbours of Cl^- ions ions are 6 Na^+ ions. Therefore, the co-ordination number of NaCl structure is 6. The distance of nearest neighbour is $r_{\text{Na}} + r_{\text{Cl}} = \frac{a}{2}$.

(b) Second nearest neighbours: There are 12 second nearest neighbours for each Na^+ and Cl^- ions at $\frac{a}{\sqrt{2}}$ distance.

(c) Number of atoms per unit cell: Total number of Cl^- ions $= 8\times\frac{1}{8}+6\times\frac{1}{2}=4$

Total number of Na^+ ions $=\frac{1}{4}\times 12+1=3+1=4$

This is also a molecular crystal structure.

(d) Atomic packing fraction: $\text{APF}=\left[\frac{n_{eff}\times\text{volume of each atom}}{\text{volume of unit cell}}\right]$

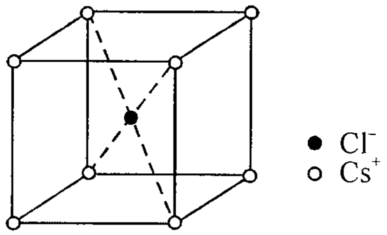
$$\therefore\quad \text{APF}=\frac{4\times\left(\frac{4}{3}\pi r_{\text{Na}}^3+\frac{4}{3}\pi r_{\text{Cl}}^3\right)}{a^3}=\frac{4\times\frac{4}{3}\pi r_{\text{Na}}^3+4\times\frac{4}{3}\pi r_{\text{Cl}}^3}{8\left(r_{\text{Na}}+r_{\text{Cl}}\right)^3}=\frac{2\pi}{3}\frac{\left(r_{\text{Na}}^3+r_{\text{Cl}}^3\right)}{\left(r_{\text{Na}}+r_{\text{Cl}}\right)^3}=\frac{2}{3}\pi\left\{\frac{1+\left(\frac{r_{\text{Cl}}}{r_{\text{Na}}}\right)^3}{\left(1+\frac{r_{\text{Cl}}}{r_{\text{Na}}}\right)^3}\right\}$$

(e) Examples : Representative crystals having NaCl structure are KCl, KBr, MgO, AgBr, etc.

Cesium chloride structure:

The space lattice of cesium chloride (CsCl) structure is simple cubic. The basis has one Cs^+ ion at (0, 0, 0) and one Cl^- ion at $\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$. Each ion is at the centre of a cube of ions of the opposite kind, the coordination number is thus eight. Obviously, the lattice points of CsCl are two interpenetrating simple cubic lattices, the corner of one sublattice is the body centre of the other. One sublattice is occupied by Cs^+ ions and the other by Cl^- ions.

The unit cell of CsCl structure is shown in the figure shown below.



The unit cell of Cesium chloride structure

(a) Number of nearest neighbours: There are 8 nearest neighbour lattice points (atoms) at distance $\left(\frac{\sqrt{3}a}{2}=r_{\text{Cs}}+r_{\text{Cl}}\right)$ in CsCl structure. For Cs^+ ion nearest neighbours are 8 Cl^- ions. Similarly, for Cl^- ion nearest neighbours are 8 Cs^+ ions.

(b) Second nearest neighbours: There are 6 second nearest neighbours at distance a.

(c) Number of atoms per unit cell : There are two ions (one Cs^+ and another Cl^-) per unit cell i.e. one molecule of CsCl.

(d) Atomic packing fraction:

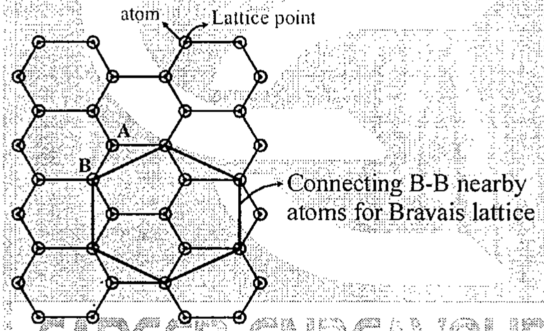
$$\text{APF} = \left[\frac{n_{eff} \times \text{volume of each atom}}{\text{volume of unit cell}}\right] = \frac{1 \bullet \frac{4}{3}\pi\left(r_{Cs}^3 + r_{Cl}^3\right)}{a^3} = \frac{\frac{4}{3}\pi\left(r_{Cs}^3 + r_{Cl}^3\right)}{\left(\frac{2}{\sqrt{3}}\left(r_{Cs} + r_{Cl}\right)\right)^3}$$

$$= \frac{\sqrt{3}}{2}\pi\frac{\left(r_{Cs}^3 + r_{Cl}^3\right)}{\left(r_{Cs} + r_{Cl}\right)^3} = \frac{\sqrt{3}}{2}\pi\frac{\left(1 + \frac{r_{Cl}^3}{r_{Cs}^3}\right)}{\left(1 + \frac{r_{Cl}}{r_{Cs}}\right)^3}$$

(e) Examples: CsCl, RbCl, LiHg etc.

Graphene:

Graphite structure is made up of graphene layers graphene has honeycomb structure i.e., carbon atoms are arranged in the honeycomb structure. In 2-dimensions one C-atom is shared by 3 nearby C-atoms. This is the graphite structure in 2-dimensional. In 3-dimensions, one C-atom is shared by 6 nearby C-atoms.



Graphite structure is layered structure but graphene is a single layered structure (single layer of carbon atom). Graphite is conductor because there is one electron free per carbonatom.

If consider lattice point on each atom position then it will not form a basis because surroundings of A and B are different.

Only A-A atoms or B-B atoms can form basis as they have same surrounding.

So either connect nearby A-A atoms or B-B atoms. This hexagon (red) is the unit cell of graphene.

(1) This unit cell is non-primitive.

(2) **Number of atoms per unit cell:**

Corners atoms $= \frac{1}{3} \times 6 = 2$

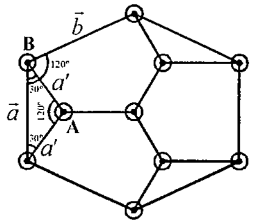
Inside atoms $= 1 \times 4 = 4$

Therefore, total numbers of atoms/unit cell = 6

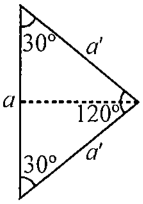
The side of unit cell is lattice parameter a. All the sides of this structure will be same.

Let lattice parameters is a. Thus, $|\vec{a}| = |\vec{b}|$, $\gamma = 120°$

(3) **Nearest neighbour distance:** $2r = a' = \frac{a}{\sqrt{3}}$.



Primitive unit cell



From figure, $\cos 30^\circ = \dfrac{a/2}{a'}$

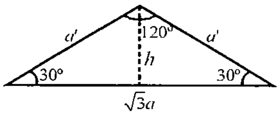
$$\Rightarrow \frac{\sqrt{3}}{2} = \frac{a}{2a'}$$

$$\Rightarrow a' = \frac{a}{\sqrt{3}}$$

(4) Number of nearest neighbours = 3.
In 2-dimensions, there will be close packing if number of nearest neighbours are 6 but here 3 so packing will be loose.

(5) Second nearest neighbour distance = a

(6) Number of second nearest neighbours = 6



(7) Area of triangle $= \dfrac{1}{2} \times \sqrt{3}a \times \dfrac{a}{2} = \dfrac{\sqrt{3}a^2}{4}$

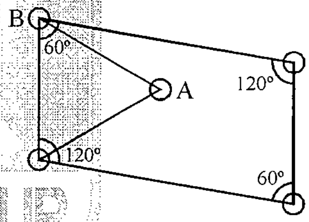
$$h^2 = a^2 - \frac{3}{4}a^2 = \frac{a^2}{4} \Rightarrow h = \frac{a}{2}$$

Area of 2 triangle $= \dfrac{\sqrt{3}a^2}{2}$

Area of unit cell $= \sqrt{3}a^2 + \dfrac{\sqrt{3}a^2}{2} = \sqrt{3}\left(1+\dfrac{1}{2}\right)a^2 = \dfrac{3\sqrt{3}a^2}{2}$

(8) Area of primitive unit cell $= \dfrac{1}{3} \times$ area of unit cell $= \dfrac{1}{3}\left(\dfrac{3\sqrt{3}}{2}\right)a^2 = \dfrac{\sqrt{3}a^2}{2}$

(9) Number of atoms per primitive unit cell = 1 + 1 = 2
Therefore, 2 atoms of graphene inside the primitive cell.
Thus, graphene has 2 atom basis (i.e., one atom of A type and one of B-type)



(10) **Number of atoms per unit area:**

Number of atoms $= \left(\dfrac{1}{6}+\dfrac{1}{3}+\dfrac{1}{3}+\dfrac{1}{6}\right) = 1$

Number of atoms per unit area $= \dfrac{6}{\left(\dfrac{3\sqrt{3}a^2}{2}\right)} = \dfrac{4}{\sqrt{3}a^2}$

(11) **Atomic radius :** $r = \dfrac{a}{2\sqrt{3}}$

(12) **Type of packing:** loose

(13) **Example :** Graphene

(14) **Atomic Packing Fraction (APF):**

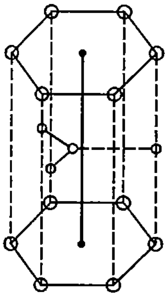
$$\text{APF} = \frac{\text{area of the atoms in unit cell}}{\text{area of unit cell}} = \frac{6 \times \pi r^2}{\dfrac{3\sqrt{3}a^2}{2}} = \frac{12\pi r^2}{3\sqrt{3} \times 4 \times 3 \times r^2} = \frac{\pi}{3\sqrt{3}} = 0.604.$$

$$\left\{a = 2\sqrt{3}r \text{ and area of the atoms} = \pi r^2\right\}$$

Therefore, APF $\simeq$ 60 %

Graphite : Graphite is a form of carbon and crystalises in simple hexagonal structure. It contains 4 atoms per primitive unit cell.

Number of atom per unit cell: Number of atoms per unit cell $= \frac{1}{6}\times 12 + \frac{1}{3}\times 3 = 3 + 1 = 4$



Height of the unit cell = 4.8 times the nearest neighbours distance

Nearest neighbour distance implies $2r = a$, a is lattice parameter

Number of nearest neighbour = 3.

One C-atom can make covalent bond with 3 nearby atoms. C-atoms of one layer can not make bond with C-atom of second layer. That's why graphite sheets slides on each other. Therefore, graphite is soft because layer do not have bonding.

Since, semiconductor are not conducting at $T = 0$ K and metals are conducting at $T = 0$ K

Graphite is a semi-metal.

Relation between lattice constant and density:

Let the lattice parameter of the unit cell is a. Therefore, the volume of the unit cell is a^3.

If ρ be the density of the unit cell then the mass of each unit cell $= a^3\rho$ ...(1)

If M is the molecular weight and N is the Avogadro number.

$\therefore$ Mass of each molecule $= \frac{M}{N}$

As if n is the number of molecules per unit cell, then the mass in each unit cell $= n\times\frac{M}{N}$ ... (2)

From equations (1) and (2), we get

$$a^3\rho = n\frac{M}{N} \quad\Rightarrow\quad a = \left(\frac{nM}{N\rho}\right)^{1/3}$$

Atomic density: The number of atoms per unit area of a lattice plane is called as atomic density.

Atomic density $(\rho) = \frac{nd}{V}$, where V = volume of unit cell and n is the number of atoms in it and d is the interplanar spacing.

$$\therefore \quad \rho = \frac{nd}{abc}$$

If $n = 1 \Rightarrow \rho = \frac{d}{abc}$

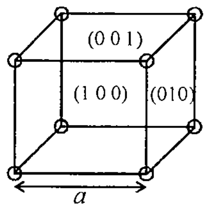
Planes in different crystals:

1. **Simple cubic structure :**
   Plane analysis in semiconductor
   First consider (1 0 0) plane
   In planes (0 1 0), (0 0 1), (1 0 0), the properties are same.

Number of atoms on (1 0 0) type plane $= \frac{1}{4} \times 4 = 1$

Area of plane $= a^2$

Therefore, atomic density = number of atoms per unit area $= \frac{1}{a^2}$

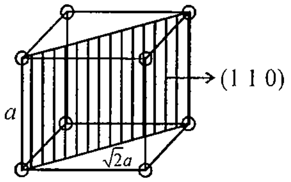


For a plane, consider atoms as disc (in two dimension)

(1 1 0) plane :

Number of atoms $= \frac{1}{4} \times 4 = 1$.

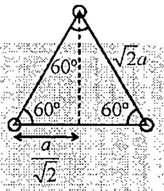
Area $= \sqrt{2}a^2$ (Since, shape is rectangular)

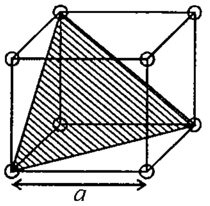


Therefore, atomic density $= \frac{1}{\sqrt{2}a^2}$

(1 1 1) plane :

Number of atoms $= \frac{1}{6} \times 3 = \frac{1}{2}$





Area $= \frac{1}{2} \times \text{base} \times \text{height}$ (Since, shape is triangular)

$$= \frac{1}{2}\left(\sqrt{2}a\right)\sqrt{\frac{3}{2}}\,a = \frac{\sqrt{3}\,a^2}{2}$$

$$h^2 = 2a^2 - \frac{a^2}{2}$$

$$h^2 = \frac{3a^2}{2} \Rightarrow h = \sqrt{\frac{3}{2}}\,a$$

Therefore, atomic density $= \frac{1}{\sqrt{3}a^2}$

Example: Which plane in simple cubic is closed pack plane ?

In. Plane having (1 0 0) largest atomic density will be closed packed plane.

Example: Which direction is closest pack in simple cubic lattice ?

In. In one dimensions, atom is considered as a line
In 2 dimensions, atom is considered as a disc
In three dimensions, atom is considered as a sphere
Directions having largest line density will be closest pack

Simple cubic structure:

Among planes (1 0 0), (1 1 0), (1 1 1), $(1\,\bar{1}\,0)$, line density of (1 0 0) is largest.
Therefore, (1 0 0) is the closest pack direction.

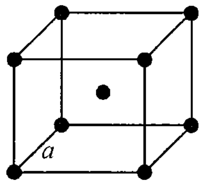
[In distance a there are 1 atom $\left(\frac{1}{2} + \frac{1}{2} = 1\right)$]. (i.e., line density $= 1/a$)

Body Centred Cubic Structure (BCC):

(1 0 0) plane :

Number of atoms = 1 (no new atom on faces)
Area $= a^2$

Plannar density $= \frac{1}{a^2}$

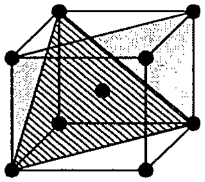


Line density $= \frac{1}{a}$

(1 1 0) plane :

Number of atoms = 2

Area $= \sqrt{2}a^2$



Planner density $= \frac{\sqrt{2}}{a^2}$

Line density $= \frac{1}{\sqrt{2}}a$

(1 1 1) plane :

Number of atoms $= \frac{1}{2}$

(Body centered atom will not contribute to (1 1 1) plane)

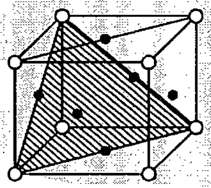
Area $= \frac{\sqrt{3}a^2}{2}$; Planner density $= \frac{1}{\sqrt{3}a^2}$; Line density $= \frac{2}{\sqrt{3}a}$

For BCC closest pack plane is (1 1 0).

Face Centred Cubic Structure (FCC):

(1 0 0) plane :

Number of atoms $= \frac{1}{4} \times \overset{\text{corners}}{4} + \overset{\text{face}}{1} = 2$



Planner density $= \frac{2}{a^2}$

Line density $= \frac{1}{a}$

(1 1 0) plane :

Number of atoms $= \frac{1}{4} \times 4 + \frac{1}{2} \times 2 = 2$

Planner density $= \frac{\sqrt{2}}{a^2}$

Line density $= \frac{\sqrt{2}}{a}$

(1 1 1) plane :

Number of atoms $= \frac{1}{6} \times 3 + \frac{1}{2} \times 3 = 2$

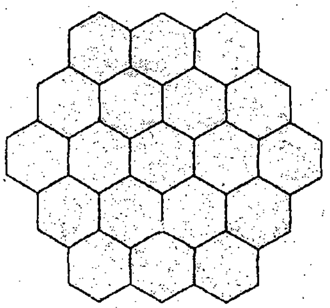
Planner density $= \frac{4}{\sqrt{3}a^2}$

Line density $= \frac{1}{\sqrt{3}a}$

For FCC closest pack plane is (1 1 1).

SOLVED PROBLEMS

In a single layer of graphite, called graphene, the carbon atoms from a hexagonal lattice (see figure). How many carbon atoms are there in a unit cell of the lattice? **[JNU 2010]**



(a) one (b) two (c) three (d) six

In. Each atom is shared by three unit cells.

Therefore, number of atoms/unit cell$=\frac{1}{3}\times 6 = 2$

Correct option is (b)

Helium atoms at low temperatures make a perfect closed pack structure of hexagonal lattice with parameters $a = 0.36$ nm and $c = 0.59$ nm. The density of the crystal is approximately **[JNU 2012]**

(a) 2000 kg/m³ (b) 100 kg/m³ (c) 123 kg/m³ (d) 200 kg/m³

In. In a conventional cell of hexagonal structure number of atoms present (N) = 6.

Volume of conventional unit cell (V) $= 6\frac{\sqrt{3}}{4}a^2\cdot c$

Number density $(n) = \frac{\text{N}}{\text{V}} = \frac{6}{\text{V}} = \frac{6}{6\times\frac{\sqrt{3}}{4}a^2.c} = \frac{4}{\sqrt{3}\,a^2c}$

Mass density = Mass of one helium atom × number density

$$= 4\times 1.67\times 10^{-27}\times\frac{4\times 10^{27}}{\sqrt{3}\times(0.36)^2\times 0.59} = 200.88 \text{ kg/m}^3$$

Correct option is (d)

Consider a regular arrangement of identical spheres in a face-centred cubic (fcc) structure in which the centres of the respective spheres are located at each of the eight corners and the centres of the six surfaces of a unit cube. The fraction of each cubic unit cell occupied by the spheres in the close-pack configuration is **[JNU 2013]**

(a) 0.50 (b) 0.62 (c) 0.74 (d) 0.88

In. $\text{APF} = \frac{\text{Number of atom}\times\text{volume of sphere}}{\text{volume}} = \frac{4\times\frac{4}{3}\pi r^3}{a^3}$

$$= \frac{4\times\frac{4}{3}\pi\left(\frac{a}{2\sqrt{2}}\right)^3}{a} = 0.74$$

Correct answer is (c)

4. The density of copper, which has a face-centred cubic (fcc) lattice structure, is 8.9 gm/cm^3. Given that the molar mass of copper is 63.55, the length of a side of the cubic cell is **[JNU 2015]**

(a) 0.45 nm (b) 0.57 nm (c) 0.29 nm (d) 0.23 nm (e) 0.36 nm

Soln. We know that,

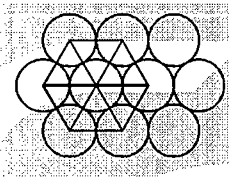
$$\rho=\frac{n\times M}{Na^3}\Rightarrow a=\left(\frac{nM}{N\times\rho}\right)^{1/3}=\left(\frac{4\times 63.55}{6.023\times10^{-26}\times 8900}\right)^{1/3}=0.36\ nm \quad \left[\because \rho=\frac{8.9\times10^{-3}}{\left(10^{-2}\right)^3}=8900\ kg/m^3\right]$$

Correct answer is (e)

5. A flat surface is covered with non-overlapping disks of same size. What is the largest fraction of the area that can be covered ?

(a) $\frac{3}{\pi}$ (b) $\frac{5\pi}{6}$ (c) $\frac{6}{7}$ (d) $\frac{\pi}{2\sqrt{3}}$ **[JEST-2013]**

Soln. Let the radius of the circular disk is a. Then, area of the circular disk $=\pi r^2$



Here, centers of three disks make an equilateral triangle with area $=\frac{\sqrt{3}a^2}{4}=\frac{\sqrt{3}(2r)^2}{4}$.

Therefore, packing fraction of the triangular lattice $=\left(\frac{\frac{3}{6}\times\pi r^2}{\frac{\sqrt{3}a^2}{4}}\right)=\left(\frac{\frac{3}{6}\times\pi r^2}{\frac{\sqrt{3}\times(2r)^2}{4}}\right)=\frac{\pi}{2\sqrt{3}}$.

Therefore, the fractional area of the flat surface that can be covered by these circular discs $=\frac{\pi}{2\sqrt{3}}$.

Hence, correct option is (d).

6. A metal suffers a structural phase transition from face-centered cubic (FCC) to the simple cubic (SC) structure. It is observed that this phase transition does not involve any change of volume. The nearest neighbour distances d_{fcc} and d_{sc} for the FCC and the SC structures respectively are in the ratio (d_{fcc}/d_{sc}) [Given: $2^{1/3}$ = 1.26]

(a) 1.029 (b) 1.122 (c) 1.374 (d) 1.130 **[JEST-2013]**

Soln. For FCC crystal strucutre, the nearest neighbour distance, $d_{fcc}=\frac{\sqrt{2}}{2}a_{fcc}$ ... (i)

For SC crystal strucutre, the nearest neighbour distance, $d_{sc}=a_{sc}$ ... (ii)

Since, volume does not change in this phase transformation from FCC to SC structure.
Therefore, density of FCC structure = density of SC structure

i.e., $\frac{4}{a_{fcc}^3}=\frac{1}{a_{sc}^3}$

$$\Rightarrow \frac{a_{fcc}^3}{a_{sc}^3}=4 \Rightarrow \frac{a_{fcc}}{a_{sc}}=(4)^{1/3}=(2)^{2/3} \quad \text{... (iii)}$$

Now, from equations (i) and (ii), we get

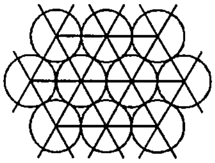
$$\frac{d_{fcc}}{d_{sc}} = \frac{\left(\frac{\sqrt{2}}{2}a_{fcc}\right)}{a_{sc}} = \frac{1}{\sqrt{2}}\times\frac{a_{fcc}}{a_{sc}} = \frac{1}{\sqrt{2}}\times(2)^{2/3} = \frac{(2)^{2/3}}{(2)^{1/2}} = (2)^{\frac{2}{3}-\frac{1}{2}} = (2)^{\frac{1}{6}} = (2^{\frac{1}{3}})^{\frac{1}{2}} = (1.26)^{\frac{1}{2}} = 1.122.$$

Hence, correct option is (b).

Circular discs of radius 1 m each are placed on a plane so as to form a closely packed triangular lattice. The number of discs per unit area is approximately equal to

(a) 0.86 m^{-2} (b) 0.43 m^{-2} (c) 0.29 m^{-2} (d) 0.14 m^{-2} **[JEST-2014]**

ln. Given, radius of circular disc $r = 1$ m



The circular discs will form a equilateral triangular lattice of side $a = 2r = 2\times 1\text{ m} = 2\text{ m}$, where r is the radius of the circular discs.

Now, area of the triangular lattices $= \frac{\sqrt{3}}{4}a^2 = \frac{\sqrt{3}}{4}\times(2r)^2 = \sqrt{3}r^2 = \sqrt{3}(1)^2 = \sqrt{3}\text{ m}^2$.

Number of circular discs inside the triangular lattice $= \frac{1}{6}\times 3 = 0.5$.

Therefore, number of discs/area $= \frac{0.5}{\sqrt{3}} = \frac{0.5}{\sqrt{3}}\text{m}^{-2} = 0.28867 = 0.29\text{ m}^{-2}$

Hence, correct option is (c).

The total number of Na^+ and Cl^- ions per unit cell of NaCl is, **[JEST 2015]**

(a) 2 (b) 4 (c) 6 (d) 8

ln. The total number of Na^+ and Cl^- ions per unit cell of NaCl is 8.

Correct option is (d)

The number of different Bravais lattices possible in two dimension is

(a) 2 (b) 3 (c) 5 (d) 6 **[JEST-2016]**

ln. The number of different Bravais lattices possible in two dimension is 5. The lattices are oblique $(a \neq b, \gamma \neq 90°)$, square $(a = b, \gamma = 90°)$, primitive rectangular $(a \neq b, \gamma = 90°)$, non-primitive rectangular $(a \neq b, \gamma = 90°)$ and hexagonal $(a = b, \gamma = 120°)$.

Hence, correct option is (c).

In a simple cubic lattice of lattice constant 0.287 nm, the number of atoms per mm^2 along the 111 plane is

(a) 2.11×10^{13} (b) 1.73×10^{13} (c) 1.29×10^{13} (d) 1.21×10^{13} **[TIFR 2016]**

ln. **Given:** Lattice parameter, $a = 0.287\text{ nm} = 2.87\times 10^{-10}$ m

Number of atoms on (111) plane, $n = 3\times\frac{1}{6} = 0.5$

Area of the (111) place in simple cubic lattice

$$= \frac{\sqrt{3}\left(\sqrt{2}a\right)^2}{4} = \frac{\sqrt{3}\times 2a^2}{4} = \frac{2\times 1.732\times\left(2.87\times 10^{-10}\right)^2}{4} = 7.133\times 10^{-20}\,m^2 = 7.133\times 10^{-14}\,mm^2$$

Therefore, number of atoms per mm^2

$$= \frac{\text{number of atoms on } (111) \text{ plane}}{\text{area of the } (111) \text{ plane}} = \frac{0.5}{7.133\times10^{-14}} = \frac{0.5}{7.133}\times10^{14} = 0.7\times10^{13} \text{ atoms/mm}^2$$

No option is matched.

11. If the molecular weight and density of rock salt (NaCl) with fcc structure are 60 and 2160 kg/m³, respectively, the lattice constant in Å is about: **[DU 2014]**

(a) 6.78 (b) 5.69 (c) 4.51 (d) 3.10

Soln. Molecular weight $(M) = 60$, density $(\rho) = 2160 \text{ kg/m}^3$

Now, $a^3 = \frac{nM}{N\rho} = \frac{4\times60}{6.023\times10^{26}\times2160} = 0.01845\times10^{-26}\text{ m}^3 = 184.5\times10^{-30}\text{ m}^3$

$\Rightarrow a = 5.69\times10^{-10}\text{ m}$

$\Rightarrow a = 5.69\text{ Å}$

Hence, correct option is (b).

12. Packing fraction of an fcc lattice is **[BHU 2012]**

(a) 0.52 (b) 0.32 (c) 0.74 (d) 0.62

Soln. Packing fraction of an fcc lattice is 0.74

Hence, correct option is (c).

13. In a simple cubic lattice $a = 2$ Å, then planar atomic density of (1, 0, 0) plane is **[BHU 2012]**

(a) 10^{15} atoms/cm² (b) 2.5×10^{15} atoms/cm²

(c) 5×10^{15} atoms/cm² (d) 7.5×10^{15} atoms/cm²

Soln. $a = 2$ Å, the planar atomic density of (100) plane is d_{100}/a^3

$$= \frac{a}{a^3} = \frac{1}{a^2} = \frac{1}{(2\times10^{-10})^2}\text{ atoms/m}^2$$

$$= \frac{1}{4}\times10^{20}\text{ atoms/m}^2 = \frac{0.25\times10^{20}}{10^4}\text{ atoms/cm}^2 = 2.5\times10^{15}\text{ atoms/cm}^2.$$

Hence, correct option is (b).

14. The crystal structure of diamond is **[BHU 2012]**

(a) fcc with two atom basis of (0 0 0) and $\frac{a}{4}\left(\hat{i}+\hat{j}+\hat{k}\right)$

(b) simple cubic with two atom basis of (0 0 0) and $\frac{a}{2}\left(\hat{i}+\hat{j}+\hat{k}\right)$

(c) fcc with two atom basis of (0 0 0) and $\frac{a}{2}\left(\hat{i}+\hat{j}+\hat{k}\right)$

(d) bcc with one atom basis

Soln. The crystal structure diamond is fcc with two atom basis of (0 0 0) and $\frac{a}{4}\left(\hat{i}+\hat{j}+\hat{k}\right)$

Hence, correct option is (a).

. For an FCC lattice the ratio of $d_{200} : d_{220} : d_{222}$ is **[BHU 2013]**

(a) $\sqrt{3}:\sqrt{6}:\sqrt{2}$ (b) $1:\sqrt{2}:\sqrt{6}$ (c) $1:2:3$ (d) $\sqrt{6}:\sqrt{3}:\sqrt{2}$

ln. $d_{200} : d_{220} : d_{222} = \dfrac{a}{\sqrt{2^2+0^2+0^2}} : \dfrac{a}{\sqrt{2^2+2^2+0^2}} : \dfrac{a}{\sqrt{2^2+2^2+2^2}}$

$$= \frac{a}{\sqrt{4}} : \frac{a}{\sqrt{8}} : \frac{a}{\sqrt{12}} = \frac{a}{2} : \frac{a}{2\sqrt{2}} : \frac{a}{2\sqrt{3}} = \sqrt{6}:\sqrt{3}:\sqrt{2}.$$

Hence, correct option is (d).

. Magnesium has h.c.p. structure. The radius of magnesium atom is 0.1605 nm. The volume of unit cell of magnesium is **[BHU 2013]**

(a) $0.7\times10^{-28}\ \text{m}^3$ (b) $2.8\times10^{-28}\ \text{m}^3$ (c) $1.4\times10^{-28}\ \text{m}^3$ (d) $0.35\times10^{-28}\ \text{m}^3$

ln. $r = 0.1605$ m, For fcc, $a = 2r = 2\times0.1605\text{m} = 0.321$ nm and $\dfrac{c}{a} = \sqrt{\dfrac{8}{3}} \Rightarrow c = a\times\sqrt{\dfrac{8}{3}} = 0.5242$ nm.

Thus, volume of the unit cell $V = \dfrac{6\times\sqrt{3}a^2\times c}{4} = \dfrac{6\times\sqrt{3}\times(0.321)^2\times(0.5242)}{4}\times10^{-27}\ \text{m}^3$

$$= 0.14\times10^{-27}\ \text{m}^3 = 1.4\times10^{-28}\ \text{m}^3.$$

Hence, correct option is (c).

. The spacing d_{hkl} of the planes (hkl) in a tetragonal crystal is **[BHU 2013]**

(a) $\left[\dfrac{h^2}{a^2}+\dfrac{k^2}{b^2}+\dfrac{l^2}{c^2}\right]^{-1/2}$ (b) $\left[\dfrac{h^2+k^2}{a^2}+\dfrac{l^2}{c^2}\right]^{-1/2}$

(c) $\left[\dfrac{4}{3}\left(\dfrac{h^2+hk+k^2}{a^2}\right)+\dfrac{l^2}{c^2}\right]^{-1/2}$ (d) $a\left[h^2+k^2+l^2\right]^{-1/2}$

ɔln. For tetragonal crystal, $a = b \neq c,\ \alpha = \beta = \gamma = 90°$

The spacing d_{hkl} of the planes (hkl) in a tetragonal crystal is given by

$$d_{hkl} = \left[\frac{h^2+k^2}{a^2}+\frac{l^2}{c^2}\right]^{-1/2}$$

Hence, correct option is (b).

3. Diamond has the following crystal structure **[BHU 2013]**

(a) hexagonal (b) simple cubic

(c) face-centered cubic (d) body-cented cubic

ɔln. Diamond has fcc crystal structure.

Hence, correct option is (c).

). A crystal system whose unit cell is specified by $a \neq b \neq c,\ \alpha = \gamma = 90 \neq \beta$ is known as **[BHU 2013]**

(a) monoclinic (b) rhombohedral (c) tetragonal (d) orthorhombic

ɔln. The crystal system whose unit cell is specified by $a \neq b \neq c,\ \alpha = \gamma = 90° \neq \beta$ is known as monoclinic crystal.

Hence, correct option is (a).

20. For hexagonal close-packed structure, the ratio of lattice parameters a and c, i.e., c/a is given by

(a) $\frac{1}{2}\left(\frac{8}{3}\right)^{1/2}$ (b) $\left(\frac{3}{8}\right)^{1/2}$ (c) $\left(\frac{8}{3}\right)^{1/2}$ (d) $\frac{1}{2}\left(\frac{3}{8}\right)^{1/2}$ **[BHU 2013]**

Soln. For HCP structure, the ratio of a and c is c/a is given by $\frac{c}{a}=\sqrt{\frac{8}{3}}$.

Hence, correct option is (c).

21. The number of ions in the unit cell of CsCl crystal is **[BHU 2013]**

(a) 1 (b) 2 (c) 3 (d) 4

Soln. The total number of ions in the unit cell of CsCl crystal is 2.
Hence, correct option is (b).

22. The Miller indices of the plane parallel to the x and y axes are **[BHU 2013]**

(a) (100) (b) (111) (c) (001) (d) (010)

Soln. Let the miller indices are (hkl). Thus, $h:k:l=\frac{a}{OA}:\frac{b}{OB}:\frac{c}{OC}=\frac{a}{\infty}:\frac{b}{\infty}:\frac{c}{c}$, $h:k:l=0:0:1$

Therefore, miller indices are (001).
Hence, correct option is (c).

23. The number of lattice points in a primitive cell is **[BHU 2013]**
(a) 2 (b) 3 (c) 1 (d) 4

Soln. The number of lattice points in a primitive cell is 1.
Hence, correct option is (c).

24. If n is the number of atoms in the unit cell of the cubic system, N and M are the Avogadro's number and atomic weight respectively and ρ is the density of the element, then the lattice constant a is given by

(a) $\left[\frac{M\rho}{nN}\right]^{1/3}$ (b) $\left[\frac{nM}{N\rho}\right]^{1/3}$ (c) $\left[\frac{nN}{M\rho}\right]^{1/3}$ (d) $\left[\frac{\rho N}{Mn}\right]^{1/3}$ **[BHU 2013]**

Soln. For unit cell of cubic system, the density is given by $\rho=\frac{nM}{Na^3}\Rightarrow a^3=\frac{nM}{N\rho}\Rightarrow a=\left[\frac{nM}{N\rho}\right]^{1/3}$.

Hence, correct option is (b).

25. ABABA...... represents an arrangement of layers called : **[BHU 2014]**
(a) hexagonal closed packing (b) cubic closed packing
(c) body centered cubic packing (d) flourite close packing

Soln. ABABA...... represents an arrangement of layers called hexagonal closed packing.
Hence, correct option is (a).

26. In diamond the coordination number of carbon is : **[BHU 2014]**
(a) 4 and its unit cell has 8 carbon atoms (b) 4 and its unit cell has 6 carbon atoms
(c) 6 and its unit cell has 4 carbon atoms (d) 4 and its unit cell has 4 carbon atoms

Soln. Coordination number of carbon in diamond cubic lattice is 4 and its unit cell has 8 carbon atoms.
Hence, correct option is (a).

. A compound formed by elements A and B crystallizes in cubic structure, in which atoms of A are at the corners while that of B are at the face center. The formula of the compound is **[BHU 2014]**

(a) AB_3 (b) AB (c) AB_6 (d) A_2B

ln. If atoms of element A lies at the corners and atoms of elements B are at the corners, then there will be total one atom of element A and three atoms of element B per unit cell. Thus, the formula of compound formed by elements A and B will be AB_3.

Hence, correct option is (a).

. For a given cubic crystal lattice parameter a is 3.18 Å. The d spacing for a (III) plane is :

(a) 2.25 Å (b) 1.84 Å (c) 3.18 Å (d) 3.90 Å **[BHU 2014]**

ln. Since, $d_{hkl} = \dfrac{a}{\sqrt{(h^2+k^2+l^2)}} \Rightarrow d_{111} = \dfrac{3.18\text{ Å}}{\sqrt{1^2+1^2+1^2}} = \dfrac{3.18}{\sqrt{3}}\text{ Å} = 1.84\text{ Å}$

Hence, correct option is (b).

. For a simple cubic lattice, the ratio of $d_{100} : d_{110} : d_{111}$ is **[BHU 2015]**

(a) $\sqrt{2}:\sqrt{3}:\sqrt{6}$ (b) $\sqrt{6}:\sqrt{3}:\sqrt{2}$ (c) $\sqrt{2}:\sqrt{6}:\sqrt{3}$ (d) $\sqrt{3}:\sqrt{2}:\sqrt{6}$

ln.

$$d_{100} : d_{110} : d_{111} = \frac{a}{\sqrt{1^2+0^2+0^2}} : \frac{a}{\sqrt{1^2+1^2+0^2}} : \frac{a}{\sqrt{1^2+1^2+1^2}}$$

$$= \frac{a}{\sqrt{1}} : \frac{a}{\sqrt{2}} : \frac{a}{\sqrt{3}} = \sqrt{6}:\sqrt{3}:\sqrt{2}.$$

Hence, correct option is (b).

. Given that the cube edge of diamond is 0.356 nm. The number of atoms per metre cube is

(a) 0.88×10^{29} (b) 5.31×10^{29} (c) 2.77×10^{29} (d) 1.77×10^{29} **[BHU 2015]**

ln. Cube edge of diamond (a) = 0.356 nm.

The number atoms per metre cube $= \dfrac{n}{a^3}$, where n is number of atoms per unit cell.

$$= \frac{8}{(0.356\times10^{-9})^3} = 177.3\times10^{27} = 1.77\times10^{29}\text{ atoms/m}^3.$$

Hence, correct option is (d).

. Sodium chloride has the following crystal structure **[BHU 2015]**

(a) hexagonal (b) face-centered cubic

(c) simple cubic (d) body-centered cubic

ln. Sodium chloride (NaCl) has the face-centered cubic crystal structure.

Hence, correct option is (b).

. The spacing d_{hkl} of the planes (hkl) in a hexagonal crystal is **[BHU 2015]**

(a) $\left[\dfrac{4}{3}\left(\dfrac{h^2+hk+k^2}{a^2}\right)+\dfrac{l^2}{c^2}\right]^{-1/2}$ (b) $a\left[h^2+k^2+l^2\right]^{-1/2}$

(c) $\left[\dfrac{h^2}{a^2}+\dfrac{k^2}{b^2}+\dfrac{l^2}{c^2}\right]^{-1/2}$ (d) $\left[\dfrac{h^2+k^2}{a^2}+\dfrac{l^2}{c^2}\right]^{-1/2}$

ln. For (hkl) planes, the lattice spacing (d_{hkl}) is given by

$$d_{hkl}=\left[\frac{4}{3}\left(\frac{h^2+hk+k^2}{a^2}\right)+\frac{l^2}{c^2}\right]^{-1/2}$$

Hence, correct option is (a).

33. A crystal system whose unit cell is specified by $a=b\neq c,\ \alpha=\beta=\gamma=90$ is known as **[BHU 2015]**

(a) simple cubic (b) hexagonal (c) monoclinic (d) tetragonal

Soln. The lattice parameters a, b, c and α, β, γ given by $a=b\neq c,\ \alpha=\beta=\gamma=90°$ are characterizing the tetragonal lattice.

Hence, correct option is (d).

34. The Miller indices of a set of parallel planes which makes intercepts in the ratio of $1:\frac{1}{2}:\frac{1}{4}$ is given by

(a) (421) (b) (124) (c) (122) (d) (142) **[BHU 2015]**

Soln. Intercepts $\text{OA}:\text{OB}:\text{OC}=1:\frac{1}{2}:\frac{1}{4}$

$$\therefore\ h:k:l=\frac{1}{\text{OA}}:\frac{1}{\text{OB}}:\frac{1}{\text{OC}}=\frac{1}{1}:\frac{1}{(1/2)}:\frac{1}{(1/4)}=1:2:4$$

$\therefore\ (hkl)=(124).$

Hence, correct option is (b).

35. The number of ions in the unit cell of NaCl crystal is **[BHU 2015]**

(a) 1 (b) 2 (c) 4 (d) 8

Soln. The total number of ions in the unit cell of NaCl crystal is 8.

Hence, correct option is (d).

36. The number of lattice points in a BCC unit cell is **[BHU 2015]**

(a) 1 (b) 2 (c) 3 (d) 4

Soln. The number of lattice points in a BCC unit cell is 2.

Hence, correct option is (b).

PRACTICE SET

(Multiple Choice Questions (MCQ)):

The coordination number of a simple cubic (sc) structure is

(a) 4 (b) 6 (c) 8 (d) 12

The coordination number of a bcc structure is

(a) 4 (b) 6 (c) 8 (d) 12

The coordination number of a fcc structure is

(a) 4 (b) 6 (c) 8 (d) 12

The coordination number of a hcp structure is

(a) 3 (b) 6 (c) 9 (d) 12

The number of two dimensional lattices are

(a) 3 (b) 5 (c) 7 (d) 9

The number of macroscopic symmetry elements in a cube is

(a) 13 (b) 31 (c) 23 (d) 32

The number of identical sphere arranged in simple cubic form of side a is

(a) $\frac{a}{2}$ (b) $\frac{a}{2\sqrt{2}}$ (c) $\frac{\sqrt{3}a}{2}$ (d) $\frac{\sqrt{3}a}{4}$

The number of molecules per unit cell of NaCl structure is

(a) 1 (b) 2 (c) 4 (d) 8

The radius of identical spheres arranged in bcc form of side a is

(a) $\frac{a}{2}$ (b) $\frac{a}{2\sqrt{2}}$ (c) $\frac{\sqrt{3}a}{2}$ (d) $\frac{\sqrt{3}a}{4}$

The radius of identical spheres arranged in fcc form of side a is

(a) $\frac{a}{2}$ (b) $\frac{a}{2\sqrt{2}}$ (c) $\frac{\sqrt{3}a}{2}$ (d) $\frac{\sqrt{3}a}{4}$

The pure rotational symmetry which is not possible in crystalline solid is

(a) 2-fold (b) 3-fold (c) 4-fold (d) 5-fold

If 0.28 nm is the interionic distance in NaCl crystal, the lattice parameter is

(a) 0.14 nm (b) 0.56 nm (c) 0.08 nm (d) 0.28 nm

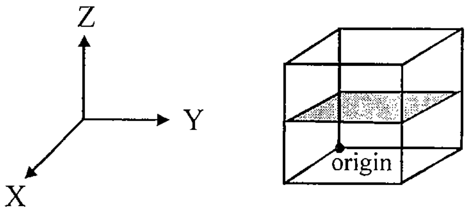
The maximum radius of the interstitial sphere that can just fit into the void between the body centred atom of bcc structure is

(a) $r\left[\left(2/\sqrt{3}\right)-1\right]$ (b) $r\left[\left(\sqrt{3}/2\right)-1\right]$ (c) $r\left[\sqrt{3}-1\right]$ (d) $r\left[\sqrt{2}-1\right]$

Magnesium crystallizes in hcp structure. If the lattice constant is 0.32 nm, the nearest neighbour distance in magnesium is

(a) 0.32 nm (b) 0.16 nm (c) 0.64 nm (d) none of these

15. The miller indices of the shaded plane



are

(a) (001) (b) (002) (c) $\left(00\frac{1}{2}\right)$ (d) none

(Multiple Select Questions (MSQ)):

1. Indicate the ***INCORRECT*** statement(s) about the simple cubic structure
   (a) Atoms/lattice points per unit cell is one.
   (b) If r is the radius of atom, the volume of unit cell is equal to $8r^3$.
   (c) The packing fraction of simple cubic structure is 68%.
   (d) In a simple cubic lattice there is one atom at each centre and one at body centre.
2. For closed-packed structure, the correct statement(s) is/are
   (a) the packing fraction should be less than 74%.
   (b) the packing fraction should be larger than or equal to 74%.
   (c) the coordination number should be 12.
   (d) the coordination number should be 8.
3. For NaCl structure, the correct statement(s) is/are
   (a) each ion is surrounded by six nearest neighbours of the opposite kind.
   (b) the coordination number is six.
   (c) representative crystals having NaCl structure are KCl, KBr, MgO, AgBr.
   (d) There are four molecules of NaCl in one unit cell.

4. Indicate the ***CORRECT*** statement(s) about the diamond cubic structure
   (a) the space lattice of dimaond is fcc with a basis of two carbon atoms assocaited with each lattice point.
   (b) The number of atoms per unit cell is 4.
   (c) the packing fraction is 34 %
   (d) the lattice parameter (a) and radius (r) are related as $a = \frac{8r}{\sqrt{3}}$
5. Indicate the ***INCORRECT*** statement(s) about the hexagonal close packed structure
   (a) the ratio of height(c) to lattice parameter (a) is $\frac{c}{a} = \sqrt{\frac{8}{3}}$
   (b) the total number of atoms in a unit cell is 3.
   (c) the volume of unit cell is equal to $\frac{3\times\sqrt{3}\,a^2c}{6}$
   (d) The coordination number is 8.

(Numerical Answer Type (NAT)):

Lead is a fcc centred cubic with an atomic radius of 0.1746 nm. The spacing of (2 2 0) planes (in units of nm) is ________________

If the lattice constant of bcc lattice structure is 0.32 nm. The nearest neighbour distance (in nm) in the structure is ________________

A metal crystallizes into two cubic phases fcc and bcc, whose unit cell parameters are 3.5Å and 3.0Å, respectively. The ratio of their densities is ______________

Sodium has bcc structure whose atomic weight is 23. Its density is 0.96×10^3 kg/m^3. The lattice parameter (in Å) is ___________

Silver crystallizes into fcc structure, where the distance between the nearest neighbour silver atoms is 2.87Å. The atomic weight of silver is 107.88. The density (in units of kg/m^3) is ______________$\times10^3$.

ANSWER KEY						
Multiple Choice Questions (MCQ)						
Questions	1	2	3	4	5	6
Answers	b	c	d	d	b	c
Questions	7	8	9	10	11	12
Answers	a	c	d	b	d	b
Questions	13	14	15			
Answers	a	a	b			
Multiple Select Questions (MSQ)						
Questions	1	2	3	4	5	
Answers	c,d	b,c	a,b,c,d	a,c,d	b,c,d	
Numerical Answer Type (NAT)						
Questions	1	2	3	4	5	
Answers	0.0873	0.277	1.259	4.3	10.71	

Chapter 2

RECIPROCAL LATTICE & X-RAY DIFFRACTION

Reciprocal lattice:

Every crystal has two lattices associated with it, the crystal lattice (or direct space lattice) and the reciprocal lattice.

The concept of reciprocal lattice was introduced for the purpose of tabulating two important properties of crystal planes : their slope and their interplanar spacing.

Each set of parallel planes in a direct lattice can be represented by a normal to these planes having length equal to the reciprocal of the interplanar spacing. The normals are drawn with reference to any arbitrary origin and points are marked at their ends. These points form a regular arrangement which is called reciprocal lattice. Thus each point in a reciprocal lattice is a representative point of a particular set of parallel planes. It is easier to deal with such points than to deal with set of planes. That's why we use reciprocal lattice.

Thus reciprocal lattice vector is a vector whose magnitude is equal to the reciprocal of the interplanar spacing i.e., $\frac{1}{d_{hkl}}$ and direction is parallel to the normal to $(h\,k\,l)$ plane.

Why reciprocal lattice is so named? The reciprocal lattice is so named because the length assigned to each normal representing the reciprocal lattice is proportional to the reciprocal of the interplanar spacing of that plane in ordinary space.

Primitive translation vectors of reciprocal lattice:

Let $\vec{a}, \vec{b}, \vec{c}$ be the primitive translation vectors of a direct space lattice for a crystal forming a primitive unit cell. The volume of this unit cell is given by

$$V = \vec{a}.\left(\vec{b}\times\vec{c}\right) = \vec{b}.\left(\vec{c}\times\vec{a}\right) = \vec{c}.\left(\vec{a}\times\vec{b}\right)$$

The vectors given by $\vec{A} = 2\pi\frac{\vec{b}\times\vec{c}}{\vec{a}.\left(\vec{b}\times\vec{c}\right)};\ \vec{B} = 2\pi\frac{\vec{c}\times\vec{a}}{\vec{b}.\left(\vec{c}\times\vec{a}\right)};\ \vec{C} = 2\pi\frac{\vec{a}\times\vec{b}}{\vec{c}.\left(\vec{a}\times\vec{b}\right)}$

are known as reciprocal lattice vectors or primitive translation vectors of reciprocal lattice.

$$\vec{A}.\vec{a} = 2\pi,\ \vec{A}.\vec{b} = 0 \text{ and } \vec{A}.\vec{c} = 0$$

$$\Rightarrow \quad \vec{B}.\vec{a} = 0,\ \vec{B}.\vec{b} = 2\pi \text{ and } \vec{B}.\vec{c} = 0$$

$$\Rightarrow \quad \vec{C}.\vec{a} = 0,\ \vec{C}.\vec{b} = 0 \text{ and } \vec{C}.\vec{c} = 2\pi$$

Choosing $\vec{A}$, $\vec{B}$ and $\vec{C}$ as the primitive translation vectors of the reciprocal lattice (also known as reciprocal axes) we can construct the reciprocal lattice points or reciprocal lattice vectors given by

$$\vec{G} = h\vec{A} + k\vec{B} + l\vec{C}$$

where h, k, l are intergers and define the coordinate of the point in the reciprocal space.

The crystal lattice is a lattice in real or ordinary space whereas reciprocal lattice is a lattice in the reciprocal space or associated $\vec{k}$ -space or Fourier space. The wave vector $\vec{k}$ is always drawn in Fourier space. Vectors in the crystal lattice have dimensions of length i.e., $\left[L^1\right]$ and vectors in the reciprocal lattice have dimensions of $\frac{1}{(\text{Length})}$ i.e., $[L^{-1}]$.

Properties of reciprocal lattice:

(i) The direct lattice is the reciprocal lattice to its own reciprocal lattice. Simple cubic lattice is self reciprocal whereas BCC and FCC lattices are reciprocal of each other.

(ii) Each point in a reciprocal lattice corresponds to a particular set of parallel planes of the direct lattice.

(iii) The distance of a reciprocal lattice point from an arbitrarily fixed origin is inversely proportional to the interplanar spacing of the corresponding parallel planes of the direct lattice.

(iv) The unit cell of the reciprocal lattice is not necessarily a parallelopiped. We deal with the Wigner Seitz cell of the reciprocal lattice which constitutes the Brillouin zone.

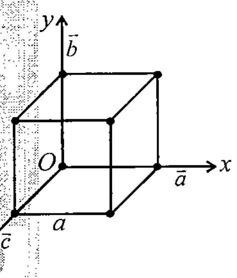
(v) The volume of a unit cell of the reciprocal lattice is inversely proportional to the volume of the corresponding unit cell of the direct lattice.

Reciprocal lattice of simple cubic lattice: Let the primitive translation vectors of a simple cubic cell be $\vec{a}, \vec{b}$ and $\vec{c}$. We can write the vectors as

$$\vec{a} = a\hat{i}, \vec{b} = a\hat{j} \text{ and } \vec{c} = a\hat{k}$$

The primitive translation vectors of the reciprocal of SC lattice are given by

$$\vec{A} = 2\pi \frac{\left(\vec{b} \times \vec{c}\right)}{\vec{a}.\left(\vec{b} \times \vec{c}\right)} = 2\pi \left(\frac{a\hat{j} \times a\hat{k}}{a^3}\right) = \frac{2\pi}{a}\hat{i}$$



Similarly, $$\vec{B} = 2\pi \frac{\left(\vec{c} \times \vec{a}\right)}{\vec{b}.\left(\vec{c} \times \vec{a}\right)} = 2\pi \left(\frac{a\hat{k} \times a\hat{i}}{a^3}\right) = \frac{2\pi}{a}\hat{j}$$

and $$\vec{C} = 2\pi \frac{\left(\vec{a} \times \vec{b}\right)}{\vec{c}.\left(\vec{a} \times \vec{b}\right)} = 2\pi \left(\frac{a\hat{i} \times a\hat{j}}{a^3}\right) = \frac{2\pi}{a}\hat{k}$$

Simple cubic lattice has side a and its reciprocal lattice has side $\left(\frac{2\pi}{a}\right)$.

Reciprocal of simple cubic lattice is also a simple cubic lattice with cube side $\left(\frac{2\pi}{a}\right)$ i.e., simple cubic lattice is self reciprocal lattice.

Volume of primitive unit cell of reciprocal lattice

$$V_R = \vec{A} \cdot \left(\vec{B} \times \vec{C}\right) \Rightarrow V_R = \left(\frac{2\pi}{a}\right)^3$$

Volume of reciprocal unit cell of any lattice is equal to $V_R = \frac{(2\pi)^3}{V_d}$, where V_d = volume of direct unit cell.

Reciprocal lattice of body centered cubic lattice:

Let the primitive translation vectors of a body centered cubic lattice are

$$\vec{a}' = \frac{a}{2}\left(\hat{i}+\hat{j}-\hat{k}\right),\ \vec{b}' = \frac{a}{2}\left(-\hat{i}+\hat{j}+\hat{k}\right),\ \vec{c}' = \frac{a}{2}\left(\hat{i}-\hat{j}+\hat{k}\right)$$

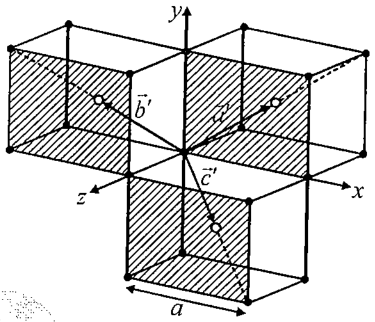
The primitive translation vectors of the reciprocal of body centered cubic lattice are given by

$$\vec{A} = 2\pi\frac{\vec{b}'\times\vec{c}'}{\vec{a}'.\left(\vec{b}'\times\vec{c}'\right)} = 2\pi\left(\frac{\frac{a^2}{2}\left(\hat{i}+\hat{j}\right)}{a^3/2}\right) = \frac{2\pi}{a}\left(\hat{i}+\hat{j}\right)$$

Similarly,

$$\vec{B} = 2\pi\frac{\vec{c}'\times\vec{a}'}{\vec{b}'.\left(\vec{c}'\times\vec{a}'\right)} = 2\pi\left(\frac{\frac{a^2}{2}\left(\hat{j}+\hat{k}\right)}{a^3/2}\right) = \frac{2\pi}{a}\left(\hat{j}+\hat{k}\right)$$

$$\vec{C} = 2\pi\frac{\vec{a}'\times\vec{b}'}{\vec{c}'.\left(\vec{a}'\times\vec{b}'\right)} = 2\pi\left(\frac{\frac{a^2}{2}\left(\hat{k}+\hat{i}\right)}{a^3/2}\right) = \frac{2\pi}{a}\left(\hat{k}+\hat{i}\right)$$



Hence, reciprocal of body centered cubic lattice is face centered cubic lattice with side $\left(\frac{4\pi}{a}\right)$.

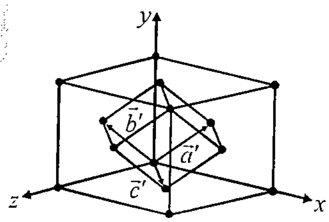
Volume of reciprocal unit cell $V_R = \vec{A}\cdot\left(\vec{B}\times\vec{C}\right) \Rightarrow V_R = 2\left(\frac{2\pi}{a}\right)^3$.

Reciprocal lattice of face centered cubic lattice: Let the primitive translation vectors of a face centered cubic lattice are $\vec{a}' = \frac{a}{2}\left(\hat{i}+\hat{j}\right),\ \vec{b}' = \frac{a}{2}\left(\hat{j}+\hat{k}\right),\ \vec{c}' = \frac{a}{2}\left(\hat{k}+\hat{i}\right)$

The primitive translation vectors of the reciprocal of face centered cubic lattice are given by

$$\vec{A} = 2\pi\frac{\vec{b}'\times\vec{c}'}{\vec{a}'.\left(\vec{b}'\times\vec{c}'\right)} = 2\pi\left(\frac{\frac{a^2}{4}\left(\hat{i}+\hat{j}-\hat{k}\right)}{\frac{a^3}{4}}\right) = \frac{2\pi}{a}\left(\hat{i}+\hat{j}-\hat{k}\right)$$

Similarly, $\vec{B} = \frac{2\pi}{a}\left(-\hat{i}+\hat{j}+\hat{k}\right)$ and $\vec{C} = \frac{2\pi}{a}\left(\hat{i}-\hat{j}+\hat{k}\right)$



Hence, reciprocal of face-centered cubic lattice is body centred cubic lattice.

X-RAY DIFFRACTION:

What are X-rays:

Like visible light, X-rays are also a form of electromagnetic radiation having wavelength of the order of 1 Å. In other words, we can say that X-rays represent a relatively high energy form of radiation. In electromagnetic spectrum, X-rays lie between ultraviolet and gamma rays.

Production of X-rays:

The X-rays are produced when a stream of high energy charged particles, e.g. a beam of electrons are allowed to strike a metal target (such as Molybdenum, Copper or Tungsten) of high melting point. In this process electrons loose some of their energy which is converted into electromagnetic radiations, i.e. some part of energy will be converted into electromagnetic radiations of high frequency according to formula $\Delta E = h\nu$. These processes give continuous X-rays.

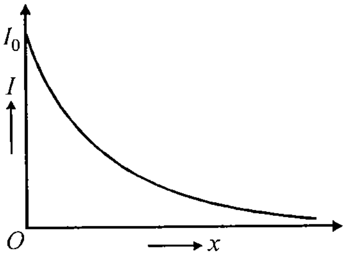
The wavelength of the X-rays can be given by $\Delta E = h\nu_{\max} = \dfrac{hc}{\lambda_{\min}}$.

$$\Rightarrow \quad \lambda_{\min} = \frac{hc}{\Delta E} = \frac{12400}{E(\text{eV})}\ \text{Å}$$

$$\Rightarrow \quad \lambda_{\min} = \frac{12400}{V\,(\text{volt})}\ \text{Å},$$

where V is the applied potential for accelerating the electrons.

Thus, we get $\lambda_{\min} \propto \dfrac{1}{V}$.



Absorption of X-rays:

When X-rays of intensity (I_0) are passed through a material of absorption coefficient μ, the intensity of the transmitted beam is given by $I_t = I_0\,e^{-\mu x}$, where x is the thickness of the material. Thus the intensity of X-ray beam decreases exponentially with the thickness of the absorbing material.

X-ray diffraction:

Similar to other electromagnetic radiatons, X-rays also interact with the electron cloud of the atoms. Since X-rays have shorter wavelengths, X-rays are scattered by adjacent atoms in crystal. The scattered X-rays interfere and give rise to diffraction phenomena. The atoms of crystal acts as a diffraction center for the X-rays like a three-dimensional diffraction grating. The X-ray diffraction pattern produced by this method can give us the information about the internal arrangement of atoms in crystals.

Consider parallel lattice planes equidistant from one another in a crystal structure, separated by a distance d. Suppose a beam of X-rays AB of wavelength λ is incident on a layer of atoms in one cleavage plane P of a crystal at a glancing angle θ at B and after reflection goes along BC. The ray DE incident on cleavage plane Q after reflection goes along EF. Draw BG perpendicular on DE and BH perpendicular on EF, then BG is the incident wave front and BH the reflected wave front.

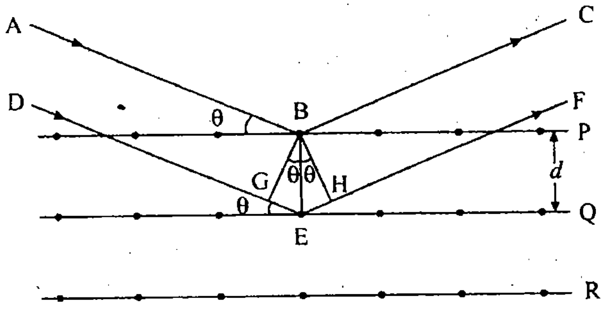
Path difference between the two wave fronts = GE + EH

Now, in ΔBGE, $\text{GE} = \text{BE}\sin\theta = d\sin\theta$ and in ΔBEH, $\text{EH} = \text{BE}\sin\theta = d\sin\theta$

Therefore, Path-difference, $\text{GE} + \text{EH} = 2d\sin\theta$

If this path-difference is an integral multiple of wavelength λ, constructive interference will take place between the reflected beams. Thus the intensity will be maximum if

$\because\ 2d\sin\theta = n\lambda$ where $n = 1, 2, 3, \ldots\ldots\ldots\ldots$



Bragg's reflection of X-rays from the atomic planes

For $n = 1$, we get the first order spectrum and for $n = 2$, we get the second order spectrum and so on. This phenomenon is similar to that of the optical diffraction with diffraction grating with only the difference that here various parallel planes of the crystal act as lines in the grating. The above equation $2d\sin\theta = n\lambda$ is Bragg's equation and represents Bragg's law.

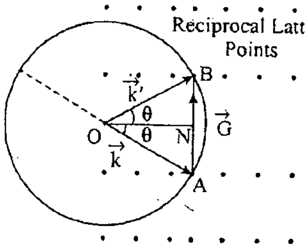
Physical significance of Bragg's law : The interpretation of Bragg's law gives that the diffraction intensities can be stronger only at certain values of θ for specific values of λ and d. Therefore, a knowledge of experimentally observed diffraction angles and wavelength of X-rays can provide the information about the interplanar spacing of the crystal. Thus, the space, shape and orientation of the unit cell can be determined from any diffraction experiment using monochromatic X-rays.

Ewald construction:

Ewald construction is a geometrical construction to get a relation between wave vector $\vec{k}$ and the direction of incident X-ray beam with the help of reciprocal lattice and thus derive Bragg's law in vector form. Draw a vector $\overline{\text{OA}} = \vec{k} = \dfrac{2\pi}{\lambda}$ in the direction of incident X-ray beam originating at O and terminating at any reciprocal lattice point A. Draw a sphere of radius $k = \dfrac{2\pi}{\lambda}$ with origin O as centre. This sphere is known as reflex sphere (The section of this sphere in a plane is a circle). Suppose this sphere intersects some point B of the reciprocal lattice whose indices are $(h'k'l')$, then $\overline{\text{AB}}$ represents the reciprocal lattice vector $\vec{G}$ and is normal to some set of direct lattice planes e.g., ON.

$$\text{AB} = \left|\vec{G}\right| = \frac{2\pi n}{d}$$

where n is the largest integral factor common to numbers $(h'k'l')$ and d is the interplanar spacing for the set of parallel planes.



Ewald construction in the reciprocal lattice

Bragg's diffraction condition in terms of reciprocal lattice:

As the wave vector $\overline{\text{OA}} = \vec{k}$ is the incident wave vector, $\overline{\text{OB}} = \vec{k}'$ is the diffracted (or reflected) wave vector and is given by

$$\vec{k}' = \vec{k} + \vec{G} \qquad \text{...(i)}$$

This shows that scattering changes only the direction of the wave vector $\vec{k}$ and the scattered wave vector differs from the incident wave vector by a reciprocal lattice vector $\vec{G}$.

For the diffraction, the magnitude of $\left(\vec{k} + \vec{G}\right)$ must be equal to the wave vector $\vec{k}'$.

i.e. $$\vec{k}'^2 = \left(\vec{k} + \vec{G}\right)^2 = \left(\vec{k} + \vec{G}\right)\cdot\left(\vec{k} + \vec{G}\right) = \vec{k}^2 + 2\vec{k}\cdot\vec{G} + \vec{G}\cdot\vec{G}$$

But the magnitude of $\vec{k}'$ = magnitude of $\vec{k}$ ($\because$ wavelength of incident and scattered beam is same)

$$\Rightarrow \qquad \cdot\,\vec{k}'^2 = \vec{k}^2$$

Hence, equation (i) becomes,

$$2\vec{k}\bullet\vec{G}+\vec{G}\bullet\vec{G}=0 \qquad \Rightarrow\ 2\vec{k}\bullet\vec{G}+G^2=0$$

$$\Rightarrow \qquad \left(\vec{k}+\frac{\vec{G}}{2}\right)\bullet\vec{G}=0$$

This is Bragg's law in vector form and gives the Bragg's diffraction conditions in terms of reciprocal lattice vector $\vec{G}$.

Brillouin zones : A Brillouin zone is the locus of all those $\vec{k}$- values in the reciprocal lattice which are Bragg reflected. For example, let us construct the Brillouin zones for a simple cubic structure with lattice parameter a. The primitive translation vectors of the simple cubic structure are

$$\vec{a}=a\hat{i},\ \vec{b}=a\hat{j} \qquad \ldots\text{(i)}$$

The corresponding translation vectors of the reciprocal lattice are

$$\vec{A}=\left(\frac{2\pi}{a}\right)\hat{i},\ \vec{B}=\left(\frac{2\pi}{a}\right)\hat{j}$$

The reciprocal lattice vector is given by

$$\vec{G}=h\vec{A}+k\vec{B}=\frac{2\pi}{a}\left(h\hat{i}+k\hat{j}\right) \qquad \ldots\text{(ii)}$$

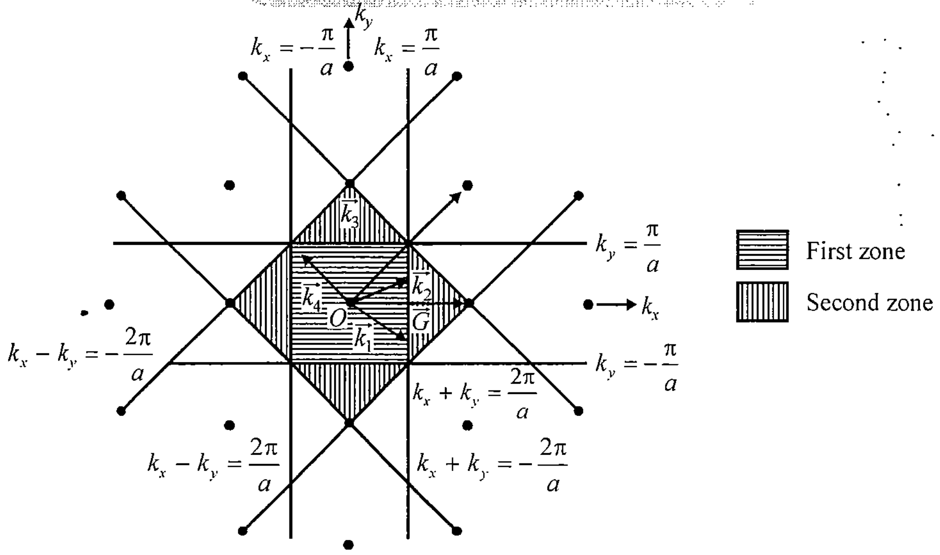
The Bragg's law in the reciprocal lattice is given by

$$2\vec{k}\cdot\vec{G}+G^2=0 \qquad \ldots\text{(iii)}$$

$$\Rightarrow \quad 2\left(\frac{2\pi}{a}\right)\left[\left(k_x\hat{i}+k_y\hat{j}\right)\bullet\left(h\hat{i}+k\hat{j}\right)\right]+\frac{4\pi^2}{a^2}\left(h^2+k^2\right)=0$$

$$\Rightarrow \quad \frac{4\pi}{a}\left[hk_x+kk_y\right]=-\frac{4\pi^2}{a^2}\left(h^2+k^2\right)$$

$$\Rightarrow \quad hk_x+hk_y=-\frac{\pi}{a}\left(h^2+k^2\right) \qquad \ldots\text{(iv)}$$


k_y
$k_x=-\frac{\pi}{a}$
$k_x=\frac{\pi}{a}$
$\vec{k}_3$
$k_y=\frac{\pi}{a}$
First zone
Second zone
$\vec{k}_4$
$\vec{k}_2$
O
$\vec{G}$
$\vec{k}_1$
k_x
$k_x-k_y=-\frac{2\pi}{a}$
$k_y=-\frac{\pi}{a}$
$k_x+k_y=\frac{2\pi}{a}$
$k_x-k_y=\frac{2\pi}{a}$
$k_x+k_y=-\frac{2\pi}{a}$

Brillouin zones of a square lattice in its reciprocal lattice

The $\vec{k}$-values which are Bragg reflected are obtained by the all possible combinations of h and k.

For $h = \pm 1$ and $k = 0$, equation (iv) gives $k_x = \pm\frac{\pi}{a}$ and k_y is arbitrary.

Also, for $h = 0$ and $k = \pm 1$, equation (iv) gives $k_y = \pm\frac{\pi}{a}$ and k_x is arbitrary.

Thus, we get a square of side $\left(\frac{2\pi}{a}\right)$ bounded by $k_x = \pm\frac{\pi}{a}$ and $k_y = \pm\frac{\pi}{a}$. This square is called as first Brillouin zone. Hence, the first zone of a square lattice of side a is a square of side $\frac{2\pi}{a}$.

In addition to this set of lines, some other sets of lines are also possible which satisfies equation (iv). For example, $h = \pm 1$ and $k = \pm 1$, equation (iv) gives the following set of lines $\pm k_x \pm k_y = \frac{2\pi}{a}$. The area bounded by these four lines are called as second Brillouin zone. Similarly, we can construct other Brillouin zones. The boundaries of Brillouin zones represent the loci of $\vec{k}$-values that are Bragg reflected. Hence we consider them as reflecting planes. The boundaries of the first zone represent the reflecting planes for the first order reflection and those of the second zone represent the reflecting planes for the second order reflection, and so on. A $\vec{k}$-vector that does not terminate at a zone boundary can not produce Bragg reflection. So, we can say that Brillouin zone pattern can be used to determine the X-ray diffraction pattern of a crystal and vice-versa. The Brillouin zones for a three dimensional cubical lattice are constructed using the generalized equation

$$hk_x + kk_y + lk_z = -\frac{\pi}{a}\left(h^2 + k^2 + l^2\right)$$

where a is the length of the cube edge.

Thus, we get first Brillouin zone as a cube of side equal to $\left(\frac{2\pi}{a}\right)$. The second zone is formed by adding pyramids to each face of the cube (first zone) as triangles are added to the square in two-dimensions, and so on.

Electron diffraction : Similar to X-rays, a beam of electron can also be used to characterize the crystal structure. A beam of electron emitted from a hot wire (as a result of thermionic emission) is accelerated through vacuum under a very large electrode potential (V) and is then passed through a collimator before it gets diffracted by the specimen to be studied.

The energy (E) acquired by these electrons will depend on accelerating voltage (V), thus, $E = eV$ ... (i)
where e is electronic charge.

According to de-Broglie hypothesis, the wavelength of the moving electrons can be given by

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE}} = \frac{h}{\sqrt{2meV}} \qquad \text{... (ii)}$$

where m is the mass of electrons.

Putting $m = 9.11 \times 10^{-31}$ kg and $e = 1.6 \times 10^{-19} C$, the wavelength λ will be $\lambda = \left(\frac{150}{V}\right)^{1/2}$ Å $= \frac{12.24}{\sqrt{V}}$ Å.

Thus, wavelength of moving electron is inversely proportional to the square root of the accelerating voltage and hence can be varied easily to get the desired wavelength of electron beam.

Thus, if we apply the accelerating voltage (V) such that the wavelength of the electron beam become equal to the order of atomic distances, then we can use electron beam diffraction.

Neutron diffraction:

Neutron diffraction is other diffraction technique that can be employed at the place of X-ray and electron diffraction.

The wavelength of neutron is given by

$$\lambda_n = \frac{h}{p} = \frac{h}{\sqrt{2m_n E}} \qquad \ldots \text{(i)}$$

Since neutrons are neutral particles, so the neutrons can not be accelerated by applying potential. The neutrons can be accelerated thermally. The kinetic energy of a neutron is given by

$$E = \frac{3}{2} k_B T \qquad \ldots \text{(ii)}$$

Thus, the wavelength $\lambda_n = \frac{h}{\sqrt{3m_n k_B T}} \Rightarrow \lambda_n \propto \frac{1}{\sqrt{T}}$.

If temperature of the neutrons is selected such that the wavelength of neutron becomes equal to the order of atomic distances, then these neutrons can be used in diffraction phenomena.

Diffraction intensity:

The different directions of the diffracted X-ray beam obtained according to Bragg's law are governed by the geometry and the size of the unit cell. These directions will not be affected by the arrangement of atoms associated with each lattice point of the crystal structure. However, the complexity of the atomic structure within the unit cell does affect the intensity of the diffracted beam. The intensity depends on the atomic scattering factor, the position of each atom in unit cell. Since, the electrons are the only component that an atom have, therefore, the X-rays scattered from one part of an atom interfere with those scattered from the other part at all angles of scattering. Since, the electrons are distributed throughout the atomic volume, the diffracted beams are obtained as a result of combination of the scattered waves from the electrons of all atoms in the unit cell. This process involves two distinct contributions.

1. Scattering from electrons in the same atom. This gives the atomic scattering factor f.
2. The summation of this scattering from all atoms in the unit cell. This gives the geometrical structure factor S.

Intensity of diffracted beam can be written as $I = F^2 = f^2 S^2$, where S is geometrical structure factor, f is atomic scattering factor and F is diffracted X-ray beam amplitude.

Thus, $F = fS$. The atomic scattering factor (f) is defined as $f = \frac{\text{diffracted amplitude by an atom}}{\text{diffracted amplitude by an electron}}$.

If the atoms are assumed to be points only, then the atomic scattering factor is just equal to the number of electrons present (i.e., the atomic number Z).

Examples: (1) the atomic scattering factor of Na ($Z = 11$) will be larger in comparison to Li ($Z = 3$).

(2) In NaCl, the atomic scattering factors of Na^+ and Cl^- ions are different. The Cl^- ion has more atomic scattering factor.

Thus, the higher scattering factor means higher number of electrons and intensity.

The geometrical structure factor (S) is given by $S = \frac{\text{diffracted amplitude by an unit cell}}{\text{diffracted amplitude by an atom}}$.

Mathematically, $S = \sum_j e^{2\pi i (u_j h + v_j k + w_j l)}$, where (h, k, l) is miller indices of the plane and (u_j, v_j, w_j) is position coordinate of the j^{th} atom.

(1) Simple cubic structure : For simple cubic structure, the effective number of atoms per unit cell is one. Let us take the position coordinate of this atom as origin (0, 0, 0).

Thus, geometrical structure factor $S = \sum_j e^{2\pi i(u_jh+v_jk+w_jl)} = e^{2\pi i(0h+0k+0l)} = e^0 = 1$

$\Rightarrow \quad S = 1$

The diffracted X-ray beam amplitude becomes $F = fS = f(1) = f$ and intensity of diffracted beam $(I) = f^2$. Hence, all the $(h\ k\ l)$ values are allowed. In other words we can say that all the reflection peaks are allowed in simple cubic structures.

(2) Body centered cubic structures:

For BCC structure, the effective number of atoms per unit cell is equal to 2. We can take the position coordinates of these atoms as (0, 0, 0) and $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$.

Thus, the geometrical structure factor (S) can be given as

$$S = \sum_j e^{2\pi i(u_jh+v_jk+w_jl)} = e^{2\pi i(0h+0k+0l)} + e^{2\pi i\left(\frac{1}{2}h+\frac{1}{2}k+\frac{1}{2}l\right)} = 1 + e^{\pi i(h+k+l)}.$$

For $(h+k+l)$ = even, $S = 1+1 = 2$

For $(h+k+l)$ = odd, $S = 1-1 = 0$

Thus, the amplitude of the diffracted X-ray beam will be given by

$$\begin{aligned} F &= 2f \quad [\text{For } (h+k+l) = \text{even}] \\ &= 0 \quad\ \ [\text{For } (h+k+l) = \text{odd}] \end{aligned}$$

and the intensity of diffracted beam becomes

$$\begin{aligned} (I) &= (2f)^2 = 4f^2, \ \text{for } (h+k+l) = \text{even} \\ &= 0, \ \text{for } (h+k+l) = \text{odd} \end{aligned}$$

Thus, the reflection peaks (1 1 0), (2 0 0), (2 1 1), (2 2 2),... are allowed for BCC structure and reflection peaks (1 0 0), (1 1 1), (3 0 0),... are absent.

(3) Face centered cubic structure:

For FCC structure, the effective number of atoms per unit cell is equal to 4. Let the position coordinates of these atoms are (0, 0, 0), $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$, $\left(0, \frac{1}{2}, \frac{1}{2}\right)$, $\left(\frac{1}{2}, 0, \frac{1}{2}\right)$.

Thus, the geometrical structure factor (S) can be given as

$$S = \sum_j e^{2\pi i(u_jh+v_jk+w_jl)} = 1 + e^{\pi i(h+k)} + e^{\pi i(k+l)} + e^{\pi i(h+l)}$$

The geometrical structure factor S will be equal to 4 if all the h, k, l are either even or odd and zero otherwise. Thus, the amplitude of the diffracted X-ray beam will be given by

$$\begin{aligned} F &= 4f, \quad h,k,l \text{ are either all odd or all even} \\ &= 0, \quad \text{otherwise} \end{aligned}$$

and the intensity of diffracted beam becomes

$$\begin{aligned} I &= (4f)^2 = 16f^2, \quad h,k,l \text{ are either all odd or all even} \\ &= 0, \quad \text{otherwise} \end{aligned}$$

Hence, reflection planes (1 1 1), (2 0 0), (2 2 0), (3 1 1),... are allowed (i.e. present) for FCC structure and reflection planes (1 0 0), (1 1 0), (2 1 1),... are not allowed (i.e. absent).

(4) Diamond cubic structure:

For diamond cubic structure, only those reflection peaks will be allowed for which

(i) $(h\,k\,l)$ are all odd, or

(ii) $(h\,k\,l)$ are all even and also $h+k+l=4n$ where $n=1,2,3,...$

i.e. if $(h\,k\,l)$ all are even and $(h+k+l)$ is divisible by 4 then this peak will be present in diamond cubic structure.

(5) Rock salt (NaCl) structure:

The selection rules for NaCl structure is same as FCC structure. There is only one exception. For KCl, the selection rules of NaCl structure is not valid.

SOLVED PROBLEMS

A beam of X-rays is incident on a BCC crystal. If the difference between the incident and scattered wave vectors is $\vec{K}=h\hat{x}+k\hat{y}+l\hat{z}$ where $\hat{x},\hat{y},\hat{z}$ are the unit vectors of the associated cubic lattice, the necessary condition for the scattered beam to give a Laue maximum is

(a) $h+k+l=$ even (b) $h=k=l$

(c) h,k,l are all distinct (d) $h+k+l=$ odd **[JEST-2012]**

ln. For BCC crystal, the effective number of atoms is two; one occupies a corner position and the other occupies the center of the cube. If the coordinates of the corner atom be arbitrary taken as (0, 0, 0), then the coordinates of the other become $\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$. Since, both atoms are identical, the diffraction amplitude (F) can be given as

$$F(hkl)=f\sum_j e^{2\pi i(u_jh+v_jk+w_jl)}=f(1+e^{i\pi(h+k+l)})$$

Above equation gives zero for odd value of $(h+k+l)$ and will give non-zero value for even values of $(h+k+l)$.

Hence, for BCC crystal, the necessary condition for the scattered beam to give a Laue maximum is $h+k+l=$ even $\therefore$

Hence, correct option is (a).

If $\vec{k}$ is the wavevector of incident light $|\vec{k}|=\frac{2\pi}{\lambda}$, λ is the wavelength of light, and $\vec{G}$ is reciprocal lattice vector, then the Bragg's law can be written as

(a) $\vec{k}+\vec{G}=0$ (b) $2\vec{k}\cdot\vec{G}+G^2=0$ (c) $2\vec{k}\cdot\vec{G}+k^2=0$ (d) $\vec{k}\cdot\vec{G}=0$ **[JEST-2016]**

ln. The Bragg's law in terms of reciprocal lattice vector G)

$\therefore\quad \vec{k}'=\vec{k}+\vec{G}$

$\Rightarrow\quad \vec{k}'\cdot\vec{k}'=(\vec{k}+\vec{G})\cdot(\vec{k}+\vec{G}) \Rightarrow |\vec{k}'|^2=|\vec{k}|^2+2\vec{k}\cdot\vec{G}+|\vec{G}|^2$

$\because\quad |\vec{k}'|^2=|\vec{k}|^2$

$\therefore\quad 2\vec{k}\cdot\vec{G}+G^2=0$

Hence, correct option is (b).

3. The second order maximum in the diffraction of X-rays of 0.20 nanometer wavelength from a simple cubic crystal is found to occur at an angle of thirty degrees to the crystal plane. The distance between the lattice planes is

(a) 1 Angstrom (b) 2 Angstrom (c) 4 Angstrom (d) 8 Angstrom **[JEST-2012]**

Soln. Given : $n = 2,\ \lambda = 0.20$ nanometer, $\theta = 30°,\ d = ?$

From Bragg's law : $2d \sin\theta = n\lambda$

$$\Rightarrow\ 2 \times d \times \sin 30° = 2 \times 0.20 \text{ nanometer}$$

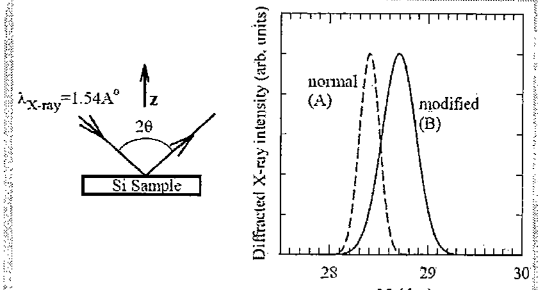
$$\Rightarrow\ d = \frac{2 \times 0.20}{2 \times \sin 30°} = \frac{2 \times 0.20}{2 \times \frac{1}{2}} = 0.40 \text{ nanometer}$$

$$\Rightarrow\ d = 4 \text{ Angstrom}$$

i.e., the distance between the lattice planes is 4 Angstrom.

Hence, correct option is (c).

4. The figure below shows the Bragg diffraction pattern for X-rays of wavelength 1.54 Å incident on two crystalline silicon thin film samples A and B. The dashed line corresponds to a normal sample A and the continuous line corresponds to another sample B, which is modified due to differences in the growth conditions.



These plots suggest that the modified sample B is

(a) stretched in all directions by 3%

(b) compressed in all directions by 3%

(c) stretched in the z-direction by 1 % and possibly compressed in x and y directions

(d) compressed in the z-direction by 1 % and possibly stretched out in x and y directions

[TIFR-2011]

Soln. From Bragg's law : $2d \sin\theta = n\lambda$

For sample (A), the maximum occurs at 28°24′ and

For sample (B), the maximum occurs at 28°42′.

For modified sample B, the diffracted X-ray intensity peak shifted by 10′ i.e., θ increases. Hence, the distance between planes will decrease in the z-direction.

$$\text{From Bragg's law } 2d \sin\theta = \lambda \ \Rightarrow\ \frac{\partial d}{d} + \cot\theta\, \partial\theta = 0 \quad (\because \lambda_{X\text{-ray}} \text{ is fixed})$$

$$\Rightarrow\ \frac{\partial d}{d} = -\cot\theta\, d\theta = -(\cot 28°18') \times (18') = -1.857 \times (5.233 \times 10^{-3}) = -9.7 \times 10^{-3}$$

$$\therefore\ \frac{\partial d}{d}\% = -9.7 \times 10^{-3} \times 100 = -0.97\% \approx -1\%.$$

Therefore, the distance between planes will decrease i.e., the modified sample is compressed in z-direction by 1 %. The sample may be streched out in x and y directions.

Hence, correct option is (d).

Metallic copper is known to form cubic crystals and the lattice constant is measured from X-ray diffraction studies to be about 0.36 nm. If the specific is 63.5, one can conclude that

(a) the crystal are of simple cubic type
(b) the crystals are of BCC type
(c) the crystals are of FCC type
(d) the crystals are a mixture of FCC and BCC types
(e) there is insufficient data to distinguish between the previous options

[TIFR-2012]

ln. Given, $a = 0.36$ nm

Specific gravity $= \rho = 8.96 \text{ g/cm}^3 = 8.96 \times 10^3 \text{ kg/m}^3$

Avogadro number $(N_A) = 6.023 \times 10^{26}$, $n = ?$

Therefore, from equation

$$\rho = \frac{nM}{N_A a^3} \Rightarrow n = \frac{\rho \cdot N_A \cdot a^3}{M} = \frac{8.96 \times 6.023 \times 10^{26} \times (0.36 \times 10^{-9})^3}{63.3} = 0.0397 \times 10^{29} \times 10^{-27}$$

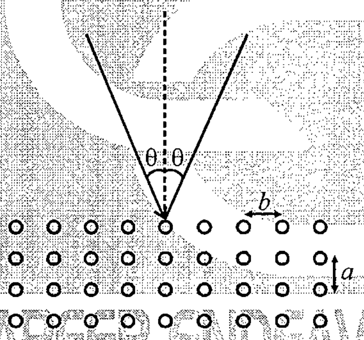
$$= 0.0397 \times 10^2 = 3.97 \cong 4$$

i.e., number of atoms per unit cell = 4.

Hence, the crystals are of FCC type

Hence, correct option is (c).

A monochromatic beam of X-rays with wavelength λ is incident at an angle θ on a crystal with lattice spacings a and b as sketched in the figure below.



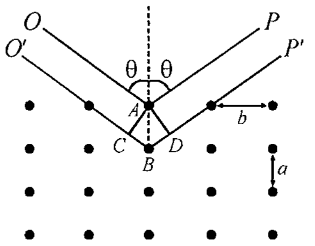
A condition for there to be a maximum in the diffracted X-ray intensity is

(a) $2\sqrt{a^2 + b^2} \sin\theta = \lambda$
(b) $2b \cos\theta = \lambda$
(c) $2a \cos\theta = \lambda$
(d) $(a + b) \sin\theta = \lambda$

[TIFR-2012]

ln. The path difference between OAP and $O'BP' = CB + BD = a\cos\theta + a\cos\theta = 2a\cos\theta$ ...(i)

For maximum intensity, this path difference should be integral multiple of λ.



$\therefore \quad 2a\cos\theta = n\lambda$, where $n = 1, 2, 3, ...$

Hence, correct option is (c).

7. In a Davisson-Germer experiment, a collimated beam of electrons of energy 54eV, at normal incidence on a given crystal, shows a peak at a reflection angle of 40°. If the electron beam is replaced by a neutron beam, and the peak appears at the same value of reflection angle, then the energy of the neutrons must be

(a) 330 eV (b) 33 eV (c) 0.3 eV (d) 0.03 eV **[TIFR 2013]**

Soln. Energy of electrons (E_e) = 54 eV, energy of neutron (E_n) = ?

Since the angle of reflection is same for electron and neutron.

Therefore, we can use Bragg's law for these cases separately.

For electron : $2d\sin\theta = \lambda_e = \dfrac{h}{\sqrt{2m_e E}}$ ... (i)

For neutron : $2d\sin\theta = \lambda_n = \dfrac{h}{\sqrt{2m_n E}}$ ... (ii)

Since, the crystal is same for both cases and θ is 40°.

From equation (i) and (ii), we get

$$\frac{h}{\sqrt{2m_e E_e}} = \frac{h}{\sqrt{2m_n E_n}} \Rightarrow 2m_e E_e = 2m_n E_n$$

$$\Rightarrow \quad 2\times 9.1\times 10^{-31}\times 54\times 1.6\times 10^{-19} = 2\times 1.6\times 10^{-27}\times E_n$$

$$\Rightarrow \quad E_n = \frac{2\times 9.1\times 10^{-31}\times 54\times 1.6\times 10^{-19}}{2\times 1.67\times 10^{-27}} = 0.03\,eV$$

Hence, correct option is (d)

8. In the X-ray diffraction pattern of a sodium metal, which has a bcc structure, the missing reflection planes will be **[DU 2016]**

(a) 101 (b) 011 (c) 020 (c) 100

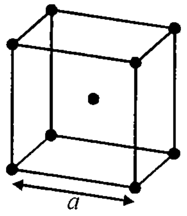
Soln. For BCC structures, $h+k+l = \text{even}$

Here in present case - All the options except (d) has $h+k+l = \text{even}$.

Hence the reflection plane (100) will be missing for BCC structure.

Hence, correct option is (d).

9. The conventional unit cell of a body-centered cubic Bravais lattice is shown in the figure below. The conventional cell has volume a^3. What is the volume of the primitive unit cell ? **[BHU 2012]**



(a) $\dfrac{a^3}{8}$ (b) $\dfrac{a^3}{4}$ (c) $\dfrac{a^3}{2}$ (d) a^3

Soln. Volume of the primitive unit cell of BCC Bravais lattice = $\dfrac{a^3}{2}$.

Hence, correct option is (c).

. The longest wavelength X-ray that can undergo Bragg diffraction in a crystal for a given family of planes of spacing d is **[BHU 2012]**

(a) $\frac{d}{4}$ (b) $\frac{d}{2}$ (c) d (d) $2d$

ln. From Bragg's law $2d\sin\theta = n\lambda$

For longest wavelength, $\theta = 90°,\ n = 1$

Thus, $2d = \lambda$

Hence, correct option is (d).

. The diffraction pattern of copper metal was measured with X-ray radiation of wavelength of 1.315 Å. The first order Bragg diffraction peak was found at an angle 2θ of 50.5°. The d-spacing between the diffracting planes in the copper metal is **[BHU 2012]**

(a) 3.25 Å (b) 1.54 Å (c) 0.41 Å (d) 4.2 Å

ln. $\lambda = 1.315\,\text{Å},\ n = 1,\ 2\theta = 50.5° = 25.25°,\ d = ?$

From Bragg's law $2d\sin\theta = n\lambda$

$$\Rightarrow\ 2\times d\times\sin 20.25° = 1\times 1.315$$

$$\Rightarrow\ d = \frac{1.315}{2\times\sin 25.25°} = 1.54\,\text{Å}.$$

Hence, correct option is (b).

:. The continuous X-ray spectrum is the result of the **[BHU 2012]**

(a) photoelectric effect (b) inverse photoelectric effect

(c) compton effect (d) auger effect

ln. The continuous X-ray spectrum is the result of the inverse photoelectric effect.

Hence, correct option is (b).

. The short wavelength limit of X-ray depends upon **[BHU 2013]**

(a) nature of the target (b) potential difference across the X-ray tube

(c) nature of the filament used (d) none of these

ln. The short wavelength $\boxed{\lambda \propto \frac{1}{V}}$. Thus, wavelength depends on potential difference across the X-ray tube.

Hence, correct option is (b).

. The de-Broglie wavelength associated with an electron of mass m and accelerated by a potential V is

(a) $\frac{h}{\sqrt{2mVe}}$ (b) $\frac{\sqrt{2mVe}}{h}$ (c) $\frac{h}{\sqrt{Vem}}$ (d) $\frac{h}{2Vem}$ **[BHU 2013]**

ln. For an electron, the de-Broglie wavelength is given by $\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE}} = \frac{h}{\sqrt{2meV}}$, where $E = \text{eV}$ is the energy of electron.

Hence, correct option is (a).

15. According to Moseley's law, the relation between the atomic number Z and frequency ν is given by **[BHU 2013]**

(a) $\nu \propto (Z-b)$ (b) $\nu \propto Z$ (c) $\nu \propto (Z-b)^3$ (d) $\nu \propto (Z-b)^2$

Soln. According to Moseley's law, the relation between the atomic number Z and frequency ν is given by $\nu \propto (Z-b)^2$

Hence, correct option is (d).

16. In the X-ray diffraction of a set of crystal planes having d equal to 0.18 nm, first order reflection is found to be at an angle of 22°. The wavelength of X-ray is : (sin 22° = 0.208) : **[BHU 2014]**

(a) 0.0749 nm (b) 0.0374 nm (c) 0.749 nm (d) 0.374 nm

Soln. $\text{d} = 0.18 \text{ nm}, n = 1, \theta = 22^\circ$

From Bragg's law: $2d\sin\theta = n\lambda$

$\Rightarrow 2 \times 0.18 \times \sin 22^\circ = 1 \cdot \lambda$

$\Rightarrow \lambda = 0.779 \text{ nm}$

Hence, correct option is (a).

17. During X-ray emission if the voltage is increased : **[BHU 2014]**

(a) minimum wavelength decreases (b) minimum wavelength increases
(c) intensity increases (d) intensity decreases

Soln. Since $\lambda_{\min} \propto \frac{1}{V}$

Hence, with increase in voltage, the minimum wavelength decreases.

Hence, correct option is (a).

18. The amplitude of scattering of X-rays scatered by a single atom is generally denoted as **[BHU 2014]**

(a) Structure factor (b) Polarization factor (c) Form factor (d) Fractional coordination

Soln. The amplitude of scattering of X-rays scattered by a simple atom in generally denoted as structure factor.

Hence, correct option is (a).

19. X-ray diffraction can be applied to **[BHU 2014]**

(a) liquids only (b) solid, crystalline materials only
(c) all liquids, solids and crystalline materials (d) gaseous or vapour materials only

Soln. X-ray diffraction can be applied to all liquids, solids and crystalline materials.

Hence, correct option is (c).

20. Characteristic X-rays are the characteristic of which of the following : **[BHU 2014]**

(a) Cathode materials (b) Anode materials
(c) Accelerating voltage (d) Tube current

Soln. Characteristic X-ray are the characteristic of the anode material.

Hence, correct option is (b).

21. FCC lattice is the reciprocal lattice of the **[BHU 2014]**

(a) BCC lattice (b) SC lattice (c) HCP lattice (d) both of the BCC and SC lattices

Soln. FCC lattice is the reciprocal of BCC lattice.

Hence, correct option is (a).

Because of which property of the crystal X-rays can be diffracted from crystals : **[BHU 2014]**

(a) random arrangement of atoms (b) colour of the crystals

(b) periodic array of atom (d) transparency of crystals

ln. Due to periodic array of atoms in crystals the X-rays can be diffracted from crystal.

Hence, correct option is (c).

According to Mosley's law the frequency of the characteristic X-radiation is proportional to the square of

(a) atomic weight of the element (b) atomic number of the element **[BHU 2014]**

(c) number of neutrons (d) square of atomic number

ln. From Mosley's law; $\sqrt{\nu} \propto Z$ where ν is frequency and z is atomic number.

$\Rightarrow \nu \propto Z^2$

Thus, the frequency of the characteristic X-radiation is proportional to the square of atomic number of the element.

Hence, correct option is (b).

The angle of diffracted X-rays is 90° from the incident direction of X-rays. The interplanar separation of the planes responsible for the diffraction is given by **[BHU 2015]**

(a) $\frac{\lambda}{\sqrt{2}}$ (b) $\frac{\lambda}{2}$ (c) $\sqrt{2}\lambda$ (d) $\frac{\sqrt{3}}{2}\lambda$

ln. $\theta = 90^\circ,\ n = 1$

From Bragg's law : $2d\sin\theta = n\lambda$

$\Rightarrow 2\times d\times\sin 90^\circ = 1\cdot\lambda \Rightarrow 2\times d\times 1 = \lambda \Rightarrow d = \lambda/2.$

Hence, correct option is (b).

Which of the statements about the spectrum of continuous X-rays is false ?

(a) There is no long wavelength cutoff limit **[BHU 2015]**

(b) Continuous X-rays are produced due to deceleration of electrons

(c) The maximum intensity of continuous X-rays depends upon potential difference across the X-ray tube

(d) Short wavelength limit of X-rays depends upon the nature of filament

ln. Out of above options, only option (d) short wavelength limit of X-rays depends upto the nature of filament, is false.

Hence, correct option is (d).

The de Broglie wavelength λ, mass m and charge e of an electron are related to the accelerating potential V by **[BHU 2015]**

(a) $V = \frac{\lambda^2}{2meh^2}$ (b) $V = \frac{h}{2me\lambda^2}$ (c) $V = \frac{h^2}{2me\lambda^2}$ (d) $V = \frac{h^2\lambda^2}{2me}$

ln. For an electron, $\lambda = \frac{h}{p} = \frac{h}{\sqrt{2me}} = \frac{h}{\sqrt{2meV}}$

Squaring both sides, we get $\lambda^2 = \frac{h^2}{2meV}$

$\Rightarrow V = \frac{h^2}{2me\lambda^2}$

Hence, correct option is (c).

27. The reciprocal lattice of an FCC crystal is **[BHU 2015]**

(a) orthorhombic (b) simple cubic (c) hexagonal (d) BCC

Soln. The reciprocal lattice of an FCC crystal is BCC.

Hence, correct option is (d).

28. Because of which property of the crystals X-rays can be diffracted from the crystals **[BHU 2015]**

(a) colour of the crystals (b) sharp edges of the crystals

(c) periodic array of atoms (d) random arrangement of atoms

Soln. Due to periodic array of atoms in crystal, the X-rays can be diffracted from the crystals.

Hence, correct option is (c).

29. X-rays can be deflected by **[BHU 2015]**

(a) magnetic field only

(b) electric field only

(c) simultaneously applying electric and magnetic field

(d) neither by electric nor magnetic field

Soln. X-rays can not be deflected by electric or magnetic field.

Hence, correct option is (d).

PRACTICE SET

(Multiple Choice Questions (MCQ)):

The distance between the adjacent atomic planes in $CaCO_3$ is 0.3 nm. The smallest Bragg scattering for 0.03 nm X-ray is

(a) 2.9° (b) 1.5° (c) 0.29° (d) 5.8°

For a NaCl crystal, the cell-edge $a = 0.563$ nm. The smallest angle at which Bragg reflection can occur corresponds to a set of planes whose indices are

(a) (100) (b) (110) (c) (111) (d) (200)

Silicon has diamond structure with until-cell-edge $a = 0.542$ nm. The interatomic separation is

(a) 0.122 nm (b) 0.234 nm (c) 0.383 nm (d) 0.542 nm

A crystal belongs to a face centered cubic lattice with four atoms in the unit cell. The size of the crystal is 1 cm and unit cell dimension is 1 nm. f is the scattering factor of the atom. The number of atoms in the crystal is

(a) 2×10^{21} (b) 4×10^{21} (c) 2×10^{23} (d) 4×10^{24}

The structure factors for (010) and (200) reflections respectively, in face centered cubic lattice are:

(a) $2f$ and zero (b) zero and $4f$ (c) $2f$ and $2f$ (d) zero and zero

The primitive translation vectors of the body centered cubic lattice are $\vec{a} = \frac{a}{2}(\hat{x} + \hat{y} - \hat{z})$, $\vec{b} = \frac{a}{2}(-\hat{x} + \hat{y} + \hat{z})$ and $\vec{c} = \frac{a}{2}(\hat{x} - \hat{y} + \hat{z})$. The primitive translation vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ of the reciprocal lattice are

(a) $\vec{A} = \frac{2\pi}{a}(\hat{x} - \hat{y})$; $\vec{B} = \frac{2\pi}{a}(\hat{y} + \hat{z})$; $\vec{C} = \frac{2\pi}{a}(\hat{x} + \hat{z})$

(b) $\vec{A} = \frac{2\pi}{a}(\hat{x} - \hat{y})$; $\vec{B} = \frac{2\pi}{a}(\hat{y} - \hat{z})$; $\vec{C} = \frac{2\pi}{a}(\hat{x} + \hat{z})$

(c) $\vec{A} = \frac{2\pi}{a}(\hat{x} + \hat{y})$; $\vec{B} = \frac{2\pi}{a}(\hat{y} + \hat{z})$; $\vec{C} = \frac{2\pi}{a}(\hat{x} - \hat{z})$

(d) $\vec{A} = \frac{2\pi}{a}(\hat{x} + \hat{y})$; $\vec{B} = \frac{2\pi}{a}(\hat{y} + \hat{z})$; $\vec{C} = \frac{2\pi}{a}(\hat{x} + \hat{z})$

The structure factor of a single cell of identical atoms (form factor f) is given by

$$S_{hkl} = f\sum_j \exp\left[-i2\pi\left(x_j h + y_j k + z_j l\right)\right]$$

where (x_j, y_j, z_j) is the coordinate of an atom, and h, k, l are the miller indices. Which one of the following statements is correct for the diffraction peaks of body-centered cubic (bcc) and face-centered cubic (fcc) lattices ?

(a) bcc : (200); (110); (222)
fcc : (111); (311); (400)

(b) bcc : (210); (110); (222)
fcc : (111); (311); (400)

(c) bcc : (200); (110); (222)
fcc : (111); (211); (400)

(d) bcc : (200); (210); (222)
fcc : (111); (211); (400)

8. Consider X-ray diffraction from a crystal with a face centered cubic (fcc) lattice. The lattice plane for which there is no diffraction peak is

(a) (2, 1, 2) (b) (1, 1, 1) (c) (2, 0, 0) (d) (3, 1, 1)

9. A metal with body centered cubic (bcc) structure shows the first (i.e. smallest angle) diffraction peak at a Bragg angle of $\theta = 30°$. The wavelength of X-ray used is 2.1 Å. The volume of the unit cell of the metal is

(a) 26.2 $(\text{Å})^3$ (b) 13.1 $(\text{Å})^3$ (c) 9.3 $(\text{Å})^3$ (d) 4.6 $(\text{Å})^3$

10. A lattice has the following primitive vectors (in Å), $\vec{a} = 2(\hat{j}+\hat{k})$, $\vec{b} = 2(\hat{k}+\hat{i})$, $\vec{c} = 2(\hat{i}+\hat{j})$. The reciprocal lattice corresponding to the above lattice is

(a) bcc lattice cube edge of $(\pi/2)\ \text{Å}^{-1}$ (b) bcc lattice with cube edge of $(2\pi)\ \text{Å}^{-1}$

(c) fcc lattice with cube edge of $(\pi/2)\ \text{Å}^{-1}$ (d) fcc lattice with cube edge of $(2\pi)\ \text{Å}^{-1}$

11. The Bragg angle corresponding to the first order reflection from (111) plane in a crystal is 30°. When *X*-ray of wavelength 0.175 nm are used, the interatomic spacing is

(a) 0.6062 nm (b) 0.3031 nm (c) 0.20205 nm (d) 0.1010 nm

12. The volume of primitive unit cell defined by axes $(\hat{k}+\hat{i})$ Å, $(\hat{i}+\hat{j})$ Å and $(\hat{j}+\hat{k})$ Å is

(a) 1 Å^3 (b) 3 Å^3 (c) 2Å^3 (d) 4Å^3

13. If 50 kV is the applied potential in an X-ray tube, then the maximum wavelength of X-rays produced is

(a) 0.2 nm (b) 2 nm (c) 0.2 Å (d) 2 Å

14. The primitive translation vectors of the face centered cubic (fcc) lattice are

$$\vec{a}_1 = \frac{a}{2}(\hat{j}+\hat{k}),\ \vec{a}_2 = \frac{a}{2}(\hat{i}+\hat{k}),\ \vec{a}_3 = \frac{a}{2}(\hat{i}+\hat{j}).$$

The primitive translation vectors of the fcc reciprocal lattice are

(a) $\vec{b}_1 = \frac{2\pi}{a}(-\hat{i}+\hat{j}+\hat{k})$; $\vec{b}_2 = \frac{2\pi}{a}(\hat{i}-\hat{j}+\hat{k})$; $\vec{b}_3 = \frac{2\pi}{a}(\hat{i}+\hat{j}-\hat{k})$

(b) $\vec{b}_1 = \frac{\pi}{a}(-\hat{i}+\hat{j}+\hat{k})$; $\vec{b}_2 = \frac{\pi}{a}(\hat{i}-\hat{j}+\hat{k})$; $\vec{b}_3 = \frac{\pi}{a}(\hat{i}+\hat{j}-\hat{k})$

(c) $\vec{b}_1 = \frac{\pi}{2a}(-\hat{i}+\hat{j}+\hat{k})$; $\vec{b}_2 = \frac{\pi}{2a}(\hat{i}-\hat{j}+\hat{k})$; $\vec{b}_3 = \frac{\pi}{2a}(\hat{i}+\hat{j}-\hat{k})$

(d) $\vec{b}_1 = \frac{3\pi}{a}(-\hat{i}+\hat{j}+\hat{k})$; $\vec{b}_2 = \frac{3\pi}{a}(\hat{i}-\hat{j}+\hat{k})$; $\vec{b}_3 = \frac{3\pi}{a}(\hat{i}+\hat{j}-\hat{k})$

15. The volume of primitive cell of the fcc reciprocal lattice is

(a) $4\left(\frac{2\pi}{a}\right)^3$ (b) $4\left(\frac{\pi}{a}\right)^3$ (c) $4\left(\frac{\pi}{2a}\right)^3$ (d) $4\left(\frac{3\pi}{a}\right)^3$

(Multiple Select Questions (MSQ)):

The plane(s) that will show allowed reflection in SC, BCC and FCC lattices all are
(a) (200) (b) (222) (c) (111) (d) (110)

The correct statement(s) about the reciprocal lattice is/are
(a) Every reciprocal lattice vector is normal to a set of parallel planes of the crystal lattice.
(b) When no common factor exists among the components of $\vec{G}$ (recirpocal lattice vector), then |G| is inversely proportional to the spacing of the lattice planes normal to $\vec{G}$.
(c) The direct lattice is the reciprocal of its own reciprocal lattice.
(d) The unit cell of the reciprocal lattice need not be a parallelopiped.

The correct statement(s) about the reciprocal lattice is/are
(a) The simple cubic structure is self reciprocal.
(b) The reciprocal lattice for a bcc lattice is an fcc structure.
(c) The reciprocal lattice for an fcc lattice is a bcc structure
(d) The reciprocal lattice for an hcp lattice is a bcc structure

For the X-ray diffraction, the correct statement(s) is/are
(a) Bragg's law is given by $2d\sin\theta = n\lambda$
(b) Bragg's law is given by $2d\sin\theta = (2n+1)\lambda$
(c) For first order reflection $n = 1$
(d) For diffraction, the wavelength of X-ray should be of the order of lattice parameter.

For the X-ray, the correct statement(s) is/are
(a) The intensity of X-ray beam transmitted through a given material decreases exponentially.
(b) The maximum frequency of X-ray corresponding to $\lambda_{\min}$ is $\frac{c}{\lambda_{\min}}$.
(c) The short wavelength limit $\lambda_{\min}$ is inversely proportional to the electrode potential.
(d) The diffraction of X-rays is governed by Bragg's law, $2d\sin\theta = n\lambda$

PART-C (Numerical Answer Type (NAT)):

The lattice parameter of copper (fcc) is 3.61Å. The first order (111) plane appear at an angle of 21.7°. The wavelength of the X-ray (in Å) used is ____________

The lattice parameter (in Å) of nickel (fcc) if the Bragg's angle for its (220) reflection is 38.2° and the wavelength of the X-ray used is 1.54 Å, is ______________

The glancing angle (in degree) on the plane (110) of a cube of rock salt (a = 2.81 Å) corresponding to second order diffraction maximum for the X-rays of wavelength 0.71Å, is _____________

The shortest wavelength (in Å) that is present in the X-rays produced at an accelerating potential of 50 kV is ________________

Copper (fcc) has density of 8960 kg/m^3. The unit cell dimension (in Å) is (given the atomic mass of Cu as 63.54 amu) __________

ANSWER KEY						
Multiple Choice Questions (MCQ)						
Questions	1	2	3	4	5	6
Answers	a	c	c	b	b	d
Questions	7	8	9	10	11	12
Answers	a	a	b	a	b	c
Questions	13	14	15			
Answers	c	a	a			
Multiple Select Questions (MSQ)						
Questions	1	2	3	4	5	
Answers	a,b	a,b,c,d	a,b,c	a,c,d	a,b,c,d	
Numerical Answer Type (NAT)						
Questions	1	2	3	4	5	
Answers	1.54	3.54	20.9	0.525	3.61	

Chapter 3

MISCELLANEOUS

LATTICE VIBRATIONS

The ability to hold the atoms or ions together is called bonding.

Types of bonding:

The individual atoms of a crystalline solid are held together as one system with different types of bonds existing between them. These bonds are classified into five idealised types.

(i) Ionic bonding : Ionic bond is formed by the complete transfer of electron from one atom to another atom.

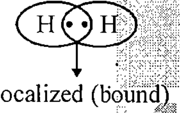
Example : NaCl. In NaCl, Na has 11 electrons and Cl has 17 electrons. Bond in NaCl crystal is formed by complete transfer of electrons from Na to Cl.

$$\overset{\curvearrowright}{\underset{11}{\mathrm{Na}}\quad\underset{17}{\mathrm{Cl}}} \longrightarrow \mathrm{Na}^{+}-\mathrm{Cl}^{-}$$

"Ionic bond is formed between electropositive and electronegative atoms". This bond has spherically symmetric charge distribution. Hence, it is not directional in nature.

(ii) Covalent bonding : Covalent bond is formed by sharing of electrons between two atoms

Examples: H_2, diamond, SiC.



In this type of bonding, charge distribution is not spherically symmetric. Charge density is maximum in the direction of bond formation. Hence, this bond is highly directional in nature.

(iii) Metallic bonding : This type of bond is also formed by sharing of electrons but shared electrons pair is not localized. Shared electron pair can move within the solid.

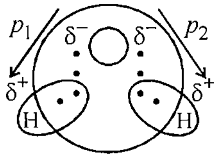
Examples: Cu, Ag, Fe etc.

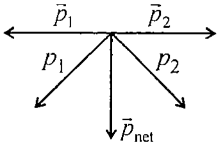
(iv) Van der Waal bonding : This type of bonding is generally formed between those atoms or molecules which do not have tendency to share or transfer of electrons among them. Van der waal bonding is very weak bonding. This bonding arises due to induced dipole moment. This is slightly directional.

Examples: Inert gases like He, Ne, Ar, solid Argon.

(v) Hydrogen bonding : When a covalent bond is formed between hydrogen atom and highly electronegative atom like oxygen, Cl, etc. then shared electron pair attracted more towards electronegative atoms. This causes the formation of permanent dipole.

Example: H_2O.





Permanent electric dipole moment is the source of H-bonding.

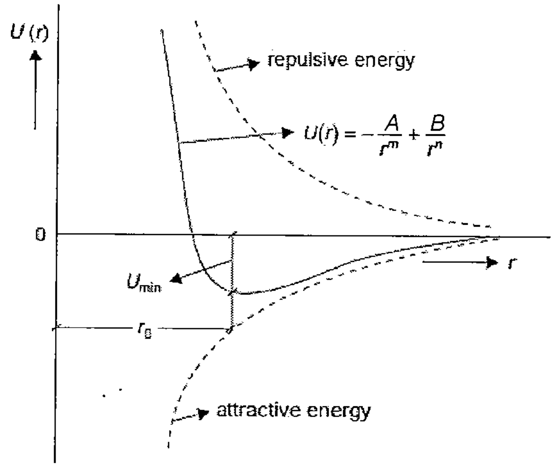
The reason of H-bonding is bond formation between highly electronegative atom and hydrogen. This bond is also directional in nature.

Cohesive Energy of solid :

The cohesive energy of a solid is defined as the energy which will be given out in the process of the formation of a crystal by bringing neutral atoms from infinity to the position of equilibrium separation.

If the potential energy due to attraction U_a is supposed to vary as the m^{th} power of the distance and that due to repulsion U_r as n^{th} power of r, then

$$U_a \propto \frac{1}{r^m} \Rightarrow U_a = -\frac{A}{r^m} \quad \text{and} \quad U_r \propto \frac{1}{r^n} \Rightarrow U_r = \frac{B}{r^n}$$



Therefore, resultant potential or cohesive energy, $U = U_a + U_r = -\frac{A}{r^m} + \frac{B}{r^n}$

where, A and B are constants of proportionality. The cohesive enegy is generally expressed in eV per atom.

The force between the two atoms is given by $F = \frac{-dU}{dr} = \frac{mA}{r^{m+1}} - \frac{nB}{r^{n+1}}$

At $r = r_0$ the attractive and repulsive forces balance each other and $F(r_0) = 0$. In other words at $r = r_0$, we get

$$\frac{mA}{r_0^{m+1}} - \frac{nB}{r_0^{n+1}} = 0 \quad \Rightarrow \frac{m}{n} = \frac{B}{A} \cdot r_0^{m-n} \Rightarrow r_0^{m-n} = \frac{A}{B} \cdot \frac{m}{n}$$

Hence, the potential energy at the equilibrium separation r_0 becomes

$$U(r_0) = \frac{-A}{r_0^m} + \frac{B}{r_0^n} = \frac{-A}{r_0^m}\left(1 - \frac{B}{A} r_0^{m-n}\right) = \frac{-A}{r_0^m}\left(1 - \frac{m}{n}\right)$$

Binding or cohesive energy of ionic crystals:

The calculation of binding or cohesive energy of crystals is one of the major problems of the theory of solids and reuire the knowledge of constituents. For ionic crystals the calculation is simple and in further sections we will describe the calculation of binding energy of NaCl crystal.

The calculation is based on the assumption that each ionic crystal is composed of positive and negative ions. Each of the ions have a spherical symmetric distribution as in rare gas atoms. Thus, it can be concluded that the force between the ions depends on the distance between the ions ad is independent of the direction of approach. The force between ions is assumed to be electrostatic in nature. Thus the main contribution to the binding energy arises from electrostatic interaction and is called as Madelung energy.

In ionic crystal there exist two types of interactions, one is long range Coulomb's electrostatic interaction which may be either attractive or repulsive, and the other is short range repulsive interaction.

The coulomb energy of attraction between two ions seperated by a distance r is

$$U_a = -\frac{e^2}{4\pi\varepsilon_0 r} \qquad \text{... (1)}$$

In the crystal, each ion interacts with all its neighbouring ions of the same as well as of opposite signs. To take this into account, the net energy of attraction per ion-pair may be written as

$$U_a = \frac{-\alpha e^2}{4\pi\varepsilon_0 r} \qquad \text{... (2)}$$

Where, α is the correction factor, called as Madelung constant. For simple crystals α lies between 1.6 and 1.8. For NaCl crystal, $\alpha = 1.7475$ and for CsCl crystal, $\alpha = 1.7626$.
Now, in an ionic crystal as two ions approach each other under coulomb attraction, at a certain distance apart, they begin to repel each other with a force which rise very rapidly with decreae in distance.

$$\therefore \quad U_r = \frac{B}{r^n} \qquad \text{... (3)}$$

$\therefore$ The total potential energy $U = U_a + U_r = -\alpha \dfrac{e^2}{4\pi\varepsilon_0 r} + \dfrac{B}{r^n}$ ... (4)

At equilibrium distance $r = r_0$, U_r will be minimum.

$$\therefore \quad \left.\frac{dU}{dr}\right|_{r=r_0} = 0 = \frac{\alpha^2 e^2}{4\pi\varepsilon_0 r_0^2} - \frac{nB}{r_0^{n+1}}$$

$$\Rightarrow \quad B = \frac{\alpha e^2}{4\pi\varepsilon_0 n} r_0^{n-1}$$

$$\therefore \quad U|_{r=r_0} = -\frac{\alpha e^2}{4\pi\varepsilon_0 r_0} + \frac{\alpha e^2}{4\pi\varepsilon_0 r_0 n} = \frac{-\alpha e^2}{4\pi\varepsilon_0 r_0}\left(1 - \frac{1}{n}\right) \qquad \text{... (5)}$$

SOLVED PROBLEMS

The mutual potential energy V of two particles depends on their spatial separation as follows:

$$V = \frac{a}{r^2} - \frac{b}{r};\ a > 0;\ b > 0$$ **[BHU 2012]**

For what separation are the particles in static equilibrium ?

(a) $\dfrac{a}{b}$ (b) $\dfrac{a}{2b}$ (c) $\dfrac{a^2}{b}$ (d) $\dfrac{2a}{b}$

ln. $V = \dfrac{a}{r^2} - \dfrac{b}{r};\ a > 0,\ b > 0$

For equilibrium, $\left.\dfrac{dV}{dr}\right|_{r=r_e} = 0$

$$\Rightarrow \quad a\left[\frac{-2}{r_e^3}\right] - b\left[\frac{-1}{r_e^2}\right] = 0$$

$$\Rightarrow \quad -\frac{2a}{r_e^3} + \frac{b}{r_e^2} = 0$$

$$\Rightarrow \frac{2a}{r_e^3}=\frac{b}{r_e^2} \Rightarrow r_e=\frac{2a}{b}$$

Hence, correct option is (d).

2. The origin of van der Waal's interaction in molecular crystal is **[BHU 2012]**

(a) nuclear (b) magnetic (c) ionic (d) fluctuating dipolar

Soln. The origin of Van der Waal's interaction in molecular crystal is fluctuating dipolar.
Hence, correct option is (d).

LATTICE VIBRATIONS

Lattice vibration:

A lattice may be regarded as a regular arrangement of atoms in which atoms are joined by elastic massless springs. The motion of any single atom is shared by all the atoms. The lattice can vibrate freely in its normal mode or it may experience forced vibrations. The normal mode vibrations are produced due to its internal energy while forced vibrations produced due to effect of dynamical external forces. The external forces may be of mechanical or electromagnetic type in nature. Normal mode vibrations give informations like thermal properties such as specific heat and thermal conductivity while forced vibrations give informations such as acoustical and optical properties of solids.

Vibrations of one-dimensional monoatomic lattice:

The dispersion relation $(\omega\text{-}k)$ for monoatomic lattice having atoms of mass m is given by

$$\boxed{\omega=\sqrt{\frac{4C}{m}}\sin\left|\frac{ka}{2}\right|}$$

where, C is spring constant and a is lattice parameter.

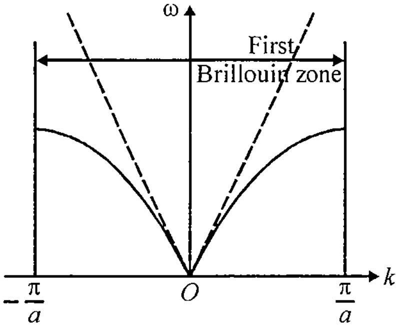
Limiting cases: (a) At lower frequencies or longer wavelength limit i.e. $\lambda\to\infty \Rightarrow k\to 0$,

$$\sin\left(\frac{ka}{2}\right)\to\frac{ka}{2}$$

Thus dispersion relation gives, $\omega=2\sqrt{\frac{C}{m}}\cdot\sin\frac{ka}{2}=2\sqrt{\frac{C}{m}}\cdot\frac{ka}{2}=a\sqrt{\frac{C}{m}}\cdot k$

The phase velocity $(v_p)=\frac{\omega}{k}=a\sqrt{\frac{C}{m}}=v_s$ and the group velocity $(v_g)=\frac{d\omega}{dk}=a\sqrt{\frac{C}{m}}=v_s$

Thus, we can say for monoatomic lattice in the long wavelength limit, phase velocity and group velocity both are same and equal to velocity of sound waves i.e., $v_p=v_g=v_s=a\sqrt{\frac{C}{m}}$

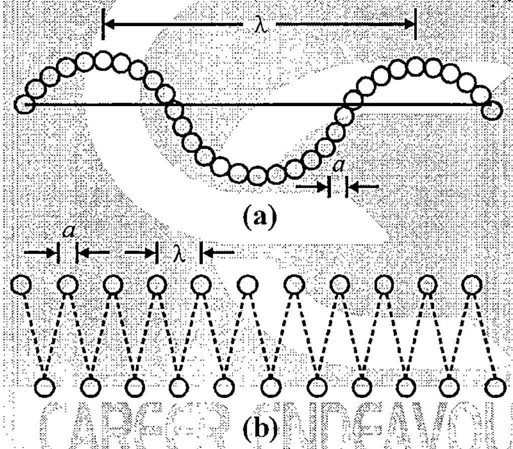


Dispersion relation for a one-dimensional monoatomic lattice

(b) At higher frequencies or smaller wavelength limit, i.e. $\lambda \to 0 \Rightarrow k \to$ higher, thus the phase velocity

$$\left(v_p\right) = \frac{\omega}{k} = \frac{2}{k}\sqrt{\frac{C}{m}}\sin\frac{ka}{2} \text{ and group velocity } \left(v_g\right) = a\sqrt{\frac{C}{m}}\cos\frac{ka}{2}.$$

Thus, both v_p and v_g are functions of frequency. Thus, for this case, this is called as phenomena of dispersion and the medium is known as dispersive medium. For long wavelength limit, $v_p = v_g = v_s$, i.e. dispersion effects are negligible and medium behaves like a homogeneous continuous medium.



A linear line of atoms consisting a wave (a) Long wavelength case
when the motion approaches that of a homogeneous line (b) standing wave

The maximum frequency $\omega = 2\sqrt{\frac{C}{m}}$ will give $k = \frac{\pi}{a}$ or $\lambda = 2a$ and from this, the group velocity $v_g = 0$ and phase velocity $v_p = \frac{2}{k}\sqrt{\frac{C}{m}} = \frac{2a}{\pi}\sqrt{\frac{C}{m}}$.

This confirms that no transfer of signal or energy take place for this frequency limit and the wave behaves like a standing wave.

Vibrations of one-dimensional diatomic lattice:

The dispersion relation $(\omega\text{-}k)$ for monoatomic lattice having atoms of masses m_1 and m_2 is given by

$$\omega^2 = C\left(\frac{1}{m_1}+\frac{1}{m_2}\right) \pm C\sqrt{\left(\frac{1}{m_1}+\frac{1}{m_2}\right)^2 - \frac{4\sin^2 ka}{m_1 m_2}}$$

This is the dispersion relation for a linear diatomic lattice.

In monoatomic lattice, there was only one value of ω for one value of wavevector k. But here in diatomic lattice, for one value of k, we have two different values of ω .

Here, plus (+) sign indicates the frequency corresponding to optical branch and minus (–) sign indicates the acoustical branch.

(I) Optical branch : The dispersion relation for optical branch is

$$\omega_+^2 = C\left(\frac{1}{m_1}+\frac{1}{m_2}\right)+C\sqrt{\left(\frac{1}{m_1}+\frac{1}{m_2}\right)^2-\frac{4\sin^2(ka)}{m_1m_2}}$$

Limiting cases : (1) At lower frequencies, longer wavelength limit i.e., $\lambda\to\infty \Rightarrow k\to 0,\ \sin(ka)\to ka$.

Thus, $$\omega_+^2 = C\left(\frac{1}{m_1}+\frac{1}{m_2}\right)+C\sqrt{\left(\frac{1}{m_1}+\frac{1}{m_2}\right)^2-\frac{4(ka)^2}{m_1m_2}}$$

$$\omega_+^2 = C\left(\frac{1}{m_1}+\frac{1}{m_2}\right)+C\sqrt{\left(\frac{1}{m_1}+\frac{1}{m_2}\right)^2-\frac{4k^2a^2}{m_1m_2}}$$

At $k=0,\ \omega_+^2 = C\left(\frac{1}{m_1}+\frac{1}{m_2}\right)+C\left(\frac{1}{m_1}+\frac{1}{m_2}\right)=2C\left(\frac{1}{m_1}+\frac{1}{m_2}\right) \Rightarrow \omega_+=\sqrt{2C\left(\frac{1}{m_1}+\frac{1}{m_2}\right)}$

Hence, the value of frequency (ω) is not zero at $k=0$.

(2) At higher frequencies, shorter wavelength limit $\lambda\to 0,\ k\to$ higher, thus $\sin^2(ka)\to 1$

Thus, $\omega_+^2 = C\left(\frac{1}{m_1}+\frac{1}{m_2}\right)+C\sqrt{\left(\frac{1}{m_1}+\frac{1}{m_2}\right)^2-\frac{4}{m_1m_2}}=C\left(\frac{1}{m_1}+\frac{1}{m_2}\right)+C\left(\frac{1}{m_1}-\frac{1}{m_2}\right)=\frac{2C}{m_1}$

$$\Rightarrow \quad \omega_+ = \sqrt{\frac{2C}{m_1}}$$

i.e., at $\sin^2(ka)=1 \Rightarrow \sin ka=\sin\frac{\pi}{2} \Rightarrow k=\frac{\pi}{2a}$ (at the boundary of the first Brillouin zone).

The frequency becomes $\omega = 2\sqrt{\frac{C}{m_1}}$.

(II) Acoustical branch : The dispersion relation for acoustical branch is

$$\omega_-^2 = C\left(\frac{1}{m_1}+\frac{1}{m_2}\right)-C\sqrt{\left(\frac{1}{m_1}+\frac{1}{m_2}\right)^2-\frac{4\sin^2 ka}{m_1m_2}}$$

Limiting cases : (1) At low frequencies longer wavelength limit i.e., $\lambda\to\infty \Rightarrow k\to 0,\ \sin(ka)\to ka$.

Thus, $$\omega_-^2 = C\left(\frac{1}{m_1}+\frac{1}{m_2}\right) - C\sqrt{\left(\frac{1}{m_1}+\frac{1}{m_2}\right)^2 - \frac{4k^2a^2}{m_1m_2}}$$

At $k=0$, $\omega_-^2 = C\left(\frac{1}{m_1}+\frac{1}{m_2}\right) - C\left(\frac{1}{m_1}-\frac{1}{m_2}\right) = 0 \Rightarrow \omega_- = 0$

Hence, the value of frequency (ω) is zero at $k=0$.

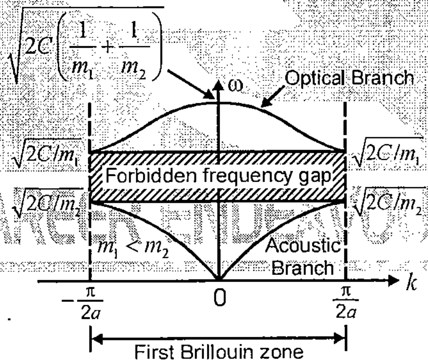
(2) At higher frequencies, shorter wavelength limit i.e., $\lambda \to 0 \Rightarrow k \to$ higher, thus $\sin^2(ka) \to 1$

Thus, $$\omega_-^2 = C\left(\frac{1}{m_1}+\frac{1}{m_2}\right) - C\sqrt{\left(\frac{1}{m_1}+\frac{1}{m_2}\right)^2 - \frac{4}{m_1m_2}} = C\left(\frac{1}{m_1}+\frac{1}{m_2}\right) - C\left(\frac{1}{m_1}-\frac{1}{m_2}\right) = \frac{2C}{m_2}$$

$$\Rightarrow \quad \omega_- = \sqrt{\frac{2C}{m_2}}$$

i.e., $\omega_- = \sqrt{\frac{2C}{m_2}}$ for $\sin ka = 1 \Rightarrow \sin ka = \sin\frac{\pi}{2} \Rightarrow k = \frac{\pi}{2a}$

i.e., at the boundary of first Brillouin zone, the frequency of acoustical branch becomes equal to $\sqrt{\frac{2C}{m_2}}$.



Thus, at the zone boundary, the frequency of acoustical branch depend upon the heavier mass (m_2). Therefore, frequency of optical branch and acoustical branch are different at zone boundary. The difference in the frequencies is known as forbidden frequency gap.

Phonons:

Since we know that the energy of a electromagnetic wave is quantized and the quantum of energy is known as photon. In the similar way, the energy of a lattice vibration of an elastic wave is also quantized and the quantum of lattice vibration is called as phonon. All types of lattice vibrations in crytals comprise phonons such as thermal vibrations are thermally excited phonons, sound waves are acoustical phonons and excitations of the branch generate optical phonons.

The energy of phonon is given by $\hbar\omega$, where ω is the angular frequency of a mode of vibration.

If n is the number of phonons in a particular mode of vibration, the total energy of that mode is $n\hbar\omega$, where n can be zero or a positive integer.

SOLVED PROBLEMS

1. Which of the following statements is NOT correct ?

(a) Acousatic branch of diatomic linear chain is similar to the monoatomic case

(b) Both group velocity and phase velocity are equal to the velocity of sound in the long wavelength limit

(c) Under the low wavelength limit, the lattice acts as a low pass filter

(d) Acoustic and optical modes in diatomic lattice cancel each other at the boundary of first Brillouin zone

Soln. For monoatomic lattice, $\omega = \sqrt{\frac{4c}{m}} \sin \left|\frac{ka}{2}\right|$, where m is mass of atom, c is force constant and k is wave vector.

For long wavelength limit $\lambda \to \infty \Rightarrow k = \frac{2\pi}{\lambda} \to 0$.

Thus, $\omega = \sqrt{\frac{4c}{m}} \sin \left|\frac{ka}{2}\right| = \sqrt{\frac{4c}{m}} \left|\frac{ka}{2}\right|$

Therefore, group velocity $(v_g) = \sqrt{\frac{4c}{m}} \left(\frac{a}{2}\right) = a\sqrt{\frac{c}{m}}$ $\left(\text{From } v_g = \frac{d\omega}{dk}\right)$

Phase velocity $(v_p) = \sqrt{\frac{4c}{m}} \left(\frac{a}{2}\right) = a\sqrt{\frac{c}{m}}$ $\left(\text{From } v_p = \frac{\omega}{k}\right)$

Also sound velocity $(v_s) = a\sqrt{\frac{c}{m}}$

Therefore, for monoatomic lattice, $v_p = v_g = v_s$; for $\lambda \to \infty$ (long wavelength limit)

The maximum frequency for the monoatomic lattice, $\omega_{\max} = \sqrt{\frac{4c}{m}} = 2\sqrt{\frac{c}{m}}$

Thus, the monoatomic lattice can pass the frequency lower than $2\sqrt{\frac{c}{m}}$. Higher frequency ($> \omega_{\max}$) is not allowed. Thus, the monoatomic lattice acts as a low pass filter. Also, acoustic branch of diatomic linear chain is similar to the monoatomic case.

The option (d) "Acoustic and optical modes in a diatomic lattice cancel each other at the boundary of first Brillouin zone" is incorrect.

Hence, correct option is (d).

SPECIFIC HEAT OF SOLIDS

The specific heat of solid:

From elementary knowledge, we know that the specific heat is the amount of heat energy that must be supplied to a mole of solid to raise its temperature by one degree. So, when some heat energy is supplied to a solid, its temperature rises and its internal energy is increased. The increase in the internal energy is manifested mainly as (i) an increase in the vibration of atoms about their mean position and (ii) an increase in the kinetic energy of free electrons. The corresponding specific heats are known as the lattice specific heat and the electronic specific heat, respectively. In order to measure the specific heat, either the volume or the pressure must be kept constant. In general, it is measured at constant volume, ie.

$$C_V = \left(\frac{dE}{dT}\right)_V$$

where E is the internal energy and T is the temperature of the system.

The classical model (Dulong-Petit's law):

From the kinetic theory of gases, we know that the kinetic energy of an atom of a monoatomic gas at temperature T is (3/2) k_BT, where k_B is the Boltzmann constant. Since an atom has three degrees of (translational) freedom, therefore, the energy (1/2) k_BT, is associated with each degree of freedom of the atom. Also, to each translational degree of freedom, there are associated two additional degrees of freedom, because energy can appear as kinetic or potential. If there are N atoms ($=N_A$, Avogadro's number for a monoatomic solid) in a mole of the solid, the total internal energy is

$$3\times 2\times N\times\frac{1}{2}k_BT = 3Nk_BT = 3RT \qquad \text{... (2)}$$

where $Nk_B = R$, the universal gas constant. Therefore, the specific heat at constant volume is

$$C_V = \left(\frac{dE}{dT}\right)_V = 3R = 5.96\ \text{cal}\cdot\text{mol}^{-1}\,\text{K}^{-1}$$

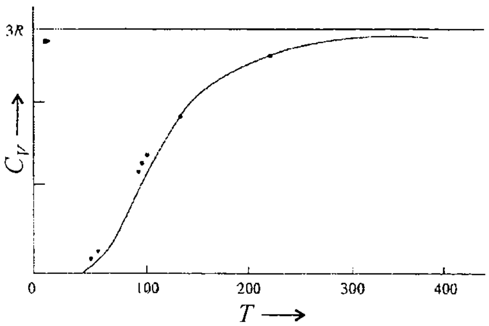
This is known as Dulong-Petit's law and is found to be generally correct at room temperature and above, for elements of atomic weight greater than 40.

The Einstein model:

Einstein was the first person to resolve the problem of specific heat of solids by the application of Planck's quantum theory. He assumed that a crystalline solid made up of N atoms (per mole) could be regarded as an array of atomic oscillators vibrating in three independent directions (as assumed for the classical calculations) with a constant frequency ν and that the allowed energy states of oscillators are integral multiples by $h\nu$.

The total internal energy of $3N$ harmonic oscillators (N atoms and 3 independent directions) is given by

$$E = \frac{3Nh\nu}{\exp\left(\frac{h\nu}{k_BT}\right)-1}$$ and the specific heat $$C_V = \left(\frac{dE}{dt}\right)_V = 3Nk_B\left(\frac{h\nu}{k_BT}\right)^2\cdot\frac{\exp\left(\frac{h\nu}{k_BT}\right)}{\left[\exp\left(\frac{h\nu}{k_BT}\right)-1\right]^2}$$



Einstein specific heat curve

The Einstein specific heat curve is fairly close to the experimental curve except at low temperature. In this region, the Einstein specific heat approaches zero more rapidly than the observed values. However, at high temperatures, it approaches the classical value. Let us consider the two temperature ranges separately.

Limiting cases:

(i) High temperature range:

Let us define a quantity $x = h\nu / k_B T$. In this temperature range, $h\nu \ll k_B T$

$$E = 3Nk_BT \text{ and } C_V = \left(\frac{dE}{dT}\right)_V = 3Nk_B = 3R$$

This is same as the classical result.

(ii) Low temperature range:

At low temperatures, $h\nu \gg k_B T$ and hence $e^x \gg 1$.

$$E = 3Nh\nu \exp\left(-\frac{h\nu}{k_BT}\right) \text{and } C_V = \left(\frac{dE}{dT}\right)_V = 3Nh\nu\left[-\exp\left(-\frac{h\nu}{k_BT}\right)\cdot\frac{h\nu}{k_BT^2}\right]$$

$$= 3Nk_B\left(\frac{h\nu}{k_BT}\right)^2 \exp\left(-\frac{h\nu}{k_BT}\right)$$

This indicates that the exponential term is more important than the term $(h\nu/k_BT)^2$ in determining the variation of C_V with temperature. Thus with decreasing temperature, C_V drops exponentially.

The Debye model:

In 1912, Debye pointed out that the source of error lies in the assumption (that all atomic oscillators vibrate independently at a constant frequency) made by Einstein. According to him, the assumption is not justifiable since all atoms are elastically coupled to its neighbours. Debye therefore, simplified the problem by considering a solid as a continuously vibrating medium (as an elastic continuum), which gives rise to a spectrum of frequencies instead of a single frequency as assumed in the Einstein's model.

The internal energy of the Debye solid is given by

$$E = \frac{9N}{\nu_D^3}\int_0^{\nu_D} \frac{h\nu\nu^2 d\nu}{\exp\left(\frac{h\nu}{k_BT}\right) - 1}$$

Let us suppose that $x = h\nu / k_BT$, so that $d\nu = (k_BT\,dx)/h$. Further, if we suppose that $x_D = (h\nu_D)/k_BT$, above equation becomes

$$E = 9N\left(\frac{k_BT}{h\nu_D}\right)^3 k_BT\int_0^{x_D}\frac{x^3}{e^x - 1}dx$$

Now, we can define the Debye temperature as θ_D according to the equation, $h\nu_D = k_B\theta_D$, then

$$\theta_D = \frac{h\nu_D}{k_B} \text{ and } x_D = \frac{\theta_D}{T}\ ;\quad E = 9Nk_BT\left(\frac{T}{\theta_D}\right)^3\int_0^{x_D}\frac{x^3}{e^x - 1}dx$$

which does not depend explicitly on the volume and looks similar to the expression obtained for Einstein's model. However, let us examine the dependence of the integral on temperature. For simplicity, we shall consider the following two temperature ranges.

High temperature range:

We assumed earlier that $x = h\nu / k_BT$. In this temperature range, $T >> \theta_D$ and hence $k_BT \gg h\nu$ even when $\nu = \nu_D$, implying that x will be small over the whole range of the integration. So, that

$$\int_0^{x_D} \frac{x^3}{e^x - 1} dx \to \int_0^{x_D} \frac{x^3}{x} dx = \frac{x_D^3}{3} = \frac{1}{3}\left(\frac{\theta_D}{T}\right)^3$$

Therefore, $E = 9Nk_BT \times \frac{1}{3} = 3Nk_BT$

which is the classical result. This implies that in the high temperature range, the specific heat of solids is the same as measured in Dulong and Petit's experiment.

Low temperature range:

In the low temperature range, $T << \theta_D$, so that $k_BT \ll h\nu_D$. In this case, the upper limit of the integration can be replaced by infinity (i.e. $x_D \to \infty$) and hence the above integral becomes

$$\int_0^{x_D} \frac{x^3}{e^x - 1} dx \to \int_0^{\infty} \frac{x^3}{e^x - 1} dx \to \frac{\pi^4}{15}$$

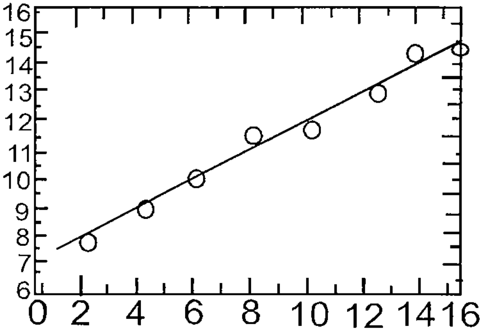
Therefore, $E = 9Nk_BT\left(\frac{T}{\theta_D}\right)^3 \times \frac{\pi^4}{15} = \frac{3}{5}\pi^4 Nk_BT\left(\frac{T}{\theta_D}\right)^3$

and $C_V = \frac{12}{5}\pi^4 Nk_B\left(\frac{T}{\theta_D}\right)^3$

This is a famous expression known as Debye T^3 law for specific heat at low temperature.

SOLVED PROBLEMS

1. The ratio of the specific heat capacity and temperature, C_υ / T, of Cu is plotted as a function of T^2, the square of the absolute temperature, in the graph below: **[JNU 2012]**



The values of γ and β (the coefficients corresponding to the electronic and the vibrational components of the specific heat) are, approximately

(a) $\gamma = 7.0\times10^{-4}\,\text{Jmol}^{-1}\text{K}^{-2}$ and $\beta = 5.0\times10^{-5}\,\text{Jmol}^{-1}\text{K}^{-4}$

(b) $\gamma = 5.0\times10^{-5}\,\text{J mol}^{-1}\,\text{K}^{-2}$ and $\beta = 7.0\times10^{-4}\,\text{J mol}^{-1}\text{K}^{-4}$

(c) $\gamma = 1.4\times10^{-3}\,\text{J mol}^{-1}\,\text{K}^{-2}$ and $\beta = 7.0\times10^{-4}\,\text{J mol}^{-1}\text{K}^{-4}$

(d) $\gamma = 5.0\times10^{-4}\,\text{J mol}^{-1}\,\text{K}^{-2}$ and $\beta = 7.0\times10^{-5}\,\text{J mol}^{-1}\text{K}^{-4}$

Soln. Specific heat varies as, $\dfrac{C}{T} = \gamma + \beta T^2$

$$\Rightarrow \quad \gamma = \text{Intercept} = 7 \times 10^{-4}\ \text{J mol}^{-1}\ \text{k}^{-2}\ \ \&\ \ \beta = \text{slope} = \frac{(9-8)\times10^{-4}}{4-2} = 5 \times 10^{-5}\ \text{J mol}^{-1}\ \text{k}^{-4}$$

Correct option is (a)

2. The Dulong-Petit law fails near room temperature (300 K) for many light elements (such as boron and beryllium) because their Debye temperature is

(a) >> 300 K (b) ~ 300 K (c) << 300 K (d) 0 K **[JEST-2012]**

Soln. The Dulong-Petit law fails near room temperature (300 K) for many light elements (such as boron and beryllium) because their Debye temperature is larger than 300 K.

Correct option is (a)

3. What is the contribution of the condition electrons in the molar entropy of a metal with electronic coefficient of specific heat ?

(a) γT (b) γT^2 (c) γT^3 (d) γT^4 **[JEST-2014]**

Soln. The contribution of the condition electrons in the molar entropy of a metal with electronic coefficient of specific heat is γT.

Correct option is (a)

4. Debye temperatures of copper and gold are 340K and 170K, respectively. At room temperature, if the specific heat for copper is 1.5 $\text{kJmol}^{-1}\text{K}^{-1}$, the specific heat for gold is about: **[DU 2014]**

(a) 0.19 $\text{kJmol}^{-1}\text{K}^{-1}$ (b) 1.2 $\text{kJmol}^{-1}\text{K}^{-1}$ (c) 1.9 $\text{kJmol}^{-1}\text{K}^{-1}$ (d) 12 $\text{kJmol}^{-1}\text{K}^{-1}$

Soln. Debye temperature for copper $(\theta_D)_{\text{Cu}} = 340\ \text{K}$

Debye temperature for gold $(\theta_D)_{\text{Au}} = 170\ \text{K}$

$(C_V)_{\text{Cu}} = 1.5\ \text{kJ-mol}^{-1}\text{K}^{-1}$

$(C_V)_{\text{Au}} = ?$

Since, $C_V \propto \left(\frac{T}{\theta_D}\right)^3 \Rightarrow \frac{(C_V)_{\text{Cu}}}{(C_V)_{\text{Au}}} = \left[\frac{(\theta_D)_{\text{Au}}}{(\theta_D)_{\text{Cu}}}\right]^3 = \left(\frac{170}{340}\right)^3 = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$

$\Rightarrow (C_V)_{\text{Au}} = 8 \times (C_V)_{\text{Cu}} = 8 \times 1.5 = 12 \text{ kJ-mol}^{-1}\text{K}^{-1}$

Hence, correct option is (d).

The Debye temperature of a metal is 450 K. Its Debye frequency in Hz is of the order of

(a) 10^{11} (b) 10^{13} (c) 10^{16} (d) 10^{20}

ln. Debye temperature of metal $\theta_D = 450\,\text{K}$ **[DU 2015]**

Since, $\theta_D = \frac{h\nu_D}{k_B} \Rightarrow \nu_D = \frac{\theta_D \times k_B}{h} = \frac{450 \times 1.38 \times 10^{-23}}{6.6 \times 10^{-34}} = 94.09 \times 10^{11}\text{ Hz} = 0.94 \times 10^{13}\text{ Hz}.$

Thus, the Debye frequency in Hz is of the order of 10^{13}.

Hence, correct option is (b).

According to Dulong-Petit's law, the specific heat of a solid **[DU 2016]**

(a) is proportional to the temperature (b) does not depend on temperature

(c) depends on square of temperature (d) is inversely proportional to temperature

ln. According to Dulong-Petit's law, the specific heat of a solid

$C_V = 3R$ ... (i)

Thus, the specific heat of a solid does not depend on temperature.

Hence, correct option is (b).

Assume that the heat capacity at constant volume of a metal varies as $aT + bT^3$ at low temperatures. The temperature dependence of entropy is given by **[BHU 2012]**

(a) $aT + bT^3$ (b) $a + bT^2$ (c) $aT + \frac{b}{3}T^3$ (d) $aT^2 + bT^4$

ln. Since $C_V = aT + bT^3$

$$\Rightarrow \left(\frac{dQ}{dT}\right)_V = aT + bT^3$$

$$\Rightarrow T\frac{dS}{dT} = aT + bT^3$$

$$\Rightarrow dS = \left(\frac{aT + bT^3}{T}\right)dt = \left(a + bT^3\right)dT$$

Integrating on both sides, we get $\Delta S = aT + \frac{bT^3}{3}$.

Hence, correct option is (c).

The specific heat C_V due to free electrons in metals varies as **[BHU 2013]**

(a) $C_V \propto T$ (b) $C_V \propto T^2$ (c) C_V is constant (d) $C_V \propto T^{-1}$

ln. The specific heat C_V due to free electrons in a metal varies as $C_V \propto T$.

Hence, correct option is (a).

9. According to Debye model heat capacity $[C_V]_{\text{lattice}}$ at low temperature varies as proportional to

(a) T (b) T^3 (c) T^{-1} (d) T^2 **[BHU 2013]**

Soln. According to Debye model heat capacity $[C_V]_{\text{lattice}}$ at low temperature is given by $[C_V]_{\text{lattice}} \propto T^3$.

Hence, correct option is (b).

10. The electronic specific heats in metals are given by : (R is gas constant, k-boltzmann constant) :

(a) $\frac{kT}{E_0}$ (b) $\frac{R}{E_0}kT$ (c) $\frac{E_0}{R}kT$ (d) $\frac{3E_0}{R}kT$ **[BHU 2014]**

Soln. The electric specific heats in metal are given by $C_e = \frac{R}{Eo}kT$

Hence, correct option is (b).

11. The lattice specific heat at constant volume C_V of a solid at lower temperature depends on temperature T as

(a) $C_V \propto T$ (b) $C_V \propto T^3$ (c) $C_V \propto T^2$ (d) $C_V \propto \frac{1}{T}$ **[BHU 2015]**

Soln. According to Debye theory, the lattice specific heat, $C_V \propto T^3$ (at low temperature).

Hence, correct option is (b).

MAGNETIC PROPERTIES OF SOLIDS

Response of substance of magnetic field:

A magnetic field can be described by either magnetic induction B (or magnetic flux density) or the field strength H. In vacuum, they are related by the equation

$$B = \mu_0 H \qquad \text{... (1)}$$

where $\mu_0 = 4\pi \times 10^{-7}\ H - m^{-1}$ and is called the permeability of the free space (vacuum).

When a substance is placed in a magnetic field, it gets magnetized and hence a magnetization M(defined as the magnetic moment per unit volume, i.e. $M = \mu_m / \Delta V$) is produced in it. The magnetic induction inside the substance is given by

$$B = \mu_0 H + \mu_0 M = \mu_0 (H + M) \qquad \text{... (2)}$$

where the first term on the right side of equation (2) is due to external field and the second term is due to the magnetization. For an isotropic medium, M and H are parallel vectors and are related to each other according to the relation

$$M = \chi H \qquad \text{... (3)}$$

where χ is the susceptibility of the medium and is a scalar quantity. Substituting the value of M from equation (3) into equation (2), we have

$$B = \mu_0 (1 + \chi) H = \mu H \qquad \text{... (4)}$$

where $\mu = \mu_0 (1 + \chi)$ ... (5)

is known as the permeability of the medium. It is often more convenient to use the relative permeability μ_r, which is defined as

$$\mu_r = \frac{\mu}{\mu_0} = (1 + \chi) \qquad \text{... (6)}$$

Magnetic susceptibility is defined as $\chi = \frac{M}{H}$.

Classification of magnetic materials:

Magnetic materials can be classified into different categories according to their χ values and the way in which these vary with the magnetic field strength and temperature. They are: (i) Diamagnetic, (ii) Paramagnetic, (iii) Ferromagnetic, (iv) Antiferromagnetic and (v) Ferrimagnetic.

Diamagnetic Materials:

The Diamagnetic property is the result of an induced magnetic moment. This arises when an atom is placed in a magnetic field. The motion of orbital electrons of the atom (analogous to a current flowing in a circuit) gets modified in such a way that weak magnetic moment opposing the field is induced. Also, if χ is negative in equation (3), the direction of magneticzation M is opposite to that of the field H. If M and H behave linearly then $\chi = \frac{M}{H}$ (for diamagnetic material)

A diamagnetic solid has a tendency to repel the magnetic lines of force due to an external applied field. A superconductor which repels all the lines of force, is an example of a perfect diamagnet.

Diamagnetic substances include inert gases (helium, argon, etc), metals (bismuth, copper, zinc, gold, silver etc.), mercury, water, glass, marble and many other organic compounds.

Diamagnetic effect arises due to change in the orbital motion of electron around the nucleus on the application of external magnetic field. These materials repel in the magnetic field. It is a very weak effect.

Thus, equation (2) $\Rightarrow$ $\mu H = \mu_0(H + \chi_m H)$

$$\Rightarrow \frac{\mu}{\mu_0} = 1 + \chi_m \Rightarrow \mu_r = 1 + \chi_m$$

Hence, $\vec{B} = \mu_0 \mu_r \vec{H}$

For diamagnetic material, $\mu_r < 1$ and $\chi_m < 0$ i.e. negative susceptibility.

Therefore, susceptibility of diamagnetic material is negative. Diamagnetic susceptibility of a solid according to quantum mechanics is given by

$$\boxed{\chi_{dia} = \frac{-\mu_0 N Z}{6m_e} e^2 \langle r^2 \rangle}$$

where, N is number of atoms per unit volume, Z is atomic number, m_e is mass of the electron and $< r^2 >$ is average value of r^2 over the particular orbit.

From above equation diamagnetic susceptibility is negative and independent of temperature.

Paramagnetic Materials:

Paramagnetism occur in those solids which have permanent magnetic moment due to spins of unpaired electrons on the application of magnetic field. These magnetic moment aligned in the direction of applied magnetic field. Hence effect is attractive type. This is strong effect as compare to diamagnetic effect. Molecular oxygen (O_2) has even number of electrons but shows paramagnetic effect (exception).

Paramagnetic suceptability of ions or atoms is given by $\chi_{ions} = \frac{C}{T}$. This is known as Curie's law.

where, C is Curie constant i.e. $\chi_{ions} \propto \frac{1}{T} \Rightarrow \boxed{\chi \propto \frac{1}{T}}$

Ferromagnetic Material :

Like paramagnetism, ferromagnetism is also associated with permanent magnetic moment but ferromagnetic materials are characterize by spontaneous magnetisation (i.e. Non-zero magnetisation even in the absence of magnetic field).

Explanation of ferromagnetism can be given in terms of formation of small regions inside the solid in which magnetisation is saturated. These small regions are called domains.

Suceptability of ferromagnetic material is given by $\chi = \frac{C}{(T - T_C)}$, where T_C is ferromagnetic curie temperature

The Weiss molecular (Exchange) field:

In order to explain the spontaneous magnetization of a ferromagnetic material, Weiss (1907) assumed that there exists an internal molecular field (also called exchange field H_{ex}) acting on a given dipole and is given by

$$H_{ex} = H + \lambda M \qquad \text{... (i)}$$

where λ is called Weiss constant (it is independent of temperature) and H is applied field. In a ferromagnet the Curie law holds for exchange field as well. Therefore,

$$\frac{M}{H_{ex}} = \frac{C}{T} \qquad \text{... (ii)}$$

Simplifying the equation (i) and (ii), we have

$$M = \frac{CH}{T - C\lambda} \qquad \text{... (iii)}$$

Therefore, the ferromagnetic susceptibility becomes,

$$\chi = \frac{M}{H} = \frac{C}{T - C\lambda} = \frac{C}{T - T_C} \qquad \text{... (iv)}$$

where $T_C = C\lambda$ is called the Curie temperature and the equation (iv) is known as Curie-Weiss law. The coresponding variation of suscepbtibility with temperature is shown in figure. As long as $T \geq T_C$, this curve and the curve shown in figure are similar. However, as $T \to T_C$. Curive Weiss susceptibility tends to infinity and for $T \leq T_C$ there exists a spontaneous magnetization in the material even in the absence of an external field.

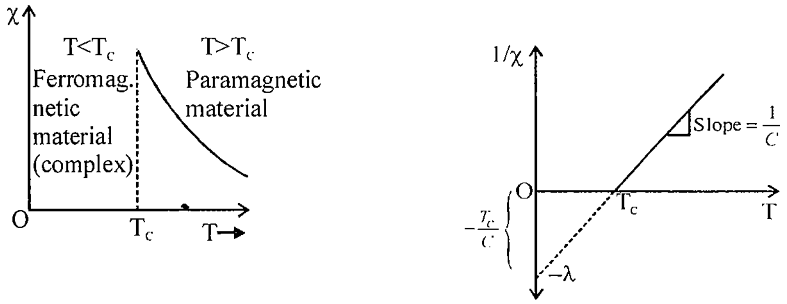


Figure : χ vs T and $\frac{1}{\chi}$ vs T for ferromagnetic materials

SOLVED PROBLEMS

A paramagnetic gas at room temperature is placed in an external magnetic field of 1.5 T (tesla). Each atom of the gas has a magnetic moment $\mu = 1.0\,\mu_{B,}$ where $\mu_B = 9.3\times10^{-24}$ J/T is the Bohr magneton. The difference in energy when an atom is aligned along the magnetic field and opposite to it, is **[JNU 2012]**

(a) 2.8×10^{-23} J (b) 1.4×10^{-23} J (c) 18.6×10^{-24} J (d) 9.3×10^{-24} J

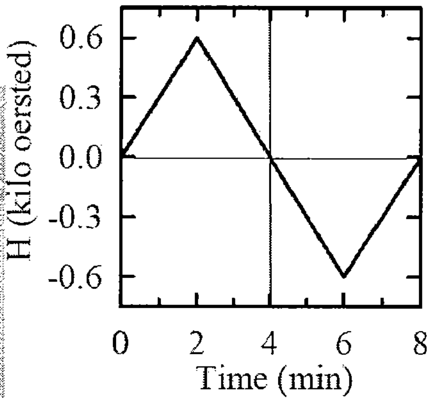
ln. When atom is along the magnetic field, $E_1 = -\mu B$

when atom is opposite to the magnetic field, $E_2 = +\mu B$

$$\therefore \quad \Delta E = E_2 - E_1 = 2\mu B = 2\times1.0\times9.3\times10^{-24}\times1.5 = 2.8\times10^{-23}\text{ J.}$$

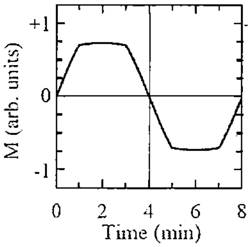
Correct option is (a)

An unmagnetised sample of iron is placed in a magnetic field H which varies with time as shown in the plot below. **[TIFR 2011]**

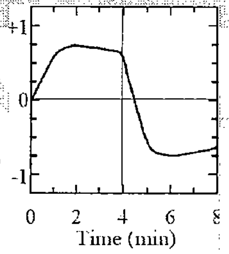


The magnetisation M of this iron sample is continuously measured and also plotted as a function of time. The appearance of this plot will be closest to

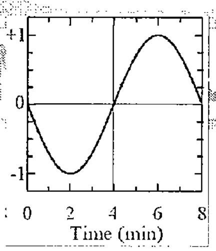
(a)



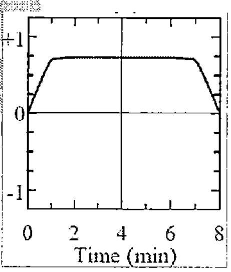
(b)



(c)



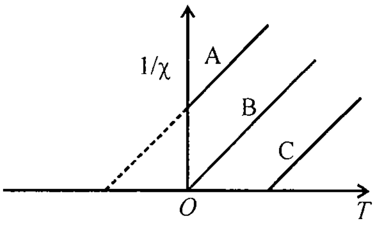
(d)



ln. The magnetisation versus time graph is closest to option (b).

Correct option is (b)

The magnetic susceptibility χ of three samples A, B and C is measured as a function of their absolute temperature T, leading to the graphs shown below.



From these graphs, the magnetic nature of the samples can be inferred to be

(a) A : anti-ferromagnet ; B : paramagnet ; C : ferromagnet
(b) A : diamagnet ; B : paramagnet ; C : anti-ferromagnet
(c) A : paramagnet ; B : anti-ferromagnet ; C : ferromagnet
(d) A : anti-ferromagnet ; B : diamagnet ; C : paramagnet **[TIFR-2013]**

Soln. The susceptibility of paramagnet is given by $\chi = \frac{C}{T} \Rightarrow \frac{1}{\chi} = \frac{T}{C}$ ... (i)

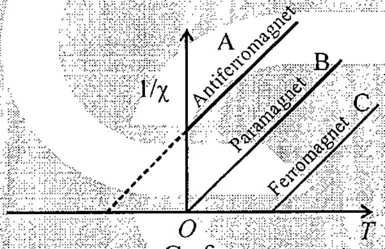
where, C is Curie constant.

The susceptibility of ferromagnet is given by $\chi = \frac{C}{(T-\theta)} \Rightarrow \frac{1}{\chi} = \left(\frac{T-\theta}{C}\right)$ ... (ii)

where, θ is Curie temperature and susceptibility of antiferromagnet is given by $\chi = \frac{C}{(T+T_N)}$, where T_N is Neel temperature.

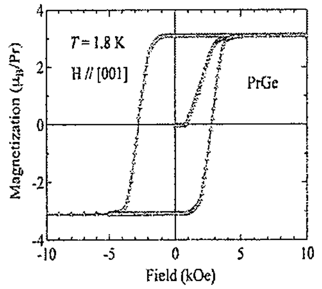
$$\Rightarrow \frac{1}{\chi} = \left(\frac{T+T_N}{C}\right)$$

Hence plot of $\frac{1}{\chi}$ versus T for paramagnet ; ferromagnet and antiferromagnet is given by



Thus, A : anti-ferromagnet ; B : paramagnet ; C : ferromagnet
Hence, correct option is (a).

4. The Curie temperature of a single crystal of PrGe is known to be 41 K. The magnetization data of this sample is measured at 1.8 K for the magnetic field applied parallel to the (001) direction is shown in the figure to the left. At a temperature of 38 K, the hysteresis loop in the figure will



(a) have the same width (b) decrease in width
(c) increase in width (d) shrink to a line **[TIFR-2013]**

Soln. The magnetization data of the sample at $T = 1.8$ K shows a M-H loop of ferromagnetic type. After Curie temperature $T = 41$ K, this will become paramagnetic type,
At $T = 38$ K, the width of hysteresis loop decreases and above $T = 41$ K, this will shrink to a line
Hence, correct option is (c).

For a paramagnetic material, the energy difference between spin magnetic dipole parallel and anti-parallel to an external field H is (μB is the Bohr magnetron) **[DU 2016]**

(a) $\mu_0 H/4\pi$ (b) $\mu_B H$ (c) $2\mu_B\mu_0 H$ (d) $\mu_B\mu_0 H$

oln. For parallel alignment of spin magnetic dipole indirection external field H, the energy $=-\mu_0\mu_B H\cos 0°$

$E_{\text{parallel}} = -\mu_0\mu_B H$ ... (i)

For antiparallel alignment of spin magnetic dipole in the direction of external field H, the energy $=-\mu_0\mu_B H\cos 180°$

$E_{\text{antiparallel}} = -\mu_0\mu_B H(-1) = \mu_0\mu_B H$ ... (ii)

From equations (i) and (ii), the energy difference,

$\Delta E = E_{\text{antiparallel}} - E_{\text{parallel}} = \mu_0\mu_B H - (-\mu_0\mu_B H) = 2\mu_0\mu_B H.$

Correct option is (c).

The susceptibility of a diamagnetic material is about **[DU 2015]**

(a) -10^{-5} (b) 10^{-7} (c) 10^2 (d) 10^3

oln. The magnetic susceptibility of the diamagnetic is always negative. Out of these options -10^{-5} will be magnetic susceptibility.

Hence, correct option is (a).

DIELECTRIC PROPERTIES OF SOLIDS

Dielectrics : Dielectrics are perfect insulators. In dielectrics electrons are very tightly bound to the atoms so that at ordinary temperatures they do not conduct any electric current.

Examples: Solids: glass, porcelin; gases: H_2, N_2; liquids: CH_3Cl, pure water

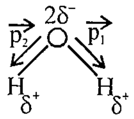
They are of two types.

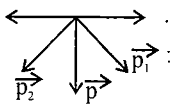
(i) Polar dielectrics

(ii) Non-polar dielectrics

Polar dielectrics are those dielectrics whose molecules contains (are characterized by) the permanent dipole moments. Ex - H_2O, NaCl, KCl.

Non-polar dielectrics are those dielectrics whose molecules do not contain the permanent dipole moments. Ex - Solid Ar.





H_2O: Electronegativity of O is higher than H so it will have more negative charge, so it contain permanent dipole moment. It is the inherent property of molecules.

If we apply external electric field on non-polar dielectrics then it will be polarized and this polarization is called induced polarisation or induced dipole momentum.

Thus, $\vec{p}_{in} \propto \vec{E}$

$$\Rightarrow \boxed{\vec{p}_{in} = \alpha\vec{E}_{loc}} \quad \text{... (1)}$$

where, α is proportionality constant and known as polarizability of atom or molecule and $\vec{E}_{loc}$ is local electric field.

Dipole moment : If we have two equal and opposite charges $+q$ and $-q$, separated by a distance r, then the dipole moment is defined as $p = qr$. The dipole moment is directed from the negative to positive charge.

Polarization: The polarization $(\vec{P})$ is defined as the dipole moment per unit volume i.e., $\vec{P} = \frac{\vec{p}}{\Delta V}$

$$\vec{P} = \chi_e \in_0 \vec{E}_{loc} \text{ also } \vec{p}_{in} = \alpha \vec{E}_{loc} \text{ and } \vec{P} = N\vec{p}_{in} = N\alpha\vec{E}_{loc} \qquad \ldots (2)$$

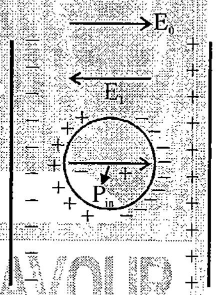
where χ_e is electric susceptibility and N is number of dipoles per unit volume.

$$\therefore \qquad \chi_e = \frac{N\alpha}{\in_0}$$

Since, $\in_r = 1 + \chi_e \qquad \therefore \in_r = 1 + \frac{N\alpha}{\in_0}$

For non-polar dielectrics: If we have a large parallel plate capacitor and we fill a non-polar dielectric between its plates. Suppose external applied field is $\vec{E}_0$. Due to this $\vec{E}_0$, non-polar dielectrics will be polarized then plates occur some bound charge and electric field develop is $\vec{E}_1$ which is less than $\vec{E}_0$.

Thus, resultant electric field, $\vec{E} = \underbrace{\vec{E}_0 + \vec{E}_1}_{E_{mac}}$



Macroscopic electric field is average electric field seen between the plates.

If there is a molecule between plates then it will behave as dipole and resultant field on this molecule is different from $\vec{E}_{mac}$ and this electric field is called local field.

$$\vec{E}_{loc} = \vec{E}_0 + \vec{E}_1 + \vec{E}_2 + \vec{E}_3$$

where $\vec{E}_0$ is externally applied electric field, $\vec{E}_1$ is depolarization field, $(\vec{E}_0 + \vec{E}_1)$ is macroscopic field, $\vec{E}_2$ is electric field due to the induce charges on the surface of the cavity (polarization field).

and $\vec{E}_3$ is electric field due to all the charges inside the cavity. Resultant of all these electric fields is the local field. Take a spherical cavity out from the surrounding of molecules. This cavity is called Lorentz cavity. If the dipole is at the centre of the cavity then charges on the cavity will be symmetric about centre.

So, $\boxed{\vec{E}_3 = 0}$ (due to symmetry)

Now, $\vec{E}_{loc} = \vec{E}_0 + \vec{E}_1 + \vec{E}_2$

$$\Rightarrow \qquad \vec{E}_{loc} = \vec{E}_{mac} + \vec{E}_2 \qquad \left(\text{since } E_2 = \frac{\vec{P}}{3\varepsilon_0}\right)$$

$$\Rightarrow \qquad \boxed{\vec{E}_{loc} = \vec{E}_{mac} + \frac{\vec{P}}{3\varepsilon_0}}$$

If solid contains N atoms (or molecules) per unit volume, each atoms due to the application of electric field will become dipole, so there will be N dipoles.

$$\vec{P} = N\vec{p}_{in}$$

$$\Rightarrow \boxed{\vec{P} = N\alpha\,\vec{E}_{loc}} \qquad \ldots (3)$$

From equation (2) and equation (3), we get

$$N\alpha\,\vec{E}_{loc} = \varepsilon_0\chi_e\vec{E}_{mac}$$

$$\Rightarrow N\alpha\,\vec{E}_{loc} = \varepsilon_0(\varepsilon_r - 1)\vec{E}_{mac} \qquad (\text{since } \chi_e = \varepsilon_r - 1)$$

$$\Rightarrow \frac{\vec{E}_{loc}}{\vec{E}_{mac}} = \frac{\varepsilon_0(\varepsilon_r - 1)}{N\alpha}$$

$$\Rightarrow \boxed{\frac{(\varepsilon_r - 1)}{(\varepsilon_r + 2)} = \frac{N\alpha}{3\varepsilon_0}}$$ where, α is electric polarizability and ε_r is dielectric constant.

This relation is known as Classius-Mossotie relation.

This is valid only for non-polar molecules.

Physical significance of Classius-Mossotie relation: Classius-Mossotie relation is a relationships between α and ε_r. α is a microscopic quantity and ε_r is macroscopic quantity (dielectric constant) because α corresponds to dipole moment and ε_r corresponds to polarisation.

For polar molecules, this equation becomes Langvein - Debye equation.

Types of polarisabilities: There are three types of polarisabilities

(i) Electronic polarisability (α_e)

(ii) Ionic polarisability (α_i)

(iii) Dipolar polarisability (α_d)

Thus total polarisability α is sum of electronic, ionic and dipolar polarisabilities.

i.e., $\alpha = \underbrace{\alpha_e + \alpha_i}_{\text{induced}} + \underbrace{\alpha_d}_{\text{Permanent}}$

(i) Electronic polarisability (α_e): This type of polarisability arises due to displacement of electron cloud of an atom relative to its nucleus in the presence of applied electric field.

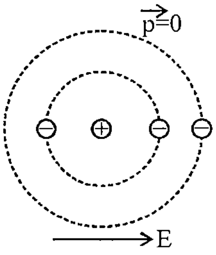
The Electronic polarisability is given by $\alpha_e = 4\pi\varepsilon_0 R^3$, where R is radius of the atom.

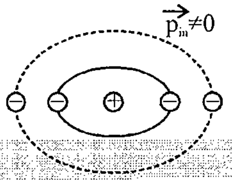
This type of polarisability is independent of temperature but depends on the frequency of applied electric field. The frequency dependence of α_e is given by

$$\boxed{\alpha_e = \frac{e^2}{m(\omega_0^2 - \omega^2)}}$$

where m is mass of the electron, e is electronic charge, ω_0 is natural frequency of oscillation of atom and ω is frequency of applied electric field.

Unit of α: In C.G.S. units, α has unit of volume (cm^3), and in M.K.S. units, α is measured in F – m^2.



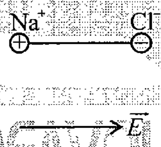


(ii) Ionic polarisability (α_i) :

This type of polarisability arises due to increase or decrease in the bond-length of ions.

This type of polarisability is independent of temperature but depends on the frequency of applied field. The ionic polarisability depends on frequency in following manner. For an ionic solid, the polarisability is given by

$$\boxed{\alpha_i = \frac{e}{\omega_0^2}\left(\frac{1}{m} + \frac{1}{M}\right)}$$



where m is mass of one ion (say Na$^+$),

M is mass of another ion (say Cl$^-$),

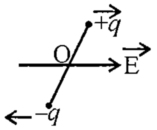
and ω_0 is natural frequency of the vibration of molecule (NaCl).

In general, $\boxed{\alpha_i \simeq \frac{\alpha_e}{10}}$.

(iii) Dipolar polarisability (α_d) :

This type of polarisability arrises due to the permanent dipole moment of the molecule. The dipolar polarisability is given by

$$\boxed{\alpha_d = \frac{p_p^2}{3k_BT}}$$



where, p_p is permanent dipole moment of the molecule, T is temperature and k_B is Boltzmann constant.

From above equation it is clear that α_d depends on both temperature and frequency of the field.

Hence total polarisability,

$$\alpha = \alpha_e + \alpha_i + \alpha_d$$

$$\Rightarrow \boxed{\alpha = \alpha_0 + \frac{p_p^2}{3k_B T}}$$

This relation is known as Langevin - Debye equation. This is valid for only polar molecules.

SOLVED PROBLEMS

Dielectric polarizability is a proportionality constant in the relation between:

(a) Electric dipole moment and local electric field (b) Displacement vector and local electric field

(c) Electric dipole moment and charge density (d) Displacement vector and charge density

ln. Electric dipole $(\vec{p}) \propto$ local electric field $(\vec{E}_{\text{local}})$ **[DU 2014]**

$\Rightarrow \quad \vec{p} = \alpha\, \vec{E}_{\text{local}}$, where α is dielectric polarizability.

Hence, correct option is (a).

What are example of piezo electric materials ? **[BHU 2014]**

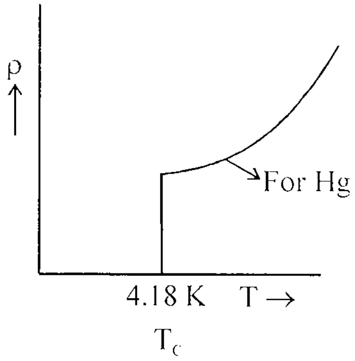
(a) Rochelle salt (b) lead zirconate (c) potassium niobate (d) barium titanate

ln. Out of the given options, Rochelle salt is the example of piezoelectric materials.

Hence, correct option is (a).

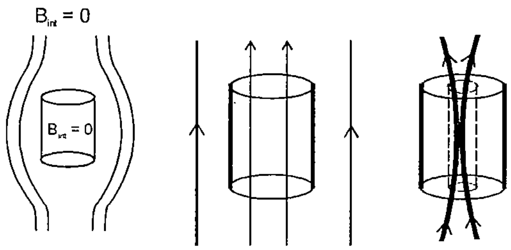
SUPERCONDUCTIVITY

In 1911, Kammerlingh Onnes discovered that the electrical resistivity of Hg (mercury) at low temperature $\simeq 4.18$ K, vanishes (zero) suddenly. This phenomenon was given the name superconductivity. The temperature at which the resistivity disappear (in zero magnetic field) is called the transition temperature or critical temperature (T_C). The transition temperature of mercury is 4.18 K. materials which shows this phenomenon are called superconducters.



Meissner's effect:

This effect is useful to make difference between superconductor and perfect (normal) conductor. When a superconductor and a perfect conductor are placed in an external magnetic field, the magnetic line of the forces do not passes through the superconductor and they are swept away. Thus making the magnetic field (B) inside the superconductor equal to zero. While line of magnetic force passes through the perfect conductor and therefore B is not equal to zero inside the perfect conductor.



As in case of superconductor, magnetic field $B = 0$

In C.G. S. units:

$$\vec{B} = \vec{H} + 4\pi M \Rightarrow \quad \frac{M}{H} = -\frac{1}{4\pi} \Rightarrow \boxed{\chi = -\frac{1}{4\pi}}$$

In M.K.S. units:

$$\Rightarrow \quad \mu_0 (H + M) = 0$$

$$\Rightarrow \quad H + M = 0 \quad \frac{M}{H} = -1 \Rightarrow \boxed{\chi = -1}$$

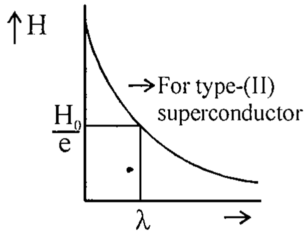
where χ is magnetic succeptibility, H is magnetic field intensity and M is magnetization.
Thus the succeptibility of a super conductor is negative which implies that a superconductor exhibits the properties of perfect diamagnetic susbtances.

Isotope effect:
It has been found that the critical temperature (T_C) of superconductor is found to be proportional to the negative half power of isotopic mass of the element i.e.

$$T_C \propto M^{-1/2} \Rightarrow \boxed{M^{1/2} T_C = \text{constant}}$$

Persistent current:
In the super conductors current contineously flows as long as the material is kept below the critical temperature (T_C). Since the current is very high and the decay formula is $i = i_0 e^{-(R/L)t}$. So once started it continue to flow indenfinitely as long as the temperature and magnetic field remain unchanged. This is called the persistent current.

The magnetic field and surface current penetrate in a superconductor to a depth of 10–100 nanaometer.



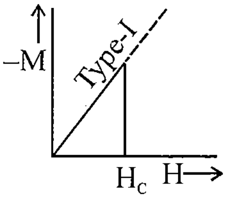
$$H = H_0 e^{-x/\lambda}$$

Similarly, $\quad J_s = J_0 e^{-x/\lambda}$

λ = penetration depth

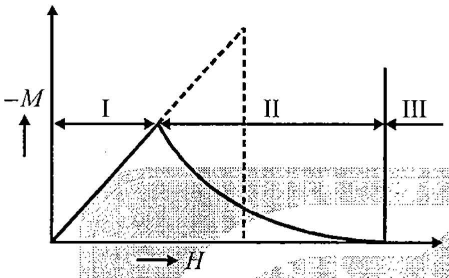
Type-I and type-II supercondutors:
Super conductors are classified into the two categories:
(i) Soft (type-I) superconductor, (ii) Hard (type-II) superconductors

(i) Type-I superconductors: "Superconductor which exhibits complete meissner effect i.e. they are completely diamagnetic are called type-I or soft superconductors". It is disappointing that the critical magnetic fields of superconductors are very low. Since the super current are started, continuously flow without heat loss.



(ii) Type-II superconductors:

These superconductors behaves differently as shown in Figure. For applied field below H_{C1}, the material is diamagnetism and hence the field is H_{C1} is called the lower critical field. Above H_{C1} magnetic flux begins to enter in the superconductor upto an upper critical field H_{C2} at which superconductivity disappear.



$$\boxed{H_C = \sqrt{H_{C_1} \times H_{C_2}}}$$

where, I = Superconducting state
II = Mixed state
III = Normal state

Critical magnetic field:

If a magnetic field is applied parallel to the length of the superconducting wire, the resistance of the wire is suddenly restored at definite magnetic field, which is a function of the temperature and is characteristic of particular metal. This field is known as the critical field magnetic field (H_C). The critical magnetic field H_C is given by the relation

$$\boxed{H_C = H_{C_0}\left[1-\left(\frac{T}{T_C}\right)^2\right]}$$

where H_{C_0} is critical magnetic field at absolute zero.

Penetration depth (λ) :

London described the Meissner effect ($B = 0$) and zero resistivity case of superconductors. He added $E = 0$ for zero resistivity.

As $J = \sigma E$ and $E = \frac{J}{\sigma} = \frac{I}{\infty} = 0$

$B = 0$ according to Meissner effect.

As we know that, $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

But $\quad \vec{E} = 0 \Rightarrow 0 = \frac{-\partial \vec{B}}{\partial t} \Rightarrow \vec{B} \neq 0$

Then London suggested that applied external magnetic field does not suddently drop to zero at the surface of superconductor but decays exponentially according to the equation.

$$H = H_0 e^{-x/\lambda}$$

where, H_0 is magnetic field intensity at the surface of the superconductor and λ is characteristic length (known as the penetration depth) at which H falls from H_0 to H_0/e.

London equation:

According to the London theory, it is assumed that there are two types of conduction electrons in a superconductor, namely, the super and the normal electrons. At 0 K a superconductor contains only super electrons but as the temperature increases the number of normal electrons increases and above the transition temperature (T_C) all the electrons are normal.

Let n_n, v_n and n_s, v_s is the electrons density, velocity in normal and superconducting state respectively. Then the average carrier density $(n) = n_n + n_s$.

The current density $\vec{J} = \vec{J}_n + \vec{J}_s = e\, n_n \vec{v}_n + e n_s \vec{v}_s$ ... (1)

For the superconducting electrons

$$m\frac{d\vec{v}_s}{dt} = e\,\vec{E} \Rightarrow \frac{d\vec{v}_s}{dt} = \frac{e\vec{E}}{m} \qquad \left[\text{as } \vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} = e\,\vec{E}\right]$$

and $\vec{J}_s = e n_s \vec{V}_s \Rightarrow$ $$\frac{d\vec{J}_s}{dt} = e\,n_s\left(\frac{e\vec{E}}{m}\right)$$

$$\Rightarrow \boxed{\frac{d\vec{J}_s}{dt} = \frac{n_s e^2 \vec{E}}{m}} \qquad \text{... (2)}$$

The equation (2) is known as the first London equation and it describes the absence of resistivity.

If $\vec{E} = 0$

$$\Rightarrow \frac{d\vec{J}_s}{dt} = 0 \Rightarrow \vec{J}_s = \text{constant}$$

Then it becomes possible to have steady (constant) current in the absence of electric field. This is the superconducting phenomenon.

The corresponding equation for the normal current is

$$\boxed{\vec{J}_n = \sigma_n \vec{E} = \frac{n_n e^2 \tau\, \vec{E}}{m}} \qquad \text{... (3)}$$

This equation shows that an electric field is necessary for establishing a steady current in normal metal. If $\vec{E} = 0$, $\vec{J}_n$ becomes zero.

Now by the Maxwell's equation

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow \vec{\nabla} \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t} \qquad \text{... (4)}$$

Now taking curl of equation (2), we get

$$\vec{\nabla} \times \frac{d\vec{J}_s}{dt} = \frac{n_s e^2 \left(\vec{\nabla} \times \vec{E}\right)}{m}$$

$$\frac{\partial}{\partial t}\left(\vec{\nabla} \times \vec{J}_s\right) = \frac{n_s e^2}{m}\left(-\mu_0 \frac{\partial \vec{H}}{\partial t}\right) = -\frac{\mu_0 n_s e^2}{m}\frac{\partial \vec{H}}{\partial t} \qquad \text{... (5)}$$

Integrating this equation, we get $\boxed{\vec{\nabla}\times\vec{J}_s \equiv -\frac{n_s\, e^2 \mu_0^{\,\cdot}\, \vec{H}}{m}}$ ... (6)

Equation (6) is second London equation and explains meissner effect $\left(i.e.\ \vec{H}=0 \Rightarrow J_s = \text{constant}\right)$.

Calculation of penetration depth (λ)**:**

Taking curl of equation (6) we get

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{J}_s\right) = -\frac{n_s\, e^2\, \mu_0}{m}\left(\vec{\nabla}\times\vec{H}\right)$$

Or, $\vec{\nabla}\left(\vec{\nabla}\cdot\vec{J}_s\right)-\left(\vec{\nabla}\cdot\vec{\nabla}\right)\vec{J}_s = -\frac{n_s\, e^2\, \mu_0}{m}\left(\nabla\times\vec{H}\right)$ (as $\vec{J}_s$ = constant)

$\Rightarrow$ $\nabla^2\vec{J}_s = \frac{\mu_0 n_s\, e^2}{m}\left(\vec{\nabla}\times\vec{H}\right)$ ... (7)

But $\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t} = \vec{J}_n + \vec{J}_s + \frac{\partial\vec{D}}{\partial T}$

where $\vec{D}$ is the electric displacement vector. For direct current case we can neglect $\vec{J}_n$ and $\frac{\partial\vec{D}}{\partial t}$, so that

$$\vec{\nabla}\times\vec{H} = \vec{J}_s$$

Therefore, equation (7) becomes

$$\nabla^2\vec{J}_s = \frac{\mu_0 n_s e^2}{m}\vec{J}_s = \frac{\vec{J}_s}{\lambda^2} \quad \text{... (8)}$$

where $\boxed{\lambda^2 = \frac{m}{\mu_0 n_s e^2}}$ ... (9)

λ has the dimension of length and is called the penetration depth.

The solution of equation (8) is

$$\boxed{J_s = A\, e^{x/\lambda} + Be^{-x/\lambda}} \quad \text{... (10)}$$

The first factor of this equation shows that J_s increases as x increases which is contrary to the fact. Then it can be left.

Therefore, equation (10) gives $J_s = B\, e^{-x/\lambda}$

at $x = 0,\ J_s = J_0 \Rightarrow\ B = J_0$

$\Rightarrow$ $\boxed{J_s = J_0 e^{-x/\lambda}}$ ... (11)

at $x = \lambda, J_s = J_0/e$

Thus, λ (penetration depth) is the distance in the superconductor at which the current value falls to $1/e$ of its value at the surface.

Drawback of London theory:

Though the London theory explains the two conditions $E=0$ and $B=0$ characterising the superconducting state but it does not tell us about the process happening inside the superconductors. The process which takes place inside the superconductors is explained by BCS theory.

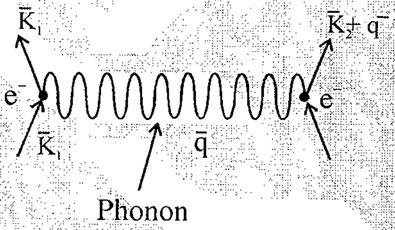
BCS (Bardeen, Cooper and Schrieffer) theory:

This theory explains the microscopic phenomenon which takes place inside the superconductor. It give, the better explanation of the superconductivity than any other theory. This theory involves the following four steps.

(i) Electron-lattice - electron interaction:

When an electron come in contact to the lattice it deforms the lattice and at the same time another electron come in contact to the deformed lattice to take the advantage of lattice deformation. In turns of it the lattice also apply a force on these electrons. Thus, there will be an interaction between electron and electron in which the phonon (lattice vibration) acts as a mediator.

(ii) Cooper pair: The fundamental postulate of BCS theory is that superconductivity phenomenon occurs, when the attractive interaction between two electrons by means of a phonon exchange dominates the usual repulsive coulomb interaction. Two such electrons which interact attractively in the phonon field are called a Cooper pair. These Cooper pairs behave like a single particle. The energy of the pair of electrons in the ground state is less than the energy of the pair in free state (electron separated). The difference of the energy of the Cooper pair and that of the energy of the Cooper pair and that of free electrons is equal to the binding energy of the Cooper pair. At temperature less than T_C electron-lattice-electron interaction is stronger than electron-electron Coulomb repulsion. Pairing is complete at $T=0$ K and is completely broken at $T=T_C$.



Electron–electron interaction through lattice phonon

(iii) Existance of Energy gap:

The energy difference between the free state of electrons and the paired state appears as the energy gap at the Fermi surface. The gap is maximum at absolute zero temperature and becomes zero at $T=T_C$.

(iv) The paired electrons (Cooper pair) are not scattered because of their special property of smooth riding over the lattice imperfection without exchanging of energy with them. As a result of this they travel a certain distance in the coupled motion. This distance is called coherence length. It is of the order of 10^{-4} cm. Coherence length $(\xi_o)=\dfrac{\hbar v_F}{\pi E_g}$; where, v_F = Velocity at Fermi-surface.

Expression for coherence length:

The London penetration depth (λ_L) is a fundamental length that characterises a superconductor. Another independent length of equal importance is the coherence length (ξ_0).

Now the modulated wavefunction corresponding to wave function

$$\psi=e^{+ikx} \text{ is } \phi=\frac{1}{\sqrt{2}}\left[e^{i(k+q)x}+e^{ikx}\right] \qquad ..(1)$$

Now the probability density corresponding to ϕ is

$$\phi^*\phi = \frac{1}{\sqrt{2}}\left[e^{-i(k+q)x}+e^{-ikx}\right].\frac{1}{\sqrt{2}}\left[e^{i(k+q)x}+e^{ikx}\right]$$

$$= \frac{1}{2}\left[1+1+e^{-iqx}+e^{iqx}\right]=\frac{1}{2}\left[2+2\cos qx\right]$$

$$\Rightarrow \qquad \phi^*\phi = \left[1+\cos qx\right] \qquad \dots (2)$$

The probability associated with the plane wave is

$$\psi^*\psi = e^{-ikx}\cdot e^{-ikx} = 1$$

The kinetic energy (E) of the wave function $\psi(x) = \dfrac{\hbar^2 k^2}{2m}$ ... (3)

While the K.E. for the modulated wave function ϕ is

$$\int\phi\left(\frac{\hat{p}^2}{2m}\right)\phi\,dx = \int\phi^*\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\right)\phi\,dx = \int\left[\frac{1}{\sqrt{2}}\left(e^{-i(k+q)x}+e^{-ikx}\right)\right]\left(\frac{-\hbar^2}{2m}\right)\frac{d^2}{dx^2}\left[\frac{1}{\sqrt{2}}\left(e^{i(k+q)x}+e^{ikx}\right)\right]dx$$

$$= +\frac{\hbar^2}{4m}\int\left[e^{-i(k+q)x}+e^{-ikx}\right]\left[(k+q)^2+k^2\right]\left[e^{i(k+q)x}+e^{ikx}\right]dx$$

$$= \frac{\hbar^2}{4m}\left[(k+q)^2+k^2\right]\int\left[e^{-i(k+q)x}+e^{-ikx}\right].\left[e^{i(k+q)x}+e^{ikx}\right]dx$$

$$= \frac{\hbar^2}{4m}\left[2k^2+2kq+q^2\right] \simeq \frac{\hbar^2}{2m}k^2+\frac{\hbar^2}{2m}kq \qquad (\text{As } q << k)$$

Thus the increase in energy $= \dfrac{\hbar^2 kq}{2m}$ ... (4)

If this increase in energy exceeds the energy gap E_g, the superconductivity will be destroyed. The critical value of q_0 of the modulation vector is given by

$$\frac{\hbar^2 k_F q_0}{2m} = E_g \qquad \dots (5)$$

The intrinsic length (ξ_0) is defined as

$$\xi_0 = \frac{1}{q_0} = \frac{\hbar^2 k_F}{2mE_g} = \frac{\hbar P_F}{2mE_g} \Rightarrow \boxed{\xi_0 = \frac{\hbar v_F}{2E_g}}$$

By BCS theory, $\boxed{\xi_0 = \dfrac{\hbar v_F}{\pi E_g}}$

$\dfrac{\lambda}{\xi_0} = k$ and $k > \dfrac{1}{\sqrt{2}}$, for type-I superconductors and $k < \dfrac{1}{\sqrt{2}}$, for type-II superconductors.

Properties which change in the superconducting transitions:

(i) All the thermoelectric effects disappear in the superconducting state.

(ii) The specific heat changes discontinuously at the transition temperature in the pressure of magnetic field.

(iii) There is also a small change in volume, in presence of magnetic field.

(iv) The thermal conductivity changes discontinuously when superconductivity is destroyed in a magnetic field.

Properties which do not change in superconducting transitions:

(i) There is no change in the reflectivity (reflection coefficient) of the metal.

(ii) The X-ray diffraction pattern is same.

(iii) The photoelectric properties, remain unchanged.

(iv) The elastic properties do not change.

(v) In the atom of a magnetic field there is no latent heat and hence no change in volume at the transition.

Superconducting band gap: $E_g = 3.53\, k_B T_C \simeq 4\, k_B T_C$

Cooper pair follow Bose-Einstein statistics. Its spin is zero.

Superconductivity is of IInd order phase transition.

Main points of BCS theory of superconductivity:

The basis of a quantum theory of superconductivity was laid by the classic 1957 papers of Bardeen, Cooper and Schrieffer. The BCS theory include:

(i) An attractive interaction between electrons can lead to a ground state seperated from excited state by an energy gap. The critical field, the thermal properties and the most of the electromagnetic properties are consequencies of the energy gap.

(ii) The electron-lattice-electron interaction leads to an energy gap of the observed magnitude. The indirect interaction proceeds when one electron interacts with the lattice and deforms it; a second electron sees the deformed lattice and adjusts itself to take advantage of the deformation to lower its energy. Thus the second electron interacts with the first electron via the lattice deformation.

(iii) The penetration depth of coherence length emerge as natural consequences of the BCS theory.

(iv) The criterion for the transition temperature T_C of an element or alloy involves the electron density of orbitals $D(E_F)$ at the Fermi level and the attractive electron-lattice-electron interaction (U) and is given by

$$T_C = 1.14\, \theta_D \exp\left[{}^{-1}\!/_{UD(E_F)}\right]$$

(v) Magnetic flux through the superconducting ring is quantized and the effective unit of charge is2e rather than e. $\phi = n\dfrac{h}{2e}, n = 1,2,3......$ The BCS ground state involves pairs of electrons, having opposite wave vectors of opposite spins: $\vec{k}_1 = -\vec{k}_2$ and $s_1 = -s_2$. Such pairs of electrons are called as the Cooper pairs.

SOLVED PROBLEMS

A solid sample has the property that, when cooled below a certain temperature, it expels any small applied magnetic field form within the material. Which of the following best describes this sample in the cooled state ? **[TIFR-2014]**

(a) Paramagnet (b) Diamagnet (c) Ferromagnet (d) Antiferromagnet

ln. Since, $B_{in} = 0 \Rightarrow \mu_0(H+M) = 0 \Rightarrow H+M=0 \Rightarrow \frac{M}{H} = -1 \Rightarrow \chi = -1$

Thus the succeptibility of a super conductor is negative which implies that a superconductor exhibits the properties of perfect diamagnetic susbtances.

Correct option is (b)

Transition temperature T_C and critical field H_C for a superconductor are related as **[DU 2015]**

(a) $H_C = H_0\left[1-\frac{T}{T_C}\right]$ (b) $H_C = H_0\left[\frac{T}{T_C}-1\right]$

(c) $H_C = H_0\left[1-\left(\frac{T}{T_C}\right)^2\right]$ (d) $H_C = H_0\left[\left(\frac{T}{T_C}\right)^2-1\right]$

ln. Transition temperature T_C and critical field H_C for a superconductor are related as

$$H_C = H_0\left[1-\left(\frac{T}{T_C}\right)^2\right]$$

Hence, correct option is (c).

In superconductors, Cooper pairs are formed due to electron-electron interactions mediated by:

(a) Phonons (b) Photons (c) Excitons (d) Polarons **[DU 2014]**

ln. In superconductors, Cooper pairs are formed due to electron-electron interactions mediated by Phonons.

Hence, correct option is (a).

Chapter 4

SEMICONDUCTOR DEVICES

Intrinsic and Extrinsic Semiconductors:

Semiconductor Material: The term conductor is applied to any material that will support a generous glow of charge when a voltage source of limited magnitude is applied across its terminals.

An insulator is a material that offers a very low level of conductivity under pressure from an applied voltage source.

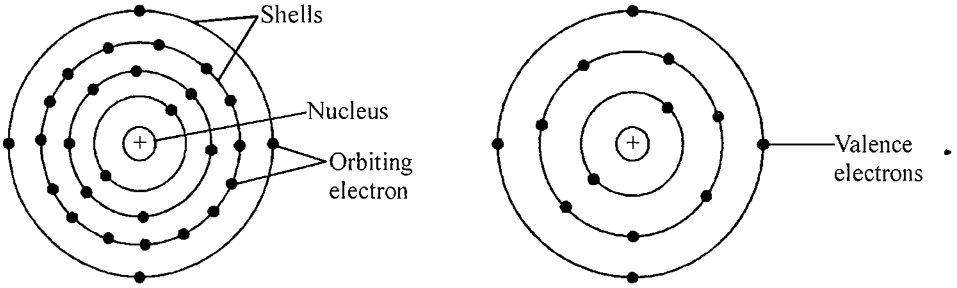
A semiconductor, therefore is a material that has a conductivity level some where between the extremes of an insulator and a conductor.

In a semiconductor current is carried by two types of carries electrons and holes whereas in metals current is carried only by electrons. Silicon (Si) and Germanium (Ge) are the two most commonly used electronic device material. Each atom of Si or Ge forms four covalent bonds with four nearest neighbouring atoms by sharing of valance electrons with opposite spin. Because of this formation of covalent bonds the valance electrons are not available and the crystal behaves as perfect insulator. However at room temperature (~ 300 K) a few of the electrons acquire sufficient kinetic energy from thermal agitation and break their covalent bonds and conduction is made possible. When an electron escape from covalent bond an electron vacancy is created in the bond and such an incomplete bond is called a hole. The hole may behave like a positively charged particle in the conduction of electrocity.

Intrinsic semiconductor materials: Intrinsic materials are those semiconductors that have been carefully refined to reduce the impurities to a very low level essentially as pure as can be made available through modern technology.

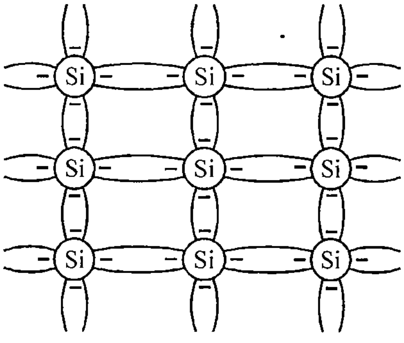
As increase in temperature of a semiconductor an result in a substantial increase in the number of free electrons and holes in the material.

Semiconductor material such as Ge and Si that show a reduction in resistance with increase in temperature are said to have a negative temperature coefficient.



(a) (b)

Atomic structure (a) Germanium (b) Silicon



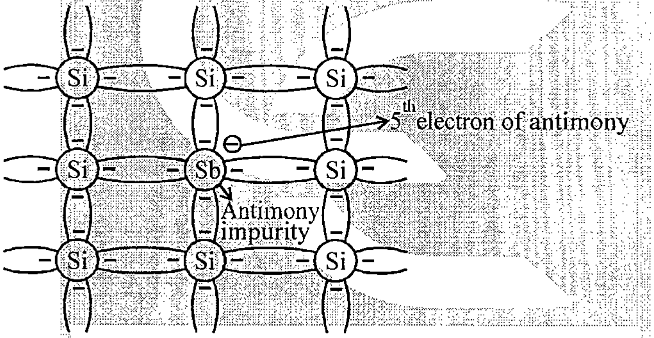
Covalent bonding of the silicon

Extrinsic semiconductor material: A semiconductor material that has been subjected to the doping process is called an extrinsic material. There are two extrinsic materials of immeasurable importance to semiconductor device fabrication *n*-type and *p*-type.

n-type materials: The n-type is created by introducing those impurity elements that have five valance electrons (pentavalent) such as antimony (Sb), arsenic (As), phosphorus (P) etc.

Diffused impurities with five valance electrons are called donor atoms.

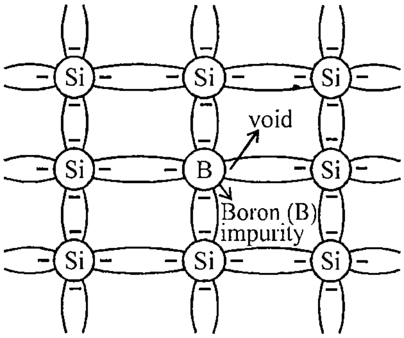
The impurity atoms displace some of the Si/Ge atoms in the lattice. Four of the five valance electrons of the impurity atoms from four covalent bonds with four nearest neighbouring Si atoms. The fifth valance electron remains very loosely bound to the impurity atom. The presence of large number of such loosely bound electron can give rise to high conduction even in room temperature (300 K).



Antimony impurity in n-type material

p-type material: A p-type material is formed by doping a pure germanium or silicon crystal with impurity atoms having three valance electrons. The elements most frequently used for this purpose are boron (B), gallium (Ga) and indium (In).

The impurity atoms displace some of the Si/ Ge atoms in the lattice. The three valance electrons of the impurity atom form three covalent bonds with three nearest neighbouring Si/ Ge atoms and impurity atom lacks by one electron to form the fourth covalent bond with its fourth neighbour. The vacancy that exists in the fourth bond constituting a hole.



Boron impurity in p-type material

"The diffused impurities with three valance electrons are called acceptor atoms".

Note: The resulting p-type material is electrically neutral for the same reasons desemiconductorribed for the n-type material.

Variation of resistivity (ρ) with temperature: Resistivity or specific resistance of material. Unit for (ρ) resistivity = Ω cm or Ω m.
For metal PTC (Positive temperature coefficient)

$$R = \rho.\frac{l}{A} \Rightarrow \boxed{R \propto \rho} \quad P\uparrow T\uparrow R\uparrow$$

In a SEMICONDUCTOR [NTC] $R\downarrow$ with $T\uparrow$

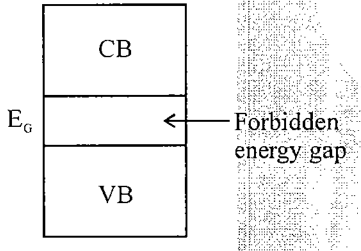
NTC → Negative temperature coefficient.

In SEMICONDUCTOR $P\downarrow$ with $T\uparrow$.

Energy Levels: The more distant the electron from the nucleus, the higher the energy state and any electron that has left its parent atom has a higher energy state than any electron in the atomic structure.

Note: Energy gap decreases with the temperature $\boxed{E_G \propto \frac{1}{Tem.}}$

Energy gap → E_G or E_g.



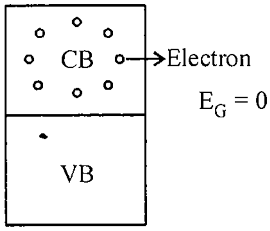
Energy band diagram of an element

E_{G0} → Energy gap at 0K

	Ge	Si
E_{G0}	0.785eV	1.21eV
E_{G300}	0.72eV	1.1eV

$$\boxed{E_{G(T)} = E_{G0} - B_0 T} \qquad B_0 = \text{a material constant} = \frac{eV}{°K} \text{ unit}$$

Energy band diagram for metals and / or conductors: All metals are good conductors of current i.e. they allow a large flow of current through them.



$E_G = 0 = 0K$

at T = 0 Kelvin

Metallic Bonding:

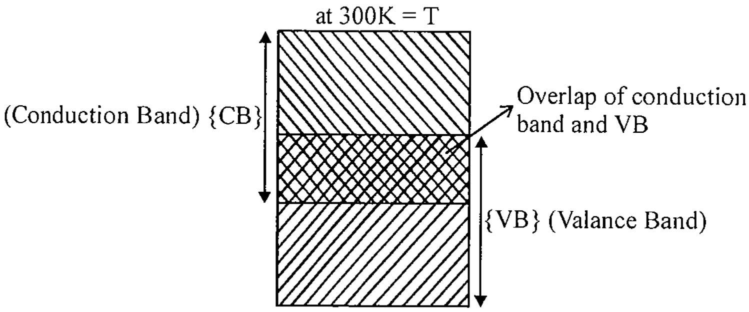
(i) Metals are unipolar, in metals only electrons are available.

(ii) In metals at 0K free electron are available.

(iii) The metals free electron concentration is independent of temperature.

(iv) The e^- concentration in metals is $n = 10^{28}/\text{m}^3$, at $t = 300\text{K}$.

(v) In metals as temperature is increasing valance sheet band and conduction band will overlap. This overlap increases with temperature.



(vi) Because of overlapping of VB and CB metals will exhibit positive temperature. coefficient of R.

{*PTC* of *R*}

i.e. resistance increases with temperature.

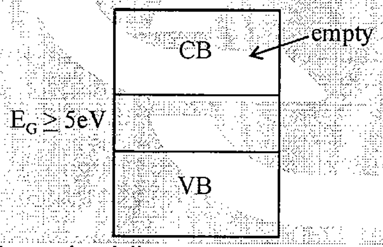
(vii) In metals there is only drift current. Eg. Gold, Silver, Cu, Iron, Al, Platinum and Tungsten.

Drift current: The current that flows through potential difference.

Energy band diagram for insulator:

(i) Insulator are bad conductors of current i.e. they do not allow any flow of current through them.

E_G - large [Energy Gap] $\geq$ 5ev.



Energy band diagram at any temperature
0→400K

(ii) Charge carrier are zero. Therefore conductivity is zero.

Ionic bonding: Insulators are negative temperature coefficient with temperature.

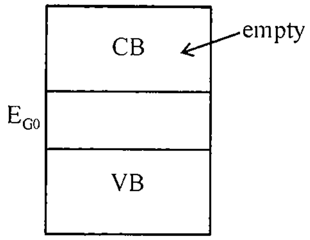
[NTC of R] i.e. $R\downarrow$ with $T\uparrow$.

E_G slightly decreases with the temperature.

e.g. SiO_2, Air, Glass, Mica, Paper, Cotton, Wood, Plastic, Ciramic, Bakelite, Rubber, Parcelain, Diamond.

Energy band diagram of SEMICONDUCTOR: In a SEMICONDUCTOR E_G is small

E_G→
- →around 1eV
- →0.7eV to 1.5eV



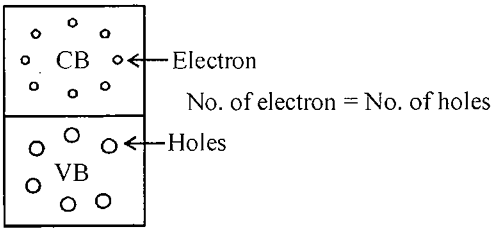
(i) Exhibit → Covalent bonding

(ii) NTC of R → ($R\downarrow$ with $T\uparrow$)

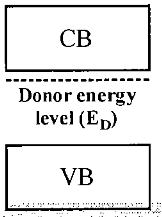
At t = 0K

At 0K carrier concentration is zero and therefore conductivity is zero. So all SEMICONDUCTOR are insulators at 0K.

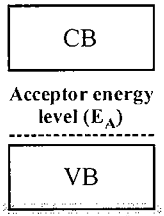
At T = 300K



SEMICONDUCTOR are bipolar {having two types of charges carrier i.e. electron and holes} at room temperature, there will be some conductivity in the semiconductor.



Energy band diagram of a *n*-type semiconductor



Energy band diagram of a *p*-type semiconductor

In case of *n*-type semiconductor the loosely bound valance electrons are formed an energy level close to the conduction band. For Si the separation between that energy level and conduction band is 0.05 eV. These new available levels are essentially discrete level because the impurity atoms are located for apart in the crystal and their interaction is small.

In case of *p*-type semiconductor this energy level is close to valance band for Si the separation is ~ 0.05 eV.

Intrinsic and extrinsic semiconductor:

Electrical properties of Ge and Si:

	Properties	*Ge*	*Si*
(1)	Atomic number	32	14
(2)	Total No. of atoms / cm^3 or Density of atoms	4.421×10^{22}	5×10^{22}
(3)	Ni at 300K .... atom/cm^3	2.5×10^{13}	1.5×10^{10}
(4)	Intrinsic resistivity Pi at 300K (Ω–cm)	45	2,30,000
(5)	Leakage current I_2	μA	nA
(6)	Maximum operating temperature	75°C	175°C
(7)	Power handling capability	Low	High
(8)	μn {mobility of electrons}	3800 cm^2/v.sec	1500 cm^2/v.sec
(9)	μp {mobility of Hole}	1800 cm^2/v.sec	500 cm^2/v.sec
(10)	Dn {electron diffusion constant at 300K}	99 cm^2/sec	34 cm^2/sec
(11)	Dp {hole diffusion constant at 300K}	47 cm^2/sec	13 cm^2/sec
(12)	E_{G0} {energy gap at 0 Kelvin}	0.785 eV	1.4 eV
(13)	E_{G300} {energy gap at 300 Kelvin}	0.72 eV	1.1 eV

Standard doping level:

) Moderate doping → 1 : (10^6 to 10^8) → P, N

) Lightly doping → 1 : (10^{11}) → P^-, N^-

) Highly doping → 1 : 10^3 → P^+, N^+

Note: The minimum doping required to convert intrinsic semiconductor into extrinsic semiconductor is (1 : 10^8).

Any doping concentration below 1 : 10^8 applied the semiconductor will remain intrinsic.

- For 1°C leakage current $\{I_0\}$ approx. increases by 1% in both Germanium and Silicon.
- Leakage current is highly sensitive to temperature.
- Leakage current doubles for every 10°C.
- Silicon have better thermal stability than Ge.

Mass action law: In a semiconductor [intrinsic and extrinsic] under thermal equilibrium the product of electrons [n] and holes [p] is always a constant and is equal to the square of intrinsic concentration (n_i).

$$\boxed{n.p = n_i^2}$$

- This law is mainly used for extrinsic semiconductor to calculate the minority concentration.

Intrinsic concentration: It is a conc. available impure SEMICONDUCTOR at a given temperature. In intrinsic semiconductor.

$$\boxed{n = p = n_i}$$

$$\boxed{n_i^2 = A_0 T^3 e^{-E_G/KT}}$$

where A_0 → Amaterial constant K = 8.62×10^{-5} eV/°K

In semiconductor {Bipolar}:

$$\boxed{\sigma = nq\mu_n + pq\mu_p}$$

where $\sigma \rightarrow$ conductivity, $n \rightarrow$ No. of electrons, $p \rightarrow$ No. of holes, $\mu_n \rightarrow$ mobility of electrons

$\mu_p \rightarrow$ mobility of holes, v → charge

Problem: A sample of n-type semiconductor has electron density of $6.25\times10^{18}/\text{cm}^3$ at 300 Kelvin. If the intrinsic concentration in the sample is $n_i = 2.5\times10^{13}/\text{cm}^3$.P =?.

ln. By mass action law $np = n_i^2$

$$P = \frac{n_i^2}{n} = 1 \times 10^8/\text{cm}^3$$

Problem: In Ge the leakage current are 5μ ampere at 10°C. Find its value when the temperature is 25°C.

ln. $I_0(T_2) = I_0(T_1)\left[2^{\left(\frac{T_2-T_1}{10}\right)}\right] = 5\times[2^{1.5}] = 14.14\ \mu$ ampere

Fermi Energy: The unit of fermi energy is eV.

Fermi energy is define as the maximum energy possessed by an electron at 0K or

Fermi energy is also define as the maximum kinetic energy possessed by the electron at 0 Kelvin.

- Fermi energy = Maximum Kinetic Energy = $\frac{1}{2}mv^2$

Maximum velocity of electron or velocity of electron $\boxed{v = \sqrt{\frac{2E_F}{m}}}$ m/s

- It is the velocity of the electron in terms of fermi energy or

Fermi energy is also define as the energy passed by fastest moving electron at zero Kelvin.

Fermi-Dirac Function : *F*(E): Also {Fermi Dirac probability function}

Fermi-Dirac function F(E) denotes the probability of electron existing as a function of (E), for a metal or a SEMICONDUCTOR.

$$\boxed{F(E) = \frac{1}{1 + e^{(E-E_F)/KT}}}$$

where E is a energy possessed by electron in eV. At T = 0K, we get two conditions

when $T = 0$

(i) If $E > E_F$, $F(E) = \frac{1}{1+e^{+\infty}} = \frac{1}{1+\infty} = 0$

It indicates at T = 0K, No. electron are available with energies $E > E_F$ and the probability of such electron existing is zero.

(ii) If $E < E_F$, $F(E) = \frac{1}{1+e^{-\infty}} = \frac{1}{1+0} = 1$ or 100%.

at $t = 0$ kelvin, the probability of electron existing is one or 100%. If the energy possessed by the electron is less than E_F.

At $T \neq 0K$ or T > 0K

If E = E_F, $F(E) = \frac{1}{1+e^0} = \frac{1}{1+1} = \frac{1}{2} = 0.5 = 50\%$

Fermi level is the characteristic level 50% probability of electron existing in the SEMICONDUCTOR. If no for bidden band exist.

- The probability of electron existing in a metal is 1 otherwise 100%.
- In metals $F(E) = 1$ or 100%. In a SEMICONDUCTOR the probability of electron existing in $F(E)$ then probability of hole existing in the SEMICONDUCTOR is 1-$F(E)$.

Fermi level in intrinsic semiconductor: In intrinsic semiconductor $n = p$

$$N_c e^{-[E_c - E_F]/KT} = N_v e^{-[E_F - E_v]/KT}$$

Here $N_c = 2\left(\frac{m_n kT}{2\pi\hbar^2}\right)^{3/2}$ and $N_v = 2\left(\frac{m_p kT}{2\pi\hbar^2}\right)^{3/2}$

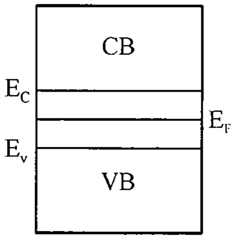
$$\frac{N_c}{N_v} = e^{-[E_F - Ev - E_c + E_F]/KT} = e^{[E_c + Ev - 2E_F]/KT}$$

Taking natural log on both the side.

$$\log_e \frac{N_c}{N_v} = \frac{E_c + E_v - 2E_F}{KT}, \quad E_c + E_v - 2E_F = KT \log e \frac{N_c}{N_v}$$

$$\boxed{E_F = \frac{E_c + E_v}{2} - \frac{KT}{2} \log_e \frac{N_c}{N_v}}$$

In intrinsic semiconductor fermi level depends only on temperature. Fermi level is a function of temperature.



Analysis: Let $m_n = m_p$

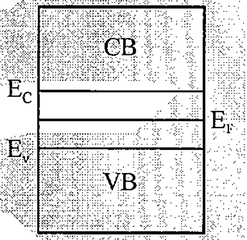
Case I: then $N_c = N_v$

$$\therefore \ \log_c \frac{N_c}{N_v} = 0$$

$$\boxed{E_F = \frac{E_c + E_v}{2}}$$

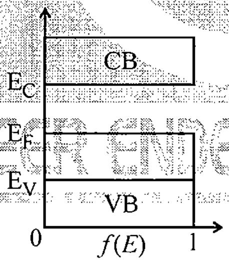
The fermi level is now existing exactly at the centre of the energy gap.

Case II: Let T = 0K



$$\boxed{E_F = \frac{E_c + E_v}{2}}$$

At 0K



$f(E) = 0$ for $E > E_F$

Thus there are no electrons in the conduction band, so the conductivity is zero and intrinsic semiconductor at $T = 0K$ working as an insulator.

Note: In intrinsic SEMICONDUCTOR fermi level will be existing exactly at the centre of the energy gap under the following three conditions.

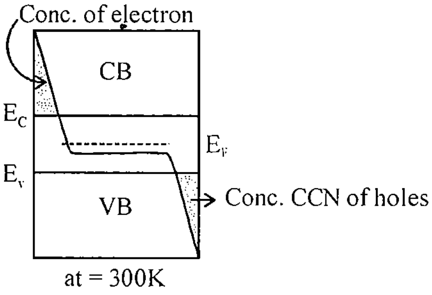
(i) $N_c = N_v$ (ii) $m_n = m_p$ (iii) $T =$ 0K

Case-III: At 300 Kelvin

$$E_F = \frac{E_c + E_v}{2} - \frac{KT}{2} \log \frac{N_c}{N_v}$$

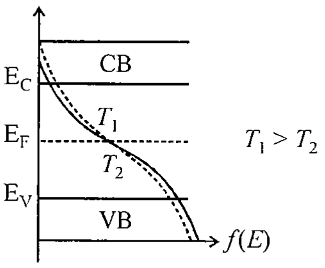
where T = 300K

electron concentration = Hole concentration



- In intrinsic semiconductor at 300K fermi level passes through the centre of the energy gap.
- At 300 Kelvin because of thermal energy a number of covalent bonds will be broken and electrons and holes are created and the conductivity of intrinsic SEMICONDUCTOR will be increasing with the temperature.

Case-IV: Fermi distribution at different temperature:



$$\text{From } f(E) = \frac{1}{1 + e^{(E - E_F)/kT}}$$

We can say that as we rise the room temperature the probability of electron existing is increases as shown in the figure. That explains why the conduction current increases as the room temperature rises.

Fermi level in n-type SEMICONDUCTOR: In n-type SEMICONDUCTOR, $n = N_D$,

$$N_c e^{-(E_c - E_F)/KT} = N_D, \quad \frac{N_c}{N_D} = e^{(E_c - E_F)/KT}$$

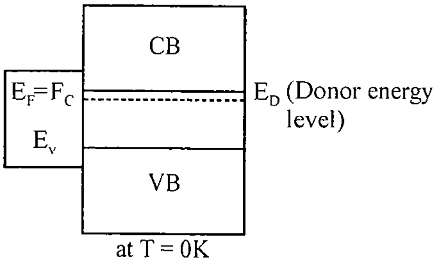
On taking natural log on both the sides.

$$\log\frac{N_C}{N_D} = \frac{[E_C - E_F]}{KT}$$

$$\boxed{[E_C - E_F] = KT\log\frac{N_C}{N_D}} \quad \boxed{E_F = E_C - KT\log_e\frac{N_C}{N_D}}$$

- In n-type SEMICONDUCTOR Fermi level depends on temperature and doping concentration.

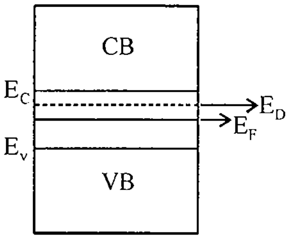
Case-I: $\boxed{\text{let } T = 0K}$, $\boxed{E_F = E_C}$



E_F considers with the edge of CB.

- At T = 0K, electron and hole concentration are zero. Therefore, conductivity is zero. Therefore n-type SEMICONDUCTOR will be working as an insulator at 0 Kelvin.

Case-II: Let T = 300K



$$E_F = E_C - KT\log\frac{N_C}{N_D}$$

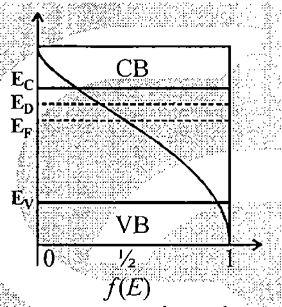
where T = 300K

In *n*-type SEMICONDUCTOR at room temperature the fermi level will be existing just below the donor energy level because $E_C - E_D = 0.01$ eV and $E_C - E_F = 0.3$eV.

In n-type SEMICONDUCTOR as temperature is increasing from 0 Kelvin to 300 Kelvin at some intermediate temperature fermi level will be considering with the donor energy level.

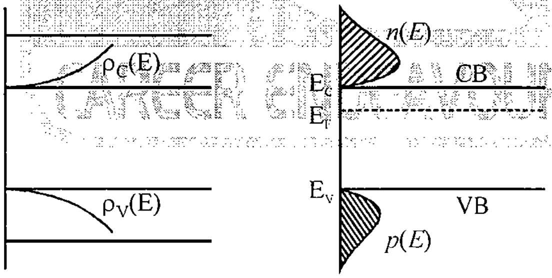
Exact Analysis:

At $T = 300\ K$



For *n*-type semiconductor

Electron concentration in conduction band, $n(E) = \rho_C(E)\,f(E)$, where $\rho_C(E)$ is the density of state in conduction band.



Hole concentration in the valance band, $p(E) = \rho_V(E)\left[1 - f(E)\right]$

Case-III: Mathematical Analysis:

$$E_C - E_F = KT\log e\frac{N_C}{N_D}$$

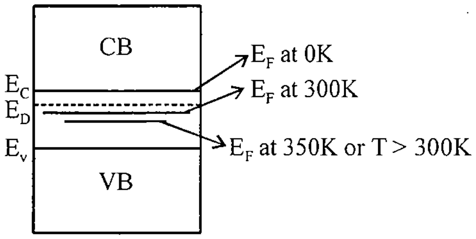
(i) Effect of Temperature: Let temperature ↑ N_C ↑ and let $N_C > N_D$

$E_C - E_F > 0$ or $\boxed{E_C > E_F}$

In type SEMICONDUCTOR as temperature increases fermi level moves away from the conduction band or fermi level moves toward the centre of energy gap.

Hence conductivity decreases with the temperature {above 300 Kelvin}

The position of fermi level for different temperature in the *n*-type SEMICONDUCTOR is given below.



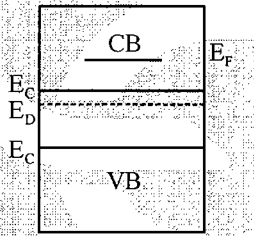
At curie temperature in n-type SEMICONDUCTOR fermi level exist at the centre of energy gap [i.e. at curie temperature *n*-type SEMICONDUCTOR will become intrinsic semiconductor and the conductivity will become small].

Effect of Doping: Let doping increasing i.e. $N_D \uparrow \left\{E_C - E_F = KT\log\frac{N_C}{N_D}\right\}$

let $N_D > N_C$, $E_C - E_F < 0$ or $E_C < E_F$

In *n*-type semiconductor doping increases fermi level moves toward the conduction bond or E_F moves away from the centre of energy gap.

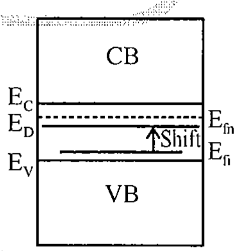
Hence $\sigma \uparrow$ with doping



N^+ semiconductor at 300K

In N-type semiconductor as doping increases fermi level takes upward shift. In a lightly doped N-type semiconductor or highly degenerate N-type semiconductor the fermi level exist in the conduction band.

Note: $\boxed{\text{Shift} = KT\log_e\frac{N_D}{n_i}}\,eV$



N^+ semiconductor at 300K

Fermi level in P-type semiconductor:

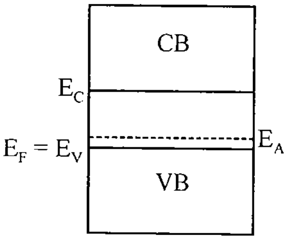
$$P \simeq N_A \qquad N_V e^{-[E_F - E_V]/KT} = N_A \Rightarrow \frac{N_V}{N_A} = e^{[E_F - E_V]/KT}$$

taking natural log on both the sides

$$\log_e \frac{N_V}{N_A} = [E_F - E_V]/KT$$

- $\boxed{[E_F - E_V] = KT\log\frac{N_V}{N_A}}$ $\boxed{E_F = E_V + KT\log\frac{N_V}{N_A}}$

In *p*-type semiconductor fermi level is a function of temperature and doping conc.



at, T = 0K

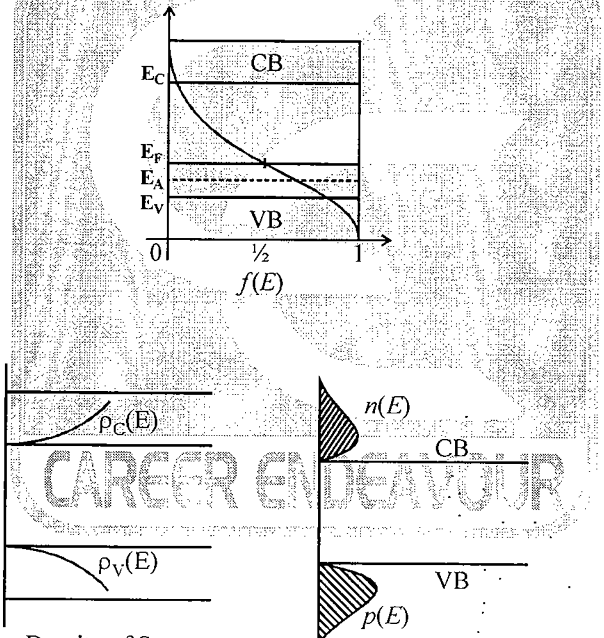
Case I: T = 0K $\boxed{E_F = E_V}$

E_F coincides with edge of VB

{electron and holes conc. are zero semiconductor is working like insulator}
At T = 0K, electrons and holes concentration are zero, therefore the conductivity is zero, therefore the *p*-type semiconductor at 0 Kelvin will be working as an insulator.

Case II: Let T = 300K, $E_F = E_V + KT\log_e\frac{N_V}{N_A}$, when T = 300K

In *p*-type semiconductor at room temperature E_F is existing just above the acceptor energy level.



Density of State

Electron and hole concentration

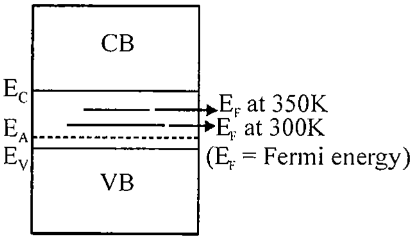
Case III: Mathematical analysis : $E_F - E_V = KT\log_e\frac{N_V}{N_A}$

(i) Effect of temperature: As temperature ↑ N_V↑

let $N_V > N_A$, $E_F - E_N > 0$ or $E_F > E_V$
In P-type semiconductor, as temperature increases E_F moves away from V_B or E_F moves towards the centre of energy gap.

Hence $\sigma\downarrow$ with T.

The position of fermi for different temperature is given below:



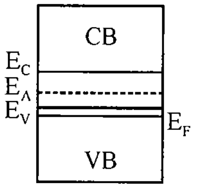
Effect of doping:

Case III: As doping increases ↑ i.e. $N_A \uparrow$

let $N_A > N_V$, $E_F - E_V < 0$ or $E_F < E_V$

In P-type semiconductor as doping N_A increases fermi level moves towards the valency bond or fermi level moves away from the centre of energy gap.

Here $\sigma \uparrow$ with doping



P^+ semi conductor at 300K

SOLVED PROBLEMS

Fermi level:

1. In an n-type semiconductor the fermi level lies 0.3eV below the conduction band at 300 Kelvin if the temperature is increased to 330K. Find the app. new position of fermi level.

Soln. The position of fermi level below the conduction band depends on (i) Temperature (ii) Donor conc. N_C changes with temperature since N_C value not given neglecting variation of N_C with temperature.

$(E_C - E_F) \propto T$

$(E_C - E_{F_1}) \propto 300$ ... (1)

$(E_C - E_{F_2}) = 330\,eV$ ... (2)

$(E_C - E_{F_2}) = \frac{330}{300} \times 0.3 = 0.33$ eV

So, the Fermi level will lie 0.33 eV below the conduction band.

2. In a *n*-type semiconductor the fermi level lies 0.4eV below the conduction band. If the conc. of donor atoms (N_D) is doubled. Find the new position of fermi level. Assume KT = 0.03eV

Soln. For N-type semiconductor

$N_D \simeq N_C\, e^{-(E_C - E_F)/KT}$, $N_D \simeq N_C\, e^{-(0.4)/0.03}$ ... (1)

$2N_D = N_C\, e^{-(E_C - E_F)/0.3}$ ... (2)

On dividing (1) and (2)

$\frac{1}{2} = e\frac{(E_C - E_F) - 0.4}{0.03}$, $\log\frac{1}{2} = \frac{(E_C - E_F) - 0.4}{0.03}$ $\Rightarrow (E_C - E_F) = 0.03\log\frac{1}{2} + 0.4 = 0.379\text{eV}$

3. In a p-type semiconductor the fermi level lies 0.4eV above the valency bond. If the conc. of acceptor atoms is increased by 3 times. Find the position of fermi level. Assume KT = 0.03eV

Soln. $N_A = N_V e^{-(E_F - E_V)/KT}$

$N_A = N_V e^{-(0.4)/0.03}$ ... (1)

$3N_A = N_V e^{-(E_F - E_V)/0.03}$ ... (2)

On dividing (1) and (2)

$\frac{1}{3} = e\frac{(E_F - E_V) - 0.4}{0.03}$, $0.03\log\frac{1}{3} = (E_F - E_V) - 0.4$

$(E_F - E_V) = 0.367eV$

In a semiconductor at room temperature the intrinsic carrier concentration and intrinsic resistivity are $1.5\times10^{16}/m^3$ and 2×10^{13} Ωm respectively it is converted into an extrinsic semi conductor with a doping concentration of $10^{20}/m^3$. Find the shift in fermi level due to doping.

ln. Shift = $KT \log_e \frac{\text{Doping conc.}}{n_i} = 8.62\times10^{-5}\times300 \log_e \frac{(10)^{20}}{1.5\times10^{16}}$ eV

= 0.227 eV

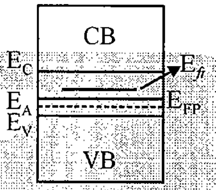
Si is dopped with boron conc. of 4×10^{17} atoms/cm³ assume $n_i = 1.5\times10^{10}$ atoms/cm³ and temperature is 27°C compare to undopped Si the fermi level of doped Si is

ln. Shift = $-KT \log_e \frac{N_A}{n_i} = -(8.62\times10^{-5}\times300)\log_e \frac{4\times10^{17}}{1.5\times10^{10}} = -0.442\,\text{eV}$

Downward shift of 0.442eV

In p-type semiconductor as doping increases, fermi level takes downward shift.

In a highly doped p-type semiconductor or highly degenerate p-type semiconductor fermi level exist in the valance band.

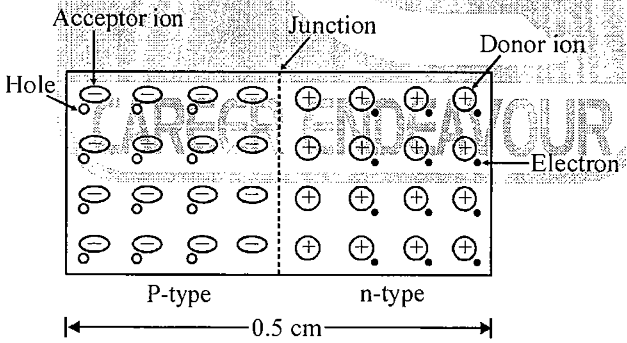


Note: $\boxed{\text{Shift} = -KT\log_e \frac{N_A}{n_i}eV}$

P-N Junction Diode:

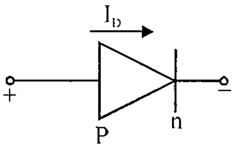
Diode is the simplest and most fundamental non-linear and active circuit element. Just like a resistor. The diode has two terminals, but unlike the register, which has a linear (straight-line) relationship between the current flowing through it and the voltage appearing across it, the diode has a non-linear i-v characteristic.

Consider a crystal of a semiconductor p-n junction is formed if donor impurities are introduced into one side and acceptors into the figure given below.



Donor ion is represented by a plus sign because after 'donating' an electron, it becomes a positive ion. Acceptor ion is indicated by a minus sign because after this atom "accepts" an electron, it converts into a negative ion.

Diode Symbol: Arrow mark indicates direction of forward current.

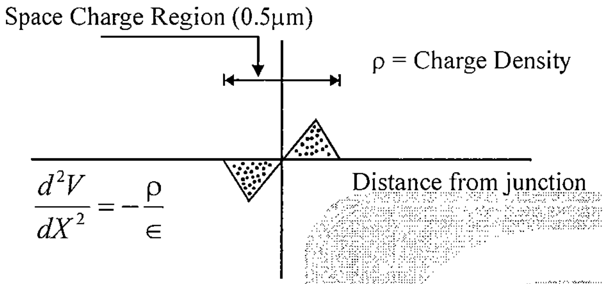


Junction Diode Characteristic:

- Some electron and holes from *n*-type and *p*-type cross into each others region and they get neutralized near junction.
- The unneutralized ions in the neighbourhood of the junction are referred to as uncovered charges.
- Since this region of the junction is depleted of mobile charges, it is called the depletion region, the space-charge region or transition region.
- The thickness of this region is of the order of wavelength of visible light (0.5 µm).
- Within this very narrow space charge layer, there are no mobile carriers.

For Step Graded p-n Junction:

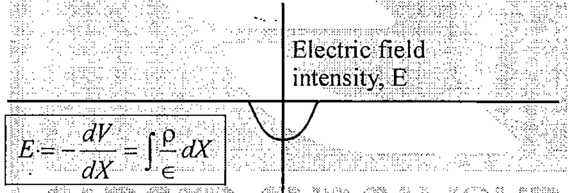
1. **Shape of charge density** ρ.


Space Charge Region (0.5µm)
ρ = Charge Density
Distance from junction
$\frac{d^2V}{dX^2} = -\frac{\rho}{\in}$

2. Corresponding field intensity curve is proportional to the integral of charge density curve.

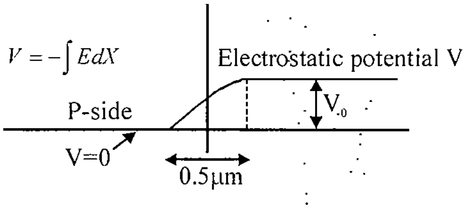
Since, $\frac{d^2V}{dX^2} = -\frac{\rho}{\in} \Rightarrow \frac{d}{dX}\left(\frac{dV}{dX}\right) = -\frac{\rho}{\in} \Rightarrow E = \int_{X_0}^{X} \frac{\rho}{\in} dX, \quad E = -\frac{dV}{dX};$

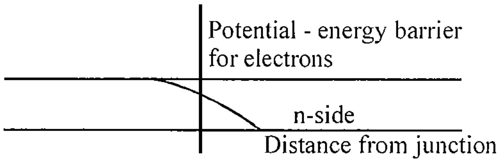
$E = 0$ at $X = X_0$


Electric field intensity, E
$E = -\frac{dV}{dX} = \int \frac{\rho}{\in} dX$

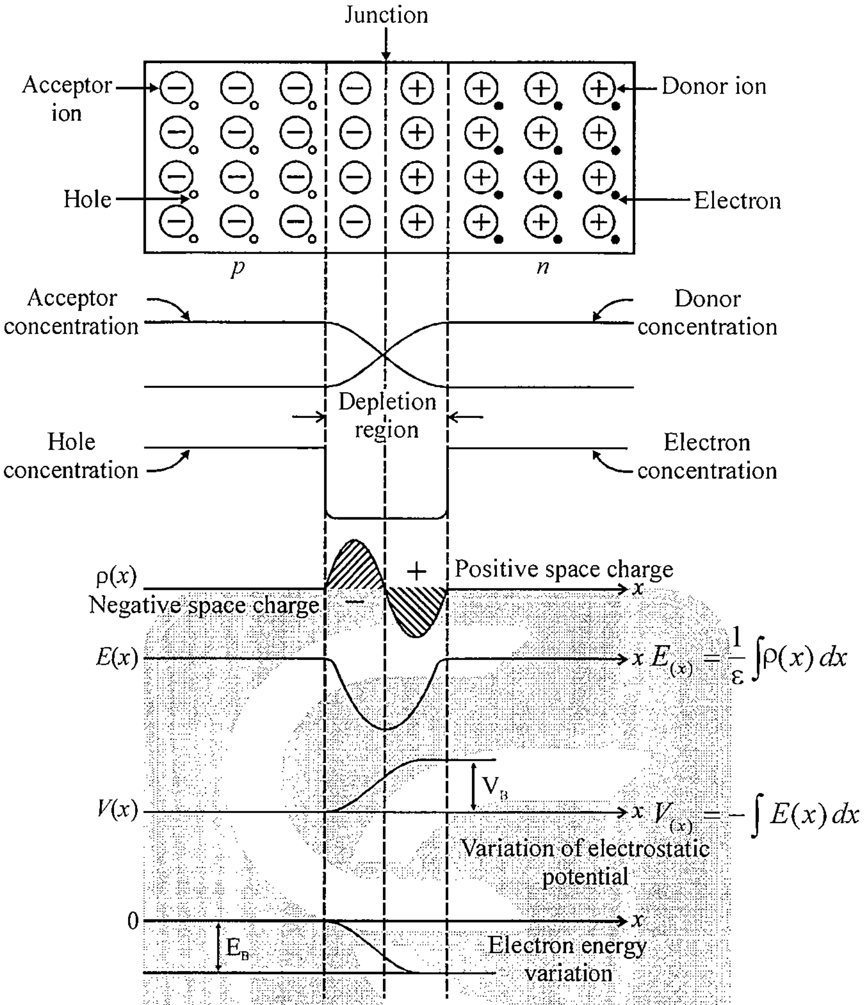
Note: Electric field intensity will be maximum at the junction.

3. **Potential:** Variation of potential in the depletion region is shown as below figure.


$V = -\int E dX$
Electrostatic potential V
P-side
V_0
V=0
0.5µm

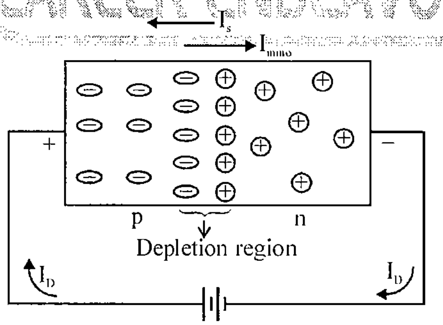

Potential - energy barrier for electrons
n-side
Distance from junction

Combination of all these diagram in depletion region shown as

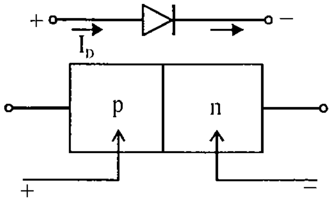

Junction
Acceptor ion
Donor ion
Hole
Electron
p
n
Acceptor concentration
Donor concentration
Depletion region
Hole concentration
Electron concentration
ρ(x)
Negative space charge
−
+
Positive space charge
x
E(x)
x
$E_{(x)} = \frac{1}{\varepsilon}\int \rho(x)\,dx$
V(x)
V_B
x
$V_{(x)} = -\int E(x)\,dx$
Variation of electrostatic potential
0
E_B
x
Electron energy variation

Forward Bias Region:

A forward biased or "on" condition is established when positive terminal of battery is connected to the p-type material and negative terminal to the n-type material as shown in figure.


I_s
I_{majo}
+
−
p
n
Depletion region
I_D
I_D

(a) Internal distribution of charge under forward bias condition.

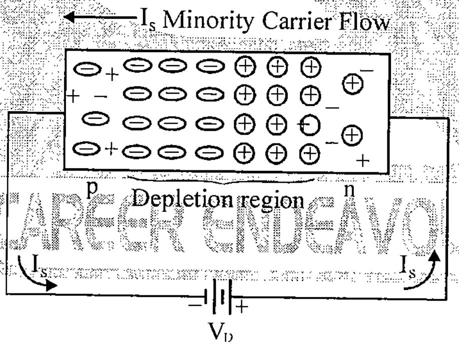


(b) Forward biased polarity.

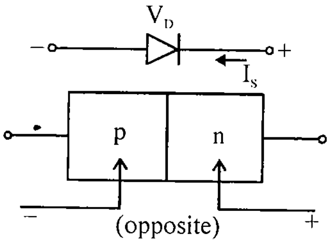
- The application of forward bias potential will "Pressure" electrons in the n-type material and holes in the p-type material to recombine with the ions near the boundary and reduce the width of the depletion region as shown in figure (a).
- The resulting minority carrier flow has not changed in magnitude (since the conduction level is controlled primarily by the limited number of impurities in the material), but the reduction in width of depletion region has resulted in a heavy majority flow across the junction.
- An electron of the n-type material now "sees" a reduced barrier at the junction due to the reduced depletion region and a strong attraction for the positive potential applied to the p-type material.
- As the applied bias increases in magnitude, the depletion region will continue to decreases in width until a flood of electron can pass through the junction, resulting in an exponential rise in current as shown in the forward bias region of the characteristic of figure (b).

Reverse Bias Condition

- If an external potential is applied across the p-n junction such that the positive terminal is connected to the n-type material and negative terminal is connected to the p-type material as shown in figure (a).
- The number of uncovered ions will increase due to the large number of free electrons / hole drawn to the respective potential of the applied voltage. The net effect, therefore is a widening of the depletion region.
- As a result majority carrier can not cross the barrier, effectively reducing the majority current flow to zero.
- The number of minority carriers however, entering the depletion region will not change, resulting in minority carrier flow across the junction.



(a) Internal distribution of charge under reverse bias condition



(b) Reverse bias polarity and direction of reverse saturation current

- The current that exists under reverse bias condition is called the reverse saturation current and is represented by I_s or I_0.

Also called as minority carrier current or leakage current or thermally generated current.

If leakage current is small, the device will have very good thermal stability.

It is highly sensitive to temperature. It doubles for every 10°C rise in temperature for both Si and Ge.

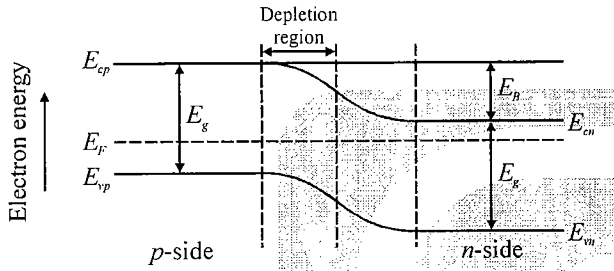
$$\boxed{I_s(T_2) = I_s(T_1)2^{\left(\frac{T_2-T_1}{10}\right)}\text{Amp}}$$

Leakage current for $\boxed{Ge\ diode = \mu A,\quad Si\ Diode = nA}$

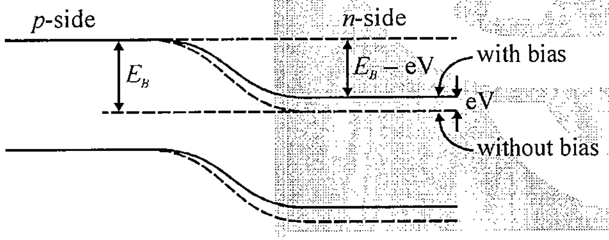
The greatest advantage of Si is smaller leakage currents.

Energy band diagram of a diode:

(*i*) Unbiased

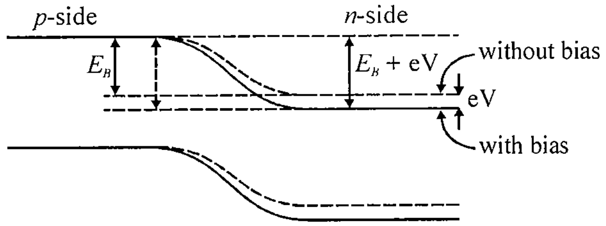

Depletion region
Electron energy
E_{cp}
E_g
E_F
E_{vp}
E_B
E_{cn}
E_g
E_{vn}
p-side
n-side

(*ii*) Forward bias


p-side
n-side
E_B
E_B – eV
with bias
eV
without bias

E_B is the energy barrier for the electron across the junction which is reduced to $(E_B - eV)$ when a forward bias V is applied.

(*iii*)Reverse bias


p-side
n-side
E_B
E_B + eV
without bias
eV
with bias

when a reverse bias V is applied the electron's energy barrier increases to $eV + E_B$.

Expression for Diode Current

The general characteristic of a semiconductor diode can be defined by Schokley's equation.

$$\boxed{I = I_0\left(e^{\frac{V_D}{\eta V_T}} - 1\right)}$$

I = Current flowing through the diode

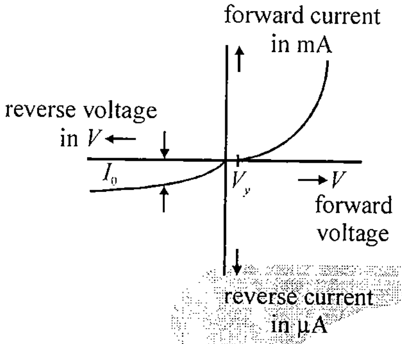
I_0 = Reverse saturation current

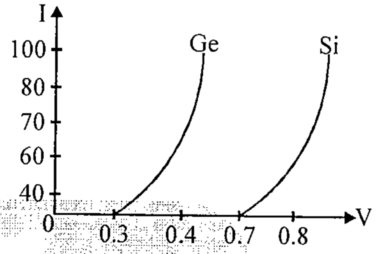
V = Voltage applied across the diode

V_T = Thermal voltage [at room temperature (i.e. T = 300°K) $V_T = 25.9\text{mV}$]

η = Ideality factor which is a function of the operating condition and physical construction. [For Ge, $\eta = 1$ and for Si, $n = 2$]

Diode V-I Characteristic:



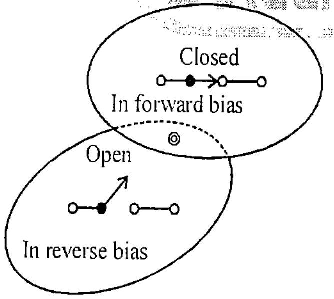


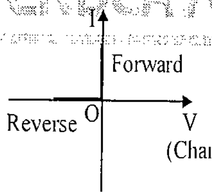
V_γ = Cut-in potential
(for Ge = 0.3V and for Si = 0.7V)

- Cut in voltage (V_γ) decreases with temperature for 1°C rise in temperature, it is reduced by 2.5 mV.

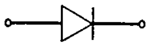
Logarithmic Characteristic: $\log I = \log I_0 + 0.434 \dfrac{V}{\eta V_T}$ at low current log I Vs. V is almost linear corresponding to $\eta = 2$. At high current diode behaves more like a resistor than a diode and current increases linearly rather than exponentially with applied voltage.

The Ideal Diode:


Closed
In forward bias
Open
In reverse bias

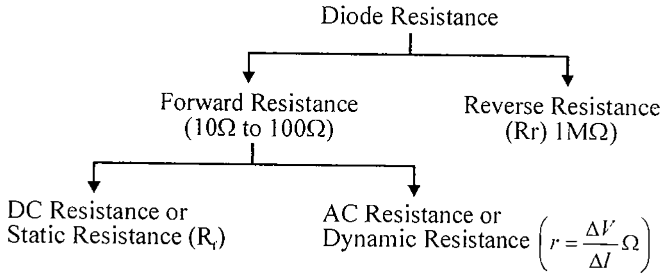


(Characteristic of ideal diode)

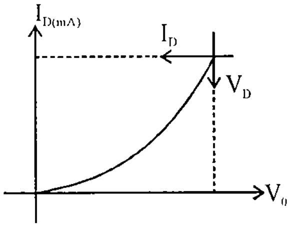


(Ideal diode)

Diode Resistance:

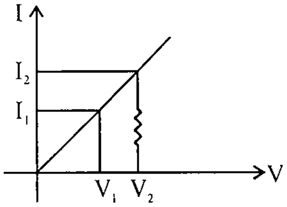

Diode Resistance
Forward Resistance
(10Ω to 100Ω)
Reverse Resistance
(Rr) 1MΩ)
DC Resistance or
Static Resistance (R_f)
AC Resistance or
Dynamic Resistance $\left(r = \frac{\Delta V}{\Delta I}\Omega\right)$

Static / DC Resistance: The static resistance of a diode is defined as the ratio V/I of the voltage to the current.



$$R_D = \frac{V_D}{I_D};$$ Typical R_D = 80 Ω (very less)

Dynamic / AC Resistance : For small signal operation current flowing through a diode varies about some dc operating current. The dynamic, or incremental, resistance r is defined as the reciprocal of the slope of the Volt-ampere characteristic, $r = \frac{dV}{dI}$. For a semiconductor diode.



$$r_d = \frac{\eta V_T}{I + I_0}$$

Dynamic conductance $g = \frac{1}{r}$ is $g = \frac{dI}{dV_D} = \frac{I_0 e^{\frac{V_D}{\eta V_T}}}{\eta V_T} = \frac{I + I_0}{\eta V_T}$

For a forward bias greater than a few tenths of a volts. I >> I_0, and r is given approximately by

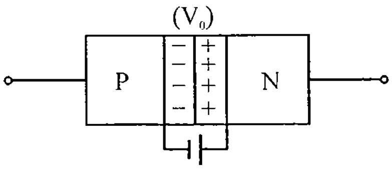
$$\boxed{r = \frac{\eta V_T}{I}}$$

Note: DC resistance > AC resistance

The dynamic resistance of Ge diode with a forward current of 26mA is 1 Ω.

Dynamic resistance in Si diode is more than in Ge diode.

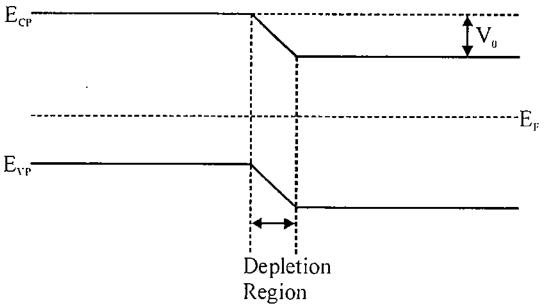
Contact Potential: (V_0)



$$V_0 = \frac{KT}{q}\ln\left(\frac{P_p}{P_n}\right), \quad V_0 = \frac{KT}{q}\ln\left(\frac{n_n}{n_p}\right)$$

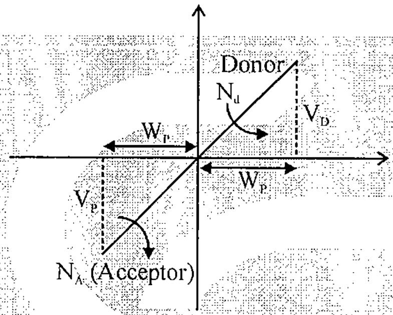
If $P_p = N_a^-$ and $\eta_n = N_d^+$

Then $\boxed{V_0 = \frac{KT}{q}\ln\left(\frac{N_a^- N_d^+}{n_i^2}\right)}$ $K \rightarrow$ Boltzman's constant



- The contact potential V_0 cannot be read by an voltmeter connected across it because of barrier voltage balance by a metal to semiconductor contact potential at the end of diode.
  Potential barrier and width of depletion layer across P-N junction:

  The barrier potential V_0 is given by $\boxed{V_0 = |V_P| + |V_n|}$



V_P is the potential full in P-region and V_n is the potential fall in N-region.

Width of depletion layer is given by $\boxed{W = W_p + W_n}$

Since crystal as a whole is electrically neutral the number of charge carriers on both side must be equal i.e.

$$N_aW_p = N_dW_n\,;\quad W = W_p + \frac{N_a}{N_d}W_p\,;\quad \boxed{W_p = \frac{W}{\left(1+\frac{N_a}{N_d}\right)}}\,;\quad \boxed{W_n = \frac{W}{\left(1+\frac{N_d}{N_a}\right)}}$$

So, it is clear that if doping is increased on one side then depletion layer width will increase on other side. In case of P–N, junction diode if doping is increased on P-side width of depletion layer will increase on N side.

Value of Width Depletion Layer:

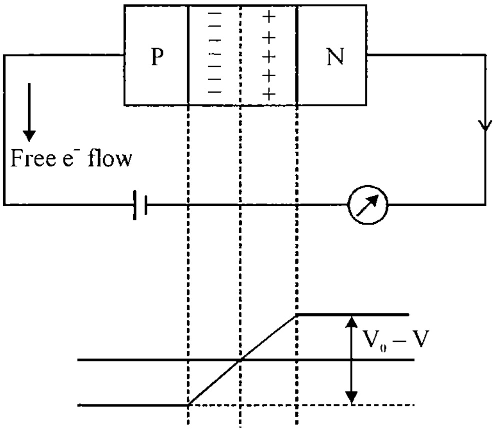
$$W_p = \sqrt{\frac{2\varepsilon V_0}{qN_a\left(1+\frac{N_a}{N_d}\right)}} \quad \text{and} \quad W_n = \sqrt{\frac{2\varepsilon V_0}{qN_d\left(1+\frac{N_a}{N_d}\right)}}$$

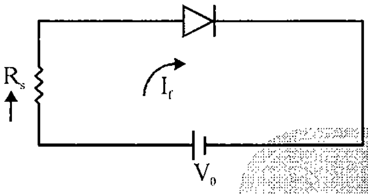
So, total width of depletion layer is given by $W = W_p - W_N$

$$\boxed{W = \frac{2\varepsilon V_0}{q}\left(\frac{N_a + N_d}{N_aN_d}\right)}$$

$\varepsilon = \varepsilon_r\varepsilon_0$; $\varepsilon_0 = 8.854\times10^{-12}\,F/m$, $\varepsilon_r = 11.7$ for Si, $\varepsilon_r = 16$ for Ge

Forward Biased P-N Junction Diode:



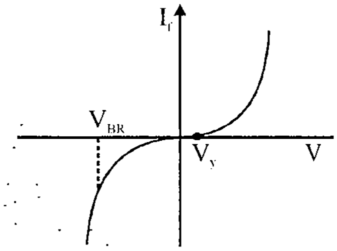


$I_f \rightarrow$ Forward current, $R_s \rightarrow$ Series resistance

When a PN junction is forward biased the width of depletion layer reduces and barrier height reduces. When a diode is properly forward the effect of barrier voltage becomes zero.

$$\boxed{I_f = I_0[e^{\frac{V_d}{\eta V_T}} - 1]}$$ $\eta = 2 \Rightarrow$ Si, $\eta = 1 \Rightarrow$ Ge

V-I Characteristic:



$V_y = 0.3V$ for Ge $= 0.7V$ for Si

Note: $V_y \rightarrow$ Cut in voltage or offset Voltage or threshold voltage

$V_y \rightarrow$ Defined as "minimum forward voltage required above which the forward current flow's through device.

In case of reverse biase current is due to flow of minority carriers. If voltage is further increased in reverse direction then reverse current does not increase. At the breakdown of junction, value of current increases abruptly.

Effect of Temperature on Forward Current:

$$\boxed{I_f = I_0 e^{\frac{V_d}{\eta V_T}}}$$

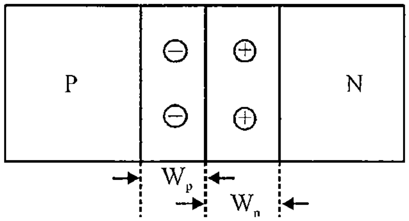
Forward current is only due to majority carrier concentration is almost independent of temperature.

Note: Cut in voltage V_y decreases with temperature. For 1°C increase in temperature V_y decreases by 2.3 milli volts.

Different Junction in Diode

1. **Step graded junction or Abrupt Junction:**

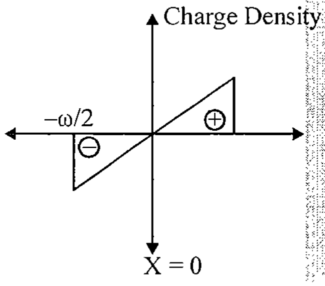
- It has abrupt p-n junction. It refer's to p^+ or PN^+ diode.
- Comparatively faster than normal diode.
- Depletion layer will always penetrate more into lightly doped region consider P^+N diode.



$\omega_N >> \omega_p$; $N_A > N_D$; $\boxed{\omega = \omega_N + \omega_P}$

$$\omega = \sqrt{\frac{2\in}{\cdot q}\left[\frac{1}{N_a}+\frac{1}{N_d}\right]}.V_0,\quad \omega = \sqrt{\frac{2\in}{q}\frac{1}{N_D}V_0}\ (N_A > N_D),\quad \boxed{V_0 = \frac{q\omega^2 N_D}{2\in}}$$

Linear Graded Junction: It is P^+ N^+ diode with equal doping level between P and N region.



$\boxed{\omega \propto (V_j)^{1/3}}$ V_j = Junction voltage

Different type of Capacitance in P-n Junction:

1. **Space Charge or Transition Capacitance:** When a P-N junction is reversed biased then depletion region acts like an insulator or dielectric material while P and N regions on either side have low resistance and acts as the plate. So this P-N junction in reverse bias may be regarded as a parallel plate capacitor if A is area of parallel plate capacitor and W is width of depletion layer then.

Transition Capacitance $C_T = \frac{\varepsilon A}{W} = C_J$

(a) For step graded junction, $\boxed{W \propto V_J^{1/2}}$

So, $\boxed{C_T \propto V_J^{-\frac{1}{2}}}$

(b) For linear graded junction $\boxed{W \propto V_J^{1/3}}$

So, $\boxed{C_T \propto V_J^{-\frac{1}{3}}}$

$V_k \rightarrow$ Knee voltage, $V_R \rightarrow$ Reversed biased voltage

$K \rightarrow$ Depends upon S.C. material

$n = \frac{1}{2}$ for step graded junction (Alloy junction)

$n = \frac{1}{3}$ for diffused junction / Linearly graded junction

So, $\boxed{C_T \propto (V_k + V_R)^{-1/2}}$ → Step graded / Alloy

$\boxed{C_T \propto (V_k + V_R)^{-1/3}}$ → Linearly graded / Diffused

Diffusion or Storage Capacitance (C_D): Diffusion capacitance dominates in a forward biased diode. It is mainly responsible for storage of minority carriers across the junction.

Mathematical Value: $C_D = \frac{dQ}{dV} = \tau.\frac{dI}{dV} = \frac{\tau}{r_d}$ $\boxed{C_D = \tau.g}$

(r_d = dynamic resistance $= \frac{dV}{dI}$)

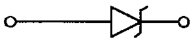
Where g_d = dynamic conductance

So, $\boxed{C_D = \frac{\tau.I_f}{\eta.V_T}}$, So $\boxed{C_D \propto I_f}$

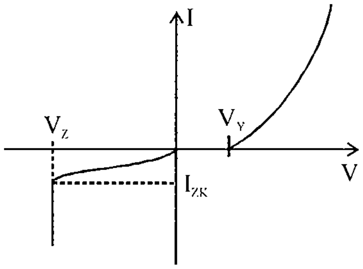
Diffusion capacitance C_D increase linearly with the forward current.

Zener Diode:

Diodes which are designed with adequate power dissipation to operate in breakdown region may be employed voltage reference or constant voltage devices. Such diodes are known as avalanche, breakdown or zener diode.

) Symbol 

i) V-I characteristic



$V_z \rightarrow$ Zener voltage or break down voltage, $V_y \rightarrow$ Cut in voltage, $I_{zk} \rightarrow$ Zener current

Zener diode is reversed biased heavily doped Si or Ge P-N junction diode which is operated in breakdown region Si is preferred to Ge due to it's high temperature and current carrying capability. Forward bias V-I characteristic of zener diode is same as ordinary diode. When reverse biased voltage applied to zener diode, on increasing the reverse voltage, zener current is increased greatly from it's normal cut off value. This voltage is V_2 or break down voltage. There are two mechanism of zener break down.

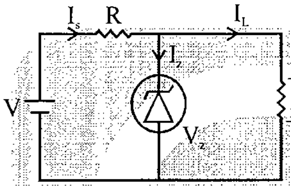
(1) (i) Zener break down / tunneling (ii) Avalanche multiplication.

(i) Zener break down / tunneling: When reverse bias is applied a very strong electric field is set up across the depletion layer and is enough to break or rupture the covalent bonds. Due to rupture of covalent bond large number of electron hole pairs are produced which constituted the reverse saturation current. It occurs in heavily doped p-n junction diode.

(ii) Avalanche break down: It occurs in case of lightly doped p-n junction diode. Here minority carriers accelerated by field applied, they collide with semiconductor atoms in the depletion region. Due to collision with valance electrons covalent bonds are broken and electron hole pairs are generated. These new carriers so produced acquire energy from applied potential and turn produce additional carriers. This forms cumulative process called as avalanche multiplication.

Temperature Characteristic of Zener Diode: If reference voltage is above 6 volt (where physical mechanism involved is avalanche multiplication) temperature coefficient is positive. But below 6 volt where zener breakdown is involved and temperature coefficient is negative.

Explanation of Above Mechanism:



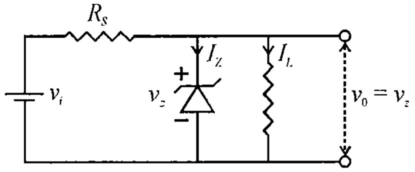
$[I_s = I_z + I_L]$

The source V and resistor R are selected in such a manner that diode is operating in breakdown region. Voltage across R_L is V_Z and diode current is I_Z. It have two type of regulation.

Zener Diode as regulator

A typical application of a zener diode as a voltage regulator is shown in the figure. In the breakdown region the the voltage across the zener diode is almost independent of current through it. This fact is very useful to design a voltage regulator. It will maintain a constant output voltage $v_0 = v_z$ independent of variations in load resistance R_L or the variation of input voltage $v_i (> v_z)$ so long as the diode remains in the breakdown region.

From the above figure we can write, $I_S = I_Z + I_L$ where, $I_S = \frac{v_i - v_z}{R_S}$ and $I_L = \frac{v_0}{R_L}$.



The minimum values of R_S that prevents the diode from being damaged is given by $R_{S,\min} = \frac{v_{i,\max} - v_z}{I_{ZM}}$,

where I_{ZM} can be obtained from $P_{Z,\max} = V_Z \cdot I_{ZM}$.

$P_{Z,\max}$ is normally specified by the manufacturer which is the maximum power that the zener diode can tolerate.

The maximum permitted value of R_S is set by the condition that the current through the zener diode must not go below I_{ZK} (it is knee current which corresponds to the knee of reverse characteristic curve).

$$R_{S,\max} = \frac{v_{i,\min} - v_z}{I_{L,\max} + I_{ZK}} \simeq \frac{v_{i,\min} - v_z}{I_{L,\max}}$$

Example: what is the maximum permissible current through a 5.6 V, 400 mW zener diode ? If the diode is used in a regulator circuit with maximum input voltage of 15 V, find the maximum value of series resistance that prevents the diode from being damaged.

$$P_{Z,\max} = I_{ZM} \cdot v_z$$

$$\therefore\ I_{ZM} = \frac{P_{Z,\max}}{v_z} = \frac{400\text{ mW}}{5.6\text{ V}} = 71.42\text{ mA}$$

current through the zener is maximum when $I_L = 0$. Therezfore, $I_S = I_{ZM}$.

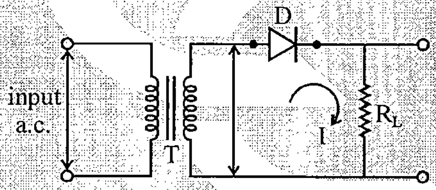
$$\therefore\ R_{S,\min} = \frac{v_{i,\max} - v_z}{I_{ZM}} = \frac{15 - 5.6\text{ V}}{71.42\text{ mA}} = 132\,\Omega. \qquad \because\ I_{zk} = 0$$

Rectifiers

The device which converts ac power into dc power by virtue of a characteristic permitting appreciable flow of current in one direction only, is called a rectifier. Diode can be used as a rectifying element in two different way.

(1) Half wave rectifier and (2) full wave rectifier.

Half wave rectifier



- In the input a ac voltage applied wave frequency ω and peak current I_0, that it can supply.
- For the positive half of the entire cycle the diode is forward biased whereas for negative half the diode is reverse biased.
- Hence the current through the diode and load R_L will be given by

$$i = I_0 \sin\omega t \text{ for } 0 \le \omega t \le \pi$$
$$= 0 \text{ for } \pi \le \omega t \le 2\pi$$

- $I_{\text{d.c.}} = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} i\,d(\omega t) = \dfrac{I_0}{\pi} \qquad V_{\text{d.c.}} = \dfrac{I_0 R_L}{\pi}$

$$I_{\text{r.m.s.}} = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} i^2\,d(\omega t)} = \frac{I_0}{2}$$

- An important quantity called efficiency of rectification (x) is defined as

$$\eta = \frac{P_{\text{d.c.}}}{P_{in}} \times 100\,\% = \frac{I_{\text{d.c.}}^2 R_L}{I_{\text{r.m.s.}}^2 (R_f + R_L)} = \frac{40.6}{\left(1 + \dfrac{R_f}{R_L}\right)}\,\%$$

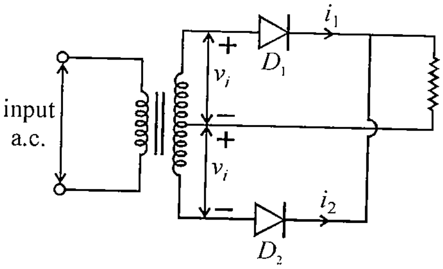
where, R_f is diode forward biased resistance.

- Another important quantity is ripple factor.

$$\nu = \frac{\text{r.m.s. value of a.c. components of load current (or voltage)}}{\text{d.c. value of load current (or voltages)}}$$

$$= \frac{I_{\text{r.m.s.}}}{I_{\text{d.c.}}} = \frac{\sqrt{I^2_{\text{r.m.s.}} - I^2_{\text{d.c.}}}}{I_{\text{d.c.}}} = \sqrt{\left(\frac{I_{\text{r.m.s.}}}{I_{\text{d.c.}}}\right)^2 - 1} = 1.21.$$

(2) Full wave rectifier



- In this case D_1 is in forward biased when the input is in positive half of a complete cycle and D_2 is forward biased when the input is in negative half.

$$\left.\begin{aligned} i_1 &= I_0 \sin \omega t \\ i_2 &= 0 \end{aligned}\right\} \text{for } 0 \le \omega t \le \pi$$

$$\left.\begin{aligned} i_1 &= 0 \\ i_2 &= -I_0 \sin \omega t \end{aligned}\right\} \text{for } \pi \le \omega t \le 2\pi$$

- $I_{\text{d.c.}} = \frac{2I_0}{\pi}$, $V_{\text{d.c.}} = \frac{2I_0 R_L}{\pi}$

$$I_{\text{r.m.s.}} = \frac{I_0}{\sqrt{2}}$$

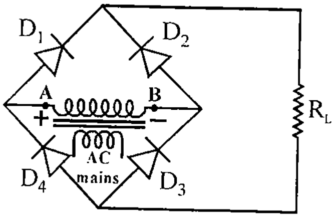
- Efficiency of rectification

$$\eta = \frac{I^2_{\text{d.c.}} R_L}{I^2_{\text{r.m.s.}}(R_f + R_L)} \times 100\,\% = \frac{81.2}{\left(1 + \frac{R_f}{R_L}\right)}\,\%$$

- Ripple factor

$$\nu = \sqrt{\frac{(I_{\text{r.m.s.}})^2}{I^2_{\text{d.c.}}} - 1} = 0.48$$

(3) Bridge rectification



- In this case when A is in positive potential with respect to B, diode D_1 and D_3 will conduct, and diode D_2 and D_4 will conduct if the point A become negative with respect to B.

- In bridge rectifier circuit a transformer without center tap at the secondary is used.
- The expression of $I_{\text{d.c.}}, I_{\text{r.m.s.}}, \eta$ and v all are same except R_f is now changed by $2R_f$, since two diodes are conducting at a time.

Example:

1) A bridge rectifier feeds a load resistance of 2500 Ω from a 30 V (rms) supply. Each diode of the rectifier has a forward resistance of 50 Ω . Calculate

(*i*) the dc load voltage

(*ii*) the ripple voltage at the output.

(*i*) The bridge circuit gives full wave rectification. So, the dc load current is $I_{\text{d.c.}} = \dfrac{2I_0}{\pi}$, where I_0 is the peak load current. Since two diodes in serier conduct simultaneously, we have

$$I_0 = \frac{V_0}{2R_f + R_L} = \frac{30\sqrt{2}}{2 \times 50 + 2500} = 0.0163 \text{ A}$$

$$\therefore \; I_{\text{d.c.}} = \frac{2I_{\text{r.m.s.}}}{\pi} = 0.0104 \text{ A}$$

The dc load voltage is $V_{\text{d.c.}} = I_{\text{d.c.}} R_L = 0.0104 \times 2500 = 26$ V

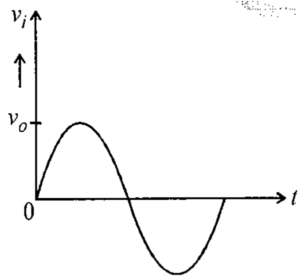
(*ii*) The ripple voltage at the output is $I'_{\text{r.m.s.}} R_L = \left(I^2_{\text{r.m.s.}} - I^2_{\text{d.c.}}\right)^{1/2} R_L$

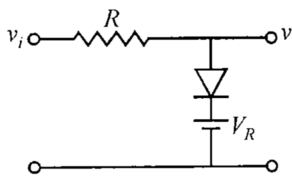
$$I_{\text{r.m.s.}} = \frac{I_0}{\sqrt{2}} = 0.0115 \text{ A}$$

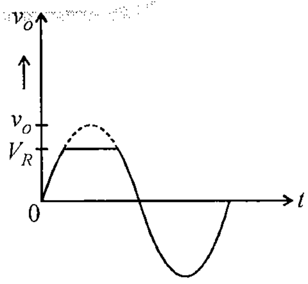
$$\therefore \; I'_{\text{r.m.s.}} R_L = (0.0115^2 - 0.0104^2)^{1/2} \times 2500 = 12.3 \text{ V}$$

Diode clipping and clamping circuit:

- Diode can be used as wave shaping circuits which can shape the voltage and current waveforms to suit various applications.
- Clipper circuits can 'clip' off a portion of the input signal. It is also referred to as voltage or current limiter amplitude selector or slicer.



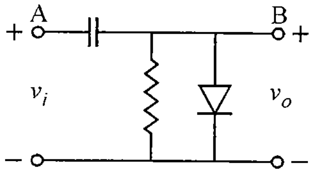


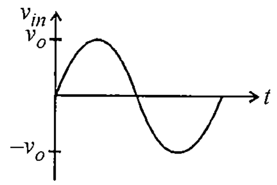


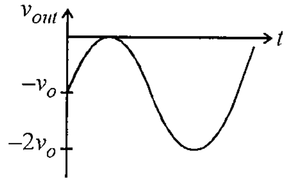
In the example shown above the circuit can 'clip' or remove portions of input waveform which lie above the reference voltage V_R. If the input $v_i < V_R$ the diode is reverse biased and does not conduct so open under ideal diode assumption. The input waveform then passes to output as it is. When v_i is greater than V_R, the diode will be forwarded biased and start to conduct which will limit the positive peak of the output voltage approximately to the level of V_R.

- The clampping circuit simply 'clamp' the input signal to a different dc level. This kind of circuit are useful for signal shaping, circuit protection and communications.

A clampping circuit (dc restor) is used to introduce a desired dc level in an ac signal. For example consider the circuit shown here, it will clamps the positive peaks to zero level.





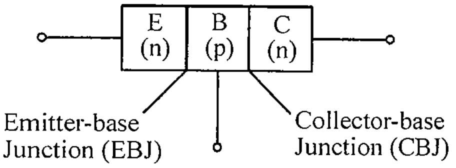


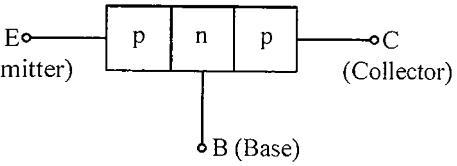
In this case the first quarter cycle charges the capacitor retains all of its charge with a voltage of v_m during subsequent cycles. Then, the output voltage is given by, $v_{out} = v_i - v_c = v_0 \sin \omega t - v_m$. By reserving the diode the negative peaks may be damped at 0 V. This is explained in the figure. Here output is given by,

$$v_{out} = v_i + v_c = v_i + v_o$$

Bipolar Junction Transistors:

- The bipolar junction transistor has three separately doped regions and contains two pn junctions.
- A single p-n junction has two modes of operation forward bias and reverse bias.
- The bipolar transistor, with two P-n junction, therefore has four possible modes of operation, depending on the bias condition of each p-n junction, which is one region for the vergatility of device.
- With three separately doped regions, the bipolar transistor is a three terminal device.
- The basic transistor principle is that the voltage between the two terminals controls the current through the third terminal.
- Current in the transistor is due to the flow of both electrons and holes, hence the name bipolar.
- The n-p-n bipolar transistor contains a thin p-region between two n-regions. In contrast the p-n-p bipolar transistor contains a thin n-region sandwiched between two p-regions.
- The three regions and their terminal connections are called the emitter, base and collector.
- The operation of the device depends, on the two p-n junction being in close proximity, so the with of the base must be very narrow, normally in the range of tenths of a micrometer (10^{-6} m).



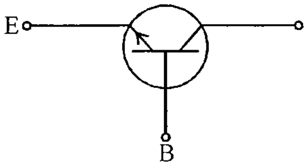


- Of the tree regions, emitter is heavily doped. The next level of doping is collector and base is less than that of collector.

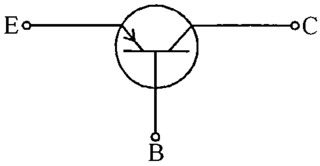
$\boxed{N_E > N_C > N_B}$

- The doping of the sandwiched layer is considerably less than that of the outer layers (typically, 10:1 or less)
- Emitter is provided with medium area ($A_C > A_E > A_B$) A; cross-sectional area
- Base is provided with smallest area to reduce the transit time.
- Collector is provided with the largest area to withstand heat dissipation.

Symbol:



(n-p-n transistor symbol)



(p-n-p transistor symbol)

Modes of Operation:

Mode of Operations	J_1 (B-E)	J_2 (C-B)	Applications
Active region	FB	RB	As an amplifier
Saturation region	FB	FB	As an electronic switch
Cut-off region	RB	RB	In digital circuit
Reverse active mode or inverted mode	RB	FB	As an amplifier with voltage and current gain to below (attenuator)

In a transistor
Emitter current - drift current
Base current - recombination current
Collector current - diffusion current

The flow of charge carriers in a transistor between base and collector is due to the diffusion of minority carriers and this action is called "transistor action". Hence, transistor action takes place in the base region.

The total current flowing into the transistor must be equal to the total current flowing out of if (applying Kirchhoff's current law to the transistor as if it were a single node). Hence, the emitter current is equal to the sum of the collector are and base currents.

$$\boxed{I_E = I_C + I_B}$$

The collector current comprise two components the majority and the minority carriers. The minority carrier current is called the leakage current and is given the symbol I_{CO} (I_C current will with emitter terminal open).

$$\boxed{I_C = I_{C\,\text{majority}} + I_{CO\,\text{minority}}}$$

I_{CO}:

It is also known as collector reverse saturation current or thermally generated current in the transistor.

I_{CO} for

Ge transistor is μA range, Si transistor is nA range.

I_{CO} is very sensitive to temperature.

I_{CO} doubles for every 10°C rise in temperature

For 1°C, I_{CO} approximately increases by 7%.

$$\boxed{I_{CO(T_2)} = I_{CO(T_1)}\, 2^{\left(\frac{T_2 - T_1}{10}\right)}}$$

I_{CO} is independent of collector supply voltage.

- The collector current is less than the emitter current. There are two reasons for this. Firstly, a part of the emitter current consists of holes that do not contribute to the collector current secondly, not all the electrons injected into the base are successful in reaching the collector.

Equation for Emitter Current:

- In a transistor under active region, emitter current is the forward current of emitter diode.

$$\boxed{I_E = I_{CO}\, e^{V_{BE}/nV_T}}$$

- The emitter current exponentially increases with base to emitter voltage of transistor.
- In a transistor under active region $V_{BE} < 1V$.

Typical value 0.2 V, For Si transistor $\boxed{V_{BE} = 0.6V \text{ to } 0.9V}$

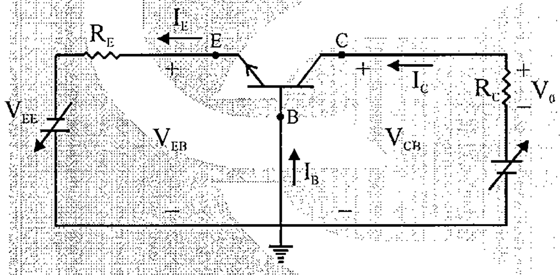
Typical value 0.7V

- V_{BE} reduced by 2.5 mV for 1°C rise in temperature, i.e.

$$\boxed{\frac{\partial V_{BE}}{\partial T} = -2.5\, mV/^\circ C}$$

Common Base Configuration:

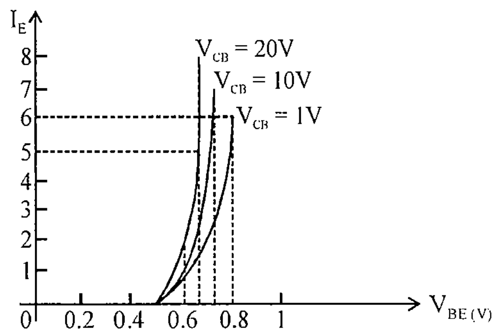
- To fully describe the behaviour of a three terminal devices such as the common base amplifier, requires two set of characteristics one for the 'driving point' or input parameters and the other for the output side.



(n-p-n transistor C-B configuration)

Input Characteristics:

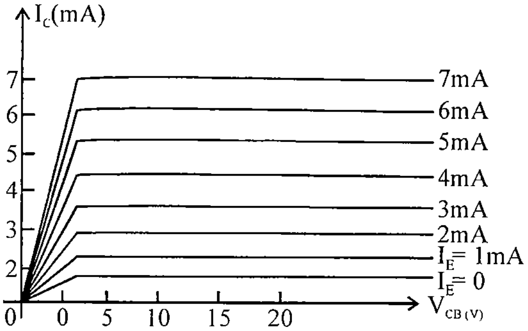
- The input set for the common-base amplifier as shown in figure relates an input current (I_E) to an input voltage (V_{BE}) for various levels of output voltage V_{CB}.



(Input or driving painting characteristic for CB silicon transistor amplifier)

Output Characteristic:

It relates I_C to V_{CB} for various levels of input current I_E as shown.



(Output or collector characteristics for a CB transistor (*n-p-n*) amplifier)

CB transistor is basically a current controlled current source (CCCS).

The curves clearly indicate that a first approximation to the relationship between I_E and I_e in the active region is given by

$$\boxed{I_C \cong I_E}$$

Properties of CB Configuration:

Lowest input resistance ($R_i < 100\Omega$)

Highest output resistance ($R_0 > 1M\Omega$)

Lowest current gain ($\alpha < 1$)

Highest voltage gain

Medium power gain (typical value 68)

Output and input voltages are in phase i.e. phase shift is 0°.

In CB amplifier current gain is loss and therefore bandwidth is large and hence CB amplifier is widely used as high frequency.

Alpha (α):

In the dc mode the levels of I_C and I_E due to the majority carriers are related by a quantity called alpha and defined by the following equation.

$$\boxed{\alpha_{dc} = \left|\frac{I_C}{I_E}\right|} \qquad \boxed{I_C = \alpha I_E + I_{CBO}}$$

Where I_{CBO} is collector to base current when emitter terminal is open.

For ac situations where the point of operation moves on the characteristic curve, an ac alpha is defined by.

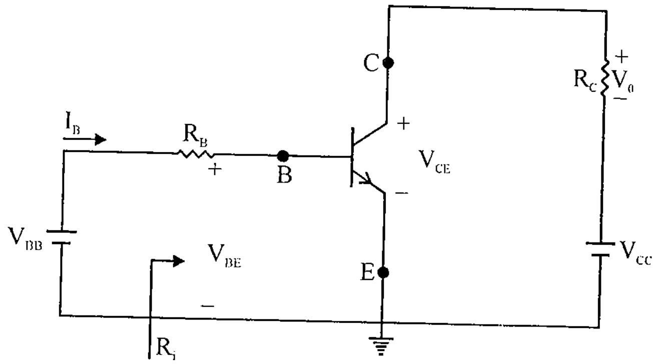
$$\boxed{\alpha_{ac} = \left.\frac{\Delta I_C}{\Delta I_E}\right|_{V_{CB}=\text{Constant}}}$$

The alpha is formally called common base amplification factor on current gain of common base transistor.

Note: The transistor's amplifying action is basically due to its capability of transfer its signal current from a low resistance circuit to high resistance circuit, contracting the two terms transfer and resistor results in the name transistor; i.e.

$$\boxed{\text{transfer} + \text{resistance} \to \text{transistor}}$$

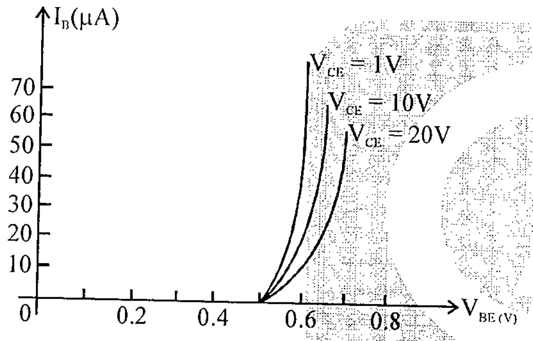
Common-Emitter Configurations:



n-p-n transistor in CE configuration

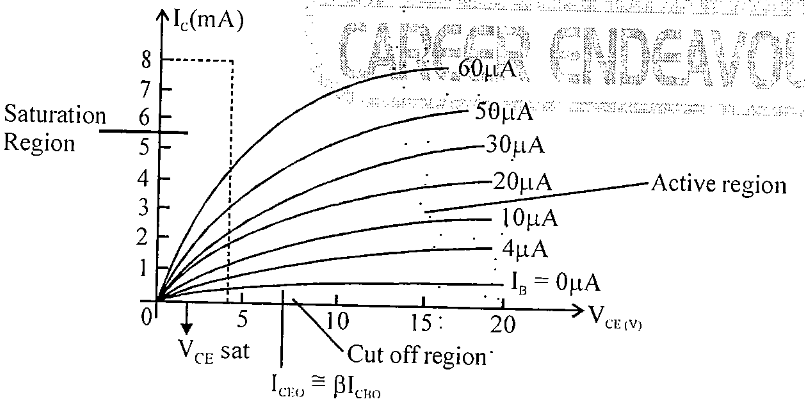
Input Characteristics:

- The input characteristics are a plot of the input current (I_B) versus the input voltage (V_{BE}) for a range of values of output voltage V_{CE}.



Output Characteristics:

- The output characteristics are a plot of the output current (I_C) versus output voltage (V_{CE}) for a range of values of input current (I_B).



Properties:

- Moderate input resistance (around 1kΩ)
- Moderate output resistance (50kΩ to 500 kΩ)
- Moderate current gain (typical value 49)
- Moderate voltage gain
- Highest power gain (typical value 4226)
- Output and input voltages are out of phase shift = 180°.
- It is most common and frequency used amplifier.

Beta (β):

In dc model $\boxed{\beta_{dc} = \frac{I_C}{I_B}}$ ($\beta > 1$) For ac $\boxed{\beta_{ac} = \left.\frac{\Delta I_C}{\Delta I_B}\right|_{V_{CE}=\text{constant}}}$

Range of β is 30 to 300. β in terms of α is $\boxed{\beta = \frac{\alpha}{1-\alpha}}$ $\boxed{\alpha = \frac{\beta}{1+\beta}}$

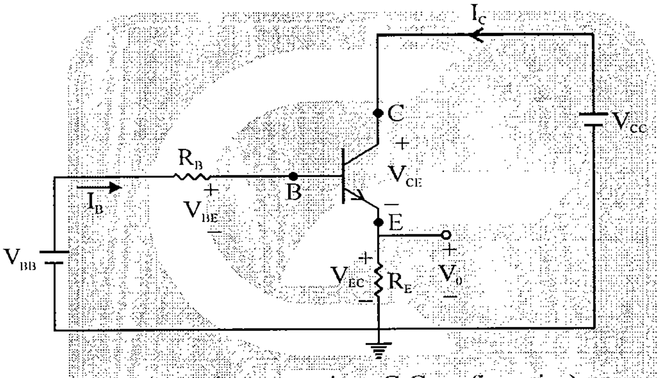
β is called the current gain of transistor in CE mode. It is the most important specification of the transistor.

β is also denoted by h_{fe} and always.

$$\boxed{I_C = \beta I_B}$$

$I_E = I_C + I_B = \beta I_B + I_B$ $\boxed{I_E = (\beta+1)I_B}$

Common Collector Configuration (C-C):



(n-p-n transistor C-C configuration)

It is popularly known as "Emitter follower".

In common collector, collector resistance must be zero and output voltage is taken across the emitter resistance.

Properties:

Highest input resistance (50kΩ to 500 kΩ)

Lowest output resistance (<100Ω)

Highest current gain

Lowest voltage gain (close to unity and ideally one)

Output and input voltages are in phase i.e. phase shift is 0°.

Emitter follower is basically a current controlled voltage source (CCVS).

Applications:

As an audio frequency power amplifier

As a buffer (impedance matching device between high resistance to low resistance)

In designing of voltage sweep circuits.

As an high input resistance devices.

As a "Boot strap emitter follower".

Gamma (γ):

$\gamma = \left|\frac{I_E}{I_B}\right| = 1+\beta$ $\boxed{\gamma = 1+\beta = \frac{1}{1-\alpha}}$ $\boxed{\alpha < \beta < \gamma}$

Note: The output characteristics of transistor in any configuration is divided in three regions.
⇒ Saturation region, ⇒ Active region, ⇒ Cut-off region

1. **Saturation Region:**

Minimum I_B required to keep the transistor under saturation region.

$$\boxed{I_{\beta\min} = \frac{I_C}{\beta_{dc}}} \quad \text{or} \quad \boxed{I_{\beta\min} = \frac{I_C}{h_{FE}}}$$

Conditions for saturation region:

$I_{B\min} \le I_B$ saturation else active

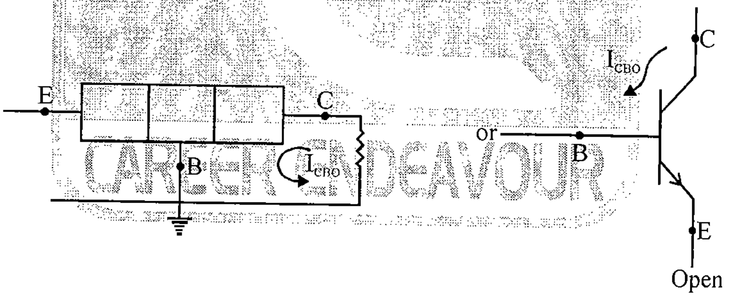
Where $I_{C\max} = \dfrac{V_{CC}}{R_C + R_E}$, if $R_E = 0$ $\boxed{I_{C\max} = \dfrac{V_{CC}}{R_C}}$

	$V_{BE\ active}$	$V_{BE\ sat}$	$V_{CE\ sat}$
Ge	$0.3V$	$0.2V$	$0.1V$
Si	$0.7V$	$0.8V$	$0.2V$

Note: Theory says under saturation $I_C \ge I_{C\max}$ but in a practical circuit the maximum collector current is $I_{C\max}$ under the saturation region i.e. $\boxed{I_C = I_{C\max}}$

I_{CBO}:

- It is the leakage current passing from collector to base with emitter open circuited, it is also called emitter cut-off current of the transistor.



At low frequencies $\boxed{I_{CBO} = I_{CO}}$

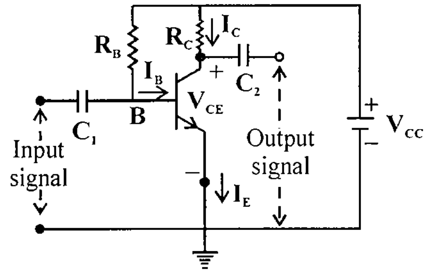
At high frequencies $\boxed{I_{CBO} = I_{CO} + \text{Surface Current}}$

Where surface current is due to skin effect. Skin effect appears only at high frequencies. Surface current is independent of temperature and depends upon only applied voltage. It increases with frequency.

D.C. Biasing Arrangement:

(1) Fixed Bias Arrangement:

Fixed bias arrangement applied to an n-p-n transistor in CE mode is shown in figure below. The current directions of figure are the actual current directions and the voltages are denoted by the standard double subscript notation i.e., the terminal marked by the first subscript is assumed to the positive with respect to the terminal marked by the second subscript. C_1 and C_2 are the so called coupling capacitors. Their reactances are chosen very small at the lowest signal frequency and hence, they practically offer short circuited path to a.c. but for d.c. they act as open circuit. These capacitors make it possible to connect the input signal source to the amplifier or the output of the amplifier to the input of another amplifier without affecting its d.c. bias. The single d.c. source V_{CC} is used to make the emitter junction forward biased and the collector junction reverse biased. R_B is called base bias resistance and R_C, the load resistance.



Determination of Q-point:

Applying KVL around base-emitter circuit loop we get

$$V_{CC} = I_B R_B + V_{BE}$$

Or, $$I_B = \frac{V_{CC} - V_{BE}}{R_B} \qquad \text{... (i)}$$

Usually, $V_{CC} > V_{BE}$ and hence,

$$I_B \simeq \frac{V_{CC}}{R_B} \qquad \text{... (ii)}$$

Thus, any change in V_{BE} due to temperature will have practically no effect on I_B.

Since, the base bias current I_B is constant, the circuit is called fixed bias circuit. Applying KVL around the collector-emitter circuit we get

$$V_{CC} = I_C R_C + V_{CE} \qquad \text{... (iii)}$$

Stability factors:

Since, the Q-piont is chosen in the active region we can write

$$I_C = I_B + (1+\beta) I_{CBO} \qquad \text{... (iv)}$$

Using equation (i),

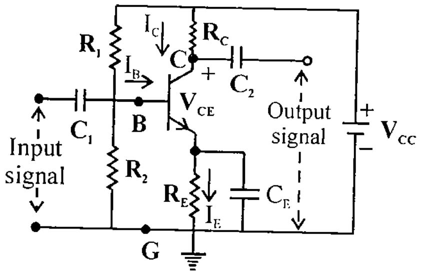
$$I_C = \beta \cdot \frac{V_{CC} - V_{BE}}{R_B} + (1+\beta) I_{CBO} \qquad \text{... (v)}$$

Now, from stability factors,

$$S = \frac{\partial I_C}{\partial I_{CBO}} = 1 + \beta$$

Voltage divider bias or self bias:

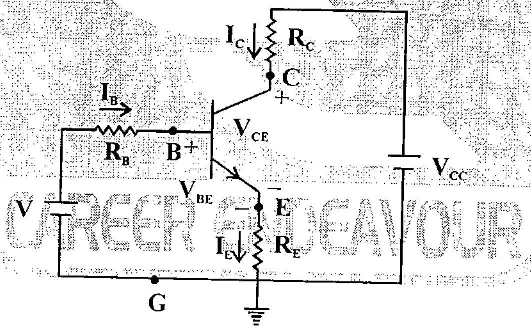
The commonly used voltage divider bias arrangement applied to an n-p-n transistor in CE mode is shown in figure below. The current directions shown in figure are the actual conventional current directions. The voltages are denoted by the standard double subscript notations. C_1 and C_2 are the so called coupling or blocking capacitors. Reactances of C_1 and C_2 are chosen very small at the lowest signal frequency and hence they allow easy flow to a.c. but blocks d.c. These capacitors make it possible to connect the input signal source or the output of the amplifier to the input of the next stage without affecting its d.c. biasing.



The capacitor C_E is known as by-pass capacitor. Its reactance is chosen very small compared to R_E. It prevents R_E from providing a.c. negative feedback gain loss but d.c. negative feedback remains. In this circuit R1 and R2 act as potential divider. the p.d. across R_E tends to reverse bias the emitter junction while p.d. across across R_2 tends to forward bias it. If temperature increases, I_{CBO} and I_C biased. The p.d. across R_E increases which makes the emitter junction less forward biased. As a result base current I_B decreases. This is turn checks the increase in I_C and it tries to bring I_C back to its original value. Thus, the stabilization of Q-point is maintained automatically by the circuit and for this the circuit is called self bias circuit.

Determination of Q-piont:

To find the d.c. operating point we can neglect the capacitors. Now, applying Thevenin's theorem to the left of the base B and the ground G in figure we above, we get the simplified circuit of figure below.



Thevinin equivalent circuit

$$V = \frac{V_{CC}}{R_1 + R_2} \cdot R_2$$

And
$$R_B = \frac{R_1 R_2}{R_1 + R_2} \qquad \text{... (1)}$$

Application of KVL around the base circuit yields,

$$V = I_B R_B + V_{BE} + I_E R_E \qquad \text{... (2)}$$

Applicatino of KVL around the collector circuit yields,

$$V_{CC} = I_C R_C + V_{CE} + I_E R_E \qquad \text{... (3)}$$

Now, we have $I_E = I_B + I_C$. Assuming $I_E \approx I_C$, we can write from equation (3)

$$V_{CE} \approx V_{CE} - I_C \left(R_C + R_E\right)$$

Stability factors:

Substituting $I_E = I_B + I_C$ in equation (2), we get

$$V = I_B\left(R_B + R_E\right) + V_{BE} + I_C R_E$$

Or,
$$I_B = \frac{V - V_{BE} - I_C R_{ER}}{R_B + R_E} \qquad \dots (4)$$

Now, in the active region,

$$I_C = \beta I_B + (1+\beta) I_{CBO}$$

Using, equation (4), we get

$$\left(1 + \frac{\beta R_E}{R_B + R_E}\right) I_C = \frac{\beta\left(V - V_{BE}\right)}{R_B + R_E} + (1+\beta) I_{CBO} \qquad \dots (5)$$

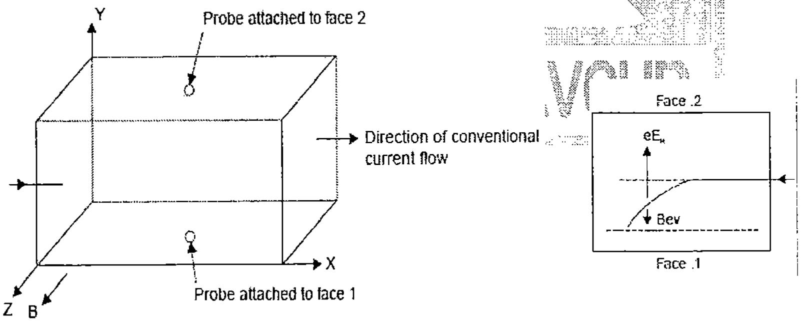
Now,

$$S = \frac{\partial I_C}{\partial I_{CBO}} = \frac{1+\beta}{1 + \dfrac{\beta R_E}{R_B + R_E}} = (1+\beta) \cdot \frac{1 + \dfrac{R_B}{R_E}}{1 + \beta + \dfrac{R_B}{R_E}}$$

Hall effect:

When a semiconductor is placed in a magnetic field $\vec{B}$ (along the z-direction) and a current density $\vec{J}$ (along x-direction) passed through it, a transverse electric field $\vec{E}$ (along y-direction) is set up given by $\vec{E} = R_H \vec{B}\,\vec{J}$ where R_H is the Hall coefficient or Hall constant.

The Hall effect may be explained with the help of following figure.



Hall effect and motion of electrons in n-type semiconductor

Assuming that the material is an n-type semiconductor, the current flow consists almost entirely of electrons moving from right to left just opposite to the conventional flow of current. If v is the velocity of electrons at right angles to the magnetic field, there is downward force on each electron of magnitude Bev. This causes the electron to be deflected in a downward direction and causes a negative charge to accumulate on the bottom face of the slab. A potential difference is therefore established from top to bottom of the specimen with bottom face negative. This potential difference causes a field E_H in the negative of y-direction, and so there is a force of eE_H acting in the upward direction on the electron. Equilibrium occurs when

$$eE_H = eBv, \quad E_H = Bv$$

If J_x is the current density in the x-direction, then $J_x = nev$ where n is the concentration of current carriers.

Thus, $E_H = \frac{BJ_x}{ne}$; $\frac{E_H}{BJ_x} = \frac{1}{ne}$

The Hall effect is described by means of the Hall coefficient R_H, defined in terms of the current density J_x by the relation $E_H = R_H J_x B \Rightarrow R_H = \frac{E_H}{J_x B} \Rightarrow R_H = \frac{1}{ne}$.

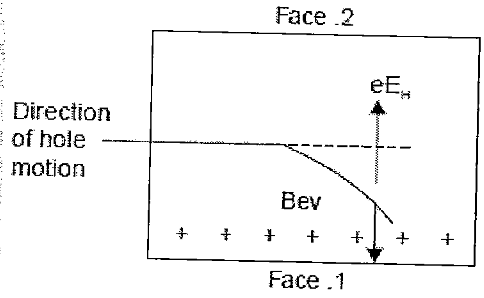
In this case $R_H = \frac{1}{ne}$.

Negative sign is used because the electronic field developed is in the negative y-direction.

i.e., $R_H = \frac{E_H}{J_x B} = -\frac{1}{ne}$.

All the quantities E_H, B and J_x can be measured, and so the Hall coefficient and carrier density n can be found out.

Figure shows the conditions that exist in a p-type semiconductor under identical conditions as that of n-type specimen described above.



Hall effect in p-type semiconductor

Experimental procedure :

The Hall coefficient is determined by measuring the Hall voltage that generates the Hall field. If V_H is the Hall voltage across the sample of thickness t, then $V_H = E_H t$.

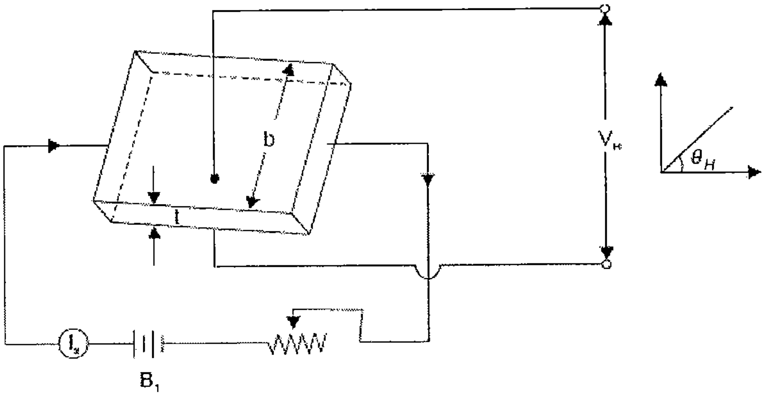
Substituting for E_H for equation $V_H = B t R_H J_x$.

If b is the width of the sample, then its cross-section will be $(b \times t)$ and the current density, $J_x = \frac{I_x}{bt}$.

$V_H = \frac{B t R_H I_x}{bt}$ and $R_H = \frac{V_H b}{I_x B}$

Note the polarity of V_H will be opposite for n and p-type.

A rectangular slab of the given material having a thickness t and width b is taken and a current I_x ampere is allowed to pass through this sample by connecting it to a battery. The slab is then placed between two pole pieces of an electromagnet such that if the current direction coincides with the x-axis, the magnetic flux density B coincides with the z-axis, as shown in the following figure.



Set up for the measurement of Hall voltage

The Hall voltage V_H is then measured by placing two probes at the centres of the bottom and top faces of the sample. If the magnetic flux density is B weber/m^2 and the Hall voltage is V_H volt, then R_H is obtained from above equations in m^3/coulomb. The analysis is given above applies only when the charge carriers are free of attractive forces in the energy bands and when they move with a steady drift velocity v_d. This is not the case in a semiconductor, and a computation of the average speed leads to the conclusion that

$$R_H = -\frac{3\pi}{8ne} = -\frac{1.18}{ne} \text{ instead of } -\frac{1}{ne}.$$

For n-type of material, the conductivity is given by $\sigma_n = ne\mu_n$ and $\mu_n = \frac{\sigma_n}{ne} = -\frac{\sigma_n R_H}{1.18}$.

For p-type material, $\mu_p = \frac{\sigma_p}{pe} = \frac{\sigma_p R_H}{1.18}$.

There is another interesting quantity called the Hall angle (θ_H) defined by the equation, $\tan\theta_H = \frac{E_H}{E_x}$

But $E_H = \frac{J_x B}{ne} = \frac{nev_x B}{ne} = \frac{v_x B E_x}{E_x}$.

Hence, $\tan\theta_H = \frac{E_H}{E_x} = \frac{v_x B}{E_x} = \mu B$, where μ is the mobility of the charge carriers.

Applications of Hall effect:

(i) Determination of type of semiconductor: For an n-type semiconductor the Hall coefficient is negative where as for a p-type semiconductor it is positive. Thus, the sign of the Hall coefficient can be used to determine whether a given semiconductor is n-type or p-type.

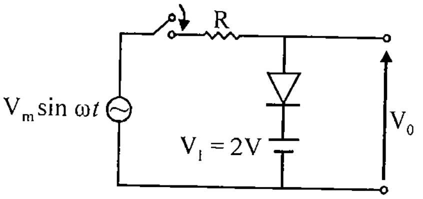
(ii) Calculation of carrier concentration (n).

(iii) Determination of mobility (μ).

(iv) Measurement of magnetic flux-density (B).

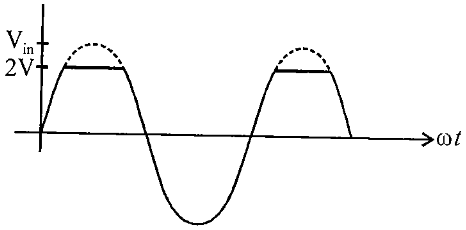
SOLVED PROBLEMS

1. Draw the output wave form of the circuit as shown in figure, when switch is closed.

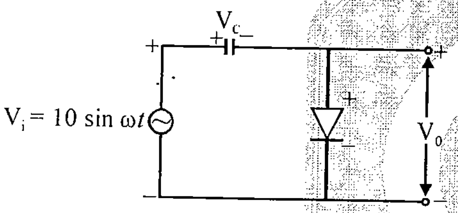


Soln. This is a simple clipping circuit

V_i	Diode	V_0
$V_i \le 2V$	OFF	V_i
$V_i \ge 2V$	ON	$2V$

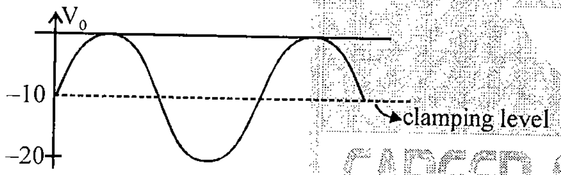


2. Draw the output waveform for the given circuit if the input is sinusoidal.



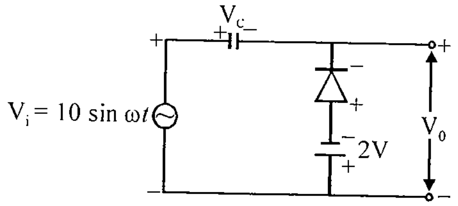
Soln. Apply KVL $V_i - V_C - V_0 = 0$, Capacitor will charge till 10V.

$\because\ V_0 = V_i - V_C,\ V_i > V_C,\ V_0 = V_i - 10$

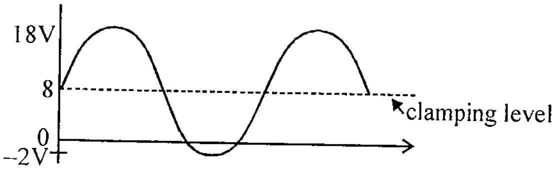


Trick input waveform is shifted to the direction indicated by the diode.

3. Draw the output waveform for the given circuit if the input is sinusoidal.



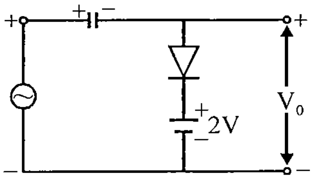
Soln. Apply kVL, $V_i - V_C - V_0 = 0,\ V_0 = V_i - V_C = V_i - (-8),\ V_0 = V_i + 8$



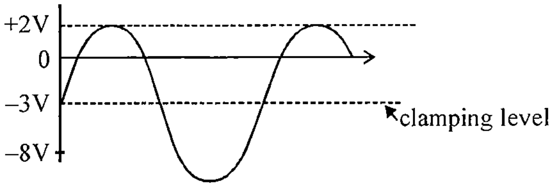
Capacitor will charge when V_i is negative

$-10 + 2 - V_i = 0,\ V_C = -8V$

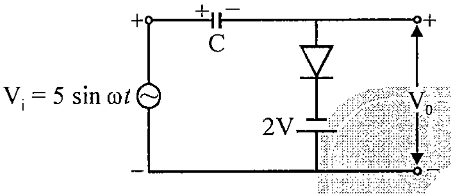
Draw the output waveform



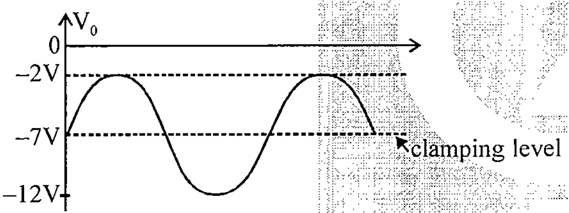
oln. Capacitor will charge when V_i is positive, apply KVL, $5 - V_C - 2 = 0$, $V_C = 3V$



Draw the output waveform

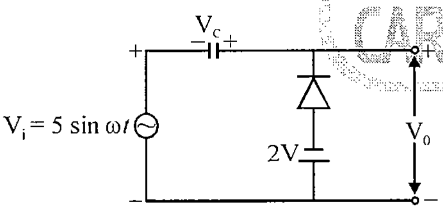

+
+
−
C
+
V0
−
$V_i = 5 \sin \omega t$
2V
−

oln. Capacitor will charge when $V_i > -2V$, apply KVL $5 - V_C + 2 = 0$, $V_i = 7V$

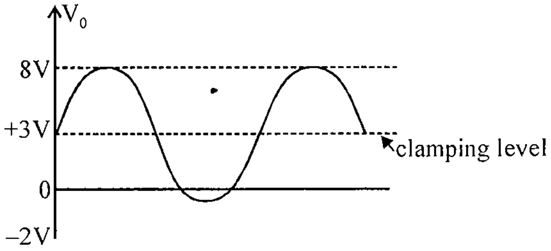


Apply KVL, $V_i - V_C - V_0 = 0$, $V_0 = V_i - V_C$

Draw the output voltage waveform


VC
−
+
+
+
$V_i = 5 \sin \omega t$
V0
2V
−
−

oln. Capacitor will charge when V_i is negative. Apply KVL, $-5 + 2 + V_C = 0$, $V_C = -3V$



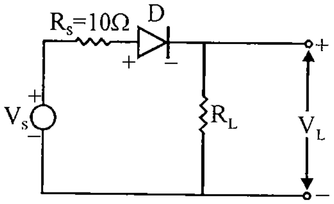
Apply KVL, $V_i - V_C - V_0 = 0$, $V_0 = V_i - V_C$, $V_0 = V_i + 3$

7. Find D.C. value and r.m.s. value of signal $X_T = 2 + 4\sin\omega t$

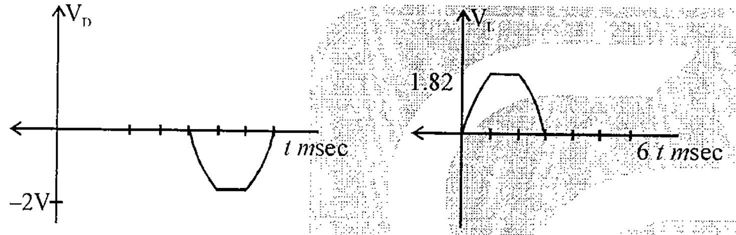
Soln. $X_{D.C.} = 2,\ X_{A.C.} = 4\sin\omega t,\ X_{A.C.(r.m.s.)} = \dfrac{4}{\sqrt{2}} = 2\sqrt{2}$

$$X_{T(r.m.s.)} = \sqrt{X^2_{D.C.} + X^2_{A.C.(r.m.s.)}} = \sqrt{2^2 + (2\sqrt{2})^2} = 2\sqrt{3}$$

8. For the circuit of figure, sketch the waveforms of V_L and V_D if the source voltage V_S is as given in figure. The diode is ideal and $R_L = 100\,\Omega$.



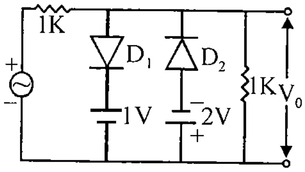
Soln.



If $V_S \geq 0, D$ conducts, so that $V_D = 0$ and $V_L = \dfrac{R_L}{R_L + R_S} V_S = \dfrac{100}{100+10} V_S = 0.91 V_S$

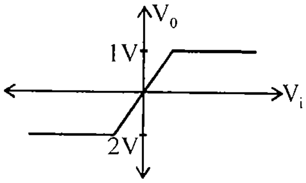
If $V_S < 0$, diode blocks, so that $V_D = V_S$ and $V_L = 0$ sketches of V_D and V_L are shown in figure.

9. Draw the transfer characteristic for the given circuit assuming ideal diodes.



Soln. Case-I: When $V_i > 1, V_1 = D_1 = \text{ON},\ D_2 = OFF$ and $V_0 = 1$ volt

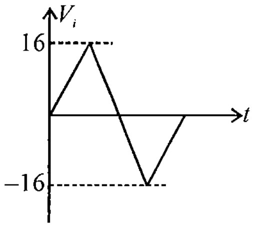
Case-II: $V_i < 2V, D_1 = \text{OFF},\ D_2 = \text{ON}$ and $V_0 = -2$ volt



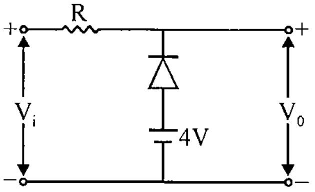
V_i	D_1	D_2	V_0
$V_i < -2V$	OFF	ON	–2V
$-2V < V_i < 1V$	OFF	OFF	$V_i/2$
$V_i > 1V$	ON	OFF	1V

Case-III: For, $-2V < Vi < 1V,\ V_0 = \dfrac{V_i}{1+1} \times 1 = \dfrac{V_i}{2}$

10. Determine V_0 for the network of figure.



Vi	Diode	V_0
$V_i > 4V$	ON	4V
$V_i > 4V$	OFF	V_i

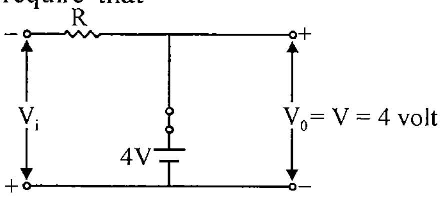


Soln. The polarity of the d.c. supply and the direction of the diode strongly suggest that the diode will be in the (on) state for the negative region of the input signal. For this region the network will appear as shown in figure where the defined terminals for V_0 require that

$V_i > 4V_1$ diode is off (O.C)

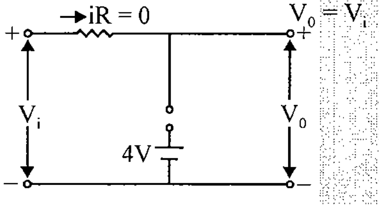
$V_i < 4V_1$ diode is on (short circuit)

For the open circuit state, $V_0 = V_i$

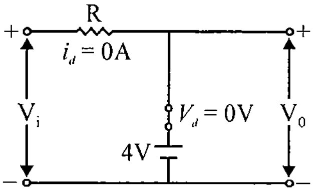


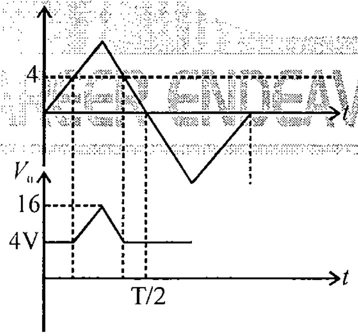
V_0 for negative region of V_i

Case-I: $V_i > 4V$, diode is off (open circuit) $V_0 = V_i$

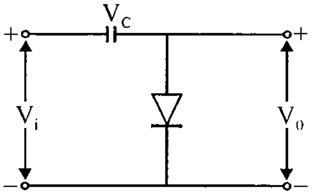


Case-II: $V_i < 4V_1$ diode is on (short circuit), V_0 = 4 volt





1. To find V_0? when (i) $0 < V_i < +5V$ (ii) $V_i = 5V$, $V_C = 5V$ (iii) $-5V < V_i < 0V$



Soln. (i) $0 < V_i < +5V$, Diode is on, it means short circuit

Capacitor charges upto $V_C = 5V$, then $V_0 = 0$

(ii) $V_i = 5V$, $V_C = 5V$

Apply KVL in input loop $V_i - V_C - V_0 = 0$, $V_0 = V_i - V_C = 0$, $V_0 = 5 - 5 = 0V$

(iii) $-5V < V_i < 0V$

Apply KVL $V_i - V_C - V_0 = 0$, $V_0 = V_i - V_C$, $V_0 = V_i - 5V$

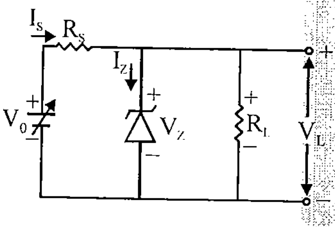
12. $X(t) = 4\sin\omega t + 2\sin(\omega t + 60°)$, find out $X_{r.m.s.}$ and $X_{d.c.}$

Soln. $X_{r.m.s.} = \sqrt{\left[X^2_{1r.m.s.} + X^2_{2r.m.s.} + 2X_{1r.m.s.}.X_{2r.m.s.}\cos\phi\right.}$

$$= \sqrt{\left(\frac{4}{\sqrt{2}}\right)^2 + \left(\frac{2}{\sqrt{2}}\right)^2 + 2\left(\frac{4}{\sqrt{2}}\right)\left(\frac{2}{\sqrt{2}}\right)\cos 60°} = \sqrt{8 + 2 + 8 \times \frac{1}{2}} = \sqrt{14}$$

and $X_{DC} = 0$

13. The zener diode in voltage regulator circuit of figure has a constant reverse breakdown voltage $V_Z = 8.2V$, for $75mA \le i_z \le 1A$ if $R_L = 9\Omega$ size R_S so that $V_L = V_Z$ is regulated to (maintained at) 8.2V while V_b varies by ±10 percent from it's nominal value of 12 volt.



Soln. By Ohm's law, $i_L = \frac{V_L}{R_L} = \frac{V_Z}{R_L} = \frac{8.2}{9} = 0.911A$

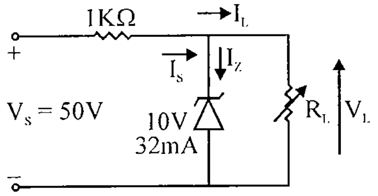
Now an application of KVL gives $R_S = \frac{V_b - V_Z}{i_Z + i_L}$

and ω_0 are to size R_S maximum zener current i_Z at the largest value of V_b.

$$R_S = \frac{(1.1)(12) - 8.2}{1 + 0.911} = 2.62\Omega$$

14. A zener regulated circuit is shown in the figure. Find the range of load resistance, load current for V_L to be maintained constant. Assume zener is ideal zone and also find the maximum power consumed.

Assuming rating of zener is 10V, 32mA



oln. Zener is ideal, means $R_Z = 0$, $V_Z = V_L = 10V$

$$I_S = \frac{50-10}{1K\Omega} = 40mA \text{ (constant)}$$

(a) R_L is minimum means $I_Z = 0$, $I_{Lmax} = I_S = 400mA$, $V_L = V_Z = I_{Lmax}.R_{Lmin}$

$$R_{Lmin} = \frac{10}{40\times10^{-3}} = 250\Omega$$

(b) R_L is maximum,
mean $I_Z = 32mA = I_{zm}$, $I_{Lmin} = I_S - I_{Zm}$
$V_L = V_Z = I_{Lmin}.\ R_{Lmax}$

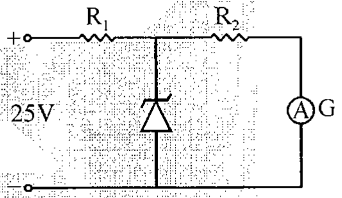
$$R_{Lmax} = \frac{V_L}{I_{Lmin}} = \frac{10}{8\times10^{-3}} = 1.25K\Omega$$

P_{max} = maximum power rating = 10 × 32 = 320m watt

5. The complete circuit shown in figure represents a 25dc voltmeter where G is a PMMC galvanometer having full scale deflection current

$I_{fsd} = 200mA$ and $R_G = 500\Omega$

and D is 20V zener diode, find R_1 and R_2.
What is the function of the diode D in this circuit.

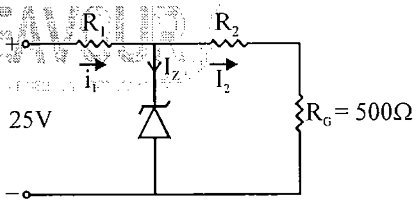


Soln. Current through galvanometer

$$I_{fsd} = I_Z = \frac{\text{Zener voltage}}{R_2 + R_G},\quad 200\times10^{-6} = \frac{20}{R_2+500}$$

As zener diode current is not specified let it be assumed zero.

$\therefore I_1 - I_2 = I_z$ or $I_1 = I_2 = 200\mu A$



Also, $I_1 = \frac{25-20}{R_1} = 200\times10^{-6}$, $R_1 = \frac{5\times10^6}{200} = 25k\Omega$

6. In a common-base connection, current amplification factor is 0.9. If the emitter current is 1mA, determine the value of base current

(a) 0.9mA (b) 1mA (c) 0.5mA (d) 0.1mA

Soln. **(d)**

7. Which one of the following statement is correct? In a transistor

(a) I_{CBO} is greater than I_{CEO} and does not depend upon temperature.
(b) I_{CEO} is greater than I_{CO} and doubles for every temperature degree rise in temperature
(c) I_{CBO} is equal to I_{CO} and double for every temperature degree rise in temperature
(d) I_{CEO} is equal to I_{CO} and doubles for every ten degrees rise in temperature

Soln. **(c)**

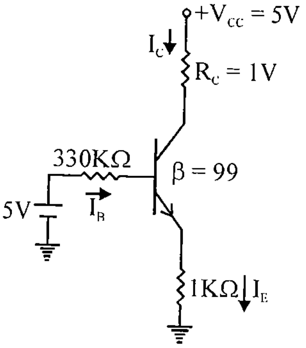
18. Find out Q point?

(a) $3mA, 1V$

(b) $-99mA, 3.02$ Volt

(c) $1.1mA, 3V$

(d) None of these



Soln. Step-I: Assume BJT is active mode

$\therefore V_{BE} = 0.7$ Volt

Step-II: Apply KVL base emitter loop

$5 - 330 \times 10^3\, I_B - 0.7 - 1 \times 10^3\, I_E = 0$

$4.3 = 330 \times 10^3\, I_B + 1 \times 10^3 (I_B + I_C) = 330 \times 10^3\, I_B + 1 \times 10^3 (I_B + \beta I_B)$

$4.3 = I_B(330 \times 10^3 + 100 \times 10^3)$, $4.3 = I_B(430 \times 10^3)$

$I_B = 10\mu A$

Step-III: $I_{C(active)} = \beta I_B = 99 \times 10 \times 10^{-6} = .99mA$

Step-IV: Apply KVL in collector-emitter loop for $I_{C(saturation)}$

$5 - I_C \times 10^3 - V_{CE(sat)} - I_E \times 10^3 = 0$

$5 - 0.2 = I_C \times 10^3 + (I_B + I_C) \times 10^3$

$4.8 = I_C(2 \times 10^3)$, $I_{C(sat)} = 2.4mA$

Step-V: $\because I_{C(active)} < I_{C(sat)}$ $\therefore$ BJT in active mode.

Step-VI: $V_{CE} = V_{CC} - I_C(R_C + R_E)$

$= 5 - 0.99 \times 10^{-3}\,(2 \times 10^3)$

$= 5 - 1.98 = 3.02$

Q point $= I_{C(active)} = 0.99mA$, $V_{CE\,(active)} = 3.02$

Correct answer is (b)

19. Find out Q-point, where $\beta = 100$

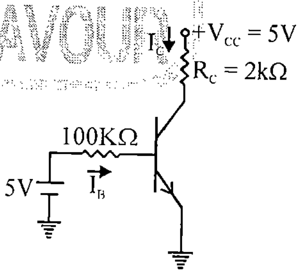
Soln. Step-I: Assuming BJT inactive mode

$\therefore V_{BE} = 0.7$ volt, For Si

Step-II: Applying KVL in base-emitter loop

$5 - I_B \times 100 \times 10^3 - V_{BE} = 0$

$I_B = 4.3 \times 10^{-5}$



Step-III: $I_{C(active)} = \beta I_B = 4.3 \times 10^{-5} \times 100 = 4.3$ mA

Step-IV: Applying KVL in collector-emitter loop for $I_{C(sat)}$

$V_{CC} - 2 \times 10^3\, I_C - V_{CE} = 0$

$\because V_{CE} = 0.2V$, this is the typically value in saturation mode for Si.

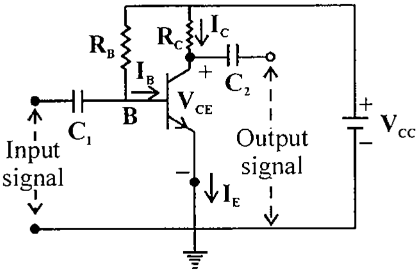
$\therefore 5 - 0.2 = 2 \times 10^3\, I_C$, $I_{C(sat)} = 2.4$ mA

Step-V: $\because I_{C(active)} > I_{C(sat)}$

$\therefore$ BJT is in saturation region

Q-Point: $[I_{C(saturation)} = 2.4$ mA, $V_{CE(sat)} = 0.2V]$

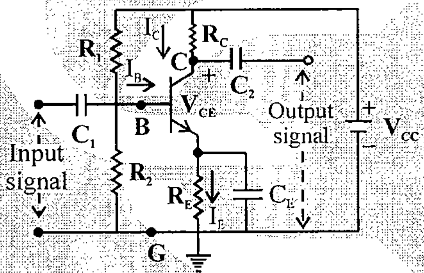
). Consider an n-p-n Si transistor used with fixed bias arrangement as shown in figure below. Assuming $\beta = 100$ and V_{CC} = 10 V, find the values of R_C and R_B required to set up the Q-point at I_C = 5 mA, V_{CE} = 5V. [Given : $I_B >> I_{CBO}$]


R_B
R_C
I_C
I_B
+
C_2
V_{CE}
B
C_1
Input signal
Output signal
−
I_E
+
−
V_{CC}

)ln. $R_c = \frac{V_{CC}-V_{CE}}{I_C} = \frac{10V-5V}{5mA} = 1\,k\Omega$

$I_B \approx \frac{I_C}{\beta} = \frac{5mA}{100} = 50\,\mu A$ and $R_B = \frac{V_{CC}-V_{BE}}{I_B} = \frac{10V-0.7V}{50\,\mu A} = 186\,k\Omega$

l. Consider the voltage divider bias, circuit of figure given below. Assuming $\beta = 260$, V_{CC} = 12V, $R_C = 560\,\Omega$, $R_E = 680\,\Omega$, $R_1 = 22k\Omega$ and $R_2 = 12\,k\Omega$ find the Q-point and the stabiity factor S.


R_1
I_C
R_C
C
I_B
+
C_2
V_{CE}
B
C_1
Input signal
Output signal
R_2
R_E
I_E
C_E
G
+
−
V_{CC}

oln. $V = \frac{V_{CC}R_2}{R_1+R_2} = \frac{12V \times 12k\Omega}{22k\Omega + 12k\Omega} = 4.24V$

$R_B = \frac{R_1R_2}{R_1+R_2} = 7.76\,k\Omega$

Now,

$$I_B = \frac{V - V_{BE}}{R_B + (1+\beta)R_E} = \frac{4.24V - 0.7V}{7.76\,k\Omega(1+260)\times 680\,\Omega} = 0.019\,mA$$

$$I_C \approx \beta I_B = 4.86\,mA$$

$$V_{CE} = V_{CC} - I_C(R_C + R_E) = 12V - 4.86\,mA(560\,\Omega + 680\,\Omega) = 5.97\,V$$

From stability factor,

$$S = (1+\beta)\cdot\frac{1+\frac{R_B}{R_E}}{1+\beta+\frac{R_B}{R_E}} = (1+260)\cdot\frac{1+7.76\,k\Omega/680\,\Omega}{1+260+7.76\,k\Omega/680\,\Omega} = 11.89$$

22. The rms value of full wave rectified waveform is : **[BHU 2014]**

(a) 0.636 times the peak value (b) 0.707 times the peak value

(c) 0.5 times the peak value (d) 0.373 times the peak value

Soln. We know that,

$$I_{rms} = \frac{I_0}{\sqrt{2}} = 0.707 \text{ times the peak value}$$

Correct option is (b)

23. The relation between current gains β and α of a transistor is : **[BHU 2014]**

(a) $\beta = \frac{\alpha}{(1+\alpha)}$ (b) $\beta = \frac{\alpha}{(1-\alpha)}$ (c) $\beta = \frac{(1+\alpha)}{\alpha}$ (d) $\beta = \frac{(1-\alpha)}{\alpha}$

Soln. We know that, $\alpha = \frac{I_C}{I_E}; \beta = \frac{I_C}{I_E}$ and $I_E = I_C + I_B$

$$\Rightarrow \quad \frac{I_E}{I_C} = 1 + \frac{I_B}{I_C} \Rightarrow \frac{1}{\alpha} = 1 + \frac{1}{\beta} \Rightarrow \frac{1}{\beta} = \frac{1}{\alpha} - 1$$

$$\Rightarrow \quad \beta = \frac{\alpha}{1-\alpha}$$

Correct option is (b)

24. The conductivity of Ge doped with phosphorus with impurity concentration $4.41\times10^{14}/\text{cm}^3$. (Given that for Ge the intrinsic carrier concentration is $2.5\times10^{13}/\text{cm}^3$, electron mobility $\mu_s = 3800$ cm²/V-sec and hole mobility $\mu_p = 1800$ cm²/V-sec) is **[BHU 2015]**

(a) 0.27 $(\Omega\text{-cm})^{-1}$ (b) 0.03 $(\Omega\text{-cm})^{-1}$ (c) 0.05 $(\Omega\text{-cm})^{-1}$ (d) 0.42 $(\Omega\text{-cm})^{-1}$

Soln. $\sigma = eNd\mu_e = 1.6\times10^{-19}\times 4.41\times10^{14}\times 3800 = 0.26(\Omega\text{-cm})^{-1}$

Correct option is (a)

25. A transistor in common base configuration has ratio of collector current to emitter current β and ratio of collector to base current α. Which of the following is true ? **[JEST 2016]**

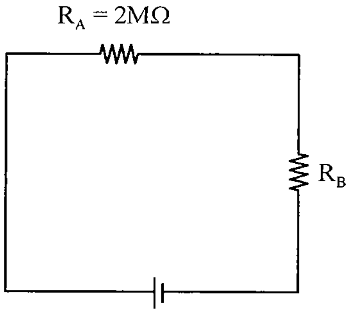
(a) $\beta = \frac{\alpha}{(\alpha+1)}$ (b) $\beta = \frac{(\alpha+1)}{\alpha}$ (c) $\beta = \frac{\alpha}{(\alpha-1)}$ (d) $\beta = \frac{(\alpha-1)}{\alpha}$

Soln. Given, $\beta = \frac{I_C}{I_E}; \alpha = \frac{I_C}{I_\beta}$

We know that, $I_C + I_\beta = I_E \Rightarrow 1 + \frac{I_\beta}{I_C} \Rightarrow \beta = \frac{\alpha}{1+\alpha}$

Correct option is (a)

26. In the circuit given below, a person measures 9.0 V across the battery, 3.0 V across the 2MΩ resistor R_A and 4.5 V across the unknown resistor R_B, using an ordinary voltmeter which has a finite input resistance r. Assuming that the battery has negligible internal resistance, it follows that (i) the resistance R_B and (ii) the input resistance r of the voltmeter are, in MΩ **[TIFR 2010]**



(a) $R_B = 3.0$, r = 1.0 (b) $R_B = 2.5$, r = 7.5 (c) $R_B = 4.0$, r = 12.0 (d) $R_B = 4.5$, r = 10.0

oln. According to given condition,

$$\frac{9\times\frac{2r}{2+r}}{R_B+\frac{2r}{2+r}}=3$$

$$\Rightarrow \quad \frac{6r}{2R_B+R_Br+2r}=1 \Rightarrow 4r=2R_B+R_Br$$

$$\Rightarrow \quad R_B=\left(\frac{4r}{2+r}\right) \qquad \ldots (1)$$

And, $$\frac{9\times\frac{R_Br}{R_B+r}}{2+\frac{R_Br}{R_B+r}}=4.5$$

$$\Rightarrow \quad \frac{2R_Br}{2R_B+2r+R_Br}=1 \Rightarrow R_Br=2R_B+2r$$

$$\Rightarrow \quad \frac{4r^2}{2+r}=2\left(\frac{4r}{2+r}+r\right) \quad \text{(Putting } R_B=\frac{4r}{2+r}\text{)}$$

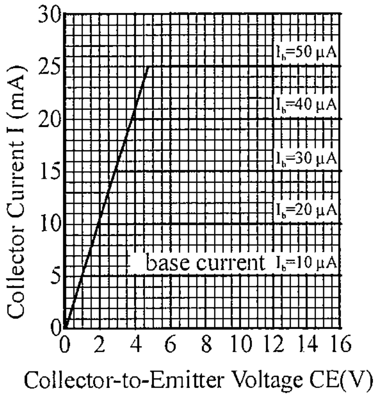
$$\Rightarrow \quad 2r^2=4r+2r+r^2 \quad \Rightarrow r^2-6r=0$$

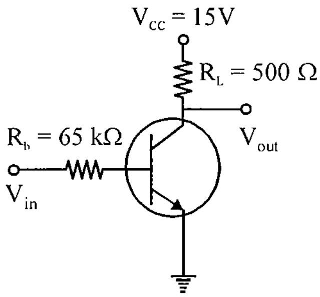
$$\Rightarrow \quad r=6 \qquad \text{since, } r\neq 0$$

$$\therefore \quad R_B=\frac{4\times 6}{2+6}=3$$

Correct option is (a)

27. A plot of the common-emitter characteristics of a silicon n-p-n transistor is shown below. Given this information, and assuming that there will be a 0.7 V drop across a forward biased silicon p-n junction, the approximate value of the output voltage V_{out} for an input voltage $V_{in} = 2$ V in the adjacent circuit will be





(a) 4 V (b) 6 V (c) 8 V (d) 10 V **[TIFR 2010]**

(e) 12 V (f) 14 V

Soln. Using KVL input,

$$V_{in} = I_B R_B + V_{BE}$$

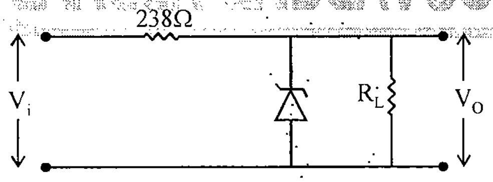
$$\Rightarrow \quad I_B = \frac{(2-0.7)}{65k\Omega} = 20\,\mu A$$

For $I_B = 20\mu A$ from graph we get $I_C = 10\,mA$

Now, $V_{CC} = I_C R_C + V_0 \Rightarrow V_0 = 15 - 10 \times 500 \times 10^{-3} = 15 - 5 = 10V$

Correct option is (d)

28. The voltage regulator circuit shown in the figure has been made with a Zener diode rated at 15V, 200mW. It is required that the circuit should dissipate 150mW power across the fixed load resistor R_L.



For stable operation of this circuit, the input voltage V_i must have a range **[TIFR 2012]**

(a) 17.5 V to 20.5 V (b) 15.5 V to 20.5 V (c) 15.5 V to 22.5 V

(d) 17.5 V to 22.5 V (e) 15.0 V to 20.5 V

Soln. From figure, we can write,

$$V_L = \frac{V_i R_L}{238 + R_L}$$

to work as voltage regulator this voltage should be greater than V_z.

$$\therefore \quad V_{i\,\min} = \frac{238 + R_L}{R_L} \times v_z = \frac{238 + 1500}{1500} \times 15 = 17.5\,V$$

$$\left[\text{Given:}\frac{15}{R_L}\times 15 = 150\times 10^{-3}\,\Omega \Rightarrow R_L = 1500\,\Omega\right]$$

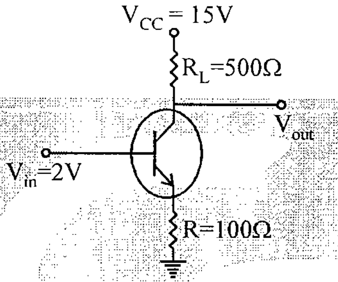
And for $V_{i\max}$,

$$V_{i\max} = I_S \times R_s + V_z = \left(I_{z\max} + I_L\right)R_s + V_z$$

$$= \left(\frac{200}{15}mA + 10\,mA\right)\times 238 + V_z = 5.5 + 15 = 20.5\,volt$$

Correct option is (a)

9. The circuit depicted on the right has been made with a silicon n-p-n transistor. Assuming that there will be a 0.7V drop across a forward-biased silicon p-n junction, the power dissipated across the transistor will be, approximately, **[TIFR 2013]**



(a) 53 mW (b) 94 mW (c) 17 mW (d) 67 mW

oln. From input loop,

$$2 = V_{BE} + I_E R_E$$

$$I_E = \frac{2-0.7}{100} = 13\,mA$$

From output loop, $V_{CC} = I_C R_C + V_{CE} + I_E R_E \quad (I_E \simeq I_C)$

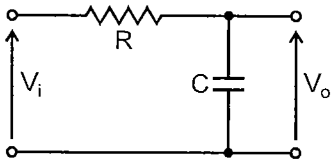
$$V_{CE} = 15 - 600\times 13\times 10^{-3} = 7.2$$

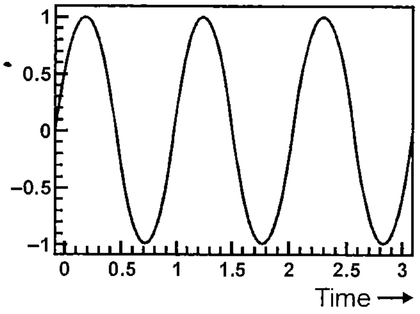
Therefore, power dessipated across the transistor

$$P = 7.2\times 13 \simeq 94\,mW$$

Correct option is (b)

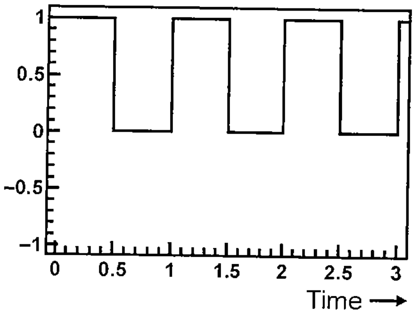
0. For the circuit depicted on the right, the input voltage V_i is a simple sinusoid as shown below, where the time period is much smaller compared to the time constant of this circuit. **[TIFR 2016]**



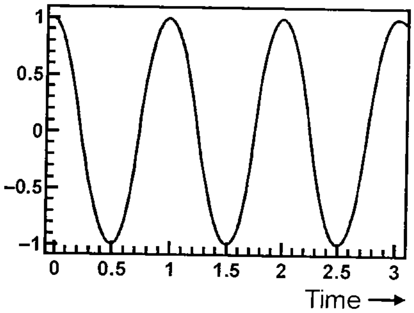


The voltage V_o across C is best represented by

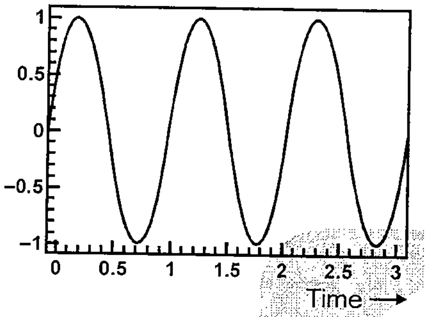
(a)



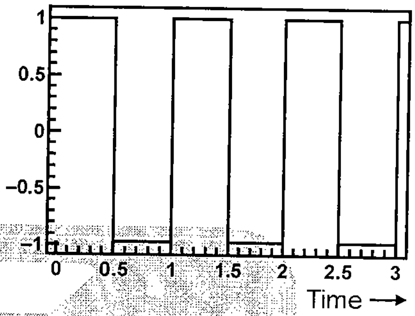
(b)



(c)



(d)



Soln. $\frac{V_0}{Vi}=\frac{1}{1+RCS}=\frac{1}{1+j\omega RC}$

$\frac{V_0}{Vi}=\frac{1}{RCS}\Rightarrow RCS>>1\Rightarrow j\omega RC>>1$

$\omega RC>>1$

$2\pi fRC>>1$

$\tau=RC$

$\boxed{\tau>>T}$ (Integrator)

Correct option is (b)

31. Consider a sawtooth waveform which rises linearly from 0 Volt to 1 Volt in 10 ns and then decays linearly to 0V over a period of 100 ns. Find the r.m.s. voltage in units of milliVolt? **[TIFR 2016]**

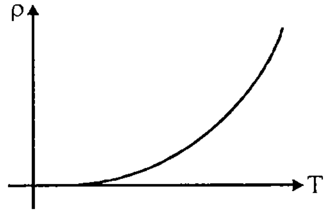
Soln. $V_{rms}=\sqrt{\frac{1}{T}\int_0^T V^2dt}=\sqrt{\frac{1}{110}\left[\int_0^{10}\left(\frac{t}{10}\right)^2dt+\int_{10}^{110}\left(\frac{110-t}{100}\right)^2dt\right]}$

$=\sqrt{\frac{1}{110}\left[\frac{1000}{300}+100\times\frac{1}{3}\right]}=\sqrt{\frac{1}{3}}=577\text{ millivolt}$

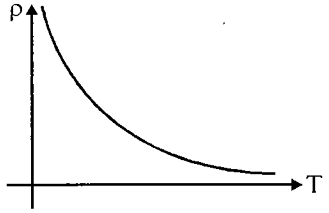
Correct answer is (577)

2. Which of the following best represents the temperature dependence of the resistivity of an undoped semiconductor ? **[BHU 2012]**

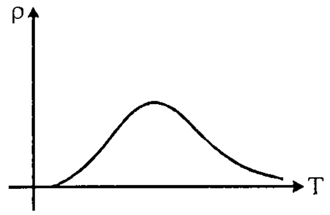
(a)



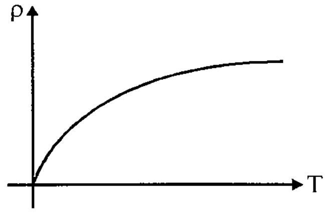
(b)



(c)



(d)



oln. The undoped semiconductor, the resistivity (ρ) decreases with increase temperature (T).
Hence, correct option is (b).

3. The mean kinetic energy of the conduction electrons in metals is ordinary much higher than kT because **[BHU 2012]**
(a) electrons have many more degrees of freedom than atoms do
(b) the electrons and the lattice are not in thermal equilibrium
(c) the electrons form a degenerate Fermi gas
(d) electrons in metals are highly relativistic

oln. The mean kinetic energy of the conduction electrons in metal is ordinary much higher than kT because the electrons form a degenerate fermi gas.
Hence, correct option is (c).

4. Which of the following statements concerning the electrical conductivities at room temperature of a pure copper sample and a pure silicon sample is NOT true ? **[BHU 2012]**
(a) If the temperature of the copper sample is increased, its conductivity will decrease
(b) If the temperature of the silicon sample is increased, its conductivity will increase
(c) The addition of an impurity in the copper sample always decreases its conductivity
(d) The addition of an impurity in the silicon always decreases its conductivity

oln. Out of above options, the option (d) "The addition of an impurity in the silicon always decreases its conductivity" is not type.
Hence, correct option is (d).

5. The Hall effect is used in solid-state physics to measure **[BHU 2012]**
(a) ratio of charge to mass
(b) magnetic susceptibility
(c) the sign of the charge carriers
(d) the width of the gap between the conduction and valence bands

oln. The Hall effect is used in solid state physics to measure the sign of the charge carriers.
Hence, correct option is (c).

36. Lattice forces affect the motion of electrons in a metallic crystal, so that the relationship between the energy E and wave number k is not the classical equation $E=\frac{\hbar^2k^2}{2m}$, where m is the electron mass. Instead, it is possible to use an effective mass m^* given by which of the following ? **[BHU 2012]**

(a) $m^*=\frac{1}{2}\hbar^2k\left(\frac{dk}{dE}\right)$ (b) $m^*=\frac{\hbar^2k}{\left(\frac{dk}{dE}\right)}$ (c) $m^*=\hbar^2k\left(\frac{d^2k}{dE^2}\right)^{1/3}$ (d) $m^*=\frac{\hbar^2}{\left(\frac{d^2E}{dk^2}\right)}$

Soln. The effective mass (m^*) is given by $m^*=\frac{\hbar^2}{\left(\frac{d^2E}{dk^2}\right)}$

Hence, correct option is (d).

37. In the Kronig-Penney model, discontinuities in E versus k curve occur for **[BHU 2012]**

(a) $k=\frac{n\pi}{a}$ (b) $k=\frac{8n\pi}{a}$ (c) $k=\frac{n^2\pi}{a}$ (d) $k=\frac{n\pi}{4a}$

Soln. In the Kronig-Penney model, discontinuities in E versus k curve occur for $k=\frac{n\pi}{a}$

Hence, correct option is (a).

38. The Fermi energy of silver is 5.51 electron volt. The average energy of the free electrons in silver at 0 °K is given by

(a) 4.205 eV (b) 0.864 eV (c) 3.306 eV (d) 9.425 eV

Soln. $E_f=5.51\,\text{eV}$

The average energy $\bar{E}_0=\frac{3}{5}E_f=\frac{3}{5}\times 5.51\,\text{eV}=3.306\,\text{eV}$

Hence, correct option is (c).

39. Which of the following statements is true about the effective mass of electron in crystal ? **[BHU 2013]**

(a) It is positive near the top of energy band (b) It is negative near the bottom of energy band
(c) It is constant through out the band (d) It is negative near the top of energy band

Soln. Out of the above options, "It is negative near the top of energy band" is correct.
Hence, correct option is (d).

40. Indicate the statement about the semiconductors which is false ? **[BHU 2013]**

(a) n-type semiconductors are obtained by doping phosphorus into silicon
(b) The conductivity of all semiconductors always increases with temperature
(c) p-type semiconductors are obtained by doping boron into silicon
(d) Intrinsic semiconductors are insulators at $T=0$ °K

Soln. Out of the above options, the option (b) the conductivity of all semiconductors always increases with temperature is false.
Hence, correct option is (b).

1. The electrical conductivity of a metal in terms of mass (m), charge (e), collision time (τ) and concentration (n) of electrons is given by **[BHU 2013]**

(a) $\frac{me\tau}{n}$ (b) $mne\tau$ (c) $\frac{ne^2\tau^2}{m}$ (d) $\frac{ne^2\tau}{m}$

oln. The electrical conductivity is given by $\sigma = \frac{ne^2\tau}{m}$.

Hence, correct option is (d).

2. Which of the statements is false ? **[BHU 2013]**

(a) A semiconductor exhibits negative temperature coefficient of resistivity

(b) In an n-type semiconductor, as the density of donor atoms is increased, the Fermi level shifts towards the valence band

(c) Mobility carriers in a p-type semiconductors are holes

(d) In an intrinsic semiconductors, the Fermi level lies midway in the forbidden gap

oln. "In an n-type semiconductor, as the density of donor atoms is increased, the fermi level shifts towards the valence band" is false.

Hence, correct option is (b).

3. Electrical conductivity of insulators is in the range : **[BHU 2014]**

(a) $10^{-10}\,(\Omega-\text{mm})^{-1}$ (b) $10^{-10}\,(\Omega-\text{cm})^{-1}$ (c) $10^{-10}\,(\Omega-\text{m})^{-1}$ (d) $10^{-8}\,(\Omega-\text{m})^{-1}$

oln. Electrical conductivity of intensity is in the range $10^{-8}\,(\Omega-\text{m})^{-1}$.

Hence, correct option is (d).

4. Energy band gap size for semiconductors is in the ranges **[BHU 2014]**

(a) 1-2 eV (b) 2-3 eV (c) 3-4 eV (d) greater than 4 eV

oln. Energy band gap size for semiconductors is in the ranges 1-2 eV.

Hence, correct option is (a).

5. Flow of electron is affected by which of the following : **[BHU 2014]**

(a) thermal vibration only (b) impurity atom only

(c) crystal defects only (d) by all of (a), (b) and (c)

oln. Flow of electron is affected by normal vibrations, impurity and crystal defects.

Hence, correct option is (d).

6. Fermi energy level for intrinsic semiconductors lies : **[BHU 2014]**

(a) at the middle of the band gap (b) close to the conduction band

(c) close to valence band (d) inside valence band

oln. Fermi-level for intrinsic semiconductors lies at the middle of the band gap.

Hence, correct option is (a).

7. In one-dimensional periodic chain of atom with lattice parameter 'a' has first brillouin zone at :

(a) a (b) $2a$ (c) π/a (d) $2\pi/a$ **[BHU 2014]**

oln. First Brillouin zone lies at π/a.

Hence, correct option is (c).

8. Mobility of holes as compared to mobility of electrons in intrinsic semiconductors is :

[BHU 2014]

(a) equal (b) greater (c) less (d) cannot be defined

oln. Mobility of holes as compared to mobility of electrons in intrinsic semiconductors is less because the mass of hole is slightly larger than electron so its velocity will be smaller.

Hence, correct option is (c).

49. According to Hall effect if a conducting materials is placed in a uniform magnetic field and a current is passed, voltage is found to develop at : **[BHU 2014]**

(a) parallel to the current
(b) parallel to the magnetic field
(c) perpendicular to the magnetic field and current
(d) 45 degrees to the magnetic field and current

Soln. According to Hall effect, if a conducting material is placed in uniform magnetic field and a current is passed, voltage is found to develop at perpendicular to the magnetic field and current.
Hence, correct option is (c).

50. In the periodic potential model, discontinuities in E versus k curve occur for **[BHU 2015]**

(a) $k=\frac{n\pi}{a}$ (b) $k=\frac{8n\pi}{a}$ (c) $k=\frac{n^2\pi}{a}$ (d) $k=\frac{n\pi}{4a}$

Soln. For periodic potential model, the discontinuous in E versus k curve occur for $k=\left(\frac{n\pi}{a}\right)$.

Hence, correct option is (a).

51. Which of the following statements is true about the velocity of electron in an energy band ? **[BHU 2015]**
(a) It is constant throughout the band
(b) It is maximum at the bottom of the band
(c) It is maximum at the inflection point of the $E(k)$ curve
(d) It is maximum at the top of the band

Soln. Out of above options, only option (c). "It is maximum at the inflection point of the $E(k)$ curve" is true for the velocity of electrons in an energy band.
Hence, correct option is (c).

52. According to the Wiedemann-Franz law, the ratio of thermal to electrical conductivity of any metal is given by **[BHU 2015]**

(a) $\left(\frac{\pi kT}{2e}\right)^2$ (b) $\frac{\pi^2}{3}\left(\frac{k}{e}\right)^2 T$ (c) $\frac{\pi^2}{3}\left(\frac{kT}{e}\right)^2$ (d) $\left(\frac{\pi k}{e}\right)^2 T$

Soln. According to Wiedemann-Franz law, $\frac{k}{\sigma}=\frac{\pi^2}{3}\left(\frac{k_B}{e}\right)^2 T$.

Hence, correct option is (b).

53. Indicate the false statement about the semiconductors **[BHU 2015]**
(a) All intrinsic semiconductors are insulators at T = 0 °K
(b) At very high temperature all semiconductors become intrinsic semiconductors
(c) The conductivity of all semiconductors always increases with temperature
(d) N-type semiconductors are obtained by doping phosphorus into silicon

Soln. Out of the given options, the option (c) the conductivity of all semiconductors always increases with temperature.
Hence, correct option is (c).

54. The density of carriers in a pure semiconductor is proportional to **[BHU 2015]**

(a) $\exp\left[-\frac{E_g}{kT}\right]$ (b) $\exp\left[-\frac{2E_g}{kT}\right]$ (c) $\exp\left[-\frac{E_g}{kT^2}\right]$ (d) $\exp\left[-\frac{E_g}{2kT}\right]$

oln. The density of carriers in a pure semiconductor is proportional to $\exp\left[-\frac{E_g}{2k_BT}\right]$.

Hence, correct option is (d).

5. The reverse saturation current in germanium p-n diode is of the order of **[BHU 2015]**
(a) nA (b) μA (c) mA (d) ampere

oln. The reverse saturation current in germanium p-n junction diode is of the order of microampere (μA)
Hence, correct option is (a).

6. The Fermi level in an n-type semiconductor at 0 °K **[BHU 2015]**
(a) lies below the donor level
(b) lies in the conduction band
(c) lies half-way between the conduction band and donor level
(d) coincides with the intrinsic Fermi level

oln. The Fermilevel in an n-type semiconductor at 0 °K lies half-way between the conduction band and donor level.
Hence, correct option is (c).

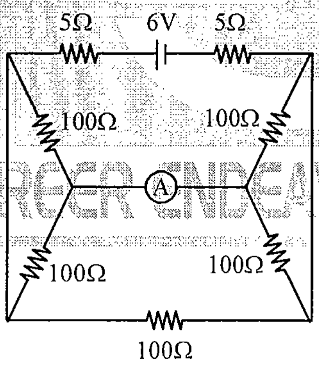
7. If the Fermi energy of a metal at 0 °K, 10 eV, the mean energy of the electrons in the metal at 0 °K is
(a) 6 eV (b) 5 eV (c) 1.5 eV (d) 2 eV

oln. The average energy of electrons in the metal at 0 °K, **[BHU 2015]**

$$\bar{E} = \frac{3}{5}E_f = \frac{3}{5}\times 10 \text{ eV} = 6 \text{ eV}.$$

Hence, correct option is (a).

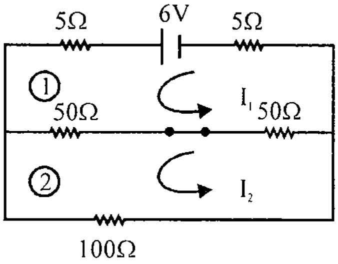
8. The current read by the ammeter (A) in the circuit given below is **[TIFR 2011]**



(a) 27.3 mA (b) 100.0 mA (c) 54.5 mA (d) 50.0 mA

oln. ⇒ 100Ω and 100Ω are in parallel

⇒ Simplify circuit
⇒ Ammeter used to measure current, make short circuit



Apply mesh analysis

Loop (1) KVL:

$$6 = 5I_1 + 50(I_1 - I_2) + 50(I_1 - I_2) + 5I_1$$

$$6 = 110I_1 - 100I_2 \qquad \dots(1)$$

Loop (2) KVL:

$$100I_2 + 50(I_2 - I_1) + 50(I_2 - I_1) = 0$$

$$200\,I_2 - 100\,I_1 = 0$$

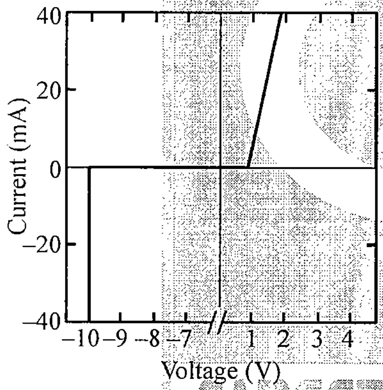
$$I_1 = \frac{1}{10};\quad I_2 = \frac{1}{20}$$

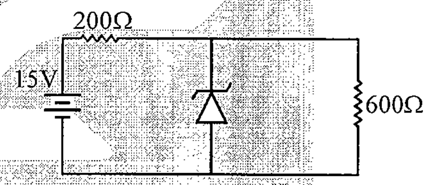
Ammeter reading = $I_1 - I_2$

$$I_1 - I_2 = 50\text{ mA}$$

Correct option is (d)

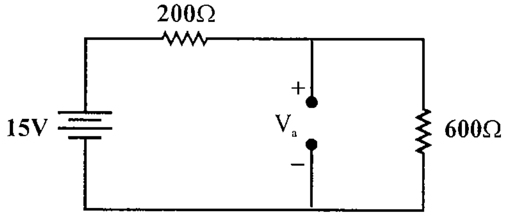
59. The figure on the right shows the current-voltage characteristics of a diode over a range of voltage and current where it is safe to operate the diode. When this diode is used in the circuit on the extreme right, the approximate current, in mA, through the diode will be **[TIFR 2013]**





(a) 0 (b) 8.3 (c) 16.7 (d) 25

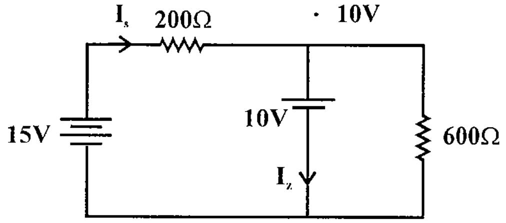
Soln.



$V_a = ?$, find V_a.

$$V_a = \frac{15 \times 600}{800}\text{V} \Rightarrow V_a = 11.25\text{V}$$

$\Rightarrow$ $V_a \geq V_z$ means zener on (since, V_z = 10 V from figure)



$$I_{600\Omega}=\frac{10}{600}\Rightarrow I_{600\Omega}=16.67\text{ mA}$$

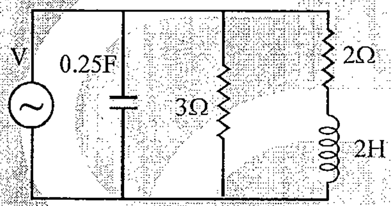
$$I_s=\frac{15-10}{200}=25\text{mA}$$

$$I_s=I_z+I_{600\Omega}$$

$$25=I_z+16.67$$

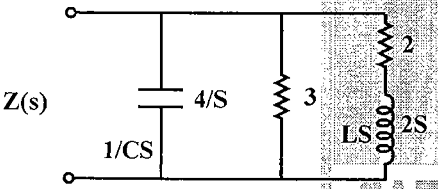
$$I_z=8.33\text{mA}$$

). Find the resonance frequency (rad/sec) of the circle shown in the figure below **[JEST 2014]**



(a) 1.41 (b) 1.0 (c) 2.0 (d) 1.73

oln.



$$\Rightarrow\quad z(s)=\frac{(2+2S)\times 3}{3+2+2S}\Rightarrow\left.\frac{6+6S}{5+2S}\right\}(3\Omega)\,||\,(2+2S)\qquad(\text{Here, } S=j\omega)$$

$$\Rightarrow\quad z(s)=\frac{\frac{4}{S}\times\left(\frac{6+6S}{5+2S}\right)}{\frac{4}{S}+\left(\frac{6+6S}{5+2S}\right)}\Rightarrow\frac{\frac{4}{S}\times\left(\frac{6+6S}{5+2S}\right)}{\frac{4(5+2S)+(6+6S)S}{S(5+2S)}}$$

$$\Rightarrow\quad z(s)=\frac{\frac{4}{S}\times\left(\frac{6+6S}{5+2S}\right)}{\frac{20+8S+6S+6S^2}{S(5+2S)}}\Rightarrow Z(S)=\frac{24+24S}{6S^2+14S+20}$$

$$\Rightarrow\quad z(s)=z(j\omega)=\frac{24+24j\omega}{-6\omega^2+14j\omega+20}\ \text{divide by (2)}$$

$$=\frac{12+12j\omega}{-3\omega^2+7j\omega+10}$$

$$\Rightarrow \qquad z(j\omega)=\frac{12+12j\omega}{10-3\omega^2+7j\omega}\left\{\frac{(10-3\omega^2)-7j\omega}{10-3\omega^2-7j\omega}\right\}$$

Considering-numerator

$$\left\{12(10-3\omega^2)+12j\omega(10-3\omega^2)-12\times 7j\omega+12\times 7\omega^2\right\}$$

Since, at resonance frequency both I and V remain same phase.
Therefore, the imginary part of the numerator should be zero.

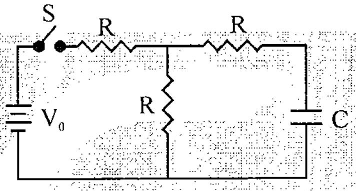
$$\Rightarrow \qquad 12j\omega(10-3\omega^2)-12\times 7j\omega=0 \Rightarrow 12(10-3\omega^2)=12\times 7 \Rightarrow 10-7=3\omega^2 \Rightarrow 3=3\omega^2$$

$$\Rightarrow \qquad \omega=\pm 1\text{ rad/sec} \quad \Rightarrow \quad \omega=1\text{ rad/sec}$$

Discard negative frequency

Correct option is (b)

61. A capacitor C is connected to a battery V_0 through three equal resistors R and a switch S as shown below: **[JEST 2012]**



The capacitor is initially uncharged. At time t = 0, the switch S is closed. The voltage across the capacitor as a function of time 't' for t > 0 is given by

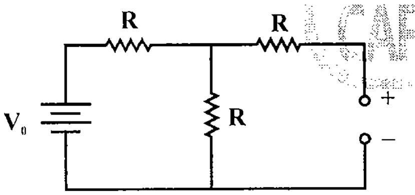
(a) $(V_0/2)\left(1-e^{-t/2RC}\right)$ (b) $(V_0/3)\left(1-e^{-t/3RC}\right)$

(c) $(V_0/3)\left(1-e^{-3t/2RC}\right)$ (d) $(V_0/2)\left(1-e^{-2t/3RC}\right)$

Soln. At t = 0, capacitor is uncharged

$$\therefore \qquad V_c(0)=0$$

At $t\to\infty$ capacitor fully charged, behaving as open circuit.



$$V_c(\infty)=\frac{V_0}{2}$$

$$\boxed{V_c(t)=V_c(\infty)+\left[V_c(0)-V_c(\infty)\right]\cdot e^{-\frac{t}{\tau}}}$$

For $t\geq 0$,

$$V_c(0^-)=V_c(0)=V_c(0^+)=0$$

Property of capacitor,

$$\tau=\text{Req}\cdot C \Rightarrow \tau=\frac{3R}{2}\cdot C \qquad \because \qquad \text{Req}=R\,||\,R+R=\frac{3R}{2}$$

$$V_c(t)=\frac{V_0}{2}+\left[0-\frac{V_0}{2}\right]\cdot e^{-\frac{t}{\frac{3R}{2}C}} \Rightarrow V_c(t)=\frac{V_0}{2}\left[1-e^{\frac{-2t}{3RC}}\right]$$

Correct option is (d)

Chapter 5

OP-AMP, SINGLE STAGE AMPLIFIERS & OSCILLATORS

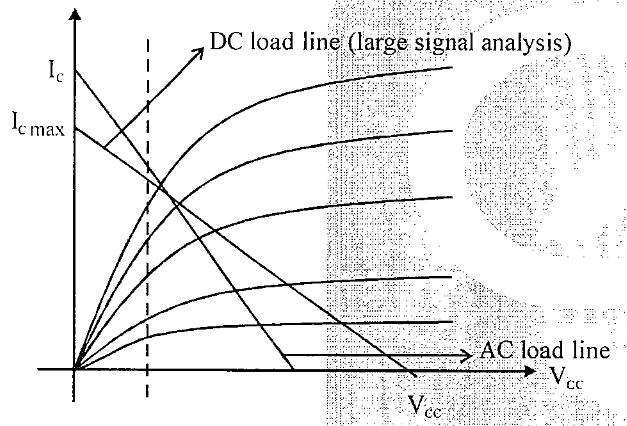
Single stage BJT amplifiers:

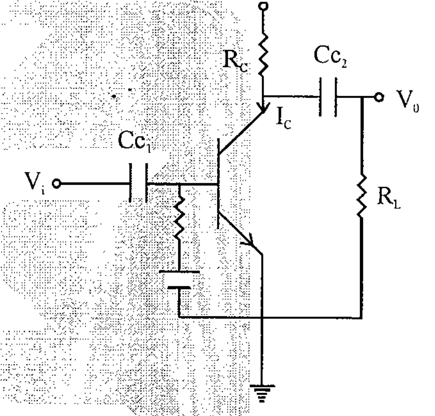
There are basically three configurations for implementing single stage BJT amplifiers

(i) The common emitter (ii) The common collector (iii) The common base configuration

All there are studied below utilizing the same basic structure with the same biasing arrangement.

Low Frequency Analysis:





DC load line →slope = $-\frac{1}{R_C}$

For AC analysis (Small signal analysis) : All the DC source are zero voltage source short circuit current source open circuit coupling capacitor and by pass capacitor are VV large.

C_{c_1}, C_{c_2}, act on AC short.

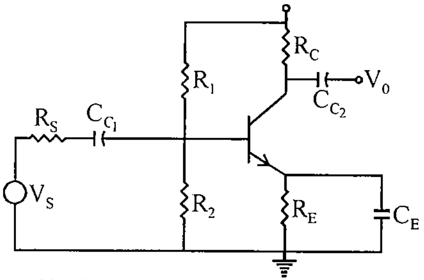
C_E by pass capacitor

$R'_L = R_C \| R_L < R_C$

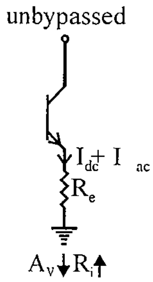
AC load line slope = $-\frac{1}{R'_L} = -\left[\frac{1}{R_C} + \frac{1}{R_L}\right]$

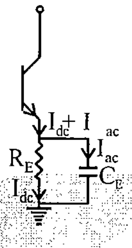
Slope of AC line > slope of DC line.

Single stage RC coupled amplifier: C_{C_1}, C_{C_2} are called coupling capacitor. They practically offer short circuited path to ac but for dc they act as open circuit.


R_C
R_1
V_0
C_{C_2}
R_S
C_{C_1}
V_S
R_2
R_E
C_E

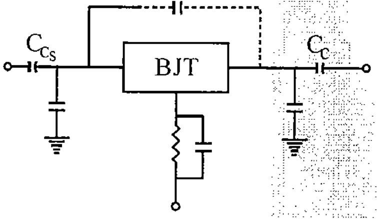
Single stage RC coupled amplifier


unbypassed
$I_{dc}+I_{ac}$
R_e
$A_V\downarrow R_i\uparrow$


$I_{dc}+I_{ac}$
I_{ac}
R_E
C_E
I_{dc}

C_E v.v.high, Z_{C_E} = v.v. low compared to R_E

C_{C_2} act as S.C. for A.C.



High $C_{C_1}=C_{C_2},C_E\to\mu F$

Low $C_D,C_T,C_W\to PF$

During A.C. Analysis:

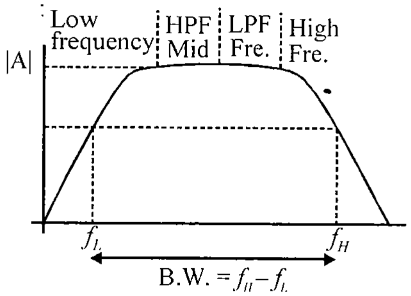
Lower frequency region	*Middle frequency region*	*High frequency region*
$F\downarrow\|Z\|\uparrow$	$\|Z_C\|$ dc decided by	$f\uparrow\|Z_C\|\downarrow$
$C\to O.C.$	value of C	$C\to S.C$
gain ↓ due to	$C_{C_1},C_{C_2},C_Z\to S.C.$	gain ↓ due to C_D,C_T,C_W
C_{C_1},C_{C_2},C_E	$C_D,C_T,C_W,C_S\to O.C$	and C_S

Note: At d.c. it will be out.

Frequency response curve gain versus frequency

At $f=f_H$ or f_L i.e. 70% of maximum

$P_{out}=\frac{P_{\max}}{2}$(Power). Half power band width



RC coupled amplifier:

1. It is audio frequency voltage amplifier
2. Operating frequency range 20Hz to 20 Khz.

- Single stage RC coupled amplifier introduce a phase shift of 180° and two stages introduces 360° of 0°.
- Coupling capacitor (C_{C_1} and C_{C_2}) is also called d.c. blocking capacitor.
- It is used to couple A.C. signal and simultaneously block d.c. signal and d.c. biasing current.
- By using emitter resistance R_E without a by pass capacitor, there will be a negative feedback across R_E and this reduces the voltage gain of amplifier.
- By pass capacitor (C_Z or C_E) is used to by pass the A.C. signal current through it and for d.c. current or biasing current C_2 is open. By using C_Z negative feedback across R_E {during A.C. analysis only} is dominated therefore voltages increases but input resistance decreases. In an amplifier for better performance always R_E and C_E combination is used.

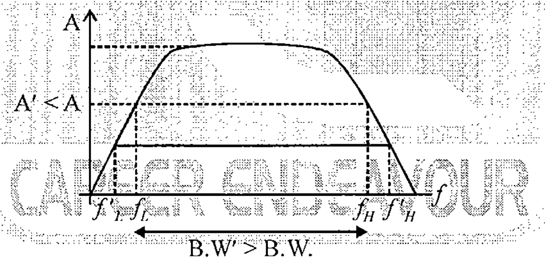
requency response curve:

- Fall of gain in low frequency region is due to C_{C_1}, C_{C_2} and C_Z
- In mid frequency region all coupling and by pass capacitor will be treated as a.c. short and all in capacitance {C_D and C_T} wiring capacitor C_W, stray capacitance C_S will be treated as A.C. open the gain of amplifier is maximum at mid frequency and it almost independent of frequency. Therefore amplifier analysis is generally done at mid frequency.
- Frequency of gain at high frequency is due to the effect of

  (i) J^n capacitor {C_D and C_T} (ii) C_W (iii) C_S (iv) Early effect
- High frequency fall is mainly due to J^n capacitance.

andwidth:

- It is the band of input signal frequency where the gain of amplifier is almost a constant.
- A larger band width indicate better reproduction of input signal
- In an amplifier gain band width product is constant i.e. one increases and other decreases.

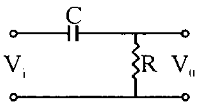


gain decreases due to negative feedback

Disadvantage: Disadvantage of single stage RC coupled amplifier is smaller gain band width product.

Low frequency analysis of RC coupled amplifier:

Amplifier behaves as a HPF



$$\frac{V_0}{V_i} = \frac{R}{R + \frac{1}{SC}} = \frac{1}{1 + \frac{1}{SCR}}$$

$$A = \frac{1}{1 - J/\omega RC} \qquad \boxed{A = \frac{1}{1 - J\omega_L/\omega} = \frac{1}{1 - Jf_L/f}}$$

$$\omega_L = 2\pi f_L = \frac{1}{RC}, \quad \phi = \tan^{-1} f_L/f \text{\{leading\} total phase shift}$$

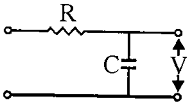
$\phi_T = 180° + \tan^{-1}\frac{f_L}{f}$ at $f = f_L$, $\phi = 180° + 45° = 225°$

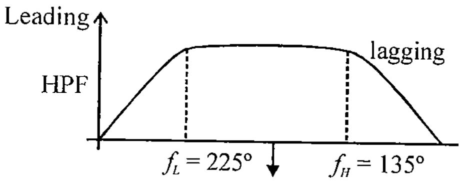
$A = \frac{V_0}{V_i} = \frac{1/SC}{R + 1/SC}$, $A = \frac{1}{1 + SCR} = \frac{1}{1 + J\omega CR}$, $A = \frac{1}{1 + J\omega/\omega H} = \frac{1}{1 + J_f/f_H}$

$\phi = -\tan^{-1} f/fH$ {lagging} total phase shift

$\phi_T = 180° - \tan^{-1} f/f_H = 180° - 45° = 135°$

High frequency region: Behaves as LPF

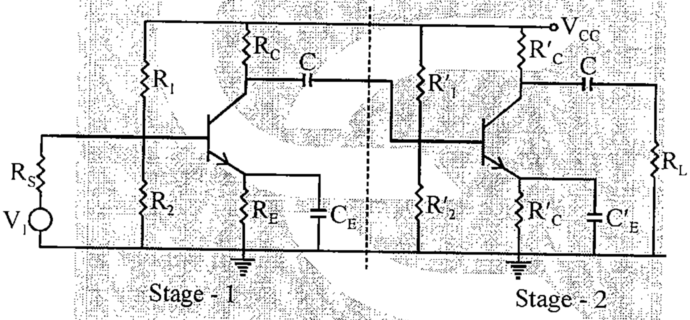




Phase distortion

Cascaded or multistage amplifier:

1. Amplifier are connected in cascade to get larger gain bandwidth product.



2. When amplifier are connected such that output of one is given to input of another they are said to be cascaded.

3. When amplifier are cascaded proper impedance matching must be provided in between the stages so that

   (i) Maximum power will be transferred from one to another stage

   (ii) Output will not be distorted.

   Note: If mismatch is more in amplifier output will be highly distorted.

4. Three different type of coupling

   (i) RC coupling {for voltage amplifier}

   (ii) Direct coupling {for direct coupled amplifier}

   (iii) Transformer coupling for proper amplifier

5. In multistage amplifier gain bandwidth product is constant.

6. In multistage amplifier bandwidth is reduced.

7. The gain bandwidth product of two stage amplifier is greater than that of single stage amplifier.

Bandwidth of multistage amplifier:

B.W.* = B.W. of multistage amplifier

B.W. = B.W. of single stage amplifier

B.W.* = $f_H^{\pm} - f_L^*$

B.W. = $f_H - f_L$

f_H^* = high 3dB cutoff frequency of multistage amplifier

f_L^* = low 3dB cutoff frequency of multistage amplifier

f_H = high 3dB cutoff frequency of single stage amplifier

f_L = low 3dB cutoff frequency of single stage amplifier

n = identical non-interacting stages of cascade.

"proper impedance matching"

Rise time: $t_r = \frac{0.35}{f_H} = \frac{0.35}{B.W.}$

Disadvantage: $t_r \uparrow$ when $B.W. \uparrow$ and response will sluggish.

$t_r^* = 1.1 \times \sqrt{t_{r_1}^2 + t_{r_2}^2 + ...t_{r_n}^2}$, t_{r_0} = Rise time of signal

$$\boxed{t_r^* = 1.1 \times \sqrt{t_{r_0}^2 + t_{r_1}^2 + t_{r_2}^2 + ... + t_{r_n}^2}}$$

Oscillators:

A transistor with proper positive feed back can act as an oscillator i.e. It can generate oscillators without any external source.

The circuit needs only a quick trigger signal to start the oscillators. Once the oscillations have started, no external signal source is needed.

In order to get continuous undamped output from the circuit, the following condition must be met

$$\boxed{\beta A = 1} \quad ...(1)$$

where, A = voltage gain of amplifier without feedback, β = feedback fraction

This relation is called Bark hauren criterion

Mathematical explanation: The voltage gain of positive feedback amplifier is given below:

$A_{Vf} = \frac{A}{1 - \beta A}$, if $\beta A = 1$, then $A_{Vf} = \infty$

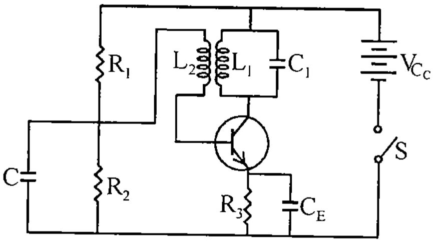
Oscillator can produces sinusoidal and non-sinusoidal [square] waves.

Different type of transistor oscillators:

1. The difference between these oscillators lies in the method by which energy is supplied to the tank circuit to meet the losses.
2. Tunned collector oscillator
3. Colpitt's oscillator
4. Hartely oscillator
5. Phase shift oscillator

6. Wien bridge oscillator
7. Crystal oscillator

Tunned collector oscillator: The frequency of oscillator depend upon the values of L_1 and C_1 and is given by $f = \frac{1}{2\pi\sqrt{L_1C_1}}$



Tunned collector oscillator

- Figure shows the circuit of tunned collector oscillator. It contains tunned circuit $L_1 - C_1$ in the collector.
- The biasing is provided by potential divider arrangement.
- The capacitor C connected in the base circuit provides low reactance path to the oscillations.

Circuit Operation:

- When switch S is closed, collector current starts increasing and charges the capacitor C_1.
- When this capacitor is fully charged, it discharges through coil L_1, setting up oscillations of frequency

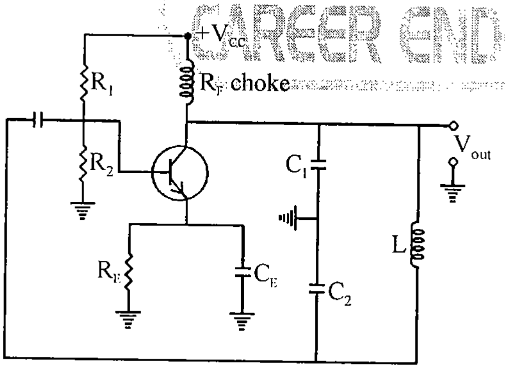
$$f = \frac{1}{2\pi\sqrt{L_1C_1}}$$

- A phase shift of 180° is created between the voltage of L_1 and L_2 due to the transformer action.
- A further phase shift of 180° takes place between base emitter and collector circuit due to transistor property. As a result, the energy feedback to the tank circuit is in phase with the generated oscillations.

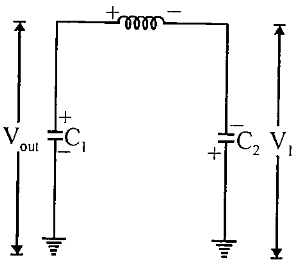
Colpitt's Oscillator:

1. The tank circuit is made up C_1, C_2 and L. The frequency of oscillations: $f = \frac{1}{2\pi\sqrt{LC_{eq}}}$

where. $C_{eq} = \frac{C_1C_2}{C_1 + C_2}$



Colpitt's oscillator



Feedback circuit

Circuit operation:

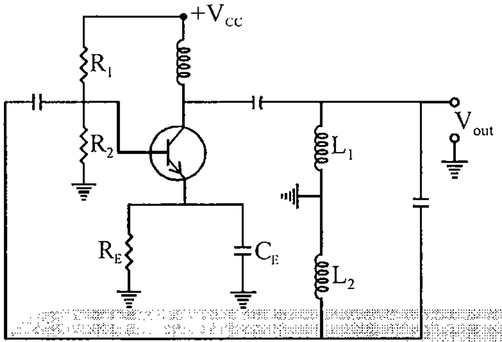
1. When the circuit is turned on, the capacitor C_1 and C_2 are charged. The capacitor discharge through L_1 setting up oscillations of frequency
2. The output voltage of the amplifier appears across C_1 and feedback voltage is developed across C_2.
3. Feedback fraction: $b = \frac{Vf}{V_{out}} = \frac{X_{C_2}}{X_{C_1}} = \frac{C_1}{C_2}$

Hartley oscillator:

The hartley oscillator is similar to colpitt's oscillator only feedback circuit changed.

The frequency of oscillations

$$\boxed{f=\frac{1}{2\pi\sqrt{CL_{eq}}}}$$ where $L_{eq}=L_1+L_2+2M$, M = mutual inductance between L_1 and L_2.



Hartley oscillator

Note: $L_1 - L_2 - C$ is also the feedback network that produces a phase shift of 180°.

Wien bridge oscillator:

The wien bridge oscillator is the most frequency used type of audio oscillator as the output is free from circuit fluctuations and ambient temperature.

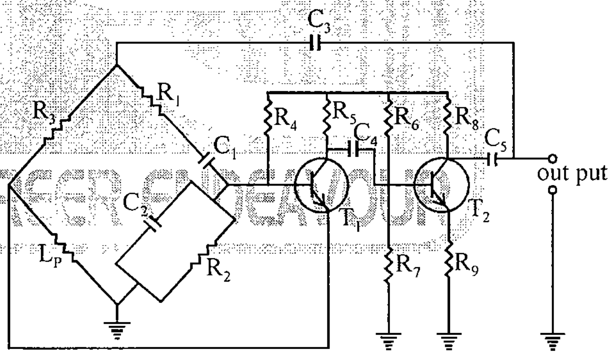
It is the standard oscillator circuit for all frequency in the range of 10Hz to about 1Mhz

The frequency of oscillations

$$f=\frac{1}{2\pi\sqrt{R_1C_1R_2C_2}},$$

if $R_1=R_2=R$ and $C_1=C_2=C$

then $f=\frac{1}{2\pi RC}$



Wien bridge oscillator

It is essential a two stage amplifier with R-C bridge circuit.

Resistance R_3 and tungsten lamp L_p are used to stabilize the amplitude of the output

Circuit operation:

The transistor T_1 serves as an oscillator and amplifier while the other transistor T_2 serves as an inverter (i.e. to produce a phase shift of 180°).

The circuit uses positive and negative feedbacks.

The positive feedback is through R_1C_1, R_2C_2 to transistor T_1.

The negative feedback is through the voltage divider to the input of transistor T_2.

The two transistors produce a total phase shift of 360° so that proper feedback is ensured.

Advantages: It gives constant output due to using temperature sensitive tungsten lamp L_P.

- The overall gain is high because of two transistors.
- The frequency of oscillators can be easily changed by using a potentiometer
- The circuit works quite easily

Disadvantages:

- It can not generate very high frequency.
- The circuit requires two transistors and a number of component

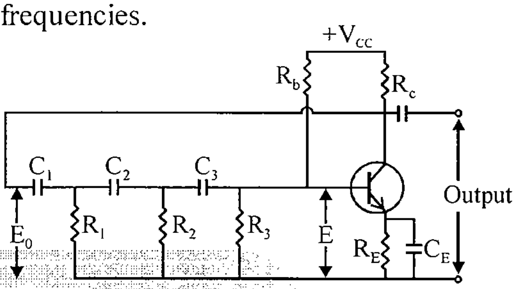
Phase shift oscillator

- Good frequency stability and wave form can be obtained from oscillators employing resistive and capacitive elements such amplifier are called R-C or phase shift oscillators and have the additional advantage that they can be used for very low frequencies.
- The frequency of oscillations is given by

$$f_0 = \frac{1}{2\pi RC\sqrt{6}},$$

where $R_1 = R_2 = R_3 = R$,

$C_1 = C_2 = C_3 = C$



Phase shift oscillator

- A phase shift of 180° is produced by the transistor amplifier. A further phase shift of 180° is produced by the RC network.
- As a result the phase shift around the entire loop is 360°
- Figure shows the three reaction of RC network.
- Each reaction produces a phase shift of 60°, consequently a total phase shift of 180° is produced.

Advantages:

- It can be used to produce very low frequencies
- The circuit provides good frequency stability
- It does not require transformer or inductors

Disadvantages:

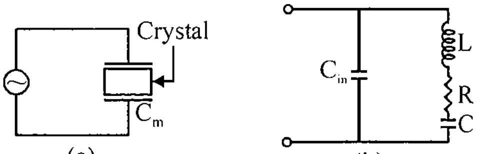
- It is difficult for the circuit to start oscillations as the feedback is generally small.
- The circuit gives small output.

Limitation of L-C and R-C:

- Due to some limitations of LC and RC oscillators, we need crystal oscillators.
- L-C oscillators are not used at low frequencies because inductor become bulky and costly.
- R-C oscillators have not constant frequency and Q-factor because of temperature variations.

Crystal oscillator:

- Electrical oscillators circuit are replaced by mechanical vibration circuit that is called crystal oscillator.
- Certain crystalline material, namely, rohelle salt, quartz and four maline exhibit the piezoelectric effect.
- Of the various piezoelectric crystals, quartz is most commonly used because it is expensive and their great mechanical strength.



(a) (b)

- When the circuit is not vibrating, it is equivalent to capacitance C_{in} because it has two metal plates separated by a dielectric.
- When a crystal vibrates, it is equivalent R-L-C series circuit.

Therefore, the equivalent circuit of a vibrating crystal is R-L-C series circuit shunted by the mounting capacitance C_m as shown in figure (b)

Q factor of crystal $Q = \frac{1}{R}\sqrt{\frac{L}{C}}$

Q factor of crystal is very high

The extremely high Q of a crystal leads to frequency stability.

Frequency response of crystal:

Case-I: When $X_C = X_L$ at series resonant

Impedance of the crystal is very low Z = R

Series resonant frequency $f_S = \frac{1}{2\pi\sqrt{LC}} Hz$

Case-II: At a slightly higher frequency, the net reactance of branch R-L-C inductive and equal to X_{cm}.

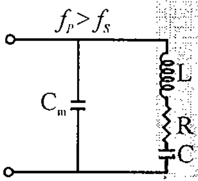
$X_L - X_{cm}$

The crystal now acts as a parallel resonant circuit

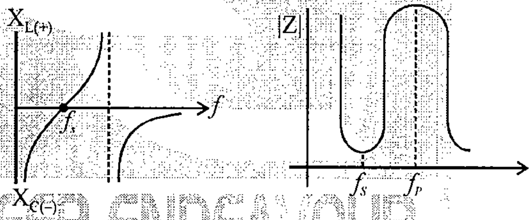
Impedance of crystal is very high

Parallel resonant frequency

$$f_P = \frac{1}{2\pi\sqrt{LC_{eq}}} \quad \text{where, } C_{eq} = \frac{C \times C_m}{C + C_m}, C_{eq} < C$$



(b)



Note: At f_S, the crystal will acts as a series resonant circuit.

At f_P, the crystal will acts as a parallel resonant circuit.

Advantages:

It has a high order of frequency stability.

The quality factor (Q) of the crystal is very high.

Disadvantages:

Hardly tunned

It is fragile and consequently can only be used in low power circuit.

Questions Based on Oscillators

1. The Darlington pair is mainly used for

(a) Impedance matching (b) Wide band voltage amplification

(c) Power amplification (d) Reducing distortion

Ans. (a)

2. Consider the following with the reference of CE transistor amplifier.

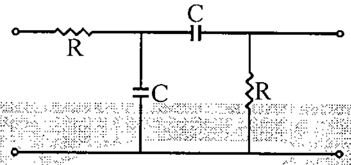
(a) The use of negative feedback (b) The concession of d.c. power to a.c.

(c) High voltage and current gains (d) The use of stop-up transformer

The power gain is due to

Ans. (a) and (c)

3. An FET oscillator uses the given phase shift network as shown below. The minimum gain required for oscillation is



(a) –29 (b) 1 (c) 3 (d) 29

Soln. Apply KVL in 1st loop

$$V_0 = I_1\left(R - \frac{J}{\omega C}\right) + I_2\left(\frac{J}{\omega C}\right) \quad \ldots(1)$$

Apply KVL in second loop

$$O = I_2\left(R - \frac{2J}{\omega C}\right) + I_1\left(\frac{J}{\omega C}\right)$$

$$I_2\left(R - \frac{2J}{\omega C}\right) = -I_1\frac{J}{\omega C} \quad \ldots(2)$$

from (1) and (2) $\frac{V_0}{I_2 R} = \frac{V_0}{V_f} = 3 + J\frac{(1-X^2)}{X}$

where $X = \frac{1}{\omega RC}$

for phase shift to be 180° (for oscillations)

$$\tan^{-1}\left(\frac{1-X^2}{3X}\right) = \pi \Rightarrow \frac{1-X^2}{3X} = 0$$

$x = 1$ i.e. $\omega = \frac{1}{RC}$ $\quad \therefore \frac{V_0}{V_y} = 3$ at $\omega = \frac{1}{RC}$ $\quad \therefore \beta = \frac{V_f}{V_0} = \frac{1}{3}$

loop gain $\beta A \geq 1$ (for oscillations)

$$\therefore A_{\text{minimum}} = \frac{1}{\beta} = 3$$

Bark hausen condition:

Every oscillator circuit to produce sustained oscillations must satisfy the following two conditions

Loop gain = –AB = 1

Gain Aj = ∞

The total phase shift suffered by the signal in travelling from input to output and back to input must introduce precisely 0° phase shift.

In practical oscillators loop gain is kept slightly greater than 1, this is oversome the circuit internal losses.

For ideal oscillator loop gain is 1.

In a practical oscillator the amplitude of oscillations are limited by on set of non-linearity.

In a practical oscillator, suppose if Bark-Hausen's criterion is failed then the circuit will stop producing the frequency of oscillators.

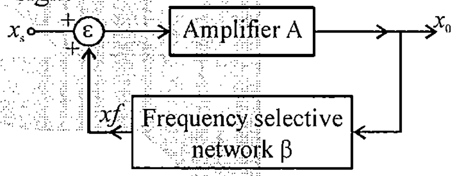
Sinusoidal oscillators:

Basic principles of sinusoidal oscillators: In spite of the name linear oscillator, some form of non-linearity has to be employed to provide central of the amplitude of the output sine wave. In fact, all oscillators are essentially non linear circuits. This complicates the task of analysis and design of oscillators, no longer is one able to apply transform (S-plane) methods directly. Nevertheless, technique have been developed by which the design of sinusoidal oscillators can be performed in two steps. The first stop is a linear one and frequency domain method of feedback circuit analysis can be readily employed. Subsequently, a non-linear mechanism for amplitude control can be provided.

The oscillator feedback loop: The basic structure of a sinusoidal oscillator consists of an amplifier and a frequency selective network connected in a positive feedback loop. Diagram is given below: Here feedback signal *xy* is summed with a positive sign.

Thus the gain with feedback is given by

$$\boxed{A_{f(s)} = \frac{A(s)}{1 - A(s)\beta(s)}}$$



Sinusoidal oscillator with positive feedback

where we note negative sign in the denominator.

For our purpose here it is more convenient to drop the minus sign and define the loop gain L(s) as:

$L(s) = A(s)\beta(s)$ ... (1)

the characteristic equation thus becomes

$1 - L(S) = 0$ ... (2)

The oscillation criterion: If at a specific frequency f_0 the loop gain Aβ is equal to unity it follows that $L(J\omega_0) = A(J\omega_0)\beta(J\omega_0) = 1$

That is at ω_0 the phase of the loop gain should be zero and the magnitude of the loop gain should be unity. This is known as Bark hausen criteria.

For this loop to produce and sustain an output x_0 with no input applied ($x_5 = 0$), the feedback signal x_y. $x_y = \beta x_0$

Should be sufficiently large that when multiplied by A it produces x_0, that is $A_{xy} = x_0$

i.e. $A\beta x_0 = x_0$ which result in $\boxed{A\beta = 1}$

An alternative approach to the study of oscillator circuits consists of examining the circuit poles, which are the roots of the characteristic equation

2. For the circuit to produce sustained oscillations at a frequency ω_0 the characteristic equation. For the circuit to produce sustained oscillations at a frequency ω_0 the characteristic equation has to have roots at $S = \pm J\omega_0$. Thus $1 - A(s)\beta(s)$ should have a factor of the form $S^2 + \omega_0^2$.

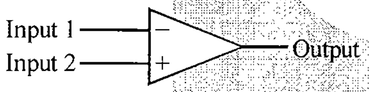
Non-linear amplitude control: The oscillation condition, the Bark hausen criteria, guarantees sustained oscillations in a mathematical sense. It is well known, however, that the parameter of any physical system cannot be obtained constant for any length of time. In otherwards, suppose we work hard to make $A\beta = 1$ at $\omega = \omega_0$ and then the temperature changes and $A\beta$ becomes slightly less than unity.

There are two basic approaches to the implementation of non-linear amplitude stabilization mechanism. The first approach makes use of a limited circuit. Oscillations are allowed to grow until the amplitude reaches the level to which the limiter is set. When the limiter comes into operation, the amplitude remains constant. Obviously, the limiter should be shoft to minimize non-linear distortion such distortion however, is reduced by the filtering action of the frequency selective network in the feedback loop.

The other mechanism for amplitude control utilizes an element whose resistance can be controlled by the amplitude of the output sinusoid. By placing this element in the feedback circuit so that its resistance determines the loop gain. The circuit can be designed to ensure that the loop gain reaches unity at the desired output amplitude. Diodes or JFETs operated in the triode region are commonly employed to implement the controlled resistance element.

Operational Amplifier:

Symbol:



(+) Non-inverting terminal, (−) Inverting terminal

Input impedance : Few mega Ω (Very high), Output impedance : Less than 100 Ω (Very low)

Differential and Common Mode Operation: One of the more important features of a differential circuit connection as provided in an op-amp is the circuit ability to greatly amplify signals that are opposite at the two inputs while only slightly amplifying signals that are common to both inputs. An op-amp provides an output component that is due to the amplification of the difference signals applied to the plus and minus input and a component due to the signals common to both inputs. Since amplification of the opposite input signals is much greater than that of common input signals the circuit provides a common-mode rejection as described by a numerical value called COMMON MODE REJECTION RATIO (CMRR).

Differential Input: When separate inputs are applied to the op-amp, the resulting difference signal is the difference between the two inputs. $\boxed{V_d = V_1 - V_2}$.

Common Input: When both input signals applied to an op-amp is common, signal element due to the two inputs can be defined as the average of the sum of the two signals. $\boxed{V_C = \left(\frac{V_1 + V_2}{2}\right)}$.

Output Voltage: Since any signal applied to op-amp in general have both in phase and out of phase components the resulting output can be expressed as $\boxed{V_0 = A_d V_d + A_c V_c}$.

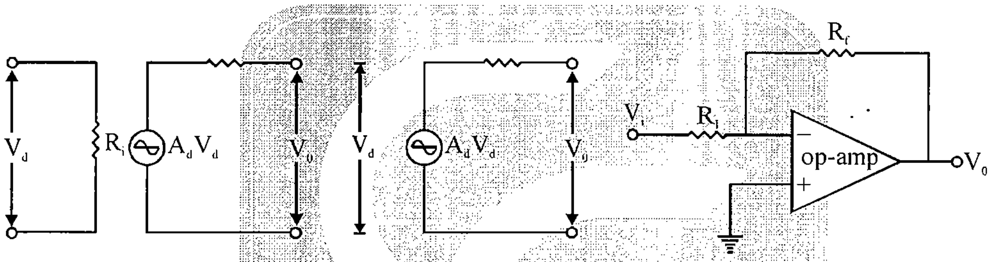
Where V_d = difference voltage, V_C = common voltage,
A_d = difference mode gain of the amplifier, A_c = Common mode gain of the amplifier.

CMRR {Common Mode Rejection Ratio}: $\boxed{CMRR = \frac{A_d}{A_c}}$

The value of CMRR can also be expressed in log term as

$$\boxed{CMRR(\text{in } d_B) = 20\log_{10}\frac{A_d}{A_c}(dB)}$$

Equivalent Circuit: While an input to the minus (–) input results in on opposite polarity output. The ac equivalent circuit of the op-amp is shown in figure. As shown the input signal applied between input terminals sees as input impedance Ri typically very high. The output voltage is shown to be the amplifier gain times the input signal taken through output impedance R_0, which is typically very low. An ideal op-amp circuit, as shown in figure would have infinite input impedance zero output impedance and infinite voltage gain.

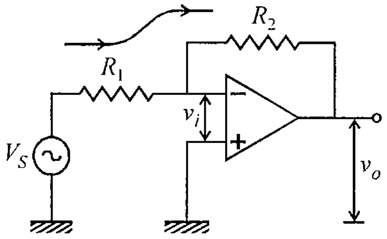


Characteristic of ideal OPAMP

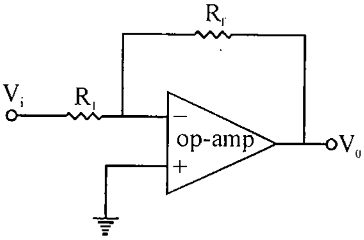
(i) Open loop voltage gain in infinite (A = ∞)
(ii) Input impedence is infinite is infinite
(iii)Output impedence is zero
(iv) Bandwidth is infinite
(v) Characteristic do not drift with temperature

Concept of virtual ground

Figure shows an circuit of inverting opamp which employs negative feedback through the resistor R_2. The negative feedback voltage R_2 tends to cancel the input signal at the point A and tries to keep the point A at ground potential. If v_i is the input voltage then output would be $v_o = -A\cdot v_i$. Since open loop gain $A \to \infty$, for finite v_o we should have $v_i \to 0$. Since, $R_i \to \infty$, no current will enter into the opamp. Thus, there is a 'virtual ground' at point A. The word virtual is used to imply that through the point A is effectively connected to ground.

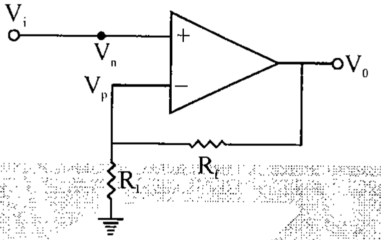


Inverting Amplifier: The most widely used constant gain amplifier circuit is the inverting amplifier.



. We can write $\boxed{V_0 = -\frac{R_f}{R_1}V_1}$

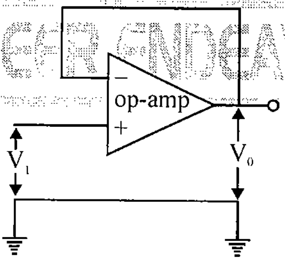
Non-inverting Amplifier: The connection of figures shows an op-amp that works as a non-inverting amplifier or constant gain multiplier. It should be noted that the inverting amplifier connection is more widely used because it has better frequency stability.



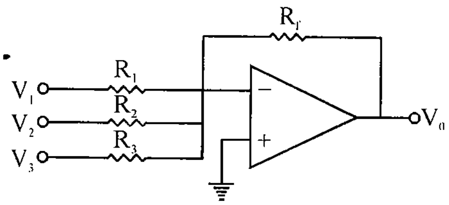
By virtual ground law: $V_n = V_p = V_i$

$$\Rightarrow V_i = \frac{R_1V_0}{R_1+R_f} \Rightarrow \frac{V_0}{V_i} = \frac{R_1+R_f}{R_1} = 1+\frac{R_f}{R_1} \Rightarrow \boxed{V_0 = \left(1+\frac{R_f}{R_1}\right)V_i}$$

Voltage Follower or Unity Follower: The unity follower circuit as shown in figure provides a gain of unity (1) with number polarity or phase reversal. From the equivalent circuit, it is clear that $\boxed{V_0 = V_1}$ and that the output is the same polarity and magnitude as the input. The circuit operates like an emitter or source follower circuit except that the gain is exactly unity.

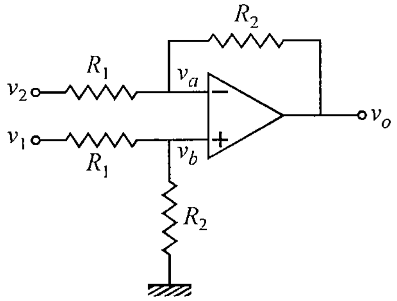


Summing Amplifier: Three input summing amplifier.



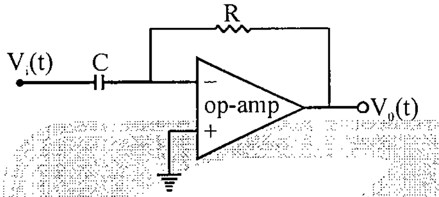
$$\boxed{V_0 = -\left(\frac{R_f}{R_1}V_1 + \frac{R_f}{R_2}V_2 + \frac{R_f}{R_3}V_3\right)}$$

Differential Amplifier


R_2
R_1
v_2
v_a
v_1
R_1
v_b
v_o
R_2

$$v_o = \frac{R_2}{R_1}(v_1 - v_2)$$

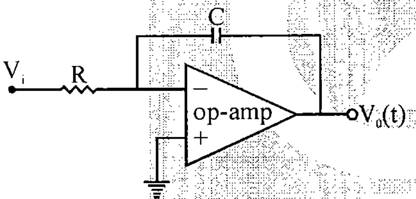
Differentiator : A differentiator circuit is shown in figure while not as useful as the circuit forms covered above the differentiator does provide a useful operation, the resulting for the circuit being


R
$V_i(t)$
C
op-amp
$V_0(t)$

$$\boxed{V_0(t) = -RC\frac{dV_1(t)}{dt}}$$

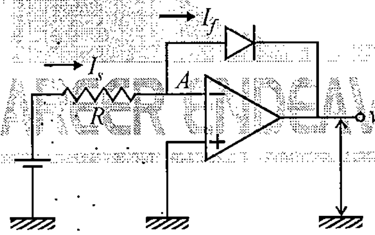
where the scale factor is –RC.

Integrator:


C
V_i
R
op-amp
$V_0(t)$

$$\boxed{V_0(t) = -\frac{1}{RC}\int V_1(t)dt}$$

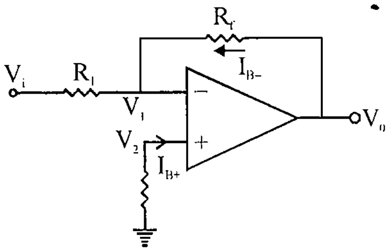
Logarithmic Amplifier



Due to virtual ground at point A, $I_S = I_f$

$$\Rightarrow \quad \frac{V_S - 0}{R} = I_0\left(e^{\frac{v_f}{\eta kT}} - 1\right) \simeq I_0\, e^{\frac{ev_f}{\eta kT}} \text{ or } v_o = -v_f = -\frac{\eta kT}{e}\ell n\frac{V_S}{I_0 R}$$

Offset Currents and Voltages {d.c. characteristic of op-amp}:


R_f
I_{B-}
V_i
R_1
V_1
V_2
I_{B+}
V_0

(1) Input bias current : $\frac{i_B^+ + i_B^-}{2}$

(2) Input offset current: $I_{0s} = |I_B^+| - |I_B^-|$

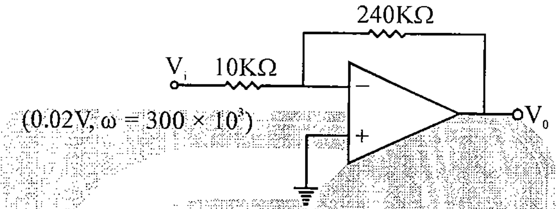
(3) Input offset voltage : $V_{0s} = V_2 - V_1$

Note: Due to mismatching between V_1 and V_2 output voltage may be positive or negative so we apply offset voltage (V_{os}).

Slew Rate: Another parameter reflecting the op-amp's ability to handling varying signal is slew rate, defined as slew rate = maximum rate at which amplifier output can change in volts per micro second.

$\boxed{SR = \frac{\Delta V_0}{\Delta t} V/\mu s}$ with t in μs.

Problem: Calculate the slew rate of given circuit.

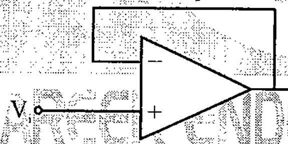


Soln. For a gain of magnitude $A_{CL} = \left|\frac{R_F}{R_1}\right| = \frac{240K\Omega}{10K\Omega} = 24$. The output voltage provides.

$K = A_{CL}, V_i = 24(0.2V) \Rightarrow 0.48V$

$\omega \le \frac{SR}{K} = \frac{0.5 v/\mu s}{0.48} = 1.1 \times 10^6$ rad/sec

Voltage Buffer: A voltage buffer circuit provides a means of isolation on input signal from a load by using a stage having unity gain with no phase or polarity inversion.



$\boxed{V_0 = V_i}$

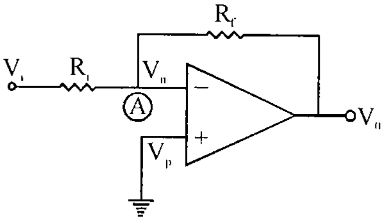
Controlled Sources: Op-amp can be used to form various types of controlled sources. An input voltage can be used to control on output voltage or current or an input current can be used to control on output voltage or current. There type of connections are suitable for use in various instrument system (circuit). It has four types:

(1) Voltage Controlled Voltage Source (2) Voltage Controlled Current Source

(3) Current Controlled Current Source (4) Current Controlled Voltage Source

(1) **Voltage Controlled Voltage Source:** An ideal form of a voltage source whose output V_0 is controlled by on input voltage V_J is shown in figure. The output voltage is seen to be independent on the input voltage. This type of circuit can be built using an op-amp as shown in figure.

(i) **Inverting op-amp:**

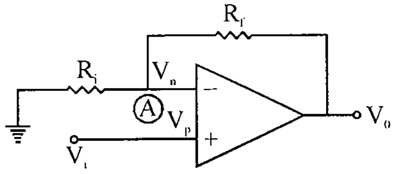


$\boxed{K = -\frac{R_f}{R_i}}$

By virtual ground condition $V_n = V_p = 0$

Now KCL at point A $\frac{V_i - 0}{R_i} = \frac{0 - V_0}{R_f} \Rightarrow \boxed{\frac{V_0}{V_i} = -\left(\frac{R_f}{R_i}\right)}$

1) **Non-inverting op-amp:**



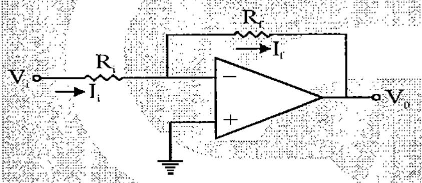
$\boxed{K = 1 + \frac{R_f}{R_i}}$

By virtual ground condition $V_p = V_n = V_i$
Now KCL at point A

$$\frac{O - V_i}{R_i} = \frac{V_i - V_o}{R_f} \Rightarrow \frac{V_0}{R_f} = \left(\frac{R_f + R_i}{R_f + R_i}\right) V_i \Rightarrow \boxed{\frac{V_0}{V_i} = 1 + \frac{R_f}{R_i}}$$

2) **Voltage Controlled Current Source:** An ideal form of circuit providing an output current controlled by an input voltage is that of figure. The output current is dependent on the input voltage.

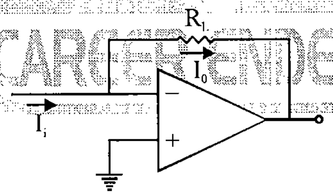
Practical Circuit:



$\boxed{I_i = \frac{V_i}{R_i}}$

3) **Current Controlled Voltage Source:** An ideal form of a voltage source controlled by a input current is shown in figure. The output voltage is dependent on the input current.

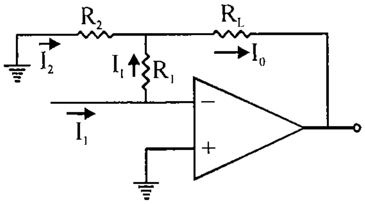
Practical Circuit:



$\boxed{V_0 = -I_i R_L}$

4) **Current Controlled Current Source:** An ideal form of a circuit providing on output current dependent on an input current is shown in figure. In this type of circuit on output current is provided dependent on the input current.

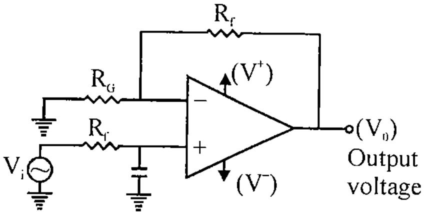
Practical Circuit:

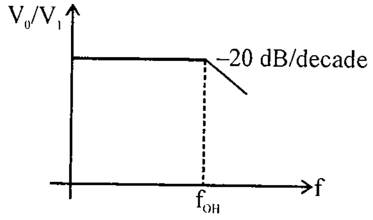


$$I_0 = I_1 + I_2 = I_1 + \frac{I_1 R_1}{R_2} = \left(1 + \frac{R_1}{R_2}\right) I_1 = K I_1$$

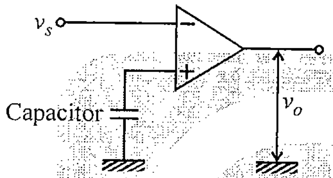
Low Pass Filter: A 1st order, low pass filter using resistor and capacitor as in figure shown has a practical slope of –20 dB per decade as shown in figure (rather them the ideal response of figure). The voltage gain below the cutoff frequency is constant at

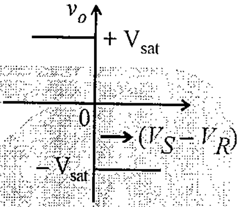
$\boxed{A_v = 1 + \frac{R_f}{R_G}}$, at a cut off frequency of $\boxed{f_{OH} = \frac{1}{2\pi R_1 C_1}}$





Voltage comparator

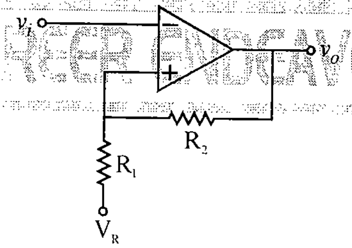




Comparator is a circuit which can compare two voltages. In the figure an opamp voltage comparator compare two voltage, input signal V_S with reference voltage V_R. When $V_S < V_R$ the output voltages goes to maximum saturation value $(+V_{sat})$ when $V_S > V_R$ the opamp output switches to the negative saturation value $(-V_{sat})$.

Schmitt Trigger

Its a comparator which uses positive or regenerative feedback and exhibits the phenomenon of hysteresis is called a regenerative comparator or more generally Schmitt trigger.



when $v_i < v_1$ the output v_o is at positive saturation v_{sat}, so we have $v_1 = \frac{R_2 V_R}{R_1 + R_2} + \frac{R_1 V_{sat}}{R_1 + R_2} = v_i$.

Now, until $v_i = v_1$, the output remains at $+v_{sat}$. A slight increase in v_i makes the output to $-v_{sat}$, and it remains their when $v_i > v_1$.

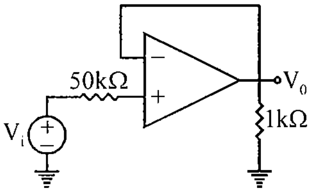
In this time, $v_1 = \frac{R_2 V_R}{R_1 + R_2} - \frac{R_1 v_{sat}}{R_1 + R_2} = v_2$

If voltage v_i now reduced the output stays at $-v_{sat}$, but when v_i just cross v_2, output switches to $+v_{sat}$.

Since, v_2 is smaller as compared to v_1 so a hysteresis will from, where $v_H = v_1 - v_2 = \frac{2R_1}{R_1 + R_2} v_{sat}$.

SOLVED PROBLEMS

. The circuit is as shown in figure



(i) The ideal closed loop voltage gain is

(a) 1 (b) −1 (c) ∞ (d) 50

oln. Voltage follower circuit, $V+=V_i$, $V-=V_i=V_0, \frac{V_0}{V_i}=1$. Hence, (a) is correct answer.

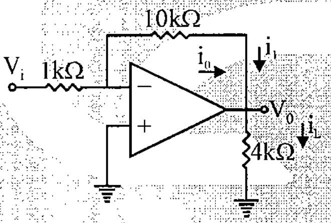
(ii) The open loop gain is A_{od} = 999, then closed loop gain is

(a) −0.999 (b) 0.999 (c) 1.001 (d) −1.001

$V+=V_i$, $V-=V_0$, $A_{od}(V_i-V_0)=V_0$, $A_{od}V_i=(A_{od}+1)V_0$, $A_{od}=999$

$$\frac{V_0}{V_i}=\frac{A_{od}}{1+A_{od}}=\frac{999}{1+999}=0.99$$

. The input to the circuit in figure is $V_i=2\sin\omega t mV$. The current I_0 is



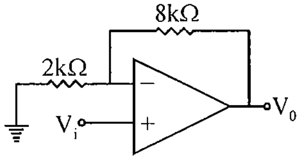
(a) $-2\sin\omega t\,\mu A$ (b) $-7\sin\omega t\,\mu A$ (c) $-5\sin\omega t\,\mu A$ (d) 0

oln. $V_0=\frac{-10}{1}(2\sin\omega t)mV=-\frac{R_2}{R_1}.V_S=-20\sin\omega t\,mV$

$i_L=\frac{V_0}{4K}=-5\sin\omega t\mu A$, $i_1=i_i=\frac{2\sin\omega t}{1K}=2\sin\omega t\mu A$

KCL at V_0, $i_0=i_L-i_1=-5\sin\omega t-2\sin\omega t=-7\sin\omega t$. Hence, (b) is correct answer.

. The op-amp of figure has a very poor open-loop voltage gain of 45 but is otherwise ideal. The closed loop gain of amplifier is



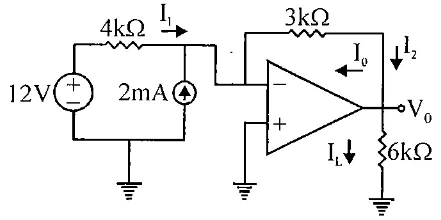
(a) 20 (b) 4.5 (c) 4 (d) 5

oln. A closed loop gain,

$$A_{CL}=\frac{V_0}{V_i}=\frac{A_{od}}{1+\beta A_{od}},\beta=\frac{2K}{8K+2K}=0.2, A_{CL}=\frac{45}{1+(45)(0.2)}=4.5$$

Hence, (b) is correct answer.

4. Find the value of current I_0?



(a) 12mA (b) 8.5mA (c) 6mA (d) 7.5mA

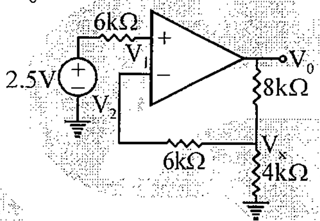
Soln. $V+ = V- = 0,\ i_1 = \dfrac{12}{4K} = 3mA$

$i_2 = 3 + 2 = 5mA,\ V_0 = -(5)(3) = -15V$

$i_2 = i_0 + i_L,\ 5 = i_0 + \dfrac{-15}{6},\ i_0 = 7.5\text{mA}$

Hence, (d) is correct answer.

5. Find the value of output voltage V_0.



(a) −7.5V (b) 7.5V (c) 8V (d) −8V

Soln. $V+ = 2.5V = V-,\ \dfrac{V_0(4)}{8+4} = 2.5,\ V_0 = 7.5V,\ V_1 = 2.5V$

(since zero current inside op-amp)

$V_1 = V_2 = 2.5V,\ V_X = 2.5V$

(zero current inside op-amp, so zero voltage drop at 6KΩ)

6. A multistage amplifier consists of three stages. The voltage gains of the stages are 30, 50 and 80 respectively. The overall voltage gain in dB will be **[BHU 2013]**

(a) 101.58 (b) 50.79 (c) 33.98 (d) 38.06

Soln. Overall gain = $Av_1 \times Av_2 \times Av_3$ = 30×50×80 = 120000

Therefore, the overall voltage gain in dB = 20 log Av_T = 20 log 120000 = 101.58

Correct option is (a)

7. An amplifier with negative feedback has a voltage gain of 100. It is found that with feedback an input signal of 0.6 V is required to produce a given output whereas without feedback the input signal must be only 50 mV for the same output. Then the voltage gain without feedback A and feedback factor β are

(a) $A = 1000,\ \beta = \dfrac{9}{1000}$ (b) $A = 1100,\ \beta = \dfrac{10}{1100}$ **[BHU 2013]**

(c) $A = 1400,\ \beta = \dfrac{13}{1200}$ (d) $A = 1200,\ \beta = \dfrac{11}{1200}$

oln. We know that gain with negative feedback is given by

$$A_f = \frac{A}{1+AB} = 100$$

The output with negative feedback is given by

$$V_0 = 100 \times 0.6 = 60V$$

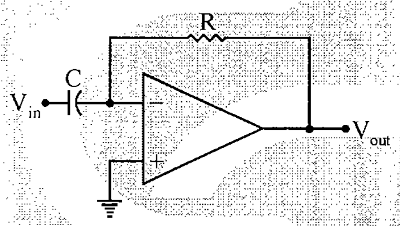
And output without negative feedback is given by

$$V_0 = A \times 50mV = 60V \Rightarrow A = \frac{60}{50} \times 10^3 = 1.2 \times 10^3 = 1200$$

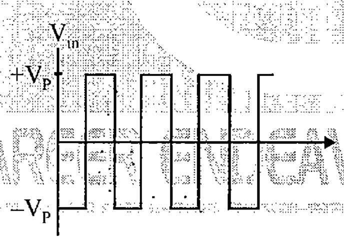
Now, $A_f = 100 = \dfrac{A}{1+A\beta} \Rightarrow \beta = \left(\dfrac{1200}{100} - 1\right) / 1200 = \dfrac{11}{1200}$

Correct option is (d)

. Consider the following circuit: **[TIFR 2012]**

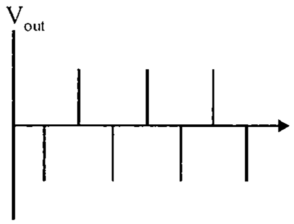


If the waveform given below is fed in at V_{in},

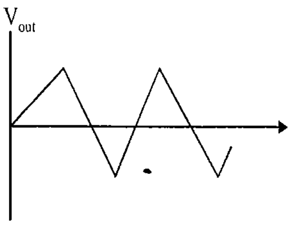


then the waveform at the output V_{out} will be

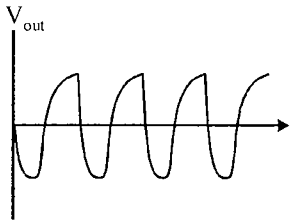
(a)



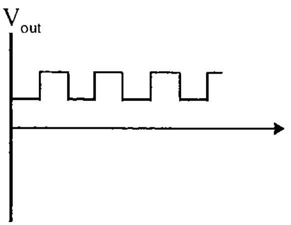
(b)



(c)



(d)



Soln. From the given circuit we may write

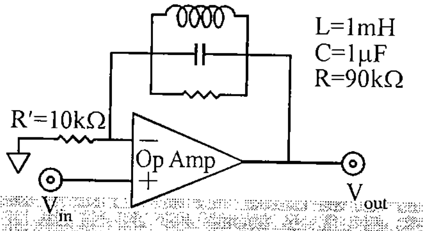
$$\frac{d}{dt}(v_i - 0)C = \frac{0 - V_0}{R}$$

$$\Rightarrow \quad V_0 = -CR\frac{dV_i}{dt}$$

So, it is a differator circuit. So, for square wave input out put will be spike (Impulse)

Correct option is (a)

9. An input of 1.0V DC is given to the ideal Op-Amp circuit depicted below. What will be the output voltage?



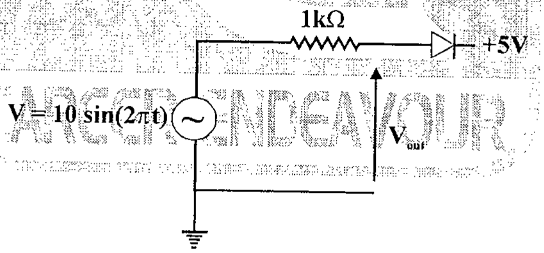
(a) 10.0 V (b) −9.0 V (c) 1.0 V (d) 0 V **[TIFR 2013]**

Soln. For DC current, $R_2 = 0$

$$\therefore \quad V_0 = V_{in}\left(1 + \frac{R_2}{R_1}\right) = 1\left(1 + \frac{0}{10}\right) = 1V$$

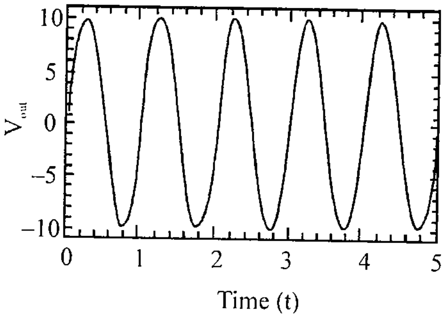
Correct option is (c)

10. Consider the following circuit **[TIFR 2014]**

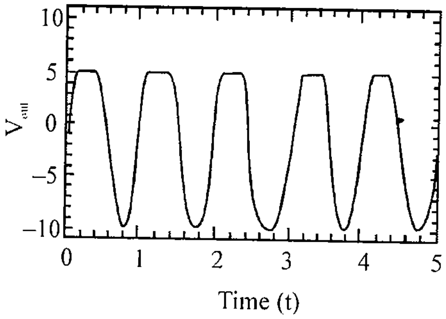


Which of the graphs given below is a correct representation of V_{out}?

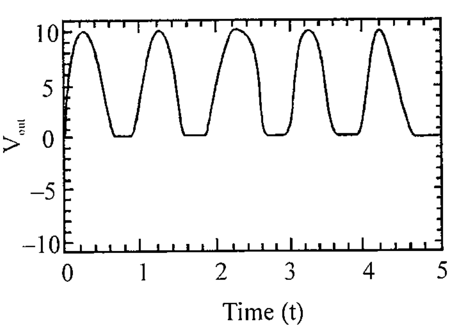
(a)



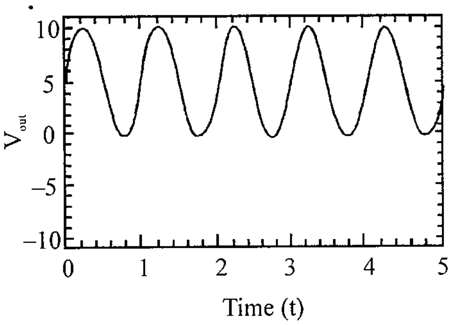
(b)



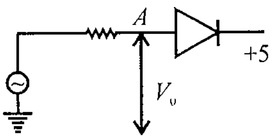
(c)



(d)



oln.

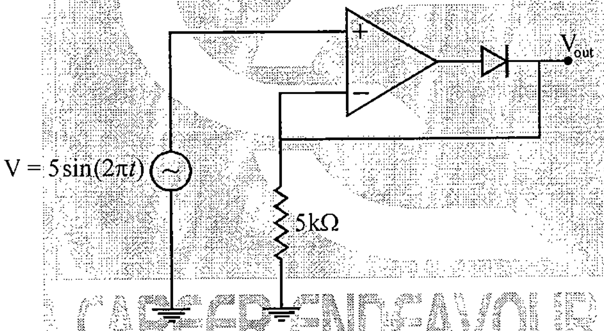


For negative half cycle output will same input as during the negative cycle diode will be in reversed biased.

But for positive half cycle, when voltage at the point A will just greater than the 5 volt diode will work in forwards baised. So, peak voltage will be +5 volt.

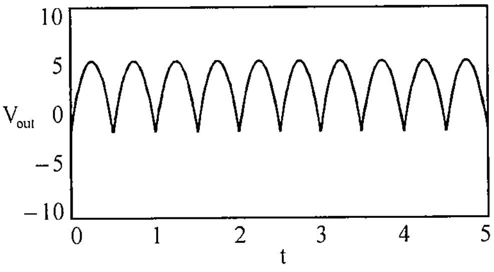
Correct option is (b)

1. In the circuit shown below, the op-amp is powered by a bipolar supply of ±10 V. **[TIFR 2015]**

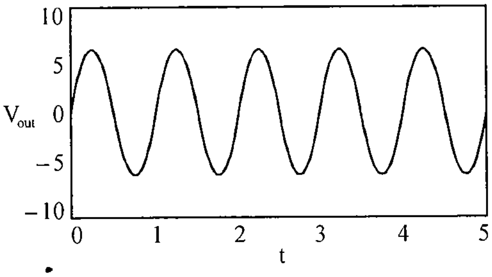


Which one of the following graphs represents V_{out} correctly ?

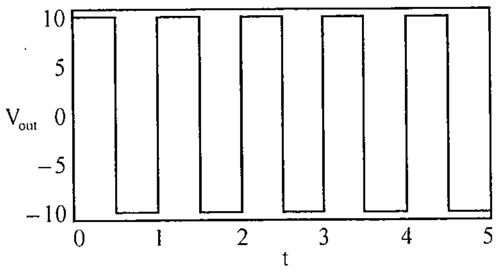
(a)



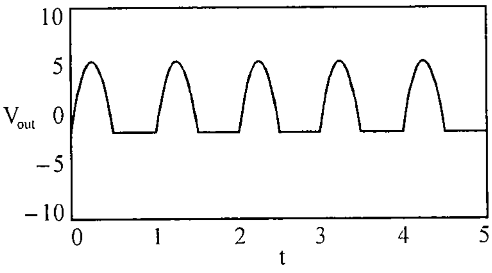
(b)



(c)



(d)



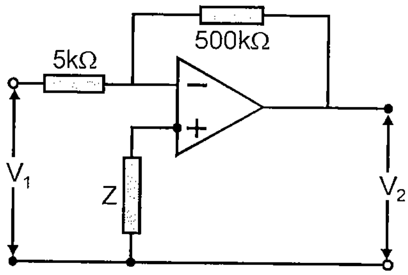
Soln. It is unit given non-inverting circuit (if don't include diode) during the positive half cycle diode will conducting. So, output will be same as input.

But, during negative half cycle diode with be in reversed baised.

Correct option is (d)

12. In the generalized operational amplifier circuit shown on the right, the amp. has a very high input impedance ($Z > 50\, M\Omega$) and an open gain of 1000 for the frequency range under consideration. Assuming that the op. amp. draws negligible current, the voltage ratio V_2/V_1 is approximately

[TIFR 2016]



(a) −190 (b) −190 (c) −90 (d) 80

Soln. $$\boxed{A_{CL} = \frac{-A_{OL}\cdot R_f}{R_f + (1 + A_{OL})R}}$$

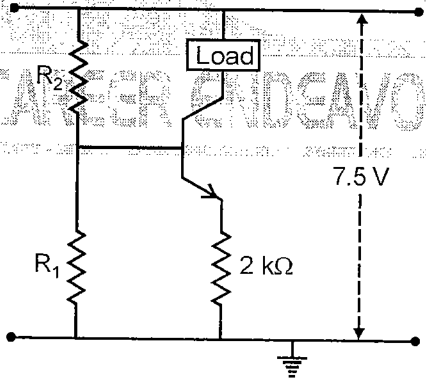
Put value

$A_{CL} = -90$

Correct option is (c)

13. In the transistor circuit shown on the right, assume that the voltage drop between the base and the emitter is 0.5V.

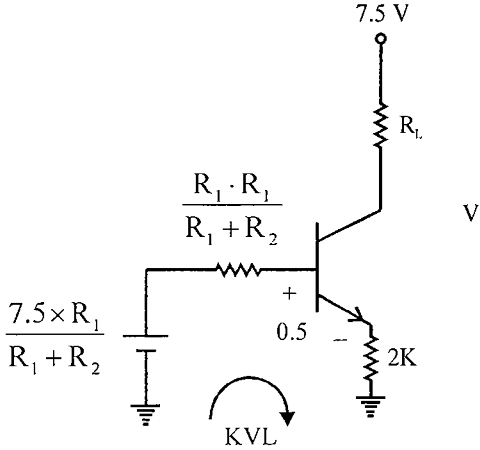
[TIFR 2016]



What will be the ratio of the resistances R_2/R_1, in order to make this circuit function as a source of constant current, $I = 1$ mA ?

(a) 4.5 (b) 3.0 (c) 2.5 (d) 2.0

oln.


7.5 V
R_L
$\frac{R_1 \cdot R_1}{R_1 + R_2}$
$\frac{7.5 \times R_1}{R_1 + R_2}$
+
0.5
–
2K
KVL

V_{be} given = 0.5

Thevenin equivalent circuit

$$I_C = I_E = 1\text{mA}$$

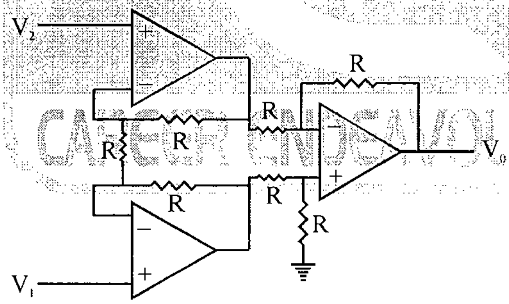
$$\frac{7.5 \times R_1}{R_1 + R_2} = I_B \times \left[\frac{R_1 \cdot R_2}{R_1 + R_2}\right] + 0.5 + 2 \times 1$$

No 'β' neglect I_B given in question

$$\frac{7.5 \times R_1}{R_1 + R_2} = 2.5 \Rightarrow \frac{R_1}{R_1 + R_2} = \frac{1}{3} \Rightarrow \frac{R_1 + R_2}{R_1} = 3 \Rightarrow \frac{R_2}{R_1} = 3 - 1 = 2$$

Correct option is (b)

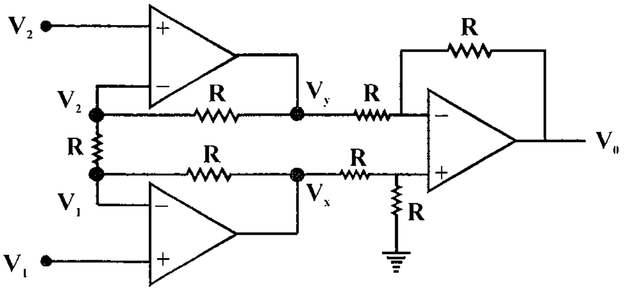
4. The classic three op-amp instrumentation amplifier configuration is shown below:


V_2
+
–
R
R
R
R
–
+
V_0
R
R
R
–
+
V_1

The op-amp are ideal and all resistors are of equal value R. The gain, defined as the output voltage V_0 divided by the differential input voltage $V_1 - V_2$, is equal to

(a) 2 (b) 3 (c) 4 (d) 6

oln. Its an instrumentation amplifier


V_2
+
–
V_2
R
V_y
R
R
R
R
R
–
+
V_0
V_1
V_x
R
–
+
V_1

KCL at V_2

$$\frac{V_2 - V_1}{R} + \frac{V_2 - V_y}{R} = 0$$

$$2V_2 - V_1 = V_y \qquad \dots (1)$$

KCL at V_1

$$\frac{V_1 - V_2}{R} + \frac{V_1 - V_x}{R} = 0$$

$$2V_1 - V_2 = V_x \qquad \dots (2)$$

$$V_0 = \frac{R}{R}\left[V_x - V_y\right]$$

Subtractor circuit, put value of V_x, V_y

$$V_0 = \frac{R}{R}\left[2V_1 - V_2 - 2V_2 + V_1\right]$$

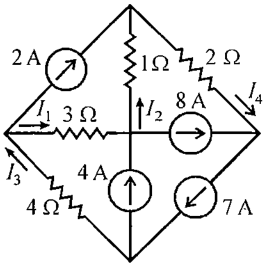
$$\frac{V_0}{V_1 - V_2} = 3$$

Correct option is (b)

PRACTICE SET

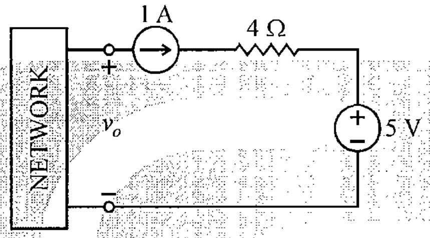
Multiple Choice Questions (MCQ):

In the given circuit of figure current I_2 is


2 A
1Ω
2 Ω
I_4
I_1 3 Ω
I_2
8 A
I_3
4 A
4 Ω
7 A

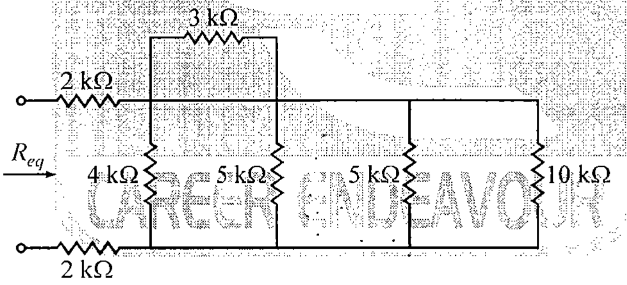
(a) 13 A (b) –3 A (c) 5 A (d) 3 A

The voltage v_0 in figure is always equal to


1 A
4 Ω
NETWORK
+
v_o
–
+
–
5 V

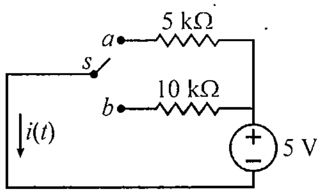
(a) 1 V (b) 5 V (c) 9 V (d) None of these

For the circuit of figure R_{eq} is


3 kΩ
2 kΩ
R_{eq}
4 kΩ
5 kΩ
5 kΩ
10 kΩ
2 kΩ

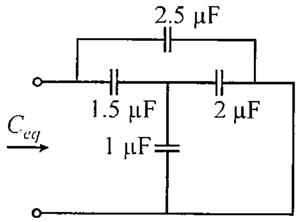
(a) 4 kΩ (b) 6 kΩ (c) 9 kΩ (d) 5.33 K

In the circuit of figure, the ideal switch S alternately stays at position A for 1 ms and at position B for 4 ms. The average value of current $i(t)$ is


5 kΩ
a
S
10 kΩ
b
$i(t)$
+
–
5 V

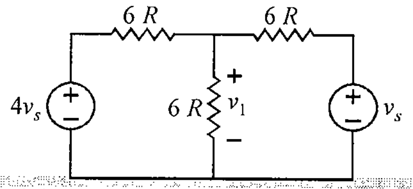
(a) 0.6 mA (b) 1.5 mA (c) 0.75 mA (d) 2.5 mA

5. For the circuit shown in figure the value of C_{eq} is


2.5 μF
1.5 μF
2 μF
C_{eq}
1 μF

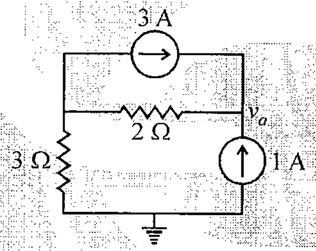
(a) 3.5 μF (b) 7 μF (c) 2.4 μF (d) 2.6 μF

6. In the following circuit the voltage v_1 is


6 R
6 R
$4v_s$
6 R
+
v_1
−
v_s

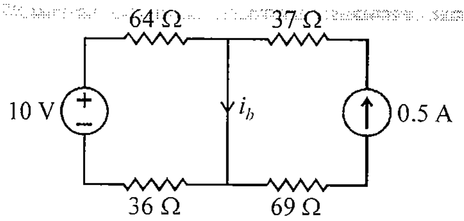
(a) $0.4v_s$ (b) $1.6v_s$ (c) $0.67v_s$ (d) $2.5v_s$

7. In the following circuit the voltage v_o is


3 A
2 Ω
v_o
3 Ω
1 A

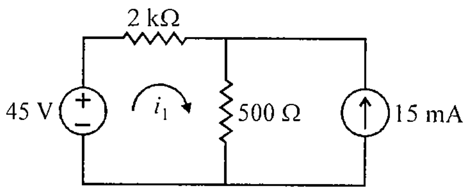
(a) −11 V (b) 11 V (c) 3 V (d) −3 V

8. In the following circuit the current i_b is


64 Ω
37 Ω
10 V
i_b
0.5 A
36 Ω
69 Ω

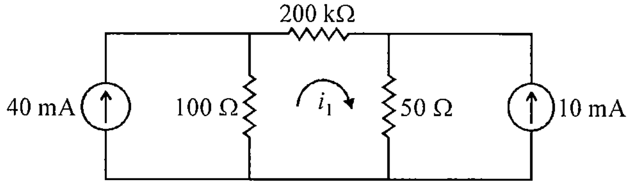
(a) 0.6 A (b) 0.5 A (c) 0.4 A (d) 0.3 A

9. In the following circuit the current i_1 is


2 kΩ
45 V
i_1
500 Ω
15 mA

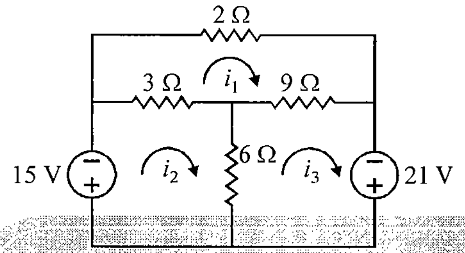
(a) 20 mA (b) 15 mA (c) 10 mA (d) 5 mA

0. In the following circuit the current i_1 is .


200 kΩ
40 mA
100 Ω
i_1
50 Ω
10 mA

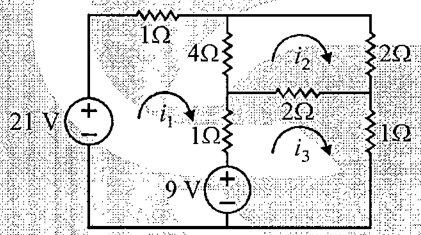
(a) 10 mA (b) –10 mA (c) 0.4 mA (d) –0.4 mA

1. In the following circuit the currents i_1, i_2 and i_3 is


2 Ω
3 Ω
i_1
9 Ω
15 V
i_2
6 Ω
i_3
21 V

(a) 3A, 2A and 4A (b) 3A, 3A and 8A (c) 1A, 3A and 4A (d) 1A, 2A and 8A

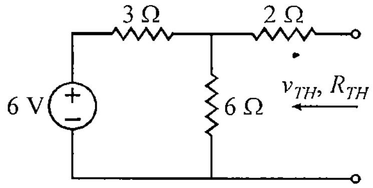
2. The mesh equation for the following circuit are


1Ω
4Ω
i_2
2Ω
21 V
i_1
2Ω
1Ω
i_3
1Ω
9 V

(a) $\begin{bmatrix} 6 & 4 & -1 \\ 4 & -8 & -2 \\ -1 & -2 & 4 \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 9 \end{bmatrix}$ (b) $\begin{bmatrix} 6 & 4 & -1 \\ 4 & 8 & -2 \\ -1 & -2 & 4 \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 9 \end{bmatrix}$

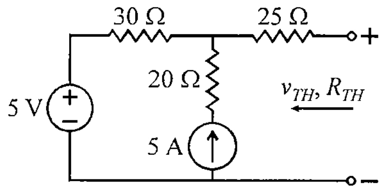
(c) $\begin{bmatrix} 6 & -4 & -1 \\ -4 & 8 & -2 \\ -1 & -2 & 4 \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 9 \end{bmatrix}$ (d) $\begin{bmatrix} -6 & 4 & 1 \\ 4 & -8 & -2 \\ 1 & 2 & -4 \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 9 \end{bmatrix}$

3. In the following circuit the value of v_{TH} and R_{TH} are


3 Ω
2 Ω
6 V
6 Ω
v_{TH}, R_{TH}

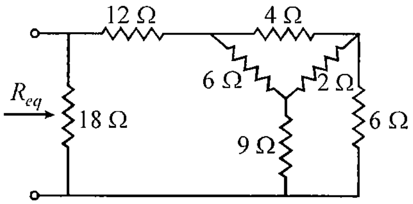
(a) 2 V, 4Ω (b) 4 V, 4Ω (c) 4 V, 5Ω (d) 2 V, 5Ω

14. For the following circuit the value of v_{TH} and R_{TH} are


30 Ω
25 Ω
+
20 Ω
5 V
5 A
v_{TH}, R_{TH}
–

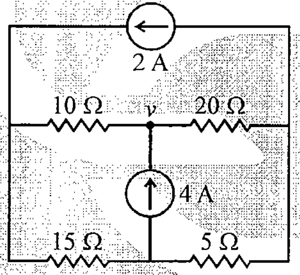
(a) −100 V, 75 Ω (b) 155 V, 55 Ω (c) 155 V, 37 Ω (d) 145 V, 75 Ω

15. In the following circuit the value of R_{eq} is


12 Ω
4 Ω
6 Ω
2 Ω
R_{eq}
18 Ω
9 Ω
6 Ω

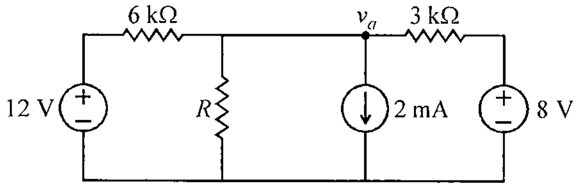
(a) 18 Ω (b) $\frac{72}{13}\,\Omega$ (c) $\frac{36}{13}\,\Omega$ (d) 9 Ω

16. In the following circuit the voltage v is


2 A
10 Ω
v
20 Ω
4 A
15 Ω
5 Ω

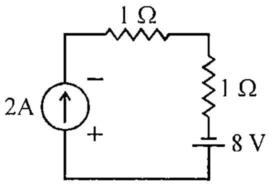
(a) 60 V (b) −60 V (c) 30 V (d) −30 V

17. The node voltage v_a in the following circuit is 4 V. The value of R would be


6 kΩ
v_a
3 kΩ
12 V
R
2 mA
8 V

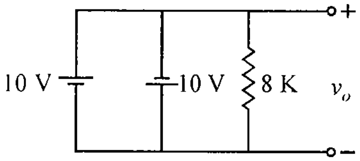
(a) 6 kΩ (b) 12 kΩ (c) 16 kΩ (d) 18 kΩ

18. Find $V_{2A} = ?$



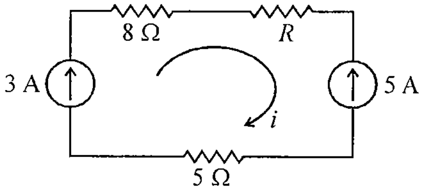
(a) 4V (b) −4V (c) 8V (d) −8V

9. Find v_o ?



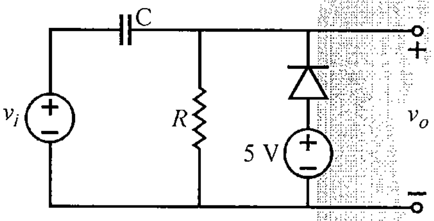
(a) 10 V (b) –10 V (c) 20 V (d) None of these

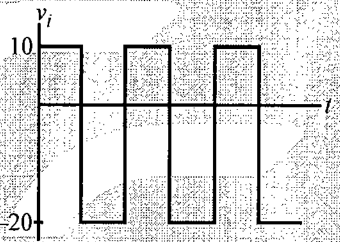
0. Find i ?



(a) –2A (b) 2A (c) Data insufficient (d) None of these

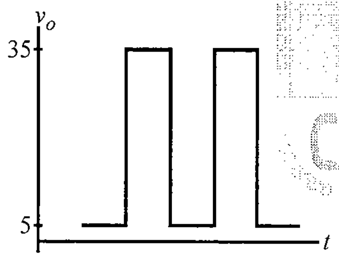
1. For the circuit shown below, the input voltage v_i is as shown in figure.



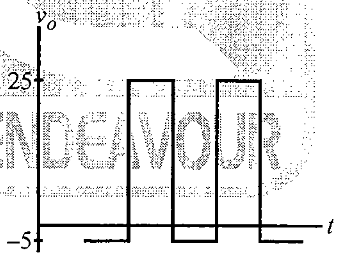


Assume the RC time constant large and cuting voltage $V_\gamma = 0$. The output voltage v_o is

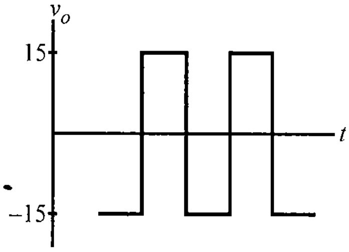
(a)



(b)

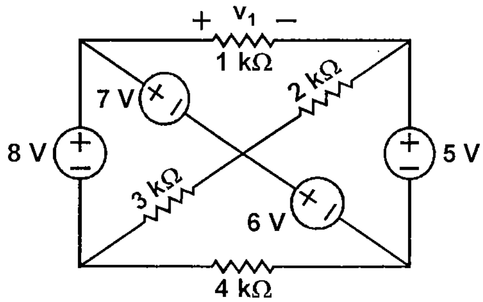


(c)



(d) None of these

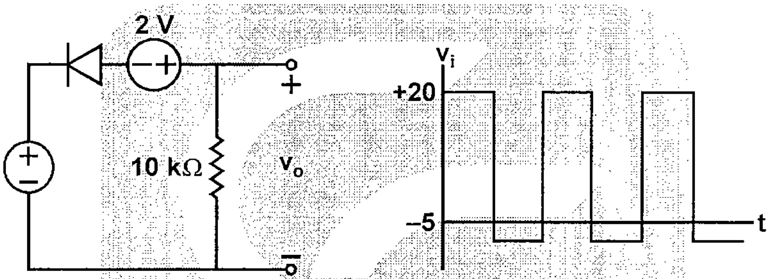
22. In the circuit of figure the voltage $v_1 = ?$

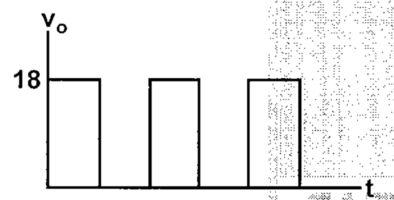

+ V_1 −
1 kΩ
7 V
2 kΩ
8 V
5 V
3 kΩ
6 V
4 kΩ

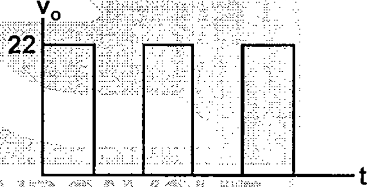
(a) −11 V (b) 5 V (c) 8 V (d) 18 V

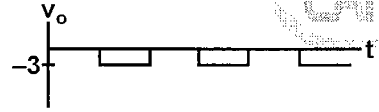
23. Consider the given a circuit and a waveform for the input voltage. The diode in circuit has cutin voltage $V_\gamma = 0$.

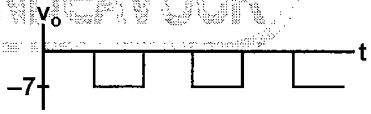
The waveform of output voltage v_0 is


2 V
V_i
10 kΩ
V_o
+
−
V_i
+20
−5
t

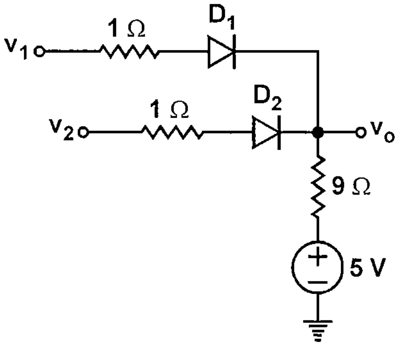
(a)


(b)


(c)


(d)


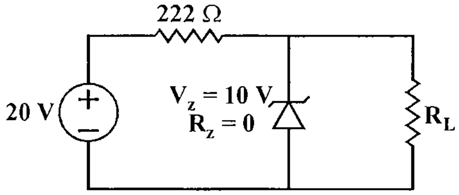
24. Consider the circuit shown below. Assume diodes are ideal.


V_1
1 Ω
D_1
V_2
1 Ω
D_2
V_o
9 Ω
5 V

If $v_1 = v_2 = 10$ V, then output voltage v_o is

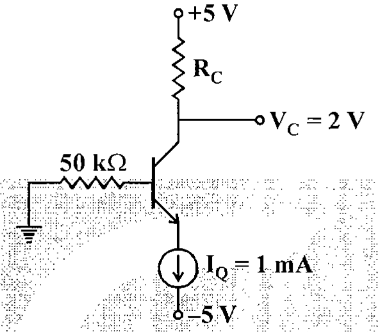
(a) 0 V (b) 9.737 V (c) 9 V (d) 9.5 V

5. In the voltage regulator circuit shown below the power rating of Zener diode is 400 mW. The value of R_L that will establish maximum power in Zener diode is



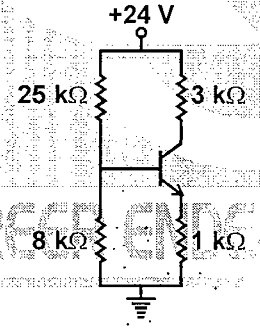
(a) 5 kΩ (b) 2 kΩ (c) 10 kΩ (d) 8 kΩ

6. The common-emitter current gain of the transistor is $\beta = 75$. The voltage V_{BE} in ON state is 0.7 V. The value of I_C and R_C is



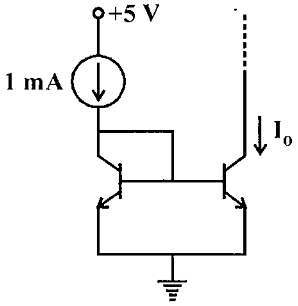
(a) 0.987 mA, 3.04 kΩ (b) 1.013 mA, 2.96 kΩ
(c) 0.946 mA, 4.18 kΩ (d) 1.057 mA, 3.96 kΩ

7. For the circuit shown below, let $\beta = 75$. The Q-point (I_{CQ}, V_{CEQ}) is



(a) (4.68 mA, 16.46 V) (b) (3.12 mA, 1.86 V)
(c) (3.12 mA, 8.46 V) (d) (4.68 mA, 5.22 V)

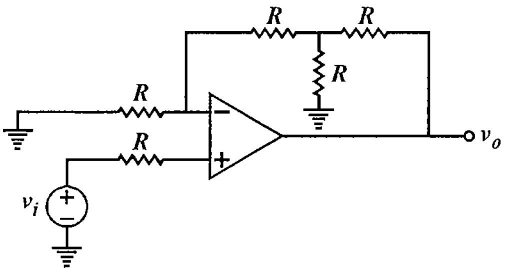
8. In the current mirror circuit shown below the transistor parameters are $V_{BE} = 0.7$ V, $\beta = 50$ and the Early voltage is infinite. Assume transistor are matched.



The output current I_o is

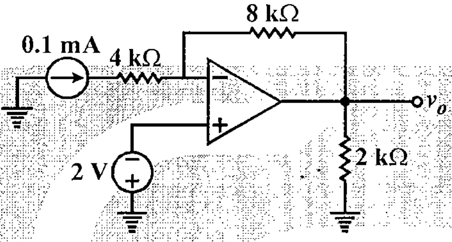
(a) 1.04 mA (b) 1.68 mA (c) 962 μA (d) 432 μA

29. For the circuit shown below the value of $A_v = \frac{v_o}{v_i}$ is



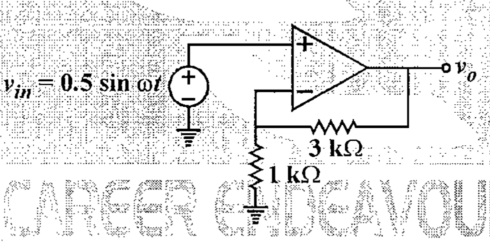
(a) 5 (b) –5 (c) 6 (d) –6

30. For the circuit shown below the value of v_o is


8 kΩ
0.1 mA
4 kΩ
v_o
2 V
2 kΩ

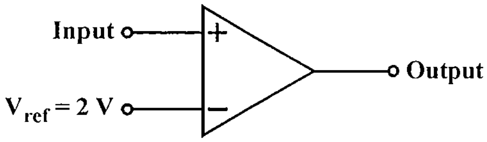
(a) –10 V (b) 18 V (c) –18 V (d) 28 V

31. The op-amp shown in figure, has a slew rate of 1V/ns, what is the highest input frequency for which no slewing occurs


$v_{in} = 0.5 \sin \omega t$
v_o
3 kΩ
1 kΩ

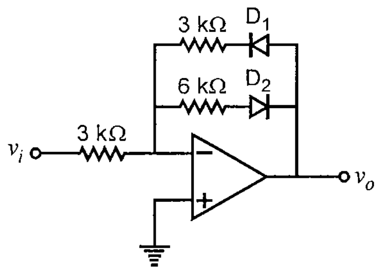
(a) 153.8 MHz (b) 63.7 MHz (c) 79.6 MHz (d) 127.4 MHz

32. If the input to the following ideal comparator is a sinusoidal signal of 8 V (peak to peak) without any DC componen., the output of the comparator has a duty cycle of


Input
Output
$V_{ref} = 2$ V

(a) 1/2 (b) 1/3 (c) 1/6 (d) 1/12

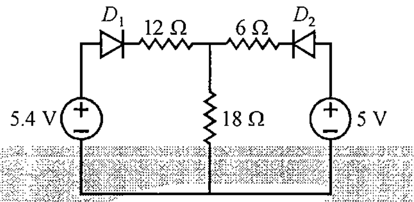
3. Consider the circuit shown below


3 kΩ
D1
6 kΩ
D2
3 kΩ
v_i
v_o
−
+

If $v_i = 2$ V, then output v_o is

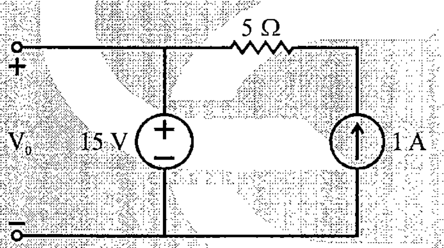
(a) 4 V (b) –4 V (c) 3 V (d) –3 V

4. In the circuit shown below diodes has cutin voltage of 0.6 V. The diode in ON state are


D_1
12 Ω
6 Ω
D_2
5.4 V
18 Ω
5 V

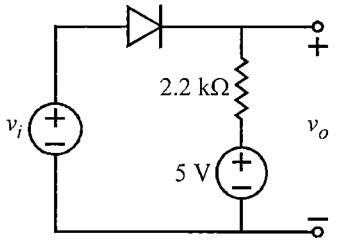
(a) only D_1 (b) only D_2 (c) both D_1 and D_2 (d) None of these

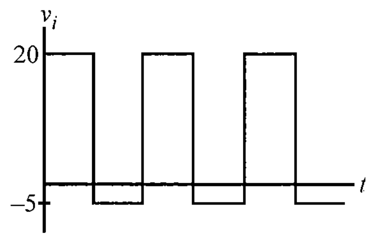
5. In the circuit of figure the value of voltage v_o is



(a) 10 V (b) 15 V (c) 20 V (d) None of these

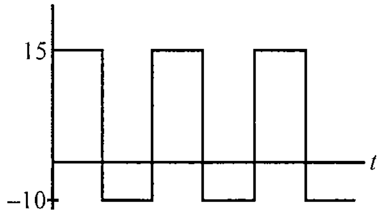
6. Consider the given a circuit and a waveform for the input voltage. The diode in circuit has cutin voltage $V_\gamma = 0$.


v_i
2.2 kΩ
5 V
+
v_o
−

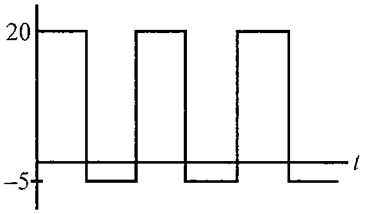


The waveform of output voltage v_o is

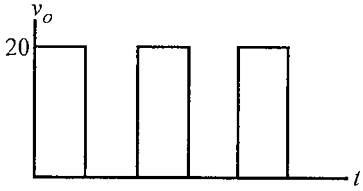
(a)



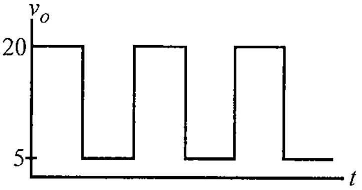
(b)



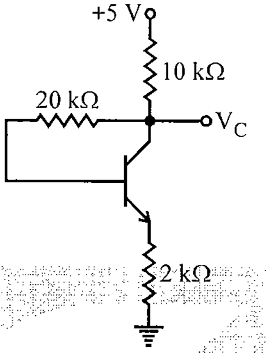
(c)



(d)

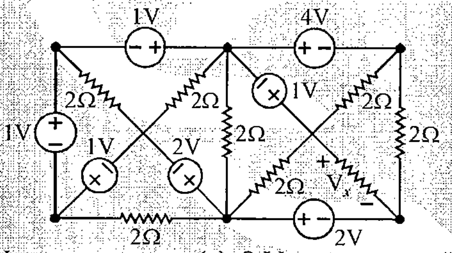


37. The common-emitter current gain of the transistor is $\beta = 75$. The voltage V_{BE} in ON state is 0.7 V. The value of V_C is


+5 V
10 kΩ
20 kΩ
V_C
2 kΩ

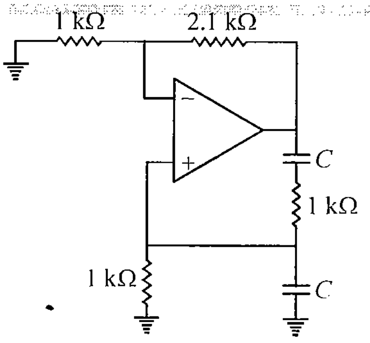
(a) 1.49 V (b) 2.9 V (c) 1.78 V (d) 2.3 V

38. In the circuit of figure the voltage V_x is


1V
4V
2Ω
2Ω
1V
2Ω
1V
1V
2V
2Ω
2Ω
+
V_x
−
2Ω
2Ω
2Ω
2V

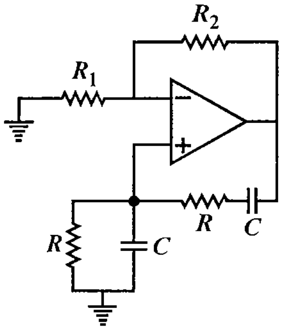
(a) 3 V (b) 2 V (c) 8 V (d) 4 V

39. The value of C required for sinusoidal oscillation of frequency 1 kHz in the following circuit is


1 kΩ
2.1 kΩ
−
+
C
1 kΩ
1 kΩ
C

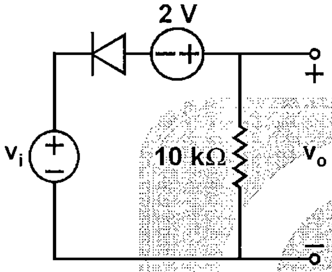
(a) $\frac{1}{2\pi}\mu F$ (b) $2\pi\ \mu F$ (c) $\frac{1}{2\pi\sqrt{6}}\mu F$ (d) $2\pi\sqrt{6}\ \mu F$

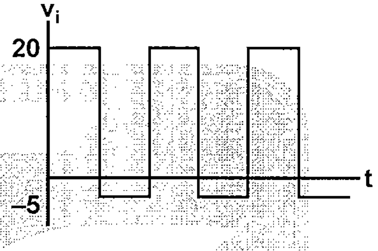
0. In the following circuit minimum required value of R_2/R_1 to sustain oscillation is



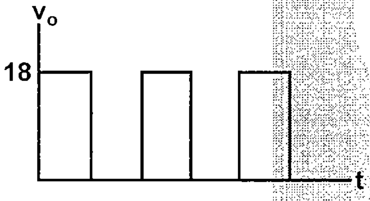
(a) 1.5 (b) 1 (c) 2 (d) 4

1. Consider the given a circuit and a waveform for the input voltage. The diode in circuit has cutin voltage $V_\gamma = 0$. The waveform of output voltage v_0 is

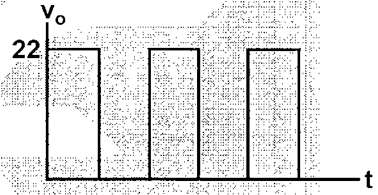




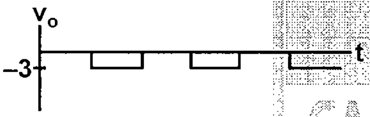
(a)



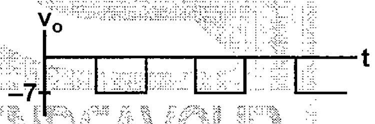
(b)



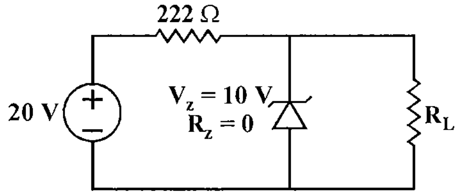
(c)



(d)

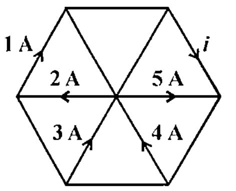


2. In the voltage regulator circuit shown below the power rating of Zener diode is 400 mW. The value of R_L that will establish maximum power in Zener diode is



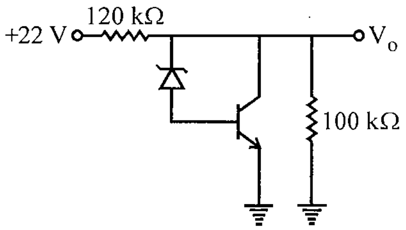
(a) 5 kΩ (b) 2 kΩ (c) 10 kΩ (d) 8 kΩ

3. For the circuit shown in figure the current i is



(a) 1 A (b) 2 A (c) 3 A (d) 4 A

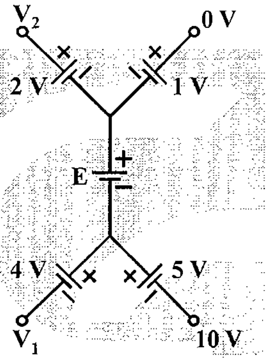
46. In the shunt regulator shown below, the $V_Z = 8.2$ V and $V_{BE} = 0.7$ V. The regulated output voltage V_o is


+22 V
120 kΩ
V_o
100 kΩ

(a) 11.8 V (b) 7.5 V (c) 12.5 V (d) 8.9 V

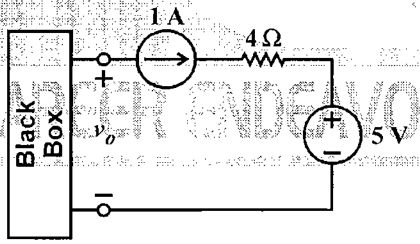
Multiple Select Questions (MSQ):

1. In the circuit of the figure, the value of the voltage source E is


V_2
0 V
2 V
1 V
E
+
−
4 V
5 V
V_1
10 V

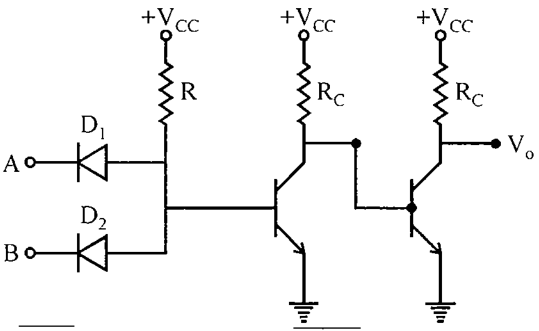
(a) −16 V (b) 4 V (c) −6 V (d) 16 V

2. The voltage v_o in figure is always equal to


1 A
4 Ω
Black Box
+
v_o
−
+
−
5 V

(a) 1 V (b) 5 V (c) 9 V (d) None of these

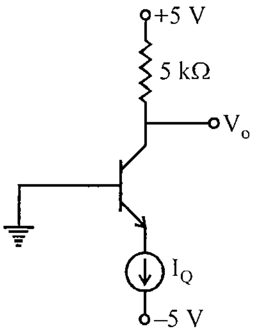
3. Below figure acts as :


$+V_{CC}$
$+V_{CC}$
$+V_{CC}$
R
R_C
R_C
D_1
D_2
A
B
V_o

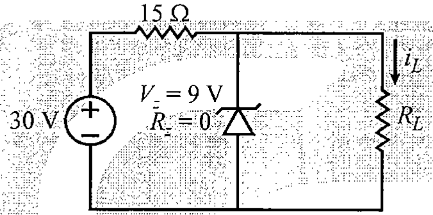
(a) $A\cdot B$ (b) $\overline{\overline{A}\cdot\overline{B}}$ (c) $\overline{\overline{A}+\overline{B}}$ (d) $A[\overline{A}+AB]$

Numerical Answer Type (NAT) :

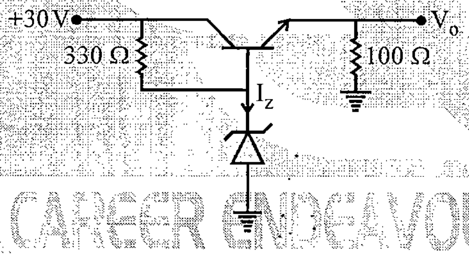
. For the circuit shown below, $V_{CB} = 0.5$ V and $\beta = 100$. The value of I_Q is __________mA.


+5 V
5 kΩ
V$_o$
I$_Q$
–5 V

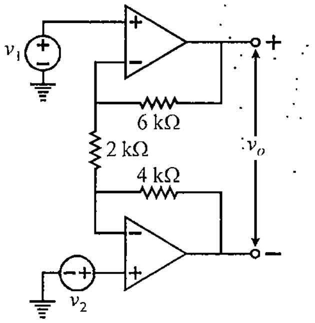
. In the voltage regulator circuit shown below the maximum load current i_L that can be drawn is ________ mA.


15 Ω
i_L
$V_z = 9$ V
$R_z = 0$
30 V
R_L

. For circuit shown in figure, the transistor has $\beta = 40$, $V_{BE} = 0.7$ V and voltage across zener diode is 15 V. The current (in mA) through the zener diode is ________________


+30 V
V$_o$
330 Ω
100 Ω
I$_z$

. For the circuit shown below the voltage gain $A_{vd} = \dfrac{v_o}{(v_1 - v_2)}$ is ______________


v_1
6 kΩ
2 kΩ
4 kΩ
v_o
v_2
+
–

Multiple Choice Questions (MCQ)

1. (b)	2. (d)	3. (d)	4. (a)	5. (a)
6. (b)	7. (b)	8. (a)	9. (b)	10. (a)
11. (a)	12. (c)	13. (b)	14. (b)	15. (d)
16. (a)	17. (a)	18. (a)	19. (b)	20. (d)
21. (a)	22. (c)	23. (c)	24. (b)	25. (b)
26. (a)	27. (d)	28. (c)	29. (a)	30. (a)
31. (c)	32. (b)	33. (b)	34. (c)	35. (b)
36. (d)	37. (a)	38. (d)	39. (a)	40. (c)
41. (c)	42. (b)	43. (a)	44. (d)	

Multiple Select Questions (MSQ)

1. (d)	2. (d)	3. (a),(c),(d)

Numerical Answer Type (NAT)

1. (0.85 – 0.95)	2. (1.3 to 1.5)	3. (40 to 43)	4. (6)

Chapter 6

DIGITAL ELECTRONICS

Boolean Algebra:

(1) $A+A=A$, (2) $A+1=1$, (3) $A+0=A$, additive identity

(4) $A+\overline{A}=1$, (5) $A.A=A$, (6) $A.1=A$ multiplicative identity

(7) $A.0=0$, (8) $A.\overline{A}=0$, (9) $\overline{\overline{A}}=A$,

(10) $A+A.B=A$, (11) $A(A+B)=A$ (12) $A.(\overline{A}+B)=AB$

(13) $A+\overline{A}B=A+B$, (14) $\overline{A}+AB=\overline{A}+B$, (15) $A(\overline{A}+B)=AB$

Note: $A+BC=(A+B)(A+C)$

Boolean Law:

) **Commutative Law:** $A+B=B+A, AB=B.A$

) **Associative Law:** $A+(B+C)=(A+B)+C$, $A.(B.C)=(A.B).C$

) **Distributive Law:** $A+B.C=(A+B)(A+C)$, $A.(B+C)=AB+AC$

) **Demorgan's Law:** $\overline{A.B}=\overline{A}+\overline{B}$,

$\overline{A+B}=\overline{A}.\overline{B}$

SOLVED PROBLEMS

The Boolean expression $(X+Y)(X+\overline{Y})(\overline{X}+Y)$ is equivalent to

(a) $\overline{X}Y$ (b) $X\overline{Y}$ (c) XY (d) $\overline{X}\overline{Y}$

oln. $(X+Y)(X+\overline{Y})(\overline{X}+Y)=(X+Y)(XY+\overline{X}\overline{Y}) = XY+XY=XY$

Given that $AB+\overline{A}C+BC=AB+\overline{A}C$, then $(\overline{A}+C)(B+C)(A+B)$ is equivalent to

(a) $(\overline{A}+B)(A+C)$ (b) $(A+B)(\overline{A}+C)$

(c) $(A+\overline{B})(\overline{A}+C)$ (d) $(A+\overline{B})(\overline{A}+\overline{C})$

oln. Using duality $(A+B)(\overline{A}+C)(B+C)=(A+B)(\overline{A}+C)$

Expression $A+\overline{A}B+\overline{A}\overline{B}C+\overline{A}\overline{B}\overline{C}D+\overline{A}\overline{B}\overline{C}\overline{D}E$ would be similar to

(a) $A+\overline{A}B+CD+E$ (b) $A+B+CDE$

(c) $A+BC+CD+DE$ (d) $A+B+C+D+E$

Soln. $F = A + \bar{A}B + \bar{A}\bar{B}C + \bar{A}\bar{B}\bar{C}(D + \bar{D}E)$

$= A + \bar{A}B + \bar{A}\bar{B}(C + \bar{C}(D+E)) = A + \bar{A}(B + \bar{B}(C + D + E)) = A + B + C + D + E$

Complement and Dual of Expression:

For calculating dual of a expression multiply is replaced by plus sign, while plus sign is replaced by multiply. After calculating dual primed variable are replaced by unpaired. Variable while unprimed by primed variables.

4. Calculate dual complement of given expression $Y(A,B,C) = A\bar{B}C + \bar{A}B\bar{C}$

$Y_D(A,B,C) = (A + \bar{B} + C)(\bar{A} + B + \bar{C})$, $\bar{Y}(A,B,C) = (\bar{A} + B + \bar{C})(A + \bar{B} + C)$

Minterms:

For n variables we have 2^n minterms. Here each minterm is obtained by from AND operation of n variables of n variables with each variable being primed if corresponding bit of binding number is zero unprimed if corresponding bit is 1. Sum of all minterms is always equal to 1.

Minterms for No. 110 is $AB\bar{C}$.

Maxterms:

It is nothing but simply complement of minterm and each minterm is complement of maxterm.

Maximum term for 011 is Maxterm for number 110 is $\bar{A} + \bar{B} + C$.

SOP → SUM OF PRODUCT, POS → PRODUCT OF SUM

Examples:

f(A, B, C) = $\sum$(0, 3, 5, 6, 7) .... SOP, f(A, B, C) = π(0, 3, 5, 6, 7) .... POS

5. f(A, B, C) = $\sum$(0, 5, 6) .... POS

Soln. $m_0 + m_5 + m_6 = \bar{A}\bar{B}\bar{C} + A\bar{B}C + AB\bar{C}$

6. f(A, B, C) = π(0, 2, 4, 7) ...... SOP = $M_0M_2M_4M_7$

$= (A + B + C)(A + \bar{B} + C)(\bar{A} + B + C)(\bar{A} + \bar{B} + \bar{C})$

Deriving sum of product (SOP) expression from the truth table

Derive a minterm expression for the following table

Input			**Output**
C	B	A	Y
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	0

$Y = \bar{A}\bar{B}C + \bar{A}BC + A\bar{B}\bar{C} + AB\bar{C}$

Deriving product of sum (POS) expression from a truth table

Derive a Maxterm expression for the following truth table

Input			**Output**
C	B	A	Y
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	0

$$Y=(A+B+C)(A+\overline{B}+C)(\overline{A}+B+C)(\overline{A}+\overline{B}+\overline{C})$$

Binary Number:

A decimal number such as 7392 represents a quantity equal to 7 thousands, plus 3 hundreds, plus 9 tens, plus 2 units. The thousands, hundreds etc. are power of 10 implied by the position of the coefficients. To be more exact, 7392 should be written as

$$7 \times 10^3 + 3 \times 10^2 + 9 \times 10 + 2 \times 10^6$$

In general, a number with a decimal point is represented by a series of coefficients as follows:

$$a_5 a_4 a_3 a_2 a_1 . a_{-1} a_{-2} a_{-3}$$

The decimal number system is said to be of base or radix, 10 because it uses ten digits and the coefficients are multiplied by the power of 10. The binary system is a different number system. The coefficients of the binary system have two possible values: 0 and 1 each coefficient multiplied by 2^J.

For example the decimal equivalent of the binary number 11010.11 is 26.75

$$1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 + 1 \times 2^{-1} \times 1 \times 2^{-2} = 26.75$$

In the hexadecimal (base 16) number system, the first term digits are borrowed from the decimal system. The letters A, B, C, D, E and F are used for digits 10, 11, 12, 13, 14 and 15 respectively. An example of a hexadecimal number is

$$(B65F)_{16} = 11 \times 16^3 + 6 \times 16^2 + 5 \times 16 + 15 = (46687)_{10}$$

Number base conversion:

A binary number can be converted to decimal by forming the sum of the powers of 2 of those coefficients whose value is 1. For example

$$(1010.011)_2 = 2^3 + 2^1 + 2^{-2} + 2^{-3} = (10.375)_{10}$$

The conversion from decimal to binary or to any other base-*r* system is more convenient if the number is separated into an integer part and fraction part and the conversion of each part done separately. The conversion of an integer from decimal to binary is best explained by example.

Integer quotient	**Remainder**	**Coefficient**
$\frac{41}{2}=20$	$+\frac{1}{2}$	$a_0=1$
$\frac{20}{2}=10$	$+0$	$a_1=0$
$\frac{10}{2}=5$	$+0$	$a_2=0$

$\frac{5}{2}=2$	$+\frac{1}{2}$	$a_3=1$
$\frac{2}{2}=1$	$+0$	$a_4=0$
$\frac{1}{2}=0$	$+\frac{1}{2}$	$a_5=1$

Answer $(41)_{10} = (a_5a_4a_3a_2a_1a_0)_2 = (101001)_2$

The arithmetic process can be manipulated more conveniently as follows:

Integer	**Remainder**
41	
20	1
10	0
5	0
2	1
1	0
0	1

Answer: 101001

The conversion from decimal integers to any base-*r* system is similar to the example, except that division is done by *r* instead of 2.

The conversion of a decimal fraction to binary is accomplished by a method similar to that used for integers. However, multiplication is used instead of division and integers are accumulated instead of remainders.

7. Convert $(0.6875)_{10}$ to binary.

	Integer	**Fraction**	**Coefficient**
Soln.	$0.6875 \times 2 = 1+$	0.3750	$a_{-1} = 1$
	$0.3750 \times 2 = 0+$	0.7500	$a_{-2} = 0$
	$0.7500 \times 2 = 1+$	0.5000	$a_{-3} = 1$
	$0.5000 \times 2 = 1+$	0.0000	$a_{-4} = 1$

$(0.6875)_{10} = (0.a_{-1}\ a_{-2}\ a_{-3}\ a_{-4}) = (0.1011)_2$

8. Convert $(0.513)_{10}$ to octal

Soln. $0.513 \times 8 = 4.104$

$0.104 \times 8 = 0.832$

$0.832 \times 8 = 6.656$

$0.656 \times 8 = 5.248$

$0.248 \times 8 = 1.984$

$0.984 \times 8 = 7.872$

The answer to seven significant figures is obtained from the integer part of the product.

$(0.513)_{10} = (0.406517....)_8$

The conversion of decimal number with both integer and fraction parts is done by converting the integer and fraction separately and then combining the two answers.

Octal and hexadecimal numbers:

The conversion from the binary to octal and hexadecimal plays an important part in digital computers. Since $2^3 = 8$ and $2^4 = 16$, each octal digit corresponds to three binary digits and each hexagonal digit corresponds to four binary digits. The conversion from binary to octal is easily accomplished by partitioning the binary number into groups of three digits each, starting from the binary point and proceeding to the left and to the right. The corresponding octal digit is then assigned to each group. The following examples illustrates the procedure.

$(\underbrace{10}\ \underbrace{110}\ \underbrace{001}\ \underbrace{101}\ \underbrace{011}\ \underbrace{111}\ \underbrace{100}\ \underbrace{000}\ \underbrace{110})_2 = (26153.7460)_8$

Conversion from binary to hexadecimal is similar, except that the binary number is divided into groups of four digits:

$(\underbrace{10}_{2}\ \underbrace{1100}_{C}\ \underbrace{0110}_{6}\ \underbrace{1011}_{B}\ \underbrace{1111}_{F}\ \underbrace{0010}_{2})_2 = (2C6B.F2)_{16}$

Conversion from octal or hexadecimal to binary is done by a procedure reverse to the above. Each octal digit is converted to its three digit binary equivalent. Similarly each hexadecimal digit is converted to its four-digit binary equivalent. This is illustrated in the following examples:

$(673.124)_8 = (\underbrace{110}_{6}\ \underbrace{111}_{7}\ \underbrace{011}_{3}\ \underbrace{001}_{1}\ \underbrace{010}_{2}\ \underbrace{100}_{4})_2$

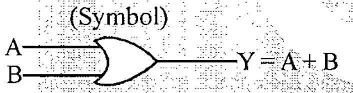
$(306.D)_{16} = (\underbrace{0011}_{3}\ \underbrace{0000}_{0}\ \underbrace{0110}_{6}\ \underbrace{1101}_{D})_2$

Logic Gates: Positive Logic: logic 0 = 0V, logic 1 = 5V

Negative Logic: logic 1 = 0V, logic 0 = 5V.

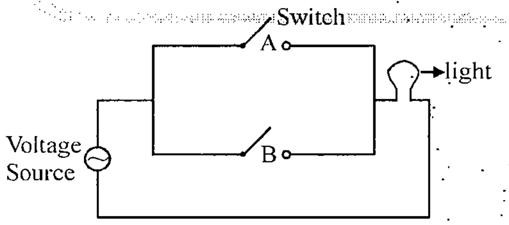
Logic-Gates: They are the most basic and most important building blocks of any digital system including computers. Three logic gates are AND, OR and NOT gates. These three gates together can be used to construct the logic circuit for any given logic or boolean expression.

OR Gates: $\boxed{Y = A + B}$



It is a logic circuit with 2 or more than two inputs and one output.

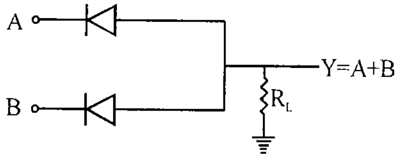
Switching Circuit of OR Gate:



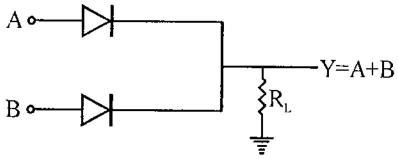
Truth Table for OR gate:

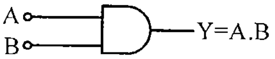
A	B	$Y = A + B$
0	0	0
0	1	1
1	0	1
1	1	1

(1) Negative Logic OR Gate:



(2) Positive Logic OR Gate:

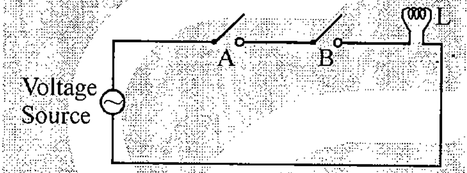


AND Gate: $\boxed{Y = A.B}$ 

{Symbol of AND Gate}

It is a logic circuit having two or more than two input and one output.

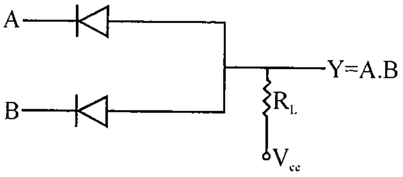
Switching Circuit of AND Gate:


A
B
L
Voltage Source

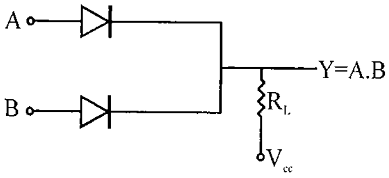
Truth Table:

A	B	$Y = A.B$
0	0	0
0	1	0
1	0	0
1	1	1

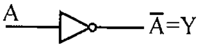
(1) Positive logic AND gate:


A
B
Y=A.B
R_L
V_{cc}

(2) Negative logic AND gate:


A
B
Y=A.B
R_L
V_{cc}

Note: Negative logic OR gate is same as positive logic AND gate and vice-versa while positive logic OR gate is same as negative logic AND gate and vice-versa.

NOT Gate: $\boxed{Y = \bar{A}}$ 

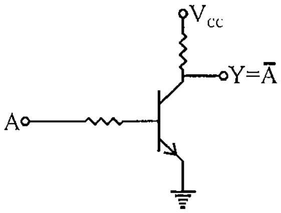
(Symbol of NOT Gate)

It is one input and one output logic gate whose output is always the complement of the Input.

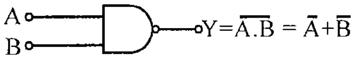
Truth Table:

A	$Y=\bar{A}$
0	1
1	0

Switch Circuit of NOT Gate:



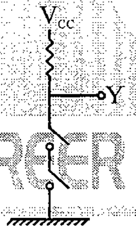
NAND Gate: NAND stands for NOT-AND. An AND gate followed by a NOT circuit makes it a NAND gate. The truth table of a NAND gate is obtained from truth table of an AND gate by complementing the output entries.


A
B
$Y=\overline{A.B} = \bar{A}+\bar{B}$

Input		**Output**
A	B	NAND
0	0	1
0	1	1
1	0	1
1	1	0

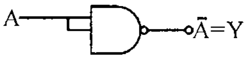
When both inputs are high in a NAND gate the output is low. NAND gate gives zero output only in high input condition.

Switching Circuit of NAND Gate:

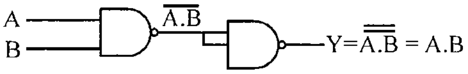


he NAND gate is called an universal gates as all functions OR, AND, NOT can be realized from t

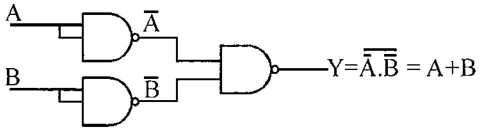
OT:



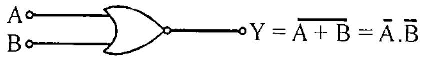
ND:


A
B
$\overline{A.B}$
$Y=\overline{\overline{A.B}} = A.B$

OR:


A
B
$\bar{A}$
$\bar{B}$
$Y=\overline{\bar{A}.\bar{B}} = A+B$

NOR Gate: NOR stands for NOT-OR. Thus the NOR Gate output is obtained by complementing the output of OR gate. Thus output of NOR gate logic 1 when all its input are logic 0. For all other cases the output is logic 0. NOR gate also called as universal gate because all the gates can be designed by this gate.


A
B
Y = $\overline{A+B}$ = $\bar{A}.\bar{B}$

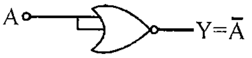
Input		**Output**
A	B	NOR
0	0	1
0	1	0
1	0	0
1	1	0

Output of NOR will gate be high when all of its input are low.

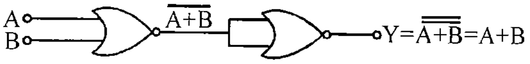
NOR gates as universal Gate:

OR, NOT, AND can be derived using only NOR gate.

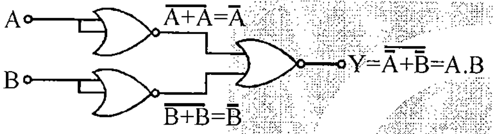
NOT:



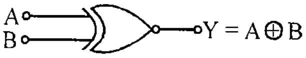
OR:


A
B
$\overline{A+B}$
Y=$\overline{\overline{A+B}}$=A+B

AND:


A
B
$\overline{A+A}$=$\bar{A}$
$\overline{B+B}$=$\bar{B}$
Y=$\overline{\bar{A}+\bar{B}}$=A.B

XOR/EX-OR Gate: The output of 2 input EX-OR gate is logic 1 when the two inputs are unequal and output is logic 0 when two inputs are equal (Odd function)



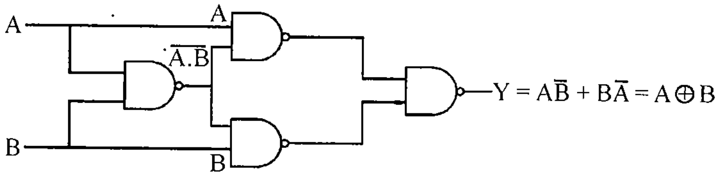
Truth Table of EX-OR Gate:

Input		**Output**
A	B	XOR
0	0	0
0	1	1
1	0	1
1	1	0

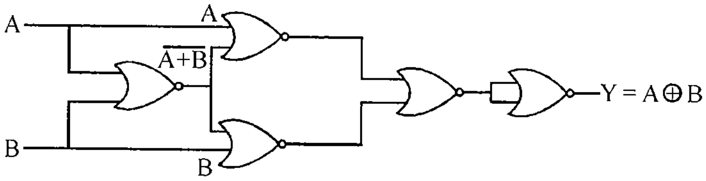
The symbol ⊕ means the terms are XORed together. $Y = A\bar{B} + \bar{A}B$ is boolean expression for EX-OR.

Implementation of EX-OR gate with 2 input NAND and NOR gate:

Using NAND Gate:


A
A
$\overline{A.B}$
B
B
Y = A$\bar{B}$ + B$\bar{A}$ = A⊕B

Using NOR Gate:

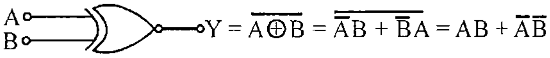

A
A
$\overline{A+B}$
B
B
Y = A⊕B

Application of EX-OR/XOR Gate:

1. XOR Gate can be used as an inequality comparator.
2. It can also be used as controlled inverter.
3. It can be used for complementing in adder circuit.
4. XOR is an important logic gate used in arithmetic operations, code conversion, and in error detector and correction circuit.

EX-NOR Gate: The output of the XOR Gate is connected to an inverter and the output taken.

Example:

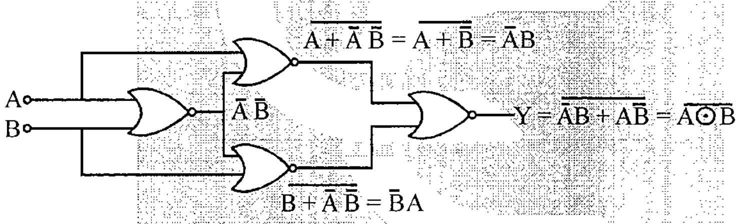

A
B
$Y = \overline{A \oplus B} = \overline{\bar{A}B + \bar{B}A} = AB + \bar{A}\bar{B}$

Truth Table:

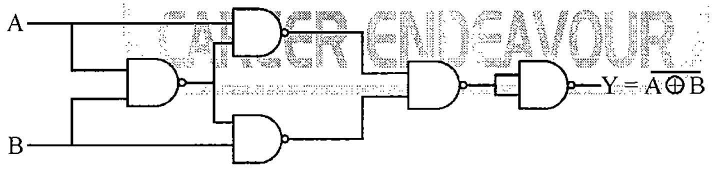
Input		**Output**
A	B	Y
0	0	1
0	1	0
1	0	0
1	1	1

So, If XOR is an odd number detector then XNOR is an even number detector.

Implementation of EX-NOR gate with 2 input NOR and NAND gate.


$\overline{A + \bar{A}\bar{B}} = \overline{A + \bar{\bar{B}}} = \bar{A}B$
A
B
$\bar{A}\bar{B}$
$Y = \overline{\bar{A}B + A\bar{B}} = A \odot B$
$\overline{B + \bar{A}\bar{B}} = \bar{B}A$

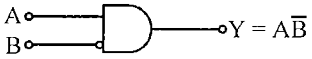
Using NOR Gate

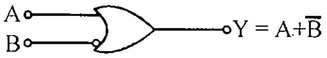

A
B
$Y = \overline{A \oplus B}$

Using NAND Gate

Logic gate	*No. of NAND gate used*	*No. of NOR gate used*
NOT	1	1
AND	2	3
OR	3	2
EX – OR	4	5
EX – NOR	5	4
NOR	4	1
NAND	1	4

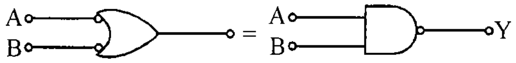
Inhibitor: If one of the inputs to an AND or OR gate is inverted it becomes an INHIBITOR. An inhibitor is also known as AND-NOT or OR-NOT gate.



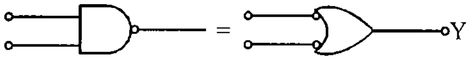


If we add on inverter at the output only the outputs get inverted. How ever, if we invert the input, the gate functions change.

1. If both the inputs to one OR gate are inverted, the gate becomes a NAND gate.



2. If both the inputs to an NAND gate are inverted, the gate becomes a NOR gate. If we invert both inputs and outputs.



If we invert both inputs and outputs.

Design of a Boolean expression by minimum number of NAND gate.

1st Step: Write boolean expression is form of AND and OR gate.

2nd Step: Place a bubble at the output of each AND gate and place a bubble at each input of OR gate.

3rd Step: NOT followed by NOT means original literal.

Demorgans Theorem:

$\overline{ABC} = \overline{A} + \overline{B} + \overline{C}$, $\overline{A+B+C} = \overline{A}.\overline{B}.\overline{C}$

1. Minimise logic expression

$(A + B)(A + C) \rightarrow A + BC$

Note: A is present in both the bracket

2. $(\overline{X} + Y)(\overline{X} + Z) \Rightarrow \overline{X} + YZ$

Note: $\overline{X}$ is present in both the bracket

3. $(A + B + C)(A + \overline{B} + C)(A + B + \overline{C})$

$\Rightarrow (A + B)(A + \overline{B} + C)$

$\Rightarrow A + B(\overline{B} + C)$

$\Rightarrow A + BC$

Concensns Theorem:

$AB + \overline{A}C + \enclose{circle}{BC} \rightarrow$ redundant term

$\Rightarrow AB + \overline{A}C + BC(A + \overline{A})$

$\Rightarrow AB + \overline{A}C + BCA + BC\overline{A}$

$\Rightarrow AB + \overline{A}C$

Note: (1) Three variable

(2) Each variable is used twice as pair

(3) Only one variable is complemented

$\overline{A}\overline{B}+\overline{B}\overline{C}+A\overline{C} \Rightarrow \overline{A}\overline{B}+A\overline{C}$

Transposition Theorem:

$(A+B)(\overline{A}+C)$

$\Rightarrow AC+\overline{A}B+BC$

$\Rightarrow AC+\overline{A}B$

SOLVED PROBLEMS

. Reduce $AB+ABC+\overline{A}B+A\overline{B}C$ using law's of Boolean algebra **[BHU 2012]**

(a) $B+AC$ (b) $A+BC$ (c) $C+AB$ (d) $A+B+C$

oln. $AB+ABC+\overline{A}B+A\overline{B}C$

$\Rightarrow \left(A+\overline{A}\right)B+AC\left(B+\overline{B}\right)$

$=B+AC \quad \left[A+\overline{A}=1\right]$

Correct option is (a)

. 10101 binary number corresponds to decimal number **[BHU 2012]**

(a) 31 (b) 21 (c) 11 (d) 3

oln. 10101

$=1\times2^4+0\times2^3+1\times2^2+0\times2^1+1\times2^0=16+4+1=21$

Correct option is (b)

. The unique output for a NAND logic gate is a 0 ; **[BHU 2014]**

(a) when all inputs are 0 (b) when all inputs are 1

(c) when any one input is 0 (d) when any one input is 1

oln. $y=\overline{A\cdot B}$

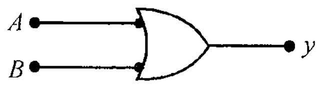
A	B	y
1	1	0
0	1	1
1	0	1
0	0	1

Therefore, correct option is (b)

. Bubbled (inputs inverted) OR gate is equivalent to **[BHU 2015]**

(a) NOR gate (b) NAND gate (c) NOT gate (d) XNOR gate

oln.



$y=\overline{A}+\overline{B}=\overline{AB}$ it is the output of NAND gate

Correct option is (b)

5. The simplified Boolean expression for the given Boolean expression $AB+(A+B)\cdot(B+C)$ is

(a) $B+AC$ (b) $C+AB$ (c) $A+BC$ (d) $AB+AC$ **[BHU 2015]**

Soln. $y=AB+(A+B)\cdot(B+C)$

$$=AB+B+AC$$
$$=(A+1)B+AC$$
$$=B+AC$$

Correct option is (a)

6. The Boolean expression $AB\overline{C}+BC\overline{A}+CA\overline{B}+ABC$ is equivalent to the simplified Boolean expression in POS form **[BHU 2015]**

(a) $(A+B)\cdot(B+C)\cdot(C+A)$ (b) $(A+\overline{B})\cdot(B+\overline{C})\cdot(C+\overline{A})$

(c) $(\overline{A}+\overline{B})\cdot(\overline{B}+\overline{C})\cdot(\overline{C}+\overline{A})$ (d) $(\overline{A}+B)\cdot(\overline{B}+C)\cdot(\overline{C}+A)$

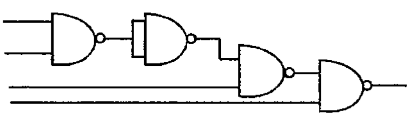
Soln. $AB\overline{C}+BC\overline{A}+CA\overline{B}+ABC$

$$=AB\overline{C}+BC\overline{A}+CA\overline{B}+ABC+ABC+ABC$$
$$=AB\left(\overline{C}+C\right)+BC\overline{A}+CA\overline{B}+ABC+ABC$$
$$=AB+BC\overline{A}+CA\overline{B}+ABC+ABC$$
$$=AB+BC+CA$$
$$=AB\left(C+1\right)+BC+CA$$
$$=ABC+AB+BC+CA$$
$$=ABC+ABC+AB+AB+BC+CA+CA$$
$$=(A+B)\cdot(B+C)\cdot(C+A)$$

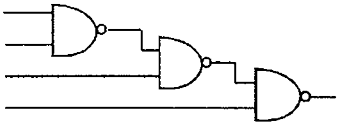
Correct option is (a)

7. Which of the following circuits will act like a 4-input NAND gate? **[JEST]**

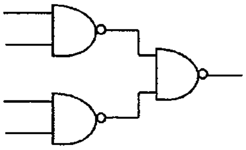
(a)



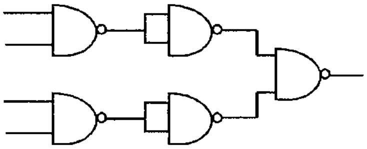
(b)



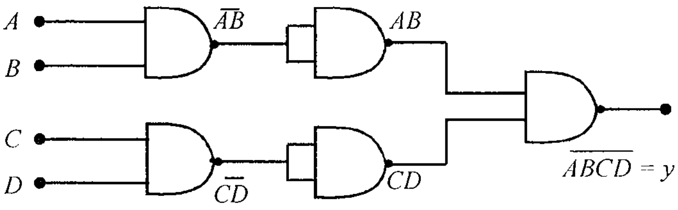
(c)



(d)

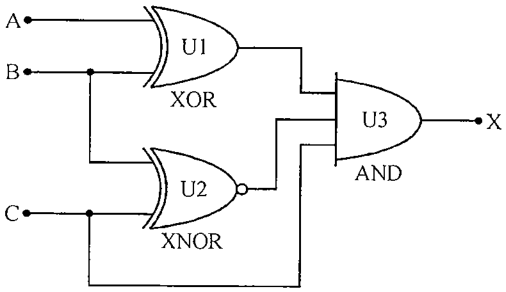


Soln.



Correct option is (d)

. For the logic circuit shown in figure 4, the required input condition (A, B, C) to make the output (X) = 1 is,

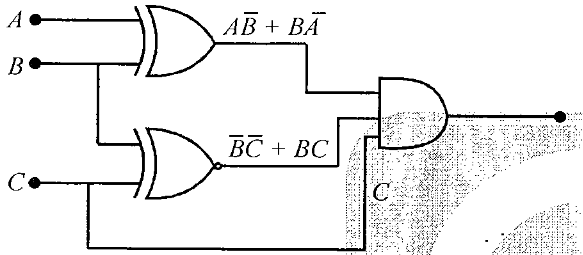

A
B
C
U1
XOR
U2
XNOR
U3
AND
X

[JEST 2015]

Figure 4

(a) 1, 0, 1 (b) 0, 0, 1 (c) 1, 1, 1 (d) 0, 1, 1

oln.


A
B
C
$A\bar{B}+B\bar{A}$
$\bar{B}\bar{C}+BC$
C

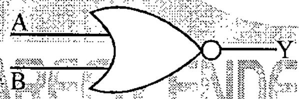
$$X=\left(\bar{A}B+\bar{B}A\right)\cdot\left(\bar{B}\bar{C}+BC\right)\cdot C$$

$$=\left(\bar{A}B+\bar{B}A\right)\left(BC\right)=\bar{A}BC$$

therefore, to get output X = 1 input will be 0, 1, 1

Correct option is (d)

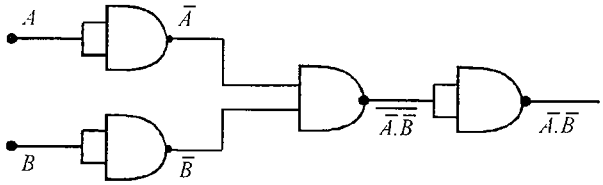
Consider the circuit shown below. **[TIFR 2012]**



The minimum number of NAND gates required to design this circuit is

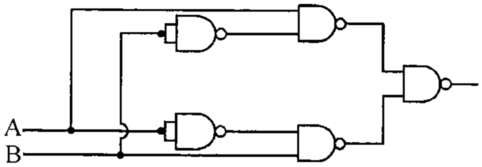
(a) 6 (b) 5 (c) 4 (d) 3

ln. $y=\overline{A+B}=\bar{A}\cdot\bar{B}$


A
$\bar{A}$
B
$\bar{B}$
$\overline{\bar{A}.\bar{B}}$
$\bar{A}.\bar{B}$

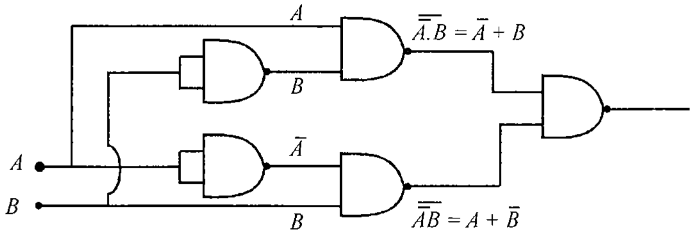
Correct option is (c)

). The circuit shown below uses only NAND gates. Find the final output. **[TIFR 2013]**



(a) A XOR B (b) A OR B (c) A AND B (d) A NOR B

Soln.


A
$\overline{\bar{A}.B} = \bar{A} + B$
B
$\bar{A}$
A
B
B
$\overline{A\bar{B}} = A + \bar{B}$

$$\overline{(\bar{A}+B)\cdot(A+\bar{B})} = \overline{\bar{A}+B} + \overline{A+\bar{B}} = A\bar{B} + \bar{A}B = A\ XOR\ B$$

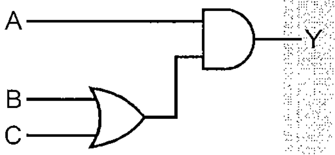
Correct option is (a)

11. In a digital circuit for three input signals (A, B and C) the final output (Y) should be such that for inputs

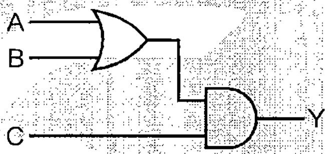
A	B	C
0	0	0
0	0	1
0	1	0

the output (Y) should be low and for all other cases it should be high. Which of the following digital circuits will give such output ? **[TIFR 2016]**

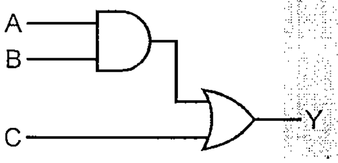
(a)



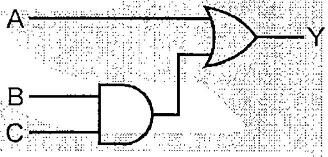
(b)



(c)



(d)

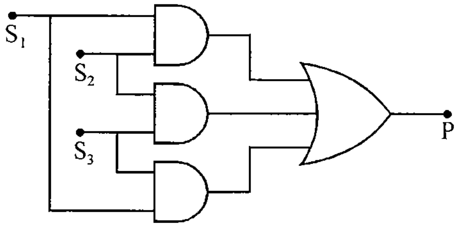


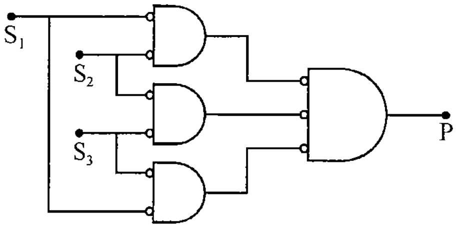
Soln.

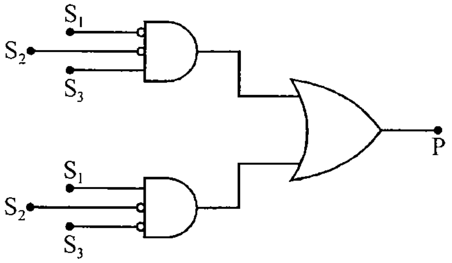
A	B	C	y
0	0	0	0
0	0	1	0
0	1	0	0
1	0	0	1

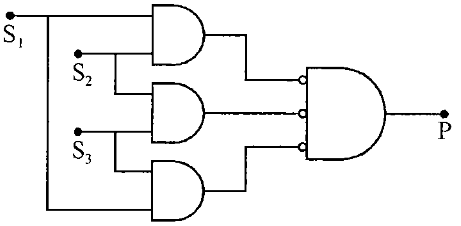
Only option (d) will give such output.
Correct option is (d)

12. A building has three overhead water tanks, each fitted with a sensor (S_1, S_2, S_3) which goes to 0 when the water level in the tank falls below a set value and remains 1 otherwise. A common pump is used to raise water from an underground storage tank to these overhead tanks. Of the following circuits, which one will turn on (P = 1) the pump only when at least two of the tanks have water level below the set value ? **[TIFR 2015]**

(a) 

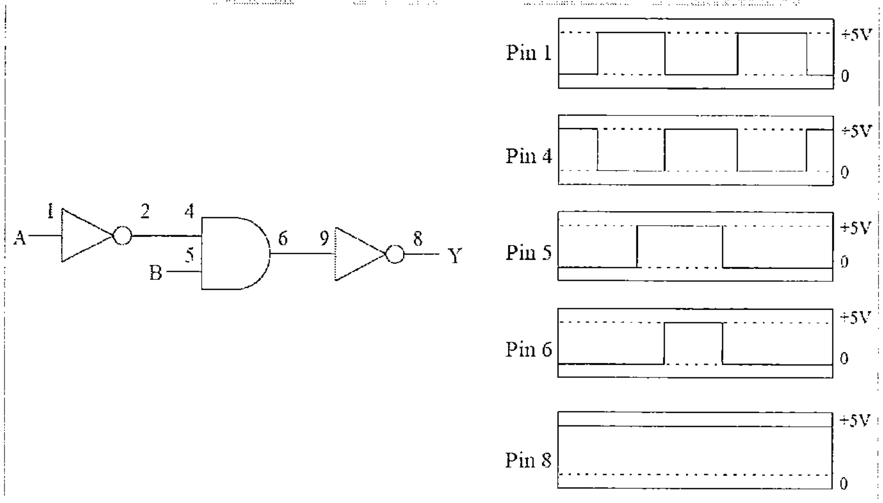
(b) 

(c) 

(d) 

oln. According to given condition, the output P will be 1 when any two input will be zero. Only option (d) satisfy the given condition.

3. The digital electronic circuit shown below (left side) has some problem and is not performing as intended. The voltage at each pin as a function of time is shown in the adjacent figures. **[TIFR 2011]**



The problem in the about circuit may be that

(a) the Pin 6 is shorted to ground (b) the input inverter is shorted

(c) the Pin 8 is clamped to +5 V (d) OR gate is used instead of AND gate

oln. ⇒ Pin–1 is like input

⇒ Pin–2 = Pin–4

⇒ $\text{Pin}-2=\overline{\text{Pin}-1}$

⇒ Inverter

⇒ here, Pin–4 = Pin–2

⇒ Pin–4 is inverted output of Pin–1, which is correct.

⇒ Pin–6 will be high when both Pin–4 and Pin–5 will be high, which is correct.

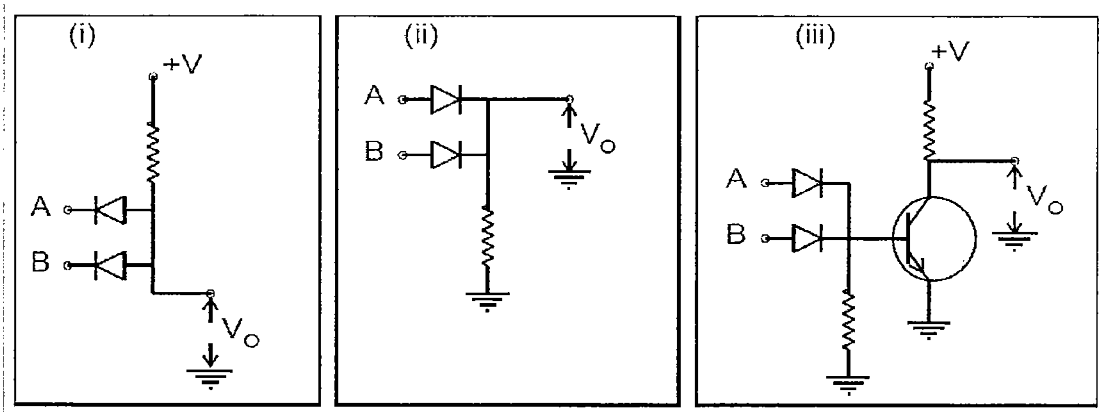
$\text{Pin}-8=\overline{\text{Pin}-9}=\overline{\text{Pin}-6}$

⇒ Pin–8 is having incorrect output

⇒ Problem is Pin–8 is clamped to +5V/high

Correct option is (c)

14. The three electronic circuits marked (i), (ii) and (iii) in the figure below can all work as logic gates, where the input signals are either 0V or 5 V and the output is V_0. **[TIFR 2011]**


(i)
+V
A
B
V_O
(ii)
A
B
V_O
(iii)
+V
A
B
V_O

Identify the correct combination of logic gates (i), (ii), (iii), in the options given below.

(a) NOR, XOR, AND

(b) OR, NAND, NOR

(c) NAND, AND, XOR

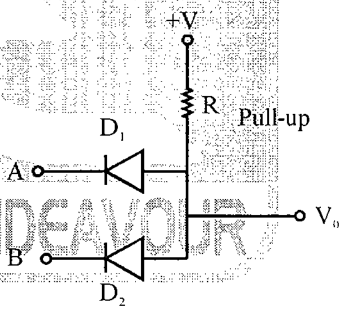
(d) XOR, AND, NAND

(e) AND, OR, NOR

(f) NOR, NAND, OR

Soln.

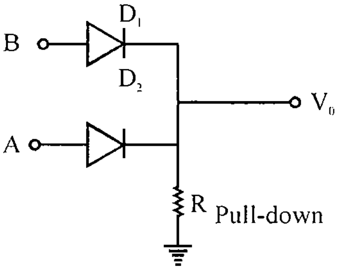
A	B	D_1	D_2	V_0
0	0	ON	ON	0
0	1	ON	OFF	0
1	0	OFF	ON	0
1	1	OFF	OFF	1

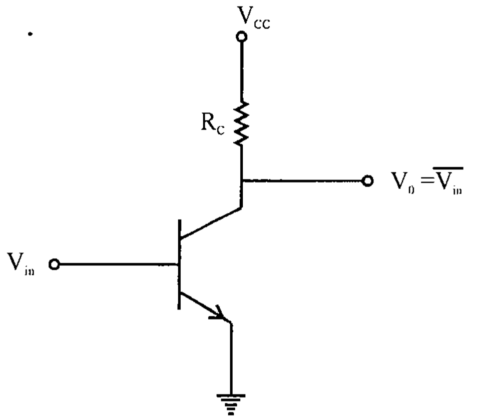

+V
R
Pull-up
D_1
A
V_0
B
D_2

Hence, it is AND gate

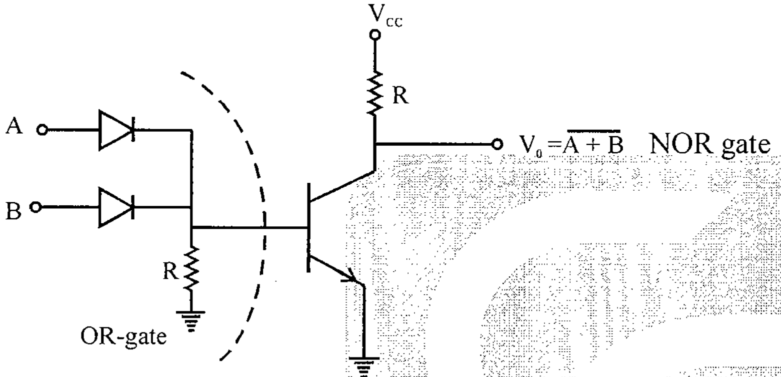
A	B	D_1	D_2	V_0
0	0	OFF	OFF	0
0	1	OFF	ON	1
1	0	ON	OFF	1
1	1	ON	ON	1

hence, OR gate


D_1
B
D_2
V_0
A
R
Pull-down


V_{cc}
R_C
$V_0 = \overline{V_{in}}$
V_{in}

V_{in}	V_0
0	1
1	0


V_{cc}
R
A
B
R
OR-gate
$V_0 = \overline{A+B}$
NOR gate

Correct option is (e)

5. To measure the voltage in the range 0 – 5 V with a precision of 5 mV, the minimum number of bits required in a digital voltmeter is **[TIFR 2015]**

(a) 9 (b) 10 (c) 11 (d) 12

oln. S = step size = 5 mA

$$S = \frac{V_H - V_L}{2^n}$$

Where, n = number of bits

V_H = high voltage value

V_L = low voltage value

$$5\text{mV} = \frac{5-0}{2^n} \Rightarrow 2^n = 1000 \Rightarrow \boxed{n = 10}$$

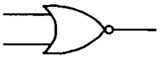
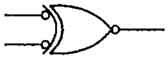
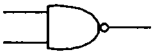
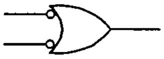
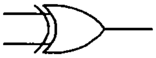
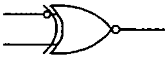
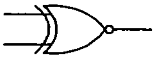
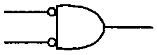
Since, n is number of bits (integer)

Correct option is (b)

PRACTICE SET

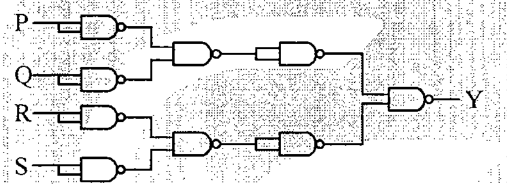
Multiple Choice Questions (MCQ):

1. Match the logic gates in Column A with their equivalents in Column B.

Column - A	Column - B
P. 	1. 
Q. 	2. 
R. 	3. 
S. 	4. 

(a) P-2, Q-4, R-1, S-3 (b) P-4, Q-2, R-1, S-3
(c) P-2, Q-4, R-3, S-1 (d) P-4, Q-2, R-3, S-1

2. For the circuit shown in Fig. the boolean expression for the output Y in terms of inputs, P, Q, R and S is

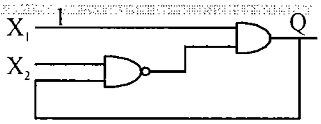


(a) $\overline{P}+\overline{Q}+\overline{R}+\overline{S}$ (b) $P+Q+R+S$ (c) $(\overline{P}+\overline{Q})(\overline{R}+\overline{S})$ (d) $(P+Q)(R+S)$

3. The minimum number of NOR gates required to implement $A(A+\overline{B})(A+\overline{B}+C)$ is equal to
(a) 0 (b) 3 (c) 4 (d) 7

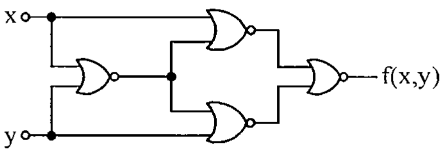
4. If $(211)_x = (152)_8$, then the value of base x is
(a) 6 (b) 5 (c) 7 (d) 9

5. In the figure, as long $X_1 = 1$ and $X_2 = 1$, the output Q remains



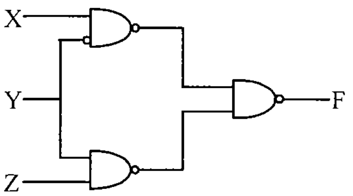
(a) at 1 (b) at 0 (c) at its initial value (d) at unstable

6. Identify the logic function performed by the circuit shown in Fig.



(a) exclusive OR (b) exclusive NOR (c) NAND (d) NOR

. Consider the following circuit.



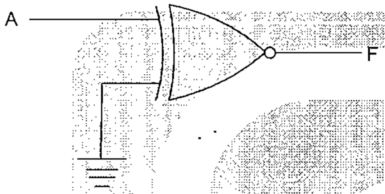
Which one of the following is **TRUE**?

(a) F is independent of X
(b) F is independent of Y
(c) F is independent of Z
(d) None of X, Y, Z is redundant

. Boolean expression for the ouput of XNOR (Equivalent) logic gate with inputs A and B

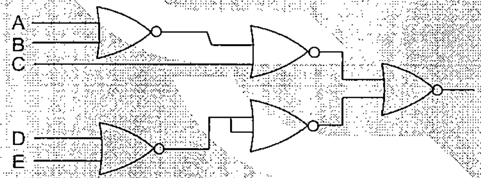
(a) $A\bar{B} + \bar{A}B$ (b) $\overline{AB} + AB$ (c) $(\bar{A} + B)(A + \bar{B})$ (d) $(\bar{A} + B)(A + B)$

. The ouput of the logic gate in fig. is



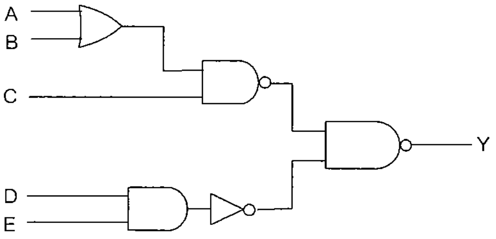
(a) 0 (b) 1 (c) $\bar{A}$ (d) A

0. The circuit shown in figure realizes the function.



(a) $A + B + C\ (D\bar{E})$
(b) $\left[\bar{A} \cdot \bar{B} + C\right]\bar{D} \cdot \bar{E}$
(c) $(A+B+C)(\bar{D}\bar{E})$
(d) $(A+B+C)\ (\bar{D}\bar{E})$

1. The ouput Y_0 the circuit shown in the figure is



(a) (A + B) C + DE
(b) AB + C (D + E)
(c) (A + B) C + D + E,
(d) (AB + C) DE

2. The output of a logic gate is '1' when all its inputs are at logic 'O' Then the gate is either
(a) a NAND or an Ex-OR gate
(b) a NOR or an EX-NOR gate
(c) an OR or an EX-NOR gate
(d) an AND or an EX-OR gate

13. Match List - I (Boolean statement/ Statements is true) with List-II (Conditions under which) and select the corrent answer using the codes given below the lists:

List - I	**List - II**
A. $A \oplus B = 0$	1. $A \neq B$
B. $\overline{A + B} = 0$	2. $A = B$
C. $\overline{A} . B = 0$	3. $A = 0$ and $B = 1$
D. $A \oplus B = 1$	4. $A = 1$ and $B = 0$

(a) A - 2, B - 3, C - 4, D - 1 (b) A - 1, B - 3, C - 4, D - 2
(c) A - 2, B - 4, C - 3, D - 1 (d) A - 1, B - 4, C - 3, D - 2

14. The minimized form of the logical expression
$(\overline{A}\overline{B}\overline{C} + \overline{A}B\overline{C} + \overline{A}BC + AB\overline{C})$ is

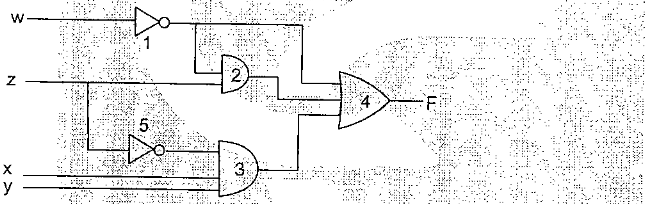
(a) $\overline{A}\overline{C} + B\overline{C} + \overline{A}B$ (b) $A\overline{C} + \overline{B}C + \overline{A}B$ (c) $\overline{A}C + \overline{B}C + \overline{A}B$ (d) $A\overline{C} + \overline{B}C + A\overline{B}$

15. For the identity, $AB + \overline{A}C + BC = AB + \overline{A}C$. the dual form is

(a) $(A + B)(\overline{A}+C)(B+C) = (A+B)(\overline{A}+C)$ (b) $(A + B)(A+\overline{C})(\overline{B}+\overline{C}) = (\overline{A}+\overline{B})(A+\overline{C})$

(c) $(A + B)(\overline{A}+C)(B+C) = (\overline{A}+\overline{B})(A+\overline{C})$ (d) $\overline{A}\overline{B} + A\overline{C} + \overline{B}\overline{C} = \overline{A}\overline{B} + A\overline{C}$

16. Consider the following gate network.



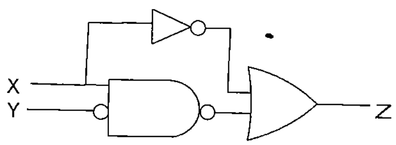
Which one of the following gates is redundant?
(a) Gate No. 1 (b) Gate No. 2 (c) Gate No. 3 (d) Gate No. 4

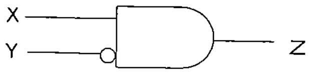
17. How is inversion achieved using EX-OR gate?
(a) Giving input signal to the two input lines of the gate tied together
(b) Giving input to one input line and logic zero to the other line
(c) Giving input to one input line and logic one to the other line
(d) Inversion cannot be achieved using EX-OR gate

18. The logic funciton $f = \overline{(x.\overline{y}) + (\overline{x} + y)}$ is the same as

(a) $f = 0$ (b) $f = \overline{(\overline{x}.\overline{y}) + (x.y)}$ (c) $f = \overline{(x.y)}\,(\overline{x}.\overline{y})$ (d) $f = \overline{x}.\overline{y} + xy$

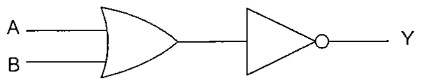
19. The logic operation of two combinational circuits shown in the figures below are





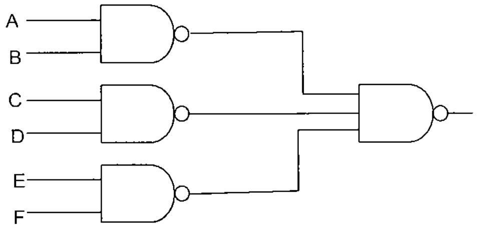
(a) Complementary (b) identical (c) Entirely different (d) None of these

0. For the circuit shown, Y is given by



(a) $\bar{A}+\bar{B}$ (b) $\bar{A}\bar{B}$ (c) $\overline{AB}$ (d) none of these

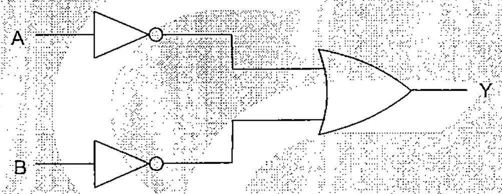
1. The output of the gated network shown in the given figure is



(a) $(\overline{AB})(\overline{CD})(\overline{EF})$ (b) $\overline{AB}+\overline{CD}+\overline{EF}$

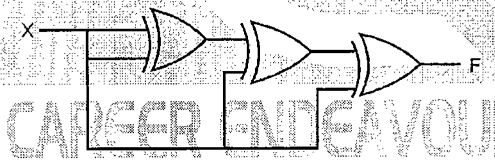
(c) $AB+CD+EF$ (d) $(A+B(C+D)(E+F)$

2. In the circuit shown, the input word is AB and ouput is Y. The value of input for Y = 0 is.



(a) 00 (b) 01 (c) 10 (d) 11

3. For the circuit shown below, output F is



(a) F = 1 (b) F = 0 (c) F = X (d) F = $\bar{X}$

4. The addition of two binary variables A and B results into a SUM and a CARRY ouput. Consider the following expressions for the SUM and CARRY ouput:

1. SUM = $A.B+\bar{A}\bar{B}$
2. SUM = $A.\bar{B}+\bar{A}.B$
3. CARRY = A.B
4. CARRY = A + B

Which of these expressions are correct?

(a) 1 and 3 (b) 2 and 3 (c) 2 and 4 (d) 1 and 4

5 Which one of the following is equivalent to AND - OR realization?

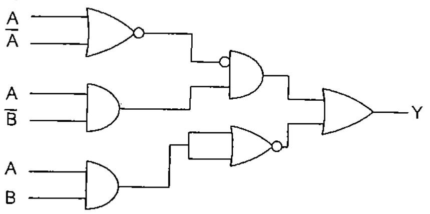
(a) NAND-NOR realizaiton

(b) NOR-NOR realizaiton

(c) NOR-NAND realization

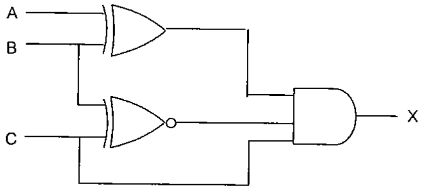
(d) NAND-NAND realization

26. The output Y for the logic circuit shown in the given figure is



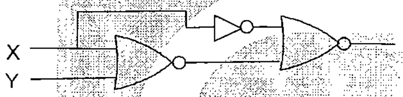
(a) $A\overline{B}$ (b) $\overline{A}+\overline{B}$ (c) $\overline{A}\overline{B}$ (d) $A+\overline{B}$

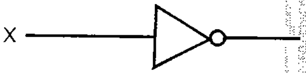
27. For the logic circuit shown in fig. the required input condition (A,B,C) to make the ouput (X) = 1 is

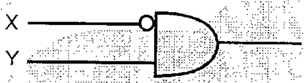


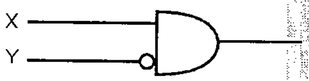
(a) 1, 0, 1 (b) 0, 0, 1 (c) 1, 1, 1 (d) 0, 1, 1

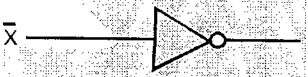
28. The logic circuit shown in the given figure can be minimized to



(a)


(b)


(c)


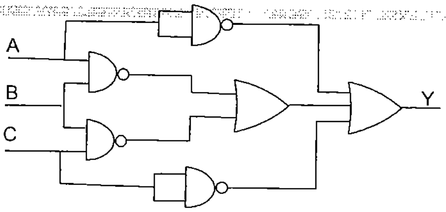
(d)


29. Which one of following sets of gates are best suited for parity checking and parity generation?

(a) AND, OR, NOT gates (b) X-OR X-NOR gate

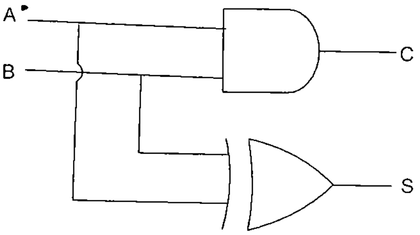
(c) NAND gates (d) NOR gates

30. For the logic circuit shown in the figure, the ouput Y is equal to



(a) $\overline{A\overline{B}C}$ (b) $\overline{A}+\overline{B}+\overline{C}$ (c) $\overline{A}\overline{B}+\overline{BC}$ (d) $\overline{A}\overline{B}+\overline{B}$

31. For the circuit shown, for AB = 00, AB = 01 C, S values respectively are



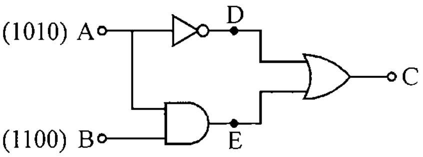
(a) 0,0, and 0.1 (b) 0,0 and 1.0 (c) 0.1 and 0.0 (d) 1.0 and 0.0

Multiple Select Questions (MSQ) :

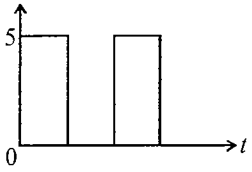
Find out the correct equivalence of the expression $A \oplus B$.

(a) $\overline{A}B + A\overline{B}$ (b) $AB + \overline{A}\overline{B}$ (c) $\overline{\overline{A+B} + \overline{\overline{A}+\overline{B}}}$ (d) $A\overline{B} + \overline{A}B$

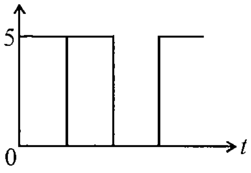
In the circuit shown below find out the correct pulse pattern. Assume 1 as 5 volt and 0 at 0 volt.



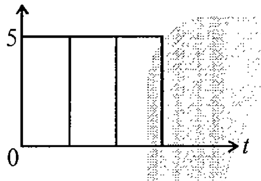
(a) At point A,



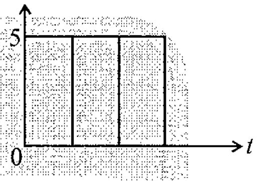
(b) At point C,



(c) At point D,

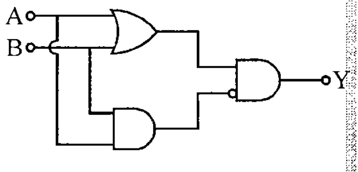


(d) At point B,



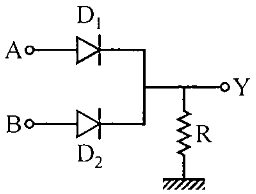
In the circuit shown below, when the inputs A = B = 0, what are the possible logic states of C and D ?

(a) C = 0, D = 1 (b) C = 1, D = 1 (c) C = 1, D = 0 (d) C = 0, D = 0



(a) The output $Y = A\overline{B} + B$ (b) The output $Y = (A + B)\overline{AB}$

(c) Representing XOR gate (d) Representing AND gate



(a) This circuit will act as an OR gate (b) when A = 0, B = 1, output Y = 1

(c) when B = 1, A = 1, output Y = 1 (d) This is an AND gate

An EX-OR gate is used

(a) as an equality detector (b) in arithmetic senction of computer

(c) as universal gate (d) as an inequality detector

A NAND can be buid up using

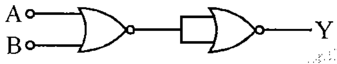
(a) five NOR gate (b) three NOR gate

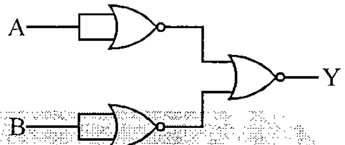
(c) two NOT gate and a OR gate (d) One XOR and one NOT gate

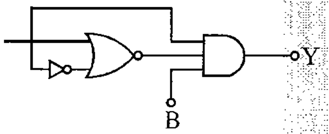
8. Out of the following statements find out the correct one, which represents De Morgan's theorem.
(a) The product of variables is equal to the complements of the sum of the variables
(b) The complement of a sum of variables is equal to the product of the complements of the individual variables
(c) The complement of a product of variables is equal to the sum of the complements of the individual variables
(d) The sum of the complements of individual variable is equal to the product of the complement of the individual variables.

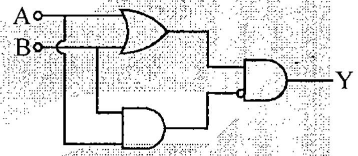
9. Simplified form of the expression, $\mathrm{ABC} + \overline{\mathrm{A}}\mathrm{BC} + \mathrm{A}\overline{\mathrm{B}}\mathrm{C} + \overline{\mathrm{A}}\mathrm{B}\overline{\mathrm{C}} + \overline{\mathrm{A}}\overline{\mathrm{B}}\mathrm{C} + \overline{\mathrm{A}}\overline{\mathrm{B}}\overline{\mathrm{C}}$
(a) $\overline{\mathrm{C}} + \mathrm{AC}$ (b) $\overline{\mathrm{A}} + \mathrm{C}$ (c) $1 + \overline{\mathrm{B}}$ (d) $\mathrm{C} + \overline{\mathrm{A}}\overline{\mathrm{C}}$

10. Which of the following is representing a AND gate ?

(a) 

(b) 

(c) 

(d) 

11. Duality theorem converts / simplify
(a) Minimum term to maximum term (b) SOP to POS
(c) Positive logic to negative logic (d) Remove redundancy

Numerical Answer Type (NAT) :

1. The AND function can be realized by using only n number of NOR gates. The n equal to ________________

2. Minimum number of two input NAND gate to design function $f = x\overline{y}z$ ______________

3. $(21)_3 = \sqrt{(100)_x}$, the value of x = ____________

Multiple Choice Questions (MCQ)

1. (d) 2. (b) 3. (a) 4. (c) 5. (d)

6. (b) 7. (d) 8. (c) 9. (c) 10. (b)

11. (a) 12. (b) 13. (a) 14. (a) 15. (a)

16. (b) 17. (c) 18. (a) 19. (a) 20. (b)

21. (d) 22. (b) 23. (b) 24. (d) 25. (b)

26. (d) 27. (d) 28. (b) 29. (b) 30. (a)

31. (d)

Multiple Select Questions (MSQ)

1. (a), (c) 2. (a), (b) 3. (a), (c) 4. (b), (c)
5. (a), (b), (c) 6. (b), (d) 7. (a), (c) 8. (b), (c)
9. (b), (d) 10. (b), (c) 11. (a), (b), (c)

Numerical Answer Type (NAT)

1. (3) 2. (5) 3. (7)